EQUAL SPLIT GUARANTEE SOLUTION IN ECONOMIES WITH INDIVISIBLE GOODS CONSISTENCY AND POPULATION MONOTONICITY*

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A B S T R A C T

We consider the problem of allocating a finite set of indivisible goods and a single infinitely divisible good among a group of agents, and we study a solution, called the *Equal Split Guarantee* solution, in the presence of *consistency* and *population monotonicity* properties. This solution is not *consistent*. We prove that its *maximal consistent subsolution* is the *No-envy* solution. Our main result is that the *minimal consistent extension* of the intersection of the *Equal Split Guarantee* solution with the *Pareto* solution is the *Pareto* solution. This result remains true in the restricted domain when all the indivisible goods are identical, but not when there is a unique indivisible good. Finally, we show that in the class of economies with a unique indivisible good, there is a selection from the *Equal Split Guarantee* solution that satisfies *population monotonicity.*

KEYWORDS: Indivisible Goods, Equal Split, Consistency.

1.- INTRODUCTION

We consider the problem of "fair allocation" in economies with a finite number of indivisible goods and a single infinitely divisible good, which have to be allocated among a group of agents in such a way that each agent receives at most one indivisible good.

The purpose of this paper is to study a solution, called the *Equal Split Guarantee* solution, in the presence of a given property of *consistency.*

A solution is *consistent*, if the recommendation it makes for any economy is never contradicted by the recommendation it makes for the "reduced" economies obtained, in turn, by imagining the departure of some of the agents with their allotted bundles.

The *Equal Split Guarantee* solution was proposed by Moulin (1990) as an extension, for economies with indivisible goods, of a solution which plays an important role in the literature of fair allocation in classical economies. This is the Individually Rational solution from equal division. The *Equal Split Guarantee* solution can be described as follows: Imagine an economy where all agents have identical preferences. In such an economy, most authors would probably recommend the feasible allocations in which each agent is indifferent about which he receives and which the others receive. Following this, given an economy in which agents have possible different preferences, the *Equal Split*

Guarantee solution selects the feasible allocations in which each agent is at least as well-off as he would be, according to the above recommendation, in a hypothetical economy where all the other agents have his preferences.

Another example of a solution is the *No-envy* solution (Foley, 1967), which selects the allocations in which no agent prefers any other agent's bundle to his own. This solution has been the object of several recent studies, including Svensson (1983,1988), Maskin (1987), Alkan, Demange and Gale (1988), Tadenuma and Thomson (1990, 1991a, 1991b), and Aragones (1992).

The *No-envy* solution is *consistent*, but the *Equal Split Guarantee* solution is not. The main purpose of this paper is to evaluate how far from being *consistent* this solution is. Thomson (1992) proposes two different ways of evaluating how serious violations of *consistency* may be.

One proposal is to reduce the solution in order to recover *consistency*. Of course, it is desirable to reduce it as little as possible. This can be done properly, providing that the solution contains at least one *consistent* subsolution, by taking the union of all of its *consistent* subsolutions. This union is called the *maximal consistent subsolution* of the solution.

We prove that the only *consistent* subsolution of the *Equal Split Guarantee* solution is the *No-envy* solution. Therefore, its *maximal consistent subsolution* is the *No-envy* solution.

Thomson's other proposal is then, to enlarge the solution. Here, it is desirable to enlarge it as little as possible. This can be done coherently by taking the intersection of all those *consistent* solutions which contain the solution. This intersection is called the *minimal consistent extension* of the solution.

Our main result is that the *minimal consistent extension* of the *Equal Split Guarantee and Pareto* solution is the *Pareto* solution. Thomson (1992) proves that, in economies with infinitely divisible goods, the *minimal consistent extension* of the Individually Rational from equal division and Pareto solution is the *Pareto* solution. Our result shows that this is also true in economies with indivisible goods.

We also study how robust our results are under variations in the initial model.

Firstly, we consider the class of economies where all the indivisible objects are identical. In this domain all of our results remain true.

Secondly, we consider the class of economies where there is a unique indivisible good. In this domain some, but not all, of our results hold. It remains true that the *minimal consistent extension* of the *Equal Split Guarantee and Pareto* solution is the *Pareto* solution. But, if we impose a certain property of monotonicity of preferences with respect to the indivisible good, this result is no longer true. Then, the *minimal consistent extension* of the *Equal Split Guarantee* and *Pareto* solution is a proper subsolution of the *Pareto* solution which we identify.

Finally, we study whether the *Equal Split Guarantee* solution is compatible with *population monotonicity*. This property requires that if the number of agents increase, with the resources kept fixed, none of the initial agents should benefit from the addition of the new agents.

Moulin (1990) proposed a selection of the *Equal Split Guarantee* solution that satisfies *population monotonocity* in economies with a single indivisible good, no money, and quasi linear preferences. We present a generalization of this selection.

2.- THE MODEL AND SOLUTIONS

We consider economies with indivisible goods, or "objects", such as jobs, houses ..., and a single infinitely divisible good, or "money", both of which have to be allocated among a group of agents in such a way that each agent receives, at the most, one object.

Let $\mathcal{Q}$ be an infinite set of agents, with members denoted by i,j,..., and $\mathcal{A}$ an infinite set of indivisible goods, with members denoted by $\alpha,\beta,\ldots$.

An **economy** is a list $e = (Q,A,M;R_Q)$, where Q is a finite set of agents drawn from $\mathcal{Q}$, A is a finite set of objects drawn from $\mathcal{A}$, $M \in \mathbb{R}$ is an amount of money, and $R_Q = (R_i)_{i\in Q}$ is a list of preference relations defined over $A\times\mathbb{R}$. Let P_i denote the strict preference relation associated with R_i, and I_i the indifference relation. Each preference relation is continuous and increasing in money. Thus, if $m > m'$ then $(\alpha,m)P_i(\alpha,m')$ for all $i \in Q$, for all $\alpha \in A$. Also it is assumed that no object is infinitely desirable or undesirable when compared with an other object. Thus, for any bundle (α,m), any $i \in Q$, and any $\beta \in Q$, there is an amount of money m' such that $(\alpha,m)I_i(\beta,m')$. For simplicity, we assume that $|Q| = |A|$. Let $\mathcal{E}$ be the class of all such economies.

A feasible allocation for $e = (Q,A,M;R_Q) \in \mathcal{E}$ is a pair $z = (\sigma,m)$, where $\sigma: Q \longrightarrow A$ is a bijection that assigns agents to objects, and $m = (m_{\sigma(i)})_{i\in Q} \in \mathbb{R}^{|Q|}$ is such that $\sum_{i\in Q} m_{\sigma(i)} = M$. For each $i \in Q$, let

$z_i = (\sigma(i), m_{\sigma(i)})$ be the consumption of agent i. Note that $m_{\sigma(i)}$ can be either positive, negative or zero. Let **Z(e)** be the set of feasible allocations for e.

A solution on $\mathcal{E}$ associates with every $e \in \mathcal{E}$ a non-empty subset of Z(e). Given two solutions, φ and φ', $\varphi\varphi'$ denotes their intersection.

The definition below is a typical example of a solution:

The Pareto solution, P: given $e = (Q,A,M;R_Q) \in \mathcal{E}$,

$$P(e) = \{ z \in Z(e) / \nexists\ z' \in Z(e) \text{ with } [\forall\ i \in Q,\ z'_i R_i z_i \text{ and } \exists\ j \in Q \text{ s.t. } z'_j P_j z_j] \}.$$

One of the solutions most studied in the literature on fair allocation is the *No-envy* solution. This solution picks the allocations at which no agent prefers any other agent's bundle to his own.

The No-envy solution, N, (Foley ,1967): given $e = (Q,A,M;R_Q) \in \mathcal{E}$,

$$N(e) = \{ z \in Z(e) \ / \ \forall\ i,j \in Q,\ z_i R_i z_j \}.$$

In economies with infinitely divisible goods, other concepts of fairness have been proposed. An important one is the Individually Rational solution from equal division, which picks the allocations that all agents prefer to equal division. This solution, which of course is meaningful only when equal division is well-defined, can not be applied to our model. Moulin (1990) proposes the *Equal Split Guarantee* solution as an extension of this solution for economies with indivisible goods.

For a formal definition, we need some additional notation.

Given $e = (Q,A,M;R_Q) \in \mathcal{E}$, for each $i \in Q$ let $e^i = (Q,A,M;R^i_Q) \in \mathcal{E}$ be such that $R^i_j = R_i$ for all $j \in Q$.

The economy e^i is obtained from e imagining that all the agents have the same preferences as agent i. For such an economy we define,

$$E(e^i) = \{ z^i \in Z(e^i) \ / \ \forall\ j \in Q,\ z^i_i I_i z^i_j \}.$$

Since $R^i_j = R_i$, it is easy to check that $E(e^i)$ is essentially single-valued, that is, for all z^i, $\bar{z}^i \in E(e^i)$, z^i, $\bar{z}^i$ are Pareto indifferent. Moreover, under our assumptions on preferences, if $z^i = (\sigma,m)$, and $\bar{z}^i = (\tau,\bar{m})$, there exists a permutation π of Q such that $\sigma(i) = \tau(\pi(i))$, and $m_{\sigma(i)} = \bar{m}_{\tau(\pi(i))}$.

Therefore, a generic element in $E(e^i)$ can be represented as $z^i = ((\alpha,m^i_\alpha(e)))_{\alpha\in A} \in \mathbb{R}^{2|A|}$ such that,

(i) $\sum_{\alpha\in A} m^i_\alpha(e) = M$

(ii) $(\alpha,m^i_\alpha(e))I_i(\beta,m^i_\beta(e))$ for each $\alpha,\beta \in A$, $\alpha \neq \beta$.

The Equal Split Guarantee solution, E, (Moulin, 1990): given $e = (Q,A,M;R_Q) \in \mathcal{E}$,

$$E(e) = \{ z \in Z(e) \ / \ \forall\ i \in Q,\ \forall\ z^i \in E(e^i),\ \ z_i R_i z^i_i \}$$

Moulin (1990) proves that in quasi-linear economies[1] with only one indivisible good, the *Equal Split Guarantee* solution contains the *No-envy* solution. Furthermore, in economies with two agents, the solutions coincide. These results extend to the general domain, as stated in the following propositions:

Proposition 1. For all $e = (Q,A,M;R_Q) \in \mathcal{E}$, $N(e) \subseteq E(e)$.

Proof. Let $e = (Q,A,M;R_Q) \in \mathcal{E}$, and $z = (\sigma,m) \in N(e)$ be given. Suppose that there exists $i \in Q$ such that for all $\alpha \in A$, $(\alpha,m^i_\alpha(e))P_iz_i$. In particular, $(\sigma(i),m^i_{\sigma(i)}(e))P_iz_i$. By monotonicity of preferences in money, $m^i_{\sigma(i)}(e) > m_{\sigma(i)}$. By feasibility of z, there exists $j \in Q$ such that $m^i_{\sigma(j)}(e) < m_{\sigma(j)}$. Then, $z_j=(\sigma(j),m_{\sigma(j)})P_i(\sigma(j),m^i_{\sigma(j)}(e))I_i(\sigma(i),m^i_{\sigma(i)}(e))P_iz_i$, in contradiction with the fact that $z \in N(e)$.■

Proposition 2. For all $e = (Q,A,M;R_Q) \in \mathcal{E}$ such that $|Q| = 2$, $N(e) = E(e)$.

