CHOOSING AMONG MAXIMALS

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A B S T R A C T

In a choice situation, it is usually assumed that the agents select the maximal elements in accordance with their preference relation. Nevertheless, there are situations in which a selection inside this maximal set is needed. In such a situation we can select randomly some of these maximal elements, or we can choose among them according to the behavior of these maximal elements. In order to illustrate this, let's imagine a preference relation $\succeq$, defined on a finite set $A = \{x_1, x_2, ..., x_n\}$, such that x_1 is indifferent to each alternative and x_2 is strictly preferred to every $x_i, i \geq 3$. Both x_1 and x_2 are maximal elements, but we can say that x_2 is a "better maximal" than x_1. In this paper we define selections of the set of maximal elements of a preference relation by choosing the "better" ones among them.

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1 Introduction

In choice theory, when we analyze the literature dealing with preferences, it is usually assumed that these preferences can be translated into choices. Indeed, a fundamental assumption consists of asking the agent to choose the maximal elements of his preference relation in every feasible set. But, sometimes, a selection within the maximal set is needed.

Examples in which the choice must select some of the maximal elements are clear. For instance (see Fine (1995)), "there may be complete indifference between the styles or colors of a number of items of clothing, but only one item will be chosen for wear". Or, for instance, there may be complete indifference between several menus in a restaurant, but only one of them will be selected for dinner. A different example is given by political elections: several candidates of the same party (which can be indifferent for an elector) run for a seat in the Spanish Senate, but only three of them can be chosen by each elector.

When the binary relation used by the agent to select is an order, the problem is solved by choosing the single maximal element (if k alternatives must be selected, the way to do it is to choose the k-first elements). So, orders represent, in this sense, the "ideal" preferences for an agent. If the preference relation is a preorder, the relationship between two maximal elements is clear: these elements are indifferent, and if one of them is preferred to some element, so is the other. So these elements are, in some sense, "identical" and, if some of them have to be chosen it must be done in a random way. But this "identicality" among the maximal elements does not hold for more general binary relations (semiorders, interval-orders, quasiorders or acyclic relations). Consider the following example (see Luce (1956)): an individual prefers a cup of coffee with 20 grams of sugar but, because of the lack of sensitivity, he is not able to distinguish between two cups with a difference of 2 grams of sugar. This preference relation is formally expressed as follows (x being the amount of sugar in a cup of coffee):

$$x \succ y \Longleftrightarrow |x - 20| < |y - 20| - 2$$

therefore the maximal elements are given by the interval [18,22]. Nevertheless, his "true" preferences have only one maximal element $x^* = 20$. Thus, the lack of perception gives rise to a difference between true preference and actual choice (see Fine (1995)).

Considering Luce's example, if there are only six possibilities to choose from (three cups of coffee with 20 grams of sugar, and three more with 15 grams), the maximal set is given by the first three, and moreover, every maximal element is "completely identical" to each other (note that, in this case, the binary relation is a preorder) and it doesn't matter which one is selected (in fact, the agent can move from one maximal to another indifferent alternative without losing utility). But if the six cups of coffee contain 20, 19,...,15 grams of sugar, (in this case, the binary relation is not a preorder), respectively, the maximal set is achieved again by the first three, but now the maximal elements are not "identical": they are physically different, and, in terms of preferences, the agent could lose if he moves from one alternative to another which is indifferent to it because, after several moves, he can end up with a non-maximal element.

The notion of "identical" alternatives (in terms of preferences), or alternatives with "identical behavior", has been well described in the definition of equivalent elements given in Fishburn (1970). If two alternatives are equivalents, it is not possible to discriminate among them, and the only way of choosing is to select randomly among the maximals. If the alternatives in the maximal set are not "identical", we can look at the behavior of these elements. For instance, coming back to Luce's example, let's analyze what happens with the maximal elements [18,22]: all of them are indifferent, but a difference with the case of preorders appears: 20 grams of sugar are preferred to 17, but 18 grams or 17 are indifferent; in other words, the behavior of the maximal elements with respect to the other elements is not identical. This fact illustrates a way of selecting, from the set of maximal elements, those which are "better", in the sense that they have a "maximal behavior" with respect to the non maximal elements.

Now, bearing in mind the above-mentioned considerations, we propose different selections of the set of maximal elements of a binary relation and analyze conditions which ensure the non-emptiness of these selections. The first selection we present is taken from Luce (1956), and it is especially indicated when the binary relation is a semiorder. After showing that this way of selecting is not adequate for more general cases of binary relations, we propose, in Section 3, the set of undominated maximals (especially indicated in the case of interval-orders), and the set of strong maximals for more general binary relations (Section 4). The analysis of the equivalence among the selected maximal elements (in Fishburn's terms, and by using a generalization of this notion) shows that, if needed, a selection with less elements than those given by undominated and strong maximals, it must be done randomly among them. We close the paper with some final comments.

2 Preliminaries

Throughout the paper $A = \{x_1, x_2, ..., x_n\}$ represents the (finite) set of alternatives and $\succeq$ a reflexive binary relation defined on A. From $\succeq$ the two following relations (the symmetric and asymmetric part, respectively) are defined as follows

indifference:	$x \sim y \Leftrightarrow x \succeq y \text{ and } y \succeq x$
strict preference:	$x \succ y \Leftrightarrow x \succeq y \text{ and } not(y \succeq x)$

The reflexive binary relation $\succeq$ is said to be a *preorder* if it is complete and whenever $x \succeq y \succeq z$ then $x \succeq z$. If in addition $x \sim y$ implies $x = y$, the binary relation is said to be an *order*. It is said to be an *interval-order* if it is complete and whenever $x \succ y \succeq z \succ t$ then $x \succ t$. If in addition whenever $x \succ y \succ z$, for any t then $x \succ t$ or $z \prec t$, the binary relation is said to be a *semiorder*.

An asymmetric binary relation $\succ$ is said to be a *quasiorder* if whenever $x \succ y \succ z$ then $x \succ z$. Finally, it is said to be *acyclic* if whenever $x_1 \succ x_2 \succ ... \succ x_k$ then $not(x_k \succ x_1)$. The *transitive closure* of a binary relation $\succeq$ is denoted by $\succ\succ$ and is defined as usual:

$$x \succ\succ y \Leftrightarrow \exists\ x_1, x_2, ..., x_{k-1}, x_k \in A \text{ such that}$$
$$x = x_1 \succ x_2 \succ ... \succ x_{k-1} \succ x_k = y$$

When the initial relation $\succeq$ is acyclic, its transitive closure turns out to be a quasiorder.

The set of maximal elements of a binary relation $\succeq$ defined on A will be denoted by

$$M(A, \succeq) = \{x^* \in A \mid \nexists\ y \in A \text{ with } y \succ x^*\}$$

Fishburn (1970) introduces an equivalence relation $\approx$ defined from $\succeq$ as follows:

$$x \approx y \text{ if and only if for all } z \in A, \begin{cases} x \succ z & if\ and\ only\ if \quad y \succ z \\ x \prec z & if\ and\ only\ if \quad y \prec z \end{cases}$$

This relation expresses the fact that two elements have identical behavior with respect to the other elements in the set of alternatives. In general, the maximal elements are not equivalent, as Luce's example shows. The next result states conditions under which the maximal elements are equivalent (the elemental proof is omitted).

Proposition 1 *Let $\succeq$ be a complete and reflexive binary relation defined on the (finite) set A. Then, if $\succeq$ is a preorder the maximal elements are equivalent.*

In Luce (1956) a way of selecting among the maximal elements of a binary relation $\succeq$ is presented by defining a new binary relation from the original one.