Proof. By Proposition 1, $N \subseteq E$. It is then sufficient to prove that $E \subseteq N$. Let $e = (Q,A,M;R_Q) \in \mathcal{E}$, let $|Q| = 2$, and let $z \in E(e)$ be given. For simplicity, suppose that $z = \{(\alpha,m_\alpha),(\beta,m_\beta)\}$. Suppose that agent 1 envies agent 2 at z. Then, since $z \in E(e)$, $(\beta,m_\beta)P_1(\alpha,m_\alpha)R_1((\beta,m^1_\beta(e))$. By monotonicity of preferences in money, $m_\beta > m^1_\beta(e)$. By feasibility of z, $m_\alpha < m^1_\alpha(e)$. Then, $(\alpha,m^1_\alpha(e))P_1z_1$, in contradiction with the fact that $z \in E(e)$.■

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(1) An economy $e = (Q,A,M;R_Q)$ is quasi-linear if each $i \in Q$ has preferences such that: for all $\alpha,\beta \in A$, for all m, m', $t \in \mathbb{R}$, $(\alpha,m)I_i(\beta,m')$ implies $(\alpha,m+t)I_i(\beta,m'+t)$.

As we know from Svenson (1983), the *No-envy* solution is a subsolution of the *Pareto* solution. By Proposition 2, in economies with two agents, so is the *Equal Split Guarantee* solution . However, in economies with more than two agents, this is no longer true as the following example shows.

Example 1. Let $e = (Q,A,M;R_Q) \in \mathcal{E}$ be such that $Q = \{1,2,3\}$, $A = \{\alpha,\beta,\gamma\}$, $M = 0$ and for all $m \in \mathbb{R}$,

$$(\alpha,m)I_1(\beta,m)I_1(\gamma,m+3),\ (\alpha,m)I_2(\beta,m)I_2(\gamma,m),\ (\alpha,m+2)I_3(\beta,m+1)I_3(\gamma,m).$$

For this economy,

$$(\alpha,-1)I_1(\beta,-1)I_1(\gamma,2),\ (\alpha,0)I_2(\beta,0)I_2(\gamma,0),\ (\alpha,1)I_3(\beta,0)I_3(\gamma,-1).$$

Let $z = \{(\alpha,0),(\gamma,0),(\beta,0)\}$, and $z' = \{(\alpha,0),(\beta,0),(\gamma,0)\}$. Clearly, z, $z' \in E(e)$, and z' Pareto dominates z.

By Proposition 1, the *No-envy* solution is a subsolution of the *Equal Split Guarantee* solution, and, since the *No-envy* solution is also a subsolution of the *Pareto* solution, we conclude that the *No-envy* solution is a subsolution of the *Equal Split Guarantee* and *Pareto* solution. Thus, the intersection of the *Equal Split Guarantee* solution with the *Pareto* solution is not empty.

3.- CONSISTENCY

In this section we study the *Equal Split Guarantee* solution in the presence of the *consistency* property.

A solution φ is said to be *consistent* if for any economy e and for any φ-optimal allocation for e, the restrictions of this allocation for all subgroups of agents are also φ-optimal allocations for the problem of allocating the resources received by these subgroups. Formally,

Consistency. The solution φ is **consistent** if for all $e = (Q,A,M;R_Q) \in \mathcal{E}$, for all $Q' \subset Q$, for all $z = (\sigma,m) \in \varphi(e)$, $z_{Q'} = (\sigma(i),m_{\sigma(i)})_{i\in Q'} \in \varphi(t^z_{Q'}(e))$, where, $t^z_{Q'}(e) = (Q',\bigcup_{Q'}\sigma(i),\sum_{Q'}m_{\sigma(i)};R_{Q'})$.

We call $t^z_{Q'}(e)$ the **reduced economy of e with respect to z and Q'**.

The *consistency* principle has been the object of several recent studies in different fields such as fair representation (Balinski and Young (1982)), cost allocation (Moulin (1985)), taxation (Aumann and Maschler (1985) and Young (1987)), games in coalition form (Peleg (1985), Hart and Mas-Colell (1989)), bargaining (Lensberg (1987)), two-sided matching (Sasaki and Toda (1992)), fair allocation in classical economies (Thomson (1988), Thomson (1992)), and fair allocation in economies with indivisible goods (Tadenuma and Thomson (1990,1991b)).

The following definition is the weakening of *consistency* obtained by applying it only to subgroups of cardinality two. Formally,

Bilateral consistency. The solution φ is **bilaterally consistent** if for all $e = (Q,A,M;R_Q) \in \mathcal{E}$, for all $Q' \subset Q$ with $|Q| = 2$, for all $z \in \varphi(e)$, $z_{Q'} \in \varphi(t^z_{Q'}(e))$.

This variant was proposed by Harsanyi (1959) in bargaining theory, and it was studied there by Lensberg (1987). It was also considered in other fields such as taxation (Young (1987)), games in coalition form (Peleg (1985)), fair allocation in classical economies (Thomson (1988)), and fair allocation in economies with indivisible goods (Tadenuma and Thomson (1990, 1991b)).

We will now consider a condition that is dual to *consistency*. According to this condition, if for some economy, a feasible allocation is such that its restrictions to all subgroups of cardinality 2 constitute recommendations for the problem of allocating the resources received by these subgroups, then it is in itself a recommendation for the whole economy[2]. Formally,

Converse consistency. The solution φ is **conversely consistent** if for all $e = (Q,A,M;R_Q) \in \mathcal{E}$ with $|Q| > 2$, for all $z \in Z(e)$, if, for all $Q' \subset Q$ with $|Q'| = 2$, $z_{Q'} \in \varphi(t^z_{Q'}(e))$, then $z \in \varphi(e)$.

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(2) This property is interesting in its own right. Although it is not central to this paper, it is necessary for the proof of some of our results.

This property has been studied in games in coalition form (Peleg (1985)), fair allocation in classical economies (Thomson (1988)), fair allocation in economies with indivisible goods (Tadenuma and Thomson (1990,1991b)), and two-sided matching (Sasaki and Toda (1992)).

Examples of *consistent* solutions are the *Feasible* solution, which associates with each economy its whole feasible set, the *Pareto* solution, and the *No-envy* solution. Tadenuma and Thomson (1991b) note that the *No-envy* solution is also *conversely consistent*, but prove that the *Pareto* solution is not. They also prove that, essentially, there are no proper *consistent* selections from the *No-envy* solution. We show next that the *Equal Split Guarantee* solution is *conversely consistent* (Proposition 3), but that it is not *consistent* (Example 2).

Proposition 3. The *Equal Split Guarantee* solution is *conversely consistent.*

Proof. Let $e = (Q,A,M;R_Q) \in \mathcal{E}$, and $z \in Z(e)$ be such that, for all $Q' \subseteq Q$ with $|Q'| = 2$, $z_{Q'} \in E(t^z_{Q'}(e))$. Since in domains of economies with two agents $N = E$, $z_{Q'} \in N(t^z_{Q'}(e))$. Since the *No-envy* solution is *conversely consistent*, $z \in N(e)$. Since $N \subseteq E$, $z \in E(e)$. ■

Example 2. Let $e = (Q,A,M;R_Q) \in \mathcal{E}$ be such that $Q = \{1,2,3\}$, $A = \{\alpha,\beta,\gamma\}$, $M = 0$, and for all $m \in \mathbb{R}$,

$$(\alpha,m)I_1(\beta,m)I_1(\gamma,m+3),\ (\alpha,m)I_2(\beta,m)I_2(\gamma,m),\ (\alpha,m+2)I_3(\beta,m+1)I_3(\gamma,m).$$

For this economy,

$$(\alpha,-1)I_1(\beta,-1)I_1(\gamma,2), \quad (\alpha,0)I_2(\beta,0)I_2(\gamma,0), \quad (\alpha,1)I_3(\beta,0)I_3(\gamma,-1).$$

Let $z = \{(\alpha,0),(\beta,1),(\gamma,-1)\}$. Clearly, $z \in E(e)$. Suppose that agent 1 leaves. Since in domains of economies with two agents N = E, and since agent 3 envies agent 2 at z, $z_{\{1,2\}} \notin E(t^z_{\{1,2\}}(e))$.

The *Equal Split Guarantee* solution is not *consistent*, and we now study how far from being *consistent* it is.

To do this, one option is to reduce the solution in order to recover the property. Ideally, it should be reduced as little as possible. This is done coherently by taking the union of all of the solution's *consistent* subsolutions. This union is a *consistent* solution because *consistency* is preserved under union.

The following definition was proposed by Thomson (1992).

Given a solution φ containing a *consistent* subsolution, its **maximal consistent subsolution, mcs(φ),** is defined by

$$mcs(\varphi) = \bigcup_{\psi \in \Psi} \psi \quad \text{where } \Psi = \{\psi \;/\; \psi \subseteq \varphi,\ \psi \text{ is } \textit{consistent}\ \}$$

The next Proposition shows that the *maximal consistent subsolution* of the Equal Split Guarantee solution is the *No-envy* solution.

Proposition 4. Let φ be a *consistent* subsolution of the *Equal Split Guarantee* solution. Then, for all $e = (Q,A,M;R_Q) \in \mathcal{E}$ $\varphi(e) \subseteq N(e)$.

Proof. Let $e = (Q,A,M;R_Q) \in \mathcal{E}$, and $z \in \varphi(e)$ be given. By *consistency*, for all $Q' \subset Q$, $z_{Q'} \in \varphi(t^z_{Q'}(e))$. In particular, this is true for all $Q' \subset Q$ such that $|Q'| = 2$. Since in domains of economies with two agents $N = E$, $z_{Q'} \in N(t^z_{Q'}(e))$. Since the *No-envy* solution is *conversely consistent*, $z \in N(e)$.■

The above result remains true for any subsolution of the *Equal Split Guarantee* solution satisfying *bilateral consistency* (this is the only property of φ used in the proof).

Propositions 3 and 4 also remain true if the *Equal Split Guarantee* solution is replaced by the intersection of the *Equal Split Guarantee* solution with the *Pareto* solution.

Another way to study how far the solution is from being *consistent* is to enlarge it. In this case, it is desirable to enlarge it as little as possible. This can be done properly by taking the intersection of all those *consistent* solutions which contain the solution. This intersection is a *consistent* solution because *consistency* is preserved under intersection.

The following definition was proposed by Thomson (1992).

Given a solution φ, its **minimal consistent extension, mce(φ),** is defined by

$$mce(\varphi) = \bigcap_{\psi \in \Psi} \psi \quad \text{where } \Psi = \{ \psi \geq \varphi, \psi \text{ is } \textit{consistent} \}$$

In economies with infinitely divisible goods, Thomson (1992) proves that the *minimal consistent extension* of the Individually Rational solution from equal division intersected with the *Pareto* solution is the *Pareto* solution itself. Proposition 5 shows that this result remains true in our model when the *Equal Split Guarantee* and *Pareto* solution is considered.