Definition 1 *(Luce, 1956). Given the binary relation $\succeq$ defined on A, the asymmetric binary relation $\succ^*$is defined as follows: $x \succ^* y$ if and only if one of the following situations is fulfilled*
i) $x \succ y$
ii) $x \sim y$ and there is some $z \in A$ such that $x \sim z, z \succ y$
iii) $x \sim y$ and there is some $z \in A$ such that $x \succ z, z \sim y$

Theorem 1 *(Luce, 1956) If the binary relation $\succeq$ defined on A is a semiorder, then $\succeq^*$ ($x \succeq^* y \Leftrightarrow not(y \succ^* x)$ is a preorder. Moreover, $\emptyset \neq M(A, \succeq^*) \subseteq M(A, \succeq)$.*

Definition 2 *Let $\succeq$ be a complete and reflexive binary relation defined on A. We define Luce's maximals as the following set*
$$LM(A, \succeq) = M(A, \succeq^*).$$

Thus, relation $\succeq^*$ gives us a way of selecting among the maximal elements and, in Luce's sugar example it provides the "true" maximal:
$$LM(A, \succeq) = \{20\}.$$

3 Undominated Maximals

The problem with Luce's selection is that Theorem 1, and therefore the way of selecting among maximals, does not remain valid when relation $\succeq$ is not a semiorder. The following example shows this fact.

Example 1 *Let $A = \{a, b, c, d, e\}$ and the acyclic binary relation $\succeq$ defined by:*

$$a \succ b, b \succ c, d \succ a$$

(being indifferences the non mentioned relationships). Then

$$M(A, \succeq) = \{d, e\}.$$

On the other hand, relation $\succeq^*$ is not a preorder (in fact it is not acyclic) and, in this example, $M(A, \succeq^*) = \emptyset$.

In order to define a nonempty selection of the set of maximal elements in general cases, we introduce the following auxiliary relation.

Definition 3 *Let $\succeq$ be a binary relation defined on A, and let $x, y \in A$. It is said that alternative x **weak-dominates** y, $x\ \mathbf{d}\ y$, if for every $z \in A$,*

$$y \succ z \Rightarrow x \succ z$$
$$y \sim z \Rightarrow x \succeq z$$

We denote by D (dominance relation) the asymmetric part of $\mathbf{d}$,

$$x\ D\ y \Leftrightarrow x\ \mathbf{d}\ y \text{ and } not(y\ \mathbf{d}\ x)$$

In other words, an alternative x weak-dominates some other alternative y if it has "better behavior" in a pairwise comparison with the other elements in the feasible set. This relation is somewhat similar to the "covering relation" defined by Miller (1977) and Fishburn (1977) for tournaments (complete asymmetric binary relations), and extended for general binary relations in Schwartz (1986).

Definition 4 *Let $\succeq$ be a complete and reflexive binary relation defined on A. It is said that an element x^* is an **undominated maximal** if*

a) $x^* \in M(A, \succeq)$

b) $x^* \in M(A, \mathbf{d})$

We will denote by $UM(A, \succeq)$ the set of undominated maximal elements. From this definition, it is obvious that every undominated maximal is a maximal element, $UM(A, \succeq) \subseteq M(A, \succeq)$. In Luce's example of sugar in a cup of coffee, there is only one undominated maximal, $x^* = 20$. In example 1, it is clear that $d\ D\ e$, therefore $UM(A, \succeq) = \{d\}$.

The existence of maximal elements on each subset of A is ensured by the acyclicity of the binary relation. The next result proves that the same condition ensures the existence of undominated maximals.

Theorem 2 *Let $\succeq$ be a complete and reflexive binary relation defined on the (finite) set A. If the binary relation is acyclic then $UM(A, \succeq) \neq \emptyset$.*

Proof. If the relation is acyclic, the set of maximal elements $M(A, \succeq)$ is not empty. On the other hand, the strict dominance relation D is acyclic because d is a transitive (not necessarily complete) binary relation; so there is a maximal element of the relation D on the set $M(A, \succeq)$. Call x^* such an element and suppose that there is some $x \in A$ such that $x\ D\ x^*$. As a result of the choosing of x^*, the element x is not maximal with respect to the relation $\succeq$, so there is $y \in A$ such that $y \succ x$; but $x^* \succeq y$ which contradicts $x\ D\ x^*$. So x^* is an undominated maximal element. ■

The following result ensures that this way of selecting among maximal elements is adequate for interval-orders because it always selects equivalent elements.