Proposition 5. The *minimal consistent extension* of the *Equal Split Guarantee and Pareto* solution is the *Pareto* solution.

The following Lemma is needed for the proof of Proposition 5.

Lemma 1. Let $e = (Q,A,M;R_Q) \in \mathcal{E}$, let $q = |Q|$, and let $z = (\sigma,m) \in P(e)$ be given. Then, there exists $e' = (Q',A',M';R'_{Q'}) \in \mathcal{E}$, and $z' \in EP(e')$ such that $e = t^{z'}_Q(e')$, and $z'_Q = z$.

Proof. Let $i_o \in \mathcal{Q}\backslash Q$ and $Q' = Q\cup\{i_o\}$. Let $\alpha_o \in \mathcal{A}\backslash A$ and $A' = A\cup\{\alpha_o\}$. For each $i \in Q$, and for each $\alpha \in A$, let $\bar{m}^i_\alpha \in \mathbb{R}$ be such that $z_i I_i(\alpha,\bar{m}^i_\alpha)$. Let $\bar{m} = \min_{i,\alpha}\{\bar{m}^i_\alpha\}$. Let $M' = (q+1)\bar{m}$. Let $R'_{Q'}$ be such that:

(i) For all $i \in Q$, $R'_{i|A\times R} = R_i$, and $z_i I'_i(\alpha_o,m_{\alpha o})$, with $m_{\alpha o} \geq \bar{m}$.

(ii) For all $i\in Q$, $(\alpha_o,M'-M)I'_{i_o} z_i$.

Let $z' = (\sigma',m')$ be such that $\sigma'_Q = \sigma$, $\sigma'(i_o) = \alpha_o$, $m'_A = m$, $m'_{\alpha o} = M'-M$. Clearly, $e = t^{z'}_Q(e')$, and $z'_Q = z$. We still have to prove that $z' \in EP(e')$.

Step 1. For all $i \in Q$, and for all $\alpha \in A'$, $z'_i R'_i(\alpha,m^i_\alpha(e'))$.

By monotonicity of preferences in money, it is sufficient to prove that $m_{\sigma(i)} \geq m^i_{\sigma(i)}(e')$. By the definition of $\bar{m}$, and the choice of m_α,

$$m_{\sigma(i)} + \sum_{\alpha\in A\setminus(\sigma(i))} \bar{m}^{i}_{\alpha} + m_{\alpha_o} \geq (q+1)\bar{m} = M' = \sum_{\alpha\in A'} m^{i_o}_{\alpha}(e').$$

Then, $m_{\sigma(i)} \geq m^{i}_{\sigma(i)}(e')$.

Step 2. $z' \in P(e')$.

Suppose that $z' \notin P(e')$. Then there exists $z'' \in Z(e')$ such that $z''_i R_i z'_i$ for all $i \in Q'$, and there exists $j \in Q'$ such that $z''_j P_j z'_j$. Since $z \in P(e)$, at z'' agent i_o must get some object different from α_o. Suppose that agent i_o gets object $\sigma(j)$ for some $j\in Q$. Then, by the choice of preferences of agent i_o, $m_{io} \geq m_{\sigma(i)}$. By definition of $\bar{m}$, $m''_{io} \geq \bar{m}$, and for all $i \in Q$, $m''_i \geq \bar{m}$, with at least a strict inequality. Then, $\sum_{i\in Q'} m''_i > (q+1)\bar{m} = M'$, in contradiction to the feasibility of z''. Thus, $z' \in P(e')$.

By Step 1, Step 2, and the choice of preferences for agent i_o, we conclude that $z' \in EP(e')$.■

Proof of Proposition 5. Let $e = (Q,A,M;R_Q) \in \mathcal{E}$, and let $z \in P(e)$ be given. And let $e' = (Q',A',M';R'_{Q'}) \in \mathcal{E}$, and $z' \in EP(e')$ be as described in Lemma 1. Let $\varphi^* = mce(EP)$. Since $EP \subseteq \varphi^*$, $z'\in \varphi^*(e')$. Since φ^* is *consistent*, $z = z'_Q \in \varphi^*(t^{z'}_Q(e')) = \varphi^*(e)$. Therefore $P \subseteq \varphi^*$. Since P is *consistent*, $P = \varphi^*$.■

Remark 1. The *minimal consistent extension* of the *Equal Split Guarantee* solution is the *Feasible* solution. The proof follows from Lemma 1 and Proposition 5.

Remark 2. Proposition 5 is obtained without imposing any kind of monotonicity conditions of preferences with respect to the indivisible goods. However, there are many situations in which some restrictions of preferences are justified. For example, consider an economy where the agents are homeless and the objects are houses. In this context it seems natural to think that, with the same amount of money, all agents prefer obtaining a house to remaining homeless. In order to formalize this concept, we introduce a "reference object" denoted by $\emptyset$. Preferences are extended to be defined on $(A\cup\{\emptyset\})\times\mathbb{R}$ so that the following holds:

Monotonicity of preferences on the indivisible goods; for all $i \in Q$, for all $\alpha \in A$, and for all $m \in \mathbb{R}$, $(\alpha,m)P_i(\emptyset,m)$.

Under this assumption on preferences, Proposition 5 remains true. It is sufficient to modify the proof in the following way: for each $i \in Q$, let $\bar{m}^i_\emptyset \in \mathbb{R}$ be such that, $z_iI_i(\emptyset,\bar{m}^i_\emptyset)$. By monotonicity of preferences on the indivisible goods, $\bar{m}^i_\emptyset > \bar{m}$. When economy e' is constructed, the preferences of agent i should be extended such that, $z_iI'_i(\alpha_o,m_{\alpha_o})$ with $\bar{m} \leq m_{\alpha_o} < \bar{m}^i_\emptyset$.

4.- SOME EXTENSIONS TO OTHER DOMAINS

In this section, we analyze two special domains. The first one is when all the objects are identical. In this domain all of our results hold. The second one is when there is only one object. In this domain some but not all of our results hold.

4.1. Economies with identical objects.

Consider the problem of allocating a finite number of identical objects among a group of agents. Now, an economy is a list $e = (Q, A\cup\{\emptyset\}, M; R_Q)$, where $Q \subseteq \mathcal{Q}$ is a finite set of agents, $A \subseteq \mathcal{A}$ is a finite set of identical objects, $\emptyset$ is the "null object", $M \in \mathbb{R}$ is the amount of money, and $R_Q = (R_i)_{i\in Q}$ is a list of preferences, one for each of the members of Q. We assume that preferences are defined on $(A\cup\{\emptyset\})\times\mathbb{R}$. We also assume that $|Q| \geq |A|$; that each object is assigned; and that no agent receives more than one object. Then, $|Q|-|A|$ agents receive the "null object". Let $\mathcal{E}^{id}$ be the class of such economies.

A *feasible allocation for* $e = (Q, A\cup\{\emptyset\}, M; R_Q)$ $\mathcal{E}^{id}$ is a pair $z = (\sigma, m)$, where $\sigma: Q \longrightarrow A\cup\{\emptyset\}$ is a function that specifies which agents receive the objects, and $m \in \mathbb{R}^{|Q|}$ is such that $\sum_{i\in Q} m_{\sigma(i)} = M$.

In order to describe the *Equal Split Guarantee* solution in this model, we will adapt the notation.

Given $e = (Q,A\cup\{\varnothing\},M,RQ) \in \mathcal{E}^{id}$, let $q = |Q|$, $a = |A|$, and for each $i \in Q$, $\alpha \in A$, let $m^i_\alpha(e)$, $m^i_\varnothing(e) \in \mathbb{R}$ be such that

(i) $am^i_\alpha(e) + (q-a)m^i_\varnothing(e) = M$,

(ii) $(\alpha, m^i_\alpha(e))I_i(\varnothing, m^i_\varnothing(e))$.

Then, on $\mathcal{E}^{id}$, the *Equal Split Guarantee* solution can be rewritten as,

$$E(e) = \{ z\in Z(e) \ / \ \forall i\in Q,\ z_iR_i(\alpha,\ m^i_\alpha(e))\}.$$

For this model, in the same way as the general one, the *minimal consistent extension* of the *Equal Split Guarantee* and *Pareto* solution is the Pareto solution. This result also holds if, in addition, we impose monotonicity of preferences on the indivisible goods. Since the proof of this result is similar to that of the general case, we give it in Proposition 6 in the appendix.

Proposition 6. On $\mathcal{E}^{id}$, the *minimal consistent extension* of the *Equal Split Guarantee* and *Pareto* solution is the Pareto solution.

4.2. Economies with one object.

Consider the special case where there is a single object which has to be attributed to one person. In this case we call the agent who receives the object the "winner" , and the others "losers" . Now, an economy is a list $e = (Q,(\Delta,M);R_Q)$, where $Q \subseteq \mathbb{N}$ is a finite set of agents, (Δ,M) are the resources, with $\Delta = 1$ if the object is present, and $\Delta=0$ otherwise, $M \in \mathbb{R}$ is the amount of money, and $R_Q = (R_i)_{i\in Q}$ is a list of preferences, one for each of the members of Q. Let $\mathcal{E}_1$ be the class of such economies.

A feasible allocation for $e = (Q,(\Delta,M);\ R_Q) \in \mathcal{E}_1$ is a pair $z = (\delta,m) \in [\{0,1\}\mathrm{x}\mathbb{R}]^{|Q|}$ such that $\sum_Q(\delta_i,m_i) = (\Delta,M)$, agent i's bundle being $z_i = (\delta_i,m_i)$: $\delta_i = 1$ if agent i is the winner and $\delta_i = 0$ otherwise; m_i is the amount of money he receives.

As before, in order to describe the *Equal Split Guarantee* solution in this model, the notation has to be adapted.

Given $e = (Q,(\Delta,M);R_Q) \in \mathcal{E}$, let $q = |Q|$, and for each $i \in Q$, let $m_w^i(e)$, $m_l^i(e) \in \mathbb{R}$ be such that

(i) $m_w^i(e) + (q-1)m_l^i(e) = M$

(ii) $(1,m_w^i(e))I_i(0,m_l^i(e))$.

Then, on $\mathcal{E}_1$, the *Equal Split Guarantee* solution can be rewritten as,

$$E(e) = \{\ z \in Z(e)\ /\ \forall\ i \in Q,\ \ z_iR_i(1,m_w^i(e))\}$$

In economies in which there is only money to distribute, the *Equal Split Guarantee* solution recommends the unique allocation at which all agents receive the same amount of money.

Once again, in this model, as in the general one, the *minimal consistent extension* of the *Equal Split Guarantee* and *Pareto* solution is the Pareto solution. Since the proof of this result is similar to that of the general case, we give it in Proposition 7 in the appendix.

Proposition 7. On $\mathcal{E}_1$, the *minimal consistent extension* of the *Equal Split Guarantee and Pareto* solution is the *Pareto* solution.