Proposition 2 *Let $\succeq$ be a complete, reflexive and acyclic binary relation defined on the (finite) set A. Then, if $\succeq$ is an interval-order the elements in $UM(A, \succeq)$ are equivalent.*

Proof. Let $x^*, y^* \in UM(A, \succeq)$. If $no(x^* \approx y^*)$ then there is $z \in A$ such that $x^* \succ z$ and $y^* \sim z$. As x^* does not dominate y^*, there is $z' \in A$ such that $y^* \succ z'$ and $x^* \sim z'$. Then, we have $x^* \succ z \sim y^* \succ z'$ and $x^* \sim z'$ contradicting the fact that the relation is an interval-order. ■

Undominated maximals solve the problems arising with Luce's maximals since we always can ensure that $UM(A, \succeq) \neq \emptyset$. Nevertheless, whenever the binary relation $\succeq$ is more general than a semiorder or an interval-order we can not ensure that we are choosing equivalent elements. The following example shows this case.

Example 2 *Let* $A = \{a, b, c, d, e\}$ *and the binary relation (quasiorder)* $\succeq$ *defined by:*

$$a \succ e, b \succ c, b \succ d, c \succ d$$

(being indifferences the non mentioned relationships). Then,

$$M(A, \succeq) = UM(A, \succeq) = \{a, b\},$$

although a *and* b *are not equivalent elements, and alternative* b *seems "better" than* a*, in the sense that* b *is preferred to more alternatives than* a *is.*

The following section introduces a more discriminating selection of the set of maximal elements.

4 Strong Maximals

Orders can be considered as the "ideal" preferences for an agent, because in this case he would be able to completely discriminate between any pair of alternatives and choose just one of them. But the lack of sensitivity or the impossibility of keeping every option in mind, leads to weaker classes of preferences. If we deal with quasiorders or acyclic relations, a possible interpretation of such a preference is to consider that the agent reveals strict preferences, but he does it only on some pairs of alternatives (in the other cases he is not able to decide, or he has lost this information) so that the elements for which no relation is known are considered indifferent in order to obtain a complete binary relation. Which should be the ideal preference (order) of the agent?. In general, it is possible to find different orders which could be considered as the origin of a binary relation. The notion of compatible order formalizes with this idea.

Definition 5 *Given a binary relation $\succeq$ defined on A, an order $\succ_i$ is said to be* ***compatible*** *with $\succeq$ if*
$x \succ y \Rightarrow x \succ_i y$

Let us denote by $\Theta(\succeq)$ the set of orders which are compatible with the binary relation $\succeq$. If a binary relation $\succeq$ is acyclic, compatible orders with it always exists, and it is easy to prove that:

$$M(A, \succeq) = \bigcup_i M(A, \succ_i) \ ; \ \succ_i \in \Theta(\succeq)$$

The notion of compatible orders is well known in the literature (see, for instance, Baneerje and Pattanaik (1996) where it is used to describe the maximal set of a quasiorder). An alternative way of interpreting a quasiorder or an acyclic relation is to consider that it is derived by aggregating several preference relations. In this sense, in Moulin (1985) it is proved that every quasiorder can be written as the Pareto relation of compatible orders. So, by looking at the maximal elements of the compatible orders we can get valuable information on how to select from the maximal set of the relation. It is obvious that a maximal element which remains maximal in every compatible ordering is a "good" candidate. But this is not the usual case (in fact, it only occurs when there is a unique maximal element of the binary relation in which case no selection is needed). One possible way to obtain a selection among the maximal elements is to aggregate the compatible orders by majority vote. Now, we formalize this idea.

Let $\succeq$ be a binary relation defined on A, an let $\Theta(\succeq) = \{\succ_i\}_{i \in I}$; let us denote by $N(\succeq; x, y) = \{i \in I \text{ such that } x \succ_i y\}$; then we define the binary relation P as:

$$x \ P \ y \Longleftrightarrow \#N(\succeq; x, y) > \#N(\succeq; y, x)$$

(where $\#S$ stands for the cardinal of the finite set S), and the weak relation R is given by

$$x \ R \ y \Longleftrightarrow \#N(\succeq; x, y) \geq \#N(\succeq; y, x) \qquad [1]$$

Relation P is not, in general (when every possible family of orders can be considered), an acyclic relation so the existence of maximal elements of P can not be ensured. In our case, since we are considering all the compatible orders with an acyclic relation, it will ensure that the set of maximal elements of P is nonempty. These maximal elements will be called strong maximals of the relation $\succeq$.

Definition 6 *Let $\succeq$ be a complete and reflexive binary relation defined on A. It is said that an element x^* is an* ***strong maximal*** *if $x^* \in M(A, R)$, where R is the majority relation defined by [1].*

We will denote by $SM(A, \succeq)$ the set of strong maximal elements. In Example 2, the majority relation defined by [1] yields the order

$$b\ P\ a\ P\ c\ P\ e\ P\ d$$

and then $SM(A, \succeq) = \{b\} \subsetneqq UM(A, \succeq) = M(A, \succeq) = \{a, b\}$.

Before proving our result on the existence of strong maximal elements, let us consider some properties of the majority relation P. First, it must be mentioned that the family of compatible orders with an acyclic relation $\succeq$ coincides with those which are compatible with its transitive closure $\succ\succ$. This fact enables us to work with quasiorders instead of acyclic relations, and therefore any result on the existence of strong maximal elements we obtain in this case, is immediately true for the general case of acyclic relations. On the other hand, if we look at P as a way of associating a binary relation $P(\succeq)$ to every acyclic relation $\succeq$, it is easy to prove that P satisfies *monotonicity* and *neutrality:*

Monotonicity: if $\succeq$ and $\succeq'$coincide on $A - \{a\}$ and

$$a \succ z \Rightarrow a \succ' z; \qquad a \sim z \Rightarrow a \succeq' z \qquad \text{then}$$

$$a\ P(\succeq)\ z \Rightarrow a\ P(\succeq')\ z$$

Neutrality: for any permutation σ of A

$$a\ R(\succeq^\sigma)\ b \Leftrightarrow \sigma^{-1}(a)\ R(\succeq)\ \sigma^{-1}(b)$$

where $\succeq^\sigma$ is defined as usually: $a \succeq^\sigma b \Leftrightarrow \sigma^{-1}(a) \succeq \sigma^{-1}(b)$.

In the next result we prove that acyclicity also ensures the existence of strong maximals.

Proposition 3 *Let $\succeq$ be a complete and reflexive binary relation defined on the (finite) set A. Then, if the binary relation is acyclic the set of strong maximals is nonempty.*

Proof. Without loss of generality we will assume that the relation $\succeq$ is a quasiorder (in other case, we will consider its transitive closure). Then, the function $u(x) = \#L(x)$, where

$$L(x) = \{z \in A \text{ such that } x \succ z\}$$

is a weak-utility function (i.e., $x \succ y \Rightarrow u(x) > u(y)$) and every alternative which maximizes this function is a maximal element. Let x^* be an alternative maximizing this function. We are going to prove that $x^* \in SM(A, \succeq)$. If not, there is some $z \in A$ such that $z\ P\ x^*$. As x^* is a maximal element, then $x^* \succeq z$; if the strict preference is fulfilled, then every compatible order puts x^* before z, so $x^*\ P\ z$, a contradiction; if $x^* \sim z$, as x^* maximizes function $u(x)$, we have two possibilities:

1) $u(x^*) = u(z)$; let $L(x^*) = \{x_1, ..., x_k\}, L(z) = \{z_1, ..., z_k\}$ and consider a permutation σ which translates $L(x^*)$ on $L(z)$, leaving invariant the other alternatives. As the number of compatible orders with $\succeq$ are the same as the ones compatible with $\succeq^\sigma$ (only changing the position of x^*, z, x_i, z_i), the majority relation will give $x^*\ P^\sigma\ z$, but this contradicts neutrality, since $\sigma(x^*) = x^*, \sigma(z) = z$.