At this point, some differences with respect to the general model should be mentioned. Firstly, in economies with one indivisible good, Tadenuma and Thomson (1990) prove that *consistent* selections from the *No-envy* solution exist . Therefore, any selection from the *Equal Split Guarantee* solution satisfying *consistency* is a selection from the *No-envy* solution. Secondly, if we impose monotonicity of preferences on the indivisible good, the *minimal consistent extension* of the *Equal Split Guarantee and Pareto* solution is a proper subsolution of the *Pareto* solution (Proposition 10).

In this model, monotonicity of preferences in the indivisible good means that for all $i \in \mathbb{N}$, and for all $m \in \mathbb{R}$, $(1,m)P_i(0,m)$.

Let $\mathcal{E}_1^{MON} \subseteq \mathcal{E}_1$ be the class of economies with preferences satisfying the above monotonicity condition. Let $EP|_{\mathcal{E}_1^{MON}}$ be the *Equal Split Guarantee and Pareto* solution defined on $\mathcal{E}_1^{MON}$. Let φ_c describe as follow:

(i) For all $e \in \mathcal{E}_1^{MON}$ such that $e = (Q,(0,M);R_Q)$, (i.e., the object is not present),

$$\varphi_c(e) = Z(e).$$

(ii) For all $e \in \mathcal{E}_1^{MON}$ such that $e = (Q,(1,M);R_Q)$, (i.e., the object is present),

$$\varphi_c(e) = \{z \in Z(e)/\ \bar{m}_w > M/q, \text{ and } \forall\ i \in Q\backslash\{w\},\ m_i > M/q\}$$

where $w \in Q$ is the winner at z, and $\bar{m}_w$ is such that $z_w I_w (0,\bar{m}_w)$.

This solution is always non-empty in this domain because it contains the *Equal Split Guarantee* solution.

Proposition 8. For all $e = (Q,(\Delta,M);R_Q) \in \mathcal{E}_1^{MON}$, $E(e) \subseteq \varphi_c(e)$.

Proof. Trivially, for all $e = (Q,(0,M);R_Q) \in \mathcal{E}_1^{MON}$, and for all $z \in E(e)$, $z \in \varphi_c(e)$. We still have to prove the inclusion for all $e = (Q,(1,M);R_Q) \in \mathcal{E}_1^{MON}$.

Step 1. Given $e = (Q,(1,M);R_Q) \in \mathcal{E}_1^{MON}$ with $|Q| = q$, for all $i \in Q$, $m_1^i(e) > M/q$. Suppose that $m_1^i(e) \leq M/q$. By feasibility, $m_w^i(e) \geq M/q$. Then, $(0,M/q)R_i(0,m_1^i(e))I_i(1,m_w^i(e))R_i(0,M/q)$. Thus, $(0,M/q)R_i(1,M/q)$ in contradiction to monotonicity of preferences on the indivisible good.

Step 2. Let $e = (Q,(1,M);R_Q) \in \mathcal{E}_1^{MON}$ with $|Q| = q$, and $z \in E(e)$ be given. Then, $\bar{m}_w \geq m_1^w(e)$, and for all $i \in Q\backslash\{w\}$, $m_i \geq m_1^i$. By Step 1, $\bar{m}_w > M/q$, and for all $i \in Q\backslash\{w\}$, $m_i > M/q$. Thus, $z \in \varphi_c(e)$.■

Proposition 9. The solution φ_c is *consistent*.

Proof. Let $e = (Q,(\Delta,M);R_Q) \in \mathcal{E}_1^{MON}$, and $z \in \varphi_c(e)$ be given. If $Q' \subseteq Q$ is such that the winner at z is not in Q', then $z_{Q'} \in \varphi_c(Q',\sum_{i\in Q'} z_i;R_{Q'})$ because Z is *consistent.* Let Q' ⊆ Q be such that |Q'| = q', and the winner at z is in Q'. We claim that for all $j \in Q'\setminus\{w\}$, $m_j > \frac{M-\sum_{i\in Q\setminus Q'} m_i}{q'}$. Since $z \in \varphi_c(e)$, for all $i \in Q$, $m_i > M/q$. Then,

$$\frac{M-\sum_{i\in Q\setminus Q'} m_i}{q'} \leq \frac{M-\sum_{i\in Q\setminus Q'} M/q}{q'} = \frac{M-(q-q')M/q}{q'} = M/q < m_i.$$

For the same reason, $\bar{m}_w > \frac{M-\sum_{i\in Q\setminus Q'} m_i}{q'}$. Thus, $z_{Q'} \in \varphi_c(Q',\sum_{i\in Q'} z_i;R_{Q'})$.■

Let $\varphi_c P$ be the intersection of φ_c with the *Pareto* solution. Clearly, $EP \subseteq \varphi_c P$. The solution $\varphi_c P$ is *consistent* because both, φ_c and P, are *consistent,* and because *consistency* is preserved under intersection.

Proposition 10. On $\mathcal{E}_1^{MON}$, the *minimal consistent extension* of the *Equal Split Guarantee and Pareto* solution is $\varphi_c P$.

The following result is necessary to obtain the proof of Proposition 10.

Lemma 4. Let $e = (Q,(\Delta,M);R_Q) \in \mathcal{E}_1^{MON}$, let $q = |Q|$, and let $z \in \varphi_c P(e)$ be given. Then, there exists $e_k = (Q_k,(\Delta,M_k);R^k_{Qk}) \in \mathcal{E}_1^{MON}$, and $z^k \in EP(e_k)$ such that $e = t^{z^k}_Q(e_k)$, and $z^k_Q = z$.

Proof. Let Q_k be such that $Q \subseteq Q_k$, and $|Q_k| = q+k$. Let $M_k = M + kM/q + k\varepsilon/q$. Let R^k_{Qk} be such that:

(i) For all $i \in Q$, $R^k_i = R_i$.

(ii) For all $j \in Q_k\setminus Q$, all $m \in \mathbb{R}$, $(1,m)I^k_j(0,m+\varepsilon)$.

Let z^k be such that $z^k_Q = z$, and $z_{Q_k \setminus Q} = ((0,M/q+\varepsilon/q),\ldots,(0,M/q+\varepsilon/q))$. Clearly, $e = t_Q^{z^k}(e_k)$. We still have to prove that $z^k \in EP(e_k)$.

Step 1. It is easy to check that for all $j \in Q_k \setminus Q$, $\frac{M}{q} + \frac{1}{q}\varepsilon - \varepsilon = m_w^j(e_k)$, and $m_l^j(e_k) = \frac{M}{q} + \frac{1}{q}\varepsilon$.

Step 2. For all $i \in Q$ $m_l^i(e_k) > \frac{M}{q} + \frac{1}{q}\varepsilon$.

Suppose that $m_l^i(e_k) \leq \frac{M}{q} + \frac{1}{q}\varepsilon$. By feasibility $m_w^i(e_k) \geq \frac{M}{q} + \frac{1}{q}\varepsilon - \varepsilon$. Then, $(0, \frac{M}{q} + \frac{1}{q}\varepsilon) R_i (0, m_l^i(e_k)) I_i (1, m_w^i(e_k)) R_i (1, \frac{M}{q} + \frac{1}{q}\varepsilon - \varepsilon)$. Then, for a sufficiently small ε, and by continuity of preferences $(0, \frac{M}{q}) R_i (1, \frac{M}{q})$. This is a contradiction because $(1,m)\ P_i\ (0,m)$ for all $m \in \mathbb{R}$, for all $i \in Q$.

Step 3. For all $i \in Q$, $m_w^i(e_k) < m_w^i(e)$.

We know that $m_w^i(e_k) + (q+k-1)m_l^i(e_k) = M + kM/q + k\varepsilon/q$. Then, by Step 2, $m_w^i(e_k) + (q-1)m_l^i(e_k) = M + kM/q + k\varepsilon/q - km_l^i(e_k) < M = m_w^i(e) + (q-1)m_l^i(e)$. Thus, $m_w^i(e_k) < m_w^i(e)$.

Step 4. If $k' > k$, then $m_w^i(e_{k'}) < m_w^i(e_k)$ for all $i \in Q$.

We omit the proof of this Step since the argument used is similar to that for Step 3.

By Step 1, Step 2, Step 3, and Step 4 we have obtained that for a sufficiently large k, $z^k \in E(e_k)$.

Step 5. $z^k \in P(e_k)$.

Suppose that $z^k \notin P(e_k)$. Then $z' \in Z(e_k)$ exists such that $z'_i R_i z^k_i$ for all $i \in Q_k$, and $j \in Q_k$ exists such that $z'_j P_j z^k_j$. Since $z \in P(e)$, at z' one of the new agents must get the indivisible good. For simplicity, suppose it is agent q+1. Then $m'_{q+1} \geq \frac{M}{q} + \frac{\varepsilon}{q} - \varepsilon$, and for all $j \in Q_k \backslash \{q+1\}$ $m'_j \geq \frac{M}{q} + \frac{\varepsilon}{q}$, with at least a strict inequality. Then $\sum_{i=1}^{q+k} m'_i > M + k\frac{M}{q} + k\frac{\varepsilon}{q}$. This is a contradiction because $z^k \in Z(e_k)$. Thus, $z^k \in EP(e_k)$.■

Proof of Proposition 10. Let $e = (Q,(\Delta,M);R_Q) \in \mathcal{E}_1^{MON}$, let $q = |Q|$, and let $z \in \varphi_c P(e)$ be given. And let $e_k = (Q_k,(\Delta,M_k);R^k_{Q_k}) \in \mathcal{E}_1^{MON}$, and $z^k \in EP(e_k)$ be as described in Lemma 4. Let $\varphi^* = mce(EP)$. Since $EP \subseteq \varphi^*$, $z^k \in \varphi^*(e_k)$. Since φ^* is *consistent*, $z = z^k_Q \in \varphi^*(t^{z^k}_Q(e_k)) = \varphi^*(e)$. Therefore $\varphi_c P \subseteq \varphi^*$. Since $\varphi_c P$ is *consistent*, $\varphi_c P = \varphi^*$.■

Remark 3. On $\mathcal{E}^{MON}$, the *minimal consistent extension* of the *Equal Split Guarantee* solution is φ_c. The proof follows directly from Lemma 4 and Proposition 10.

5.- POPULATION MONOTONICITY

In this section we will look for subsolutions of the equal split guarantee solution in the domain $\mathcal{E}_1$ with the following property: if the number of agents increases, keeping the resources fixed , we would expect each of the initial agents to suffer a welfare loss, or at least not to benefit from the addition of new agents. This property is known in the literature as *population monotonicity*. Formally,

The solution φ satisfies **population monotonicity** if for all $e = (Q,(1,M),R_Q)$, and $e' = (Q',(1,M'),R'_{Q'})$ with $Q \subset Q'$ and $(M,R_Q) = (M',R'_Q)$, for all $z \in \varphi(e)$, $z' \in \varphi(e')$, $z_i R_i z'_i$ for all $i \in Q$.