2) $u(x^*) > u(z)$; let $L(x^*) = \{x_1, ..., x_k; y_1, ..., y_p\}, L(z) = \{z_1, ..., z_k\}$, where $\{y_1,..., y_p\} \cap L(z) = \emptyset$. Being the relation a quasiorder, and $x^* \sim z$, it must be $y_i \sim z$, for all i. Consider now the binary relation $\succeq'$ defined from $\succeq$ by taking $z \succ' y_i\ , i = 1, ..., p$. By applying monotonicity, $z\ P(\succeq')\ x^*$ and we can reason as in case 1. ■

Note that the proof of the previous proposition provides an easy way of obtaining strong maximal elements: by maximizing the function $u(x)$. The following proposition proves that strong maximals are a selection of the undominated maximals and, therefore, a selection of the maximal elements.

Proposition 4 *Let $\succeq$ be a complete, reflexive and acyclic binary relation defined on the (finite) set A. Then,*

$$SM(A, \succeq) \subseteq UM(A, \succeq).$$

Proof. Let $x^* \in SM(A, \succeq)$ and suppose $x^* \notin UM(A, \succeq)$. If x^* is not a maximal element, then there is some $z \in A$ such that $z \succ x^*$, but this implies $z\ P\ x^*$, a contradiction. On the other hand, if there is some $z \in A$ such that $z\ D\ x^*$, then z gets a better position with respect to any other element than x^* gets, so in the compatible orders the majority relation will give $z\ P\ x^*$, a contradiction. ■

In general, strong maximal elements are not equivalent, as the following example shows. So, the equivalence notion must be weakened if we try to select some kind of equivalent elements in the general case.

Example 3 *Let* $A = \{a, b, c, d\}$ *and the binary relation (quasiorder)* $\succeq$ *defined by:*

$$a \succ c, b \succ d$$

(being indifferences the non mentioned relationships). Then,

$$M(A, \succeq) = UM(A, \succeq) = SM(A, \succeq) = \{a, b\},$$

although a *and* b *are not equivalent elements.*

Definition 7 *Let* $\succeq$ *be a binary relation defined on* A*, and let* $x, y \in A$*. It is said that* x *and* y *are* ***weak-equivalent*** *elements if there is a permutation* σ *of* A*, such that for all* $z \in A$

$$\begin{cases} x \succ z & \text{if and only if} \quad y \succ \sigma(z) \\ x \prec z & \text{if and only if} \quad y \prec \sigma(z) \end{cases}$$

In words, two elements are weak-equivalent if both are preferred, less preferred and indifferent to the same number of alternatives. In the next result we prove that strong maximals are weak-equivalent elements.

Proposition 5 *Let* $\succeq$ *be a complete and reflexive binary relation defined on the (finite) set* A*. Then, if* $\succeq$ *is a quasiorder, the strong maximals are weak-equivalent elements.*

Proof. We are going to prove that if x^*, z^* are strong maximals, then $u(x^*) = u(z^*)$, where $u(x)$ is the weak utility function used in the proof of Proposition 3. If this is not the case, let

$$L(z^*) = \{z_1, ..., z_k\}; \qquad L(x^*) = \{x_1, ..., x_k, w\}, w \notin L(z^*)$$

Then, for every compatible order in $\Theta(\succeq)$ such that $z^* \succ_i x^*$,

$$u_1 \succ_i ... \succ_i u_p \succ_i z^* \succ_i v_1 \succ_i ... \succ_i v_s \succ_i x^* \succ_i y_1 \succ_i ... \succ_i y_r$$

it is possible to find an order $\succ_j \in \Theta(\succeq)$ which reverses the relation between x^* and z^*, by permuting the alternatives z_i, z^* by x_i, x^*. But for the two classes of compatible orders of the form:

$$x^* \succ_1 w \succ_1 z^*$$

$$x^* \succ_2 z^* \succ_2 w$$

which put x^* ahead of z^*, only one can be found which reverses the position of x^* and z^*, because $x^* \succ w$

$$z^* \succ_3 x^* \succ_3 w$$

and then $x^*\ P\ z^*$, a contradiction. Then $u(x^*) = u(z^*)$, and

$$L(z^*) = \{z_1, ..., z_k\}; \qquad L(x^*) = \{x_1, ..., x_k\}$$

By considering a permutation which transforms $L(z^*)$ on $L(x^*)$, these elements turn out to be weak-equivalents. ■

It must be mentioned that the previous proposition is also valid for acyclic binary relations, since, on the one hand, the maximal elements of an acyclic relation coincide with those of its transitive closure (a quasiorder), and, on the other hand, the compatible orders with an acyclic relation are those which are compatible with its transitive closure.