Tadenuma and Thomson (1990) showed that if the No-envy solution is considered, the arrival of additional agents may benefit some of the agents originally present. However, it is possible to guarantee that all of the agents initially present either lose or gain. This property is a weaker version of *population monotonicity.*

The solution φ satisfies **weak population monotonicity** if for all $e = (Q,(1,M);R_Q)$, and $e' = (Q',(1,M'),R'_{Q'})$ with $Q \subset Q'$ and $(M,R_Q) = (M',R'_Q)$, for all $z \in \varphi(e)$, $z' \in \varphi(e')$, $z_i R_i z'_i$ or $z'_i R_i z_i$ for all $i \in Q$.

Tadenuma and Thomson (1990) proved that there exists a selection from the *No-envy* solution satisfying *weak population monotonicity.* This selection picks

Pareto efficient allocations at which all the agents are indifferent between their bundles and a reference bundle containimg the null object. Formally,

Given $e = (Q,(1,M);R_Q)$, let

$$\varphi^*(e) = \{ z \in P(e) / \text{ there is } m_o \in \mathbb{R} \text{ s.t. for all } i \in Q,\ z_i I_i (0,m_o) \}$$

Since the *No-Envy* solution is a subsolution of the *Equal Split Guarantee* solution, the solution φ^* is a selection from the *Equal Split Guarantee* solution which satisfies *weak population monotonicity.*

Moulin (1990) proposed a selection from the *Equal Split Guarantee and Pareto* solution satisfying *population monotonicity* in quasi-linear economies, M=0, and preferences satisfying monotonicity with respect to the indivisible good. This selection is the *Shapley solution.* In order to describe this solution we introduce the following notation.

Let $\mathcal{E}_{ql}^{MON}$ be the class of quasi-linear economies where preferences satisfies monotonicity with respect to the indivisible good.

Given an economy $e = (Q,(1,M),R_Q) \in \mathcal{E}_{ql}^{MON}$, for each $i \in Q$, let v_i be the value of the object for agent i. Let (Q,v) be the **cooperative game** defined by $v(S) = \max_{i\in S}(v_i) + M$ for all $S \subseteq Q$.

The Shapley solution, Sh: Given $e = (Q,(1,0),R_Q) \in \mathcal{E}_{ql}^{MON}$, let $q = |Q|$, and suppose that $v_1 \leq v_2 \leq \ldots \leq v_q$. Then,

$Sh(e) = \{z \in Z(e) / z_i=(0,m_i)$ s.t. $m_i=sh_i(Q,v)$ $i \neq q$, $z_q=(1,m_q)$ s.t $m_q=sh_q(Q,v)-v_q\}$

where $sh_i(Q,v)$ is the **Shapley value of agent i in the game (Q,v).**

We present a generalization of this solution in $\mathcal{E}_1^{MON}$, and $M \geq 0$. Finally, we present an impossibility result in $\mathcal{E}_1$ when $M < 0$, and when the monotonicity condition with respect to the indivisible good is not imposed on preferences.

At this point we introduce some more notation.

Given an economy $e_q = (Q,(1,M),R_Q)$, $|Q| = q$, we order the agents such that

$$m_1^1(e_q) \leq m_1^2(e_q) \leq \ldots\ldots \leq m_1^q(e_q).$$

Then, let $e_{q-1} = (Q_{-1},(1,M-m_1^1(e_q)),R_{Q-1})$, where $Q_{-1} \subseteq Q$, $|Q_{-1}| = q-1$.

Given e_{q-1}, we order the agents such that

$$m_1^2(e_{q-1}) \leq m_1^3(e_{q-1}) \leq \ldots\ldots \leq m_1^q(e_{q-1}).$$

Then, let $e_{q-2} = (Q_{-2},(1,M-m_1^1(e_q)-m_1^2(e_{q-1})),R_{Q-2})$, where $Q_{-2} \subseteq Q$, $|Q_{-2}| = q-2$.

Successively,

let $e_{q-k} = (Q_{-k},(1,M-\sum_{i=1}^{k} m_1^i(e_{q-i+1}));R_{Q-k})$, where $Q_{-k} \subseteq Q$, $|Q_{-k}| = q-k$.

Given an economy $e_q = (Q,(1,M),R_Q)$, $|Q| = q$,

$$\varphi(e_q) = \{(0,m_1),(0,m_2),\ldots,(1,m_q)\}$$

where $m_1 = m_1^1(e_q)$, $m_2 = m_1^2(e_{q-1}),\ldots,$ $m_{q-1} = m_1^{q-1}(e_2)$, $m_q = m_w^{q-1}(e_2)$.

Firstly, we prove that the solution φ is a selection of EP; secondly, we prove that satisfies *population monotonicity*. The proofs of both results are given in the Appendix.

Proposition 11. For all $e = (Q,(1,M),R_Q)$, $\varphi(e) \subseteq EP(e)$.

Proposition 12. In $\mathcal{E}_1^{MON}$, if $M \geq 0$, the solution φ satisfies *population monotonicity.*

Proposition 13. In $\mathcal{E}_1$, if $M < 0$, no selection from the *Equal Split Guarantee* solution satisfying *population monotonicity* exists.

Proof. Let $e = (\{1,2\},(1,-12);R_1;R_2)$, where R_1 is such that $(1,m)I_1(0,m+4)$, and R_2 is such that $(1,m)I_2(0,m+2)$. For this economy, $m_1^1(e) = -4$, $m_w^1(e) = -8$, $m_1^2(e) = -5$, $m_w^2(e) = -7$. Then,

$$E(e) = \{\ ((1,m_1),\ (0,m_2))/\ -8 \leq m_1 \leq -7,\ -5 \leq m_2 \leq -4\ \}$$

Let us consider a new economy, $e' = (\{1,2,3\},(1,-12);R_1;R_2;R_3)$ where R_3 is such that $(1,m)I_3(0,m+1)$. For this economy, $m_1^1(e') = -8/3$, $m_1^2(e') = -10/3$, $m_1^3(e') = -11/3$, $m_w^1(e') = -20/3$, $m_w^2(e') = -16/3$, $m_w^3(e') = -14/3$. Then,

$$E(e') = \{z' \in Z(e')/z'_1R_1(1,-20/3),\ z'_2R_2(1,-16/3),\ z'_3R_3(1,-14/3)\ \}$$

By monotonicity of preferences in money, for all $z' \in E(e')$ $z'_1 R_1 (1,-20/3)$, $z'_2 R_2 (1,-16/3) I_2 (0,-10/3) P_2 (0,-4)$. Thus, for all $z \in E(e)$, and for all $z' \in E(e')$ $z'_i P_i z_i$ for $i = 1,2$. Therefore, agents one and two gain with the entrance of agent three.■

Proposition 14. In $\mathcal{E}_1$, if $M \geq 0$, and the monotonicity condition with respect to the indivisible good is not imposed on preferences, no selection from the *Equal Split Guarantee* solution satisfying *population monotonicity* exists.

Proof. Let $e = (\{1,2\},(1,12);R_1;R_2)$, where $R_1 = R_2$ and are such that $(1,m) I_i (0,m-14)$ $i = 1,2$. For this economy $m^i_l(e) = -1$, $m^i_w(e) = 13$ $i = 1,2$. Then,

$$E(e) = \{((1,13),\ (0,-1)),\ ((0,-1),\ (1,13))\}.$$

Let us consider a new economy, $e' = (\{1,2,3\},(1,12);R_1;R_2;R_3)$ where R_3 is such that $(1,m) I_2 (0,m+2)$. For this economy $m^i_l(e') = -2/3$, $m^i_w(e') = 40/3$ $i = 1,2$, and $m^3_l(e') = 14/3$, $m^3_w(e') = 8/3$. Then,

$$E(e') = \{z' \in Z(e') / \ z'_i R_i (1,40/3),\ i = 1,2,\ z'_3 R_3 (1,8/3)\}.$$

By monotonicity of preferences in money, for all $z' \in E(e')$ $z'_i R_i (1,40/3) P_i (1,13) I_i (0,-1)$ $i = 1,2$. Thus, for all $z \in E(e)$ and for all $z' \in E(e')$ $z'_i P_i z_i$ $i = 1,2$. Therefore, agents one and two gain with the entrance of agent three.■

In Proposition 13 we have studied the case where $M < 0$, and in Proposition 14 the indivisible good is, in some sense, bad. In both cases it may not be natural to require that when new agents come in, all agents initially present lose. The right property seems to be that all they gain. However, in none of these cases it is possible to find a selection from the *Equal Split Guarantee* solution satisfying the above mentioned property. Examples 3 and 4 illustrate these facts.

Example 3. Let $e = (\{1,2\},(1,-12);R_1;R_2)$, where $R_1 = R_2$ and are such that $(1,m)I_i(0,m+16)$ $i = 1,2$. For this economy $m_l^i(e) = 2$, $m_w^i(e) = -14$ $i = 1,2$. Then,

$$E(e) = \{((1,-14),\ (0,2)),\ ((0,2),\ (1,-14))\}.$$

Let us consider a new economy, $e' = (\{1,2,3\},(1,-12);R_1;R_2;R_3)$ where $R_3 = R_1 = R_2$. For this economy $m_l^i(e') = 4/3$, $m_w^i(e') = -44/3$ $i = 1,2,3$. Then,

$$E(e') = \{\ z' \in Z(e')/\ z' \text{ is a permutacion of } ((1,-44/3),\ (0,4/3),\ (0,4/3))\}$$

Thus, for all $z' \in E(e')$ and for all $z \in E(e)$, $z_i P_i z'_i$ $i = 1,2$. Therefore, agents one and two lose with the entrance of agent three.

Example 4. Let $e = (\{1,2\},(1,12);R_1;R_2)$, where $R_1 = R_2$ and are such that $(1,m)I_i(0,m-2)$ $i = 1,2$. For this economy $m_l^i(e) = 5$, $m_w^i(e) = 7$, $i = 1,2$. Then,

$$E(e) = \{((1,7),\ (0,5)),\ ((0,5),\ (1,7))\}.$$

Let us consider a new economy, $e' = (\{1,2,3\},(1,12);R_1;R_2;R_3)$ where $R_3 = R_1 = R_2$. For this economy $m_l^i(e') = 10/3$, $m_w^i(e') = 16/3$ $i = 1,2,3$. Then,

$E(e') = \{ z' \in Z(e')/ z'$ is a permutation of ((1,16/3), (0,10/3), (0,10/3))}

Thus, for all $z' \in E(e')$ and for all $z \in E(e)$, $z_i P_i z'_i$ i = 1,2. Therefore, agents one and two lose with the entrance of agent three.

Sprumont (1990) proved that in the class of concave games[3] it is always possible to find a solution satisfying *population monotonicity.* In particular the Shapley value defined on concave games satisfies the property. Moulin (1990) proved that the game induced by any economy in $\mathcal{E}_{ql}^{MON}$ with M = 0 is a concave game, therefore the Shapley solution defined on this class of economies satisfies *population monotonicity.* Both Proposition 13 and 14 describe quasi-linear economies, given that an impossibility result it is obtained, should be the case that the games induced by the economies described in those Propositions are not concave. It is easy to see that, in both Proposition 13 and 14, $v(\{1,3\})-v(\{1\}) > v(\{3\})$.

In the next proposition we give a condition on the class of quasi-linear economies that guarantees the concavity of the induced game. Therefore, in such class of economies, the Shapley solution satisfies *population monotonicity.*

Proposition 15. Let $\mathcal{E}_{ql}^{c}$ be the class of quasi-linear economies 4 satisfying

$v_i \geq -M$ for all $i \in Q$. Then, for any economy in this class, the induced game (Q,v) is concave.

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(3) A game (Q,v) is concave if for all $S,T \subseteq Q$, such that $T \subseteq S$, for all $i \notin S$, $v(S \cup \{i\})-v(S) \leq v(T \cup \{i\})-v(T)$

Proof. We divide the proof in three steps.

Step 1. It is easy to verify that, for all $S,T \subseteq Q$ such that $T \subseteq S$, $T \neq \emptyset$, and for all $i \in Q$ such that $i \notin S$,

$$v(S\cup\{i\})-v(S) \leq v(T\cup\{i\})-v(T)$$

Step 2. For all $i,j \in Q$,

$$v(\{i,j\})-v(\{i\}) \leq v(\{j\})$$

By the definition of the game, $v(\{i,j\})-v(\{i\}) = \max(v_i,v_j)-v_i$. Suppose that $v_i = \max(v_i,v_j)$. Then, $v_i-v_i = 0 \leq v_j + M = v(\{j\})$.

Suppose that $v_j = \max(v_i,v_j)$. Then, $v_j-v_i \leq v_j + M = v(\{j\})$ because $-v_i \leq M$.

Step 3. For all $S \in Q$, for all $i \notin S$,

$$v(S\cup\{i\})-v(S) \leq v(\{i\})$$

This step is an immediate consequence of step 1 and 2.∎

Remark. Notice that $\mathcal{E}_{ql}^{MON}$ with $M \geq 0$ is contained in the class of economies described in Proposition 15.

A P P E N D I X

Proposition 6. On $\mathcal{E}^{id}$, the *minimal consistent extension* of the *Equal Split Guarantee* and *Pareto* solution is the Pareto solution.

The next result is needed for the proof of Proposition 6.

Lemma 2. Let $e = (Q, A\cup\{\varnothing\}, M; R_Q) \in \mathcal{E}^{id}$, let $q = |Q|$, $a = |A|$, and let $z = (\sigma, m) \in P(e)$ be given. Then, there exists $e' = (Q', A'\cup\{\varnothing\}, M'; R'_{Q'}) \in \mathcal{E}^{id}$, and $z' \in EP(e')$ such that $e = t_Q^{z'}(e')$, and $z'_Q = z$.

Proof. Let $i_o \in \mathcal{Q}\backslash Q$ and $Q' = Q\cup\{i_o\}$. Let $A' = A\cup\{\alpha\}$. For each $i \in Q$, let $\bar{m}_i$ be such that $z_i I_i(\varnothing, \bar{m}_i)$ if agent i receives one of the objects, and $z_i I_i(\alpha, \bar{m}_i)$ if agent i does not receive any of the objects. Let $\bar{m} = \min_i\{m_i, \bar{m}_i\}$. Let $M' = (q+1)\bar{m}$. Let $R'_{Q'}$ be such that:

(i) For all $i \in Q$, $R'_{i\,|\,A\cup\{\varnothing\}\times R} = R_i$.

(ii) for i_o, $(\alpha, M'-M) I'_{i_o} \left(\varnothing, \dfrac{M'-(a+1)(M'-M)}{q-a}\right)$.

Let $z' = (z, z_{i_o})$ be such that $z_{i_o} = (\alpha, M'-M)$. Clearly, $e = t_Q^{z'}(e')$, and $z'_Q = z$.

We still have to prove that $z' \in EP(e')$.

Step 1. For all $i \in Q$ such that $z_i = (\alpha, m_i)$, $m_i \geq m_\alpha^i(e')$.

We know that $m_i \geq \bar{m}$, and $\bar{m}_i \geq \bar{m}$. Then,

$(a+1)m_i + (q-a)\bar{m}_i \geq (q+1)\bar{m} = M' = (a+1)m_\alpha^i(e') + (q-a)m_\varnothing^i(e')$. Thus, $m_i \geq m_\alpha^i(e')$.

Step 2. For all $i \in Q$ such that $z_i = (\emptyset, m_i)$, $m_i \geq m^i_\emptyset(e')$.

We know that $m_i \geq \bar{m}$, and $\bar{m}_i \geq \bar{m}$. Then,

$(a+1)\bar{m}_i+(q-a)m_i \geq (q+1)\bar{m} = M' = (a+1)m^i_\alpha(e')+(q-a)m^i_\emptyset(e')$. Thus, $m_i \geq m^i_\emptyset(e')$.

By Step 1, Step 2, and the choice of preferences of agent i_0, $z' \in E(e')$. We still have to prove that $z' \in P(e')$.

Step 3. $z' \in P(e')$.Suppose that $z' \notin P(e')$. Then $z'' \in Z(e')$ exists such that $z''_i R_i z'_i$ for all $i \in Q'$, and $j \in Q'$ exists such that $z''_j P_j z'_j$. Since $z \in P(e)$, at z'' agent i_o must get the null object, and one of the agents in Q must get one of the objects. Then, $m''_{i_o} \geq \dfrac{M'-(a+1)(M'-M)}{q-a} \geq \dfrac{M'}{q+1}$. Since $m_i \geq \bar{m}$ and $\bar{m}_i \geq \bar{m}$ for all $i \in Q$, then at z'', $m''_i \geq \bar{m} = \dfrac{M'}{q+1}$, with at least a strict inequality. Then, $\sum_{i\in Q'} m''_i > M'$, in contradiction to feasibility of z''. Thus, $z' \in EP(e')$.∎

The proof of Proposition 6 is completed in the same way as in Proposition 5.

Proposition 7. On $\mathcal{E}_1$, the *minimal consistent extension* of the *Equal Split Guarantee and Pareto* solution is the *Pareto* solution.

The following result is necessary to obtain the proof of Proposition 7.

Lemma 3. Let $e = (Q,(\Delta,M);R_Q) \in \mathcal{E}_1$, let $q = |Q|$, and let $z \in P(e)$ be given. Then, $e'= (Q',(\Delta,M');R'_{Q'}) \in \mathcal{E}_1$, and $z' \in EP(e')$ exist such that $e = t^{z'}_Q(e')$, and $z'_Q = z$.

Proof. Let $i_o \in Q\backslash Q$ and $Q' = Q \cup \{i_o\}$. Suppose that agent 1 is the winner at z. Let $\bar{m}_1$ be such that $(1,m_1)I_1(0,\bar{m}_1)$. For all $i \in Q\backslash\{1\}$, let $\bar{m}_i$ be such that $(0,m_i)I_i(1,\bar{m}_i)$. Let $\bar{m} = \min_i\{m_i,\bar{m}_i\}$. Let $M' = (q+1)\bar{m}$. Let $R'_{Q'}$ be such that:

(i) For all $i \in Q$, $R'_i = R_i$.

(ii) For i_o, $(0,M'-M)I'_{i_o}(1,qM-(q-1)M')$.

Let $z' = (z,z_{i_o})$ be such that $z_{i_o} = (0,M'-M)$. Clearly, $e = t_Q^{z'}(e')$, and $z'_Q = z$. We still have to prove that $z' \in EP(e')$.

Step 1. $m_1 \geq m_w^1(e')$.

We know that $m_1 \geq \bar{m}$, and $\bar{m}_1 \geq \bar{m}$. Then,

$$m_1 + q\bar{m}_1 \geq \bar{m} + q\bar{m} = (q+1)\bar{m} = M' = m_w^1(e') + qm_l^1(e').$$

Thus, $m_1 \geq m_w^1(e')$.

Step 2. For all $i \in Q\backslash\{1\}$, $m_i \geq m_l^i(e')$.

We know that $m_i \geq \bar{m}$, and $\bar{m}_i \geq \bar{m}$. Then,

$$\bar{m}_i + qm_i \geq \bar{m} + q\bar{m} = (q+1)\bar{m} = M' = m_w^i(e') + qm_l^i(e').$$

Thus, $m_i \geq m_l^i(e')$.

By Step 1, Step 2, and the choice of preference of agent i_0, $z' \in E(e')$.

We still have to prove that $z' \in P(e')$.

Step 3. $M'-M \leq \bar{m}$.

Suppose that $M'-M > \bar{m}$. In other words, $(q+1)\bar{m} - M > \bar{m}$. Then, $\bar{m} > M/q$. Since for all $i \in Q$, $m_i \geq \bar{m}$, $\sum_{i \in Q} m_i > M$. This is a contradiction because $z \in Z(e)$.

Step 4. $z' \in P(e')$.

Suppose that $z' \notin P(e')$. Then $z'' \in Z(e')$ exists such that $z''_i R_i z'_i$ for all $i \in Q'$, and $j \in Q'$ exists such that $z''_j P_j z'_j$. Since $z \in P(e)$, at z'' the new agent must get the object. Then $m''_{i_o} \geq qM - (q-1)M'$. And, by Step 3, for all $i \in Q$, $m''_i \geq M'-M$, with at least a strict inequality. Thus, $\sum_{i\in Q'} m''_i > M'$. This is a contradiction because $z'' \in Z(e')$. Thus, $z' \in EP(e')$.■

The proof of Proposition 7 is completed in the same way as in Proposition 5.

Proposition 11. For all $e = (Q,(1,M),R_Q)$, $\varphi(e) \subseteq EP(e)$.

Proof. First, we prove that the solution φ is a selection from the *Equal Split Guarantee* solution. We divide the proof into three steps.

Step 1. $m_l^{k+1}(e_{q-k}) \geq m_l^k(e_{q-k+1})$.

We know that, $m_w^{k+1}(e_{q-k}) + (q-k-1)m_l^{k+1}(e_{q-k}) = M-\sum_{i=1}^{k} m_l^i(e_{q-i+1})$, and

$$m_w^k(e_{q-k+1}) + (q-k)m_l^k(e_{q-k+1}) = M-\sum_{i=1}^{k-1} m_l^i(e_{q-i+1}). \text{ Then,}$$

$$m_w^k(e_{q-k+1}) + (q-k-1)m_l^k(e_{q-k+1}) = M-\sum_{i=1}^{k} m_l^i(e_{q-i+1}).$$

Thus, $m_w^k(e_{q-k+1}) + (q-k-1)m_l^k(e_{q-k+1}) = m_w^{k+1}(e_{q-k}) + (q-k-1)m_l^{k+1}(e_{q-k})$. (1)

Suppose, by way of contradiction, that, $m_l^{k+1}(e_{q-k}) < m_l^k(e_{q-k+1})$.

We know that, $m_l^k(e_{q-k+1}) \leq m_l^{k+1}(e_{q-k+1})$, and $m_w^{k+1}(e_{q-k+1}) \leq m_w^k(e_{q-k+1})$.

Then, $m_l^{k+1}(e_{q-k}) < m_l^k(e_{q-k+1}) \leq m_l^{k+1}(e_{q-k+1})$,

and $m_w^{k+1}(e_{q-k}) \leq m_w^{k+1}(e_{q-k+1}) \leq m_w^k(e_{q-k+1})$.