5 Conclusions

In the next result we analyze the relationship between maximal, Luce's maximal, undominated maximal and strong maximal elements by depending on the type of relation we consider.

Theorem 3 *Let $\succeq$ be a complete and reflexive binary relation defined on the (finite) set A. Then,*
a) If $\succeq$ is acyclic, $M(A, \succeq) \supseteq UM(A, \succeq) \supseteq SM(A, \succeq)$
b) If $\succeq$ is an interval-order, $M(A, \succeq) \supseteq UM(A, \succeq) = SM(A, \succeq)$
c) If $\succeq$ is a semiorder, $M(A, \succeq) \supseteq LM(A, \succeq) = UM(A, \succeq) = SM(A, \succeq)$
d) If $\succeq$ is a preorder, $M(A, \succeq) = LM(A, \succeq) = UM(A, \succeq) = SM(A, \succeq)$

Proof. Part a) is Proposition 4. In order to prove part b) we only need to show that each undominated maximal is strong maximal. To prove it, consider two undominated maximals; then, as proved in Proposition 2, these elements are equivalent, so the function $u(x)$ coincide on these elements and both maximize it, so they are strong maximals. As semiorder is a stronger condition than interval-order, in order to prove part c) we only need to ensure

$$LM(A, \succeq) = UM(A, \succeq)$$

Let $x^* \in LM(A, \succeq)$; if $x^* \notin UM(A, \succeq)$ there is some z such that $z\ D\ x^*$, that is,

$$z \succ a \succeq x^* \qquad or \qquad z \succeq a \succ x^* \qquad \text{for some } a \in A$$

In both cases, $z \succ^* x^*$, a contradiction. On the other hand, if we consider $x^* \in UM(A, \succeq)$ and suppose $x^* \notin LM(A, \succeq)$, this implies the existence of some z such that:

$$z \succ x^* \qquad or \qquad z \sim y \succ x^* \qquad or \qquad z \succ y \sim x^*$$

The first two possibilities contradict that x^* is a maximal element. In the third case, as $x^* \in UM(A, \succeq)$, there is some $a \in A$ such that $x^* \succ a \sim z$ or $x^* \sim a \succ z$; in the first case we obtain $x^* \succ y$, which is not possible, and in the second we have

$$a \succ z \succ y \qquad and \qquad x^* \sim z, x^* \sim a, x^* \sim y$$

which contradicts that $\succeq$ is a semiorder.

Finally, part d) is obvious from the fact that a preorder is a semiorder such that relation $\succeq^*$coincides with $\succeq$. ■

Throughout this paper, several considerations about the maximal elements of a binary relation have been made and by means of some examples it has been shown that a selection of the maximal elements is needed in order to choose those, among the maximals, which have "better properties". If the binary relation is a complete preorder, every maximal element has the same behavior with respect to the other alternatives. So, in this case, if we need to select some elements of the maximal set, only a random process can be applied. If the binary relation is a semiorder, the way proposed in Luce (1956) for choosing among the maximals is adequate to select equivalent elements. With a similar purpose we define the undominated maximals, as a way of selecting equivalent elements among the maximal ones and to eliminate those maximals which have some other alternatives dominating them. Undominated maximals are specially adequate when the binary relation is an interval-order. When the binary relation is more general, a more discriminating selection has been introduced: strong maximals, and an easy way to compute them is given (by maximizing a real valued function). For quasiorders and acyclic relations, the existence of strong maximals has been provided, and a weak-equivalence among these elements has been shown.

On the other hand, as we have shown in Theorem 3, if the binary relation is an interval-order, then undominated and strong maximals coincide. So the two possible interpretations (those maximals which are not dominated by any other alternative; and those maximals which are the winners in a majority game among any compatible order) can be used to define the undominated maximals of an interval-order.

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