Thus, $m_w^k(e_{q-k+1}) + (q-k-1)m_1^k(e_{q-k+1}) > m_w^{k+1}(e_{q-k}) + (q-k-1)m_1^{k+1}(e_{q-k})$ in contradiction to (1).

Step 2. For all k = 1,...,q-1, $m_1^k(e_{q-k+1}) \geq m_1^k(e_q)$.

We know that, $m_w^k(e_{q-k+1}) + (q-k)m_1^k(e_{q-k+1}) = M - \sum_{i=1}^{k-1} m_1^i(e_{q-i+1})$.

By Step 1, $m_1^i(e_{q-i+1}) \leq m_1^k(e_{q-k+1})$ for all i = 1,...,k-1.

Then, $m_w^k(e_{q-k+1}) + (q-k)m_1^k(e_{q-k+1}) \geq M-(k-1)m_1^k(e_{q-k+1})$. In other words,

$$m_w^k(e_{q-k+1}) + (q-1)m_1^k(e_{q-k+1}) \geq M = m_w^k(e_q) + (q-1)m_1^k(e_q).$$

Thus, $m_1^k(e_{q-k+1}) \geq m_1^k(e_q)$.

Step 3. $m_w^{q-1}(e_2) \geq m_w^q(e_q)$.

We know that $m_w^{q-1}(e_2) \geq m_w^q(e_2)$. Then, it is sufficient to prove that $m_w^q(e_2) \geq m_w^q(e_q)$.

By definition, $m_w^q(e_2) + m_1^q(e_2) = M - \sum_{i=1}^{q-2} m_1^i(e_{q-i+1})$.

By Step 1, $m_w^q(e_2) + m_1^q(e_2) \geq M - (q-2)m_1^{q-1}(e_2) \geq M - (q-2)m_1^q(e_2)$.

Then, $m_w^q(e_2) + (q-1)m_1^q(e_2) \geq M = m_w^q(e_q) + (q-1)m_1^q(e_q)$.

Thus, $m_w^q(e_2) \geq m_w^q(e_q)$.

By Steps 1, 2 and 3 we conclude that φ is a selection from the *Equal Split Guarantee* solution. We still have to prove that φ is also a selection from the *Pareto* solution.

Step 4. $\varphi \subseteq P$.

Let $\{z\} = \varphi(e)$, and suppose that $z \notin P(e)$. Then there exists $z' \in Z(e)$ such that $z'_i R_i z_i$ for all $i \in Q$ and there exists $j \in Q$ such that $z'_j P_j z_j$. By feasibility, the existence of z' means that in z' the winner was a loser in z and one of the losers at z' was the winner at z. Let k be the winner at z' that in z was a loser, then $m'_k \geq m_w^k(e_{q-k+1})$. Let q be the winner at z that in z' is a loser, then $m'_q \geq m_l^{q-1}(e_2)$. If we prove that $m_w^k(e_{q-k+1}) \geq m_w^{q-1}(e_2)$, we obtain a contradiction to the feasibility of z'.

We know that, $m_w^{q-1}(e_2) + m_l^{q-1}(e_2) = M - \sum_{i=1}^{q-2} m_l^i(e_{q-i+1})$. By Step 1,

$$m_w^{q-1}(e_2) + m_l^{q-1}(e_2) \leq M - \sum_{i=1}^{k-1} m_l^i(e_{q-i+1}) - (q-k-1)m_l^k(e_{q-k+1}) =$$

$$= m_w^k(e_{q-k+1}) + (q-k)m_l^k(e_{q-k+1}) - (q-k-1)m_l^k(e_{q-k+1}) = m_w^k(e_{q-k+1}) + m_l^k(e_{q-k+1}).$$

Then, $m_w^{q-1}(e_2) + m_l^{q-1}(e_2) \leq m_w^k(e_{q-k+1}) + m_l^k(e_{q-k+1})$. By Step 1, we know that $m_l^{q-1}(e_2) \geq m_l^k(e_{q-k+1})$. Thus, $m_w^k(e_{q-k+1}) \geq m_w^{q-1}(e_2)$.■

Proposition 12. In $\mathcal{E}_1^{MON}$, if $M \geq 0$, the solution φ satisfies *population monotonicity.*

Proof. Let $e_q = (Q,(1,M);R_Q)$ be given, and let $q = |Q|$. Let $e'_{q+1} = (Q',(1,M),R'_{Q'})$ be such that $Q \subseteq Q'$, $|Q'| = q+1$, and $R'_Q = R_Q$. We divide the proof into different steps.

Step 1. Let $k \in Q$ be the agent who receives $m_l^k(e_{q-(k-1)})$, then

$$m_l^k(e_{q-(t-1)}) \leq m_l^k(e_{q-t}) \text{ for } t = 1,..,k-1.$$

By the definition of these numbers, we know that

$$m_w^k(e_{q-(t-1)}) + (q-t)m_l^k(e_{q-(t-1)}) = M - \sum_{i=1}^{t-1} m_l^i(e_{q-i+1})$$

$$m_w^k(e_{q-(t-1)}) + (q-t-1)m_l^k(e_{q-(t-1)}) = M - \sum_{i=1}^{t-1} m_l^i(e_{q-i+1}) - m_l^k(e_{q-(t-1)}).$$

Let $t \in Q$ be the agent who receives $m_l^t(e_{q-(t-1)})$.

Then, $m_l^t(e_{q-(t-1)}) \leq m_l^k(e_{q-(t-1)})$.

Thus,

$$m_w^k(e_{q-(t-1)}) + (q-t-1)m_l^k(e_{q-(t-1)}) \leq M - \sum_{i=1}^{t-1} m_l^i(e_{q-i+1}) - m_l^t(e_{q-(t-1)})$$

$$m_w^k(e_{q-(t-1)}) + (q-t-1)m_l^k(e_{q-(t-1)}) \leq m_w^k(e_{q-t}) + (q-t-1)m_l^k(e_{q-t}).$$

Since $(1,m_w^k(e_{q-(t-1)}))I_k(0,m_l^k(e_{q-(t-1)}))$, and $(1,m_w^k(e_{q-t}))I_k(0,m_l^k(e_{q-t}))$, then

$$m_l^k(e_{q-(t-1)}) \leq m_l^k(e_{q-t}).$$

Step 2. $m_l^k(e_q) \geq m_l^k(e'_{q+1})$ $k = 1,\dots,q$.

We Know that $m_w^k(e'_{q+1}) + qm_l^k(e'_{q+1}) = M$. Then

$$m_w^k(e'_{q+1}) + (q-1)m_l^k(e'_{q+1}) = M - m_l^k(e'_{q+1}).$$

By monotonicity of preferences in the indivisible good, and $M \geq 0$, $M - m_l^k(e'_{q+1}) \leq M$. Then,

$$m_w^k(e'_{q+1}) + (q-1)m_l^k(e'_{q+1}) \leq m_w^k(e_q) + (q-1)m_l^k(e_q). \text{ Thus, } m_l^k(e_q) \geq m_l^k(e'_{q+1})$$

Step 3. $m_l^k(e_{q-t}) \geq m_l^k(e'_{q+1-t}) \quad k = t,\ldots,q.$

We prove this Step by induction on t.

(1) $m_l^k(e_{q-1}) \geq m_l^k(e'_q) \quad k = 1,\ldots,q$

We know that $m_w^k(e_{q-1}) + (q-2)m_l^k(e_{q-1}) = M - m_l^1(e_q) \geq M - m_l^k(e_q)$.
By Step 1, $m_l^k(e_q) \leq m_l^k(e_{q-1})$, then, $m_w^k(e_{q-1}) + (q-2)m_l^k(e_{q-1}) \geq M - m_l^k(e_{q-1})$.
Thus, $m_w^k(e_{q-1}) + (q-1)m_l^k(e_{q-1}) \geq M$. By monotonicity of preferences in the indivisible good, and $M \geq 0$, $M \geq M - m_l^1(e'_{q+1}) = m_w^k(e'_q) + (q-1)m_l^k(e'_q)$.
Then, $m_l^k(e_{q-1}) \geq m_l^k(e'_q)$ $k = 1,\ldots,q$.

(2) $m_l^k(e_{q-2}) \geq m_l^k(e'_{q-1}) \quad k = 2,\ldots,q$

We Know that $m_w^k(e_{q-2}) + (q-3)m_l^k(e_{q-2}) = M - m_l^1(e_q) - m_l^2(e_{q-1})$.

$$M - m_l^1(e_q) - m_l^2(e_{q-1}) \geq M - m_l^1(e_q) - m_l^k(e_{q-1}).$$

$m_w^k(e_{q-2}) + (q-2)m_l^k(e_{q-2}) \geq M - m_l^1(e_q) = m_w^t(e_{q-1}) + (q-2)m_l^t(e_{q-1})$, $t = 1,..,q$.
Let $2 \in Q'$ be the agent who receives $m_l^2(e'_q)$.
If this agent is in Q, then $M - m_l^1(e_q) = m_w^2(e_{q-1}) + (q-2)m_l^2(e_{q-1})$. By (1),
$m_w^2(e_{q-1}) + (q-2)m_l^2(e_{q-1}) \geq m_w^2(e'_q) + (q-2)m_l^2(e'_q) = M - m_l^1(e'_{q+1}) - m_l^2(e'_q)$.
$M - m_l^1(e'_{q+1}) - m_l^2(e'_q) = m_w^k(e'_{q-1}) + (q-2)m_l^k(e'_{q-1})$. Thus,

$m_w^k(e_{q-2}) + (q-2)m_l^k(e_{q-2}) \geq m_w^k(e'_{q-1}) + (q-2)m_l^k(e'_{q-1})$. Then,

$m_l^k(e_{q-2}) \geq m_l^k(e'_{q-1})$ $k = 2,..,q$.
If this agent is not in Q, let $1 \in Q'$ be the agent who receives $m_l^1(e'_{q+1})$ this agent is in Q, and $m_l^1(e'_{q+1}) \geq m_l^1(e'_q)$. Then, by (1),

$$M - m_l^1(e_q) = m_w^1(e_{q-1}) + (q-2)m_l^1(e_{q-1}) \geq m_w^1(e'_q) + (q-2)m_l^1(e'_q),$$

$$m_w^1(e'_q) + (q-2)m_l^1(e'_q) = M - m_l^1(e'_{q+1}) - m_l^1(e'_q) \geq M - m_l^1(e'_{q+1}) - m_l^1(e'_{q+1}).$$

Since $m_l^1(e'_{q+1}) \leq m_l^2(e'_q)$, $M - m_l^1(e'_{q+1}) - m_l^1(e'_{q+1}) \geq M - m_l^1(e'_{q+1}) - m_l^2(e'_q)$.

$$M - m_l^1(e'_{q+1}) - m_l^2(e'_q) = m_w^k(e'_{q-1}) + (q-2)m_l^k(e'_{q-1}).$$

Thus, $m_w^k(e_{q-2}) + (q-2)m_l^k(e_{q-2}) \geq m_w^k(e'_{q-1}) + (q-2)m_l^k(e'_{q-1})$.

Then, $m_l^k(e_{q-2}) \geq m_l^k(e'_{q-1})$ $k = 2,..,q$.

Suppose that $m_l^k(e_{q-t}) \geq m_l^k(e'_{q+1-t})$ $k = t,...,q$. We should then prove that $m_l^k(e_{q-(t+1)}) \geq m_l^k(e'_{q+1-(t+1)})$ $k = t+1,...,q$.

We know that $m_w^k(e_{q-(t+1)}) + (q-t-2)m_l^k(e_{q-(t+1)}) = M - \sum_{i=1}^{t+1} m_l^i(e_{q-(i-1)})$.

$$M - \sum_{i=1}^{t+1} m_l^i(e_{q-(i-1)}) = M - \sum_{i=1}^{t} m_l^i(e_{q-(i-1)}) - m_l^{t+1}(e_{q-t}) \geq$$

$$\geq M - \sum_{i=1}^{t} m_l^i(e_{q-(i-1)}) - m_l^k(e_{q-t}) \geq M - \sum_{i=1}^{t} m_l^i(e_{q-(i-1)}) - m_l^k(e_{q-(t+1)})$$

Thus, $m_w^k(e_{q-(t+1)}) + (q-t-1)m_l^k(e_{q-(t+1)}) \geq M - \sum_{i=1}^{t} m_l^i(e_{q-(i-1)})$.

Let $s \in Q$ be the agent who receives $m_l^s(e'_{q+1-(h-1)})$ with $h \leq t+1$ in economy e'_{q+1}, and receives $m_l^s(e_{q-s})$ with $s \geq t$ in economy e_q. Thus, $M - \sum_{i=1}^{t} m_l^i(e_{q-(i-1)}) = m_w^s(e_{q-t}) + (q-t-1)m_l^s(e_{q-t})$. By induction hypothesis, $m_l^s(e_{q-t}) \geq m_l^s(e'_{q+1-t})$. Then,

$$m_w^s(e_{q-t}) + (q-t-1)m_l^s(e_{q-t}) \geq m_w^s(e'_{q+1-t}) + (q-t-1)m_l^s(e'_{q+1-t}).$$

$$m_w^s(e'_{q+1-t}) + (q-t-1)m_l^s(e'_{q+1-t}) = M - \sum_{i=1}^{t} m_l^i(e'_{q+1-(i-1)}) - m_l^s(e'_{q+1-t}) \geq$$

$$M - \sum_{i=1}^{t} m_l^i(e'_{q+1-(i-1)}) - m_l^{t+1}(e'_{q+1-t}) = m_w^k(e'_{q+1-(t+1)}) + (q-t-1)m_l^k(e'_{q+1-(t+1)}).$$

Thus,

$$m_w^k(e_{q-(t+1)}) + (q-t-1)m_l^k(e_{q-(t+1)}) \geq m_w^k(e'_{q+1-(t+1)}) + (q-t-1)m_l^k(e'_{q+1-(t+1)}).$$

Then, $m_l^k(e_{q-(t+1)}) \geq m_l^k(e'_{q+1-(t+1)})$ $k = t+1,\dots,q$.

Step 4. Let $k \in Q$ be the agent who receives $m_l^k(e_{q-(k-1)})$ in economy e_q, and receives $m_l^k(e'_{q+1-(t-1)})$ in economy e'_{q+1}, with $q-(k-1) < q+1-(t-1)$. Then,

$$m_l^k(e_{q-(k-1)}) \geq m_l^k(e'_{q+1-(t-1)}).$$

Since $q-(k-1) < q+1-(t-1)$, $t \leq k$, then, by Step 1, $m_l^k(e_{q-(k-1)}) \geq m_l^k(e_{q-(t-1)})$ and by Step 3, $m_l^k(e_{q-(t-1)}) \geq m_l^k(e'_{q+1-(t-1)})$.

Step 5. Let $k \in Q$ be the agent who receives $m_l^k(e_{q-(k-1)})$ in economy e_q, and receives $m_l^k(e'_{q+1-(t-1)})$ in economy e'_{q+1}, with $q-(k-1) \geq q+1-(t-1)$. Then,

$$m_l^k(e_{q-(k-1)}) \geq m_l^k(e'_{q+1-(t-1)}).$$

We know that $m_w^k(e_{q-(k-1)}) + (q-k)m_l^k(e_{q-(k-1)}) = M - \sum_{i=1}^{k-1} m_l^i(e_{q-(i-1)})$. Then,

$$m_w^k(e_{q-(k-1)}) + (q+1-t)m_l^k(e_{q-(k-1)}) = M - \sum_{i=1}^{k-1} m_l^i(e_{q-(i-1)}) - (t-k-1)m_l^k(e_{q-(k-1)}).$$

Since $m_l^k(e_{q-(k-1)}) \leq m_l^i(e_{q-(k-1)})$ for $i = k,\dots,q$, then,

$$m_w^k(e_{q-(k-1)}) + (q+1-t)m_l^k(e_{q-(k-1)}) \geq M - \sum_{i=1}^{k-1} m_l^i(e_{q-(i-1)}) - \sum_{i=k}^{t-2} m_l^i(e_{q-(k-1)}).$$

By Step 1, $m_l^i(e_{q-(k-1)}) \leq m_l^i(e_{q-(i-1)})$ for $i = k,\dots,t-2$. Then,

$$m_w^k(e_{q-(k-1)}) + (q+1-t)m_l^k(e_{q-(k-1)}) \geq M - \sum_{i=1}^{t-2} m_l^i(e_{q-(i-1)}).$$

Let $s \in Q$be the agent who receives $m_l^s(e'_{q+1-(h-1)})$ with $h \leq t-1$ in economy e'_{q+1}, and receives $m_l^s(e_{q-s})$ with $s \geq t-2$ in economy e_q. Thus, $M - \sum_{i=1}^{t-2} m_l^i(e_{q-(i-1)}) = m_w^s(e_{q-(t-2)}) + (q-t+1)m_l^s(e_{q-(t-2)})$. By Step 3 $m_l^s(e_{q-(t-2)}) \geq m_l^s(e'_{q+1-(t-2)})$. Then,

$$m_w^s(e_{q-(t-2)}) + (q-t+1)m_l^s(e_{q-(t-2)}) \geq m_w^s(e'_{q+1-(t-2)}) + (q-t+1)m_l^s(e'_{q+1-(t-2)}).$$

Thus,

$$m_w^s(e'_{q+1-(t-2)}) + (q-t+1)m_l^s(e'_{q+1-(t-2)}) = M - \sum_{i=1}^{t-2} m_l^i(e'_{q+1-(i-1)}) - m_l^s(e'_{q+1-(t-2)})$$

Then, $m_w^s(e'_{q+1-(t-2)}) + (q-t+1)m_l^s(e'_{q+1-(t-2)}) \geq M - \sum_{i=1}^{t-1} m_l^i(e'_{q+1-(i-1)})$.

$M - \sum_{i=1}^{t-1} m_l^i(e'_{q+1-(i-1)}) = m_w^k(e'_{q+1-(t-1)}) + (q-t+1)m_l^k(e'_{q+1-(t-1)})$. Thus,

$$m_w^k(e_{q-(k-1)}) + (q+1-t)m_l^k(e_{q-(k-1)}) \geq m_w^k(e'_{q+1-(t-1)}) + (q-t+1)m_l^k(e'_{q+1-(t-1)}).$$

Then, $m_l^k(e_{q-(k-1)}) \geq m_l^k(e'_{q+1-(t-1)})$.

Steps 4 and 5 show that any agent who was a loser in economy e_q and is a loser in economy e'_{q+1}, does not benefit from the addition of new agents. The next Steps show that this is also true for the agent who is a loser in one economy and a winner in the other economy, or who, in both economies, is a winner.

Step 6. Suppose that $q \in Q$ receives $m_w^{q-1}(e_2)$ in economy e_q, and receives $m_l^q(e'_{q+1-k})$ in economy e'_{q+1}. Then, $(1,m_w^{q-1}(e_2))R_q(0,m_l^q(e'_{q+1-k}))$.

Let $m' \in \mathbb{R}$ be such that $(1,m_w^{q-1}(e_2))I_q(0,m')$. We know that $m' \geq m_l^{q-1}(e_2)$, and by Step 1 in Proposition 11, $m_l^{q-1}(e_2) \geq m_l^t(e_{q-(t-1)})$ $\quad t = 1,\dots,q-1$.

Let $s \in Q$ be the agent who receives $m_l^s(e_{q-(s-1)})$ in economy e_q, and receives $m_l^s(e'_{q+1-h})$ in economy e'_{q+1} with $h \geq k$. Then, by Steps 4 and 5,

$$m_l^{q-1}(e_2) \geq m_l^s(e_{q-(s-1)}) \geq m_l^s(e'_{q+1-h}). \text{ Since } h \geq k,\ m_l^s(e'_{q+1-h}) \geq m_l^q(e'_{q+1-k}).$$

Thus, $m' \geq m_l^q(e'_{q+1-k})$. Then, $(1,m_w^{q-1}(e_2))R_q(0,m_l^q(e'_{q+1-k}))$.

Step 7. Suppose that $k \in Q$ receives $m_l^k(e_{q-(k-1)})$ in economy e_q, and receives $m_w^q(e'_2)$ in economy e'_{q+1}. Then, $(0,m_l^k(e_{q-(k-1)}))R_k(1,m_w^q(e'_2))$.

We know that $(0,m_l^k(e_{q-(k-1)}))I_k(1,m_w^k(e_{q-(k-1)}))$. By Step 1 in Proposition 11, $m_w^k(e_{q-(k-1)}) \geq m_w^s(e_{q-(s-1)})$ with $k \leq s$. Suppose that $s \in Q$ receives $m_l^s(e'_{q+1-t})$ in economy e'_{q+1}. By Steps 4 and 5, $m_l^s(e_{q-(s-1)}) \geq m_l^s(e'_{q+1-t})$. Thus, $m_w^s(e_{q-(s-1)}) \geq m_w^s(e'_{q+1-t})$. By Step 1 in Proposition 11, $m_w^s(e'_{q+1-t}) \geq m_w^q(e'_2)$. Thus, $m_w^k(e_{q-(k-1)}) \geq m_w^q(e'_2)$. Then, $(0,m_l^k(e_{q-(k-1)}))R_k(1,m_w^q(e'_2))$.

Step 8. Finally, suppose that the winner in economy e_q is the winner in economy e'_{q+1}. In other words, $k \in Q$ receives $m_w^{q-1}(e_2)$ in economy e_q, and receives $m_w^q(e'_2)$ in economy e'_{q+1}. Then, $m_w^{q-1}(e_2) \geq m_w^q(e'_2)$.

Let $q-1 \in Q$ be the agent who receives $m_l^{q-1}(e_2)$ in economy e_q, and receives $m_l^{q-1}(e'_{q+1-k})$ in economy e'_{q+1}. Then, by Steps 4 and 5,

$$m_l^{q-1}(e_2) \geq m_l^{q-1}(e'_{q+1-k}).$$

We know that

$$(1,m_w^{q-1}(e_2))I_{q-1}(0,m_l^{q-1}(e_2))R_{q-1}(0,m_l^{q-1}(e'_{q+1-k}))I_{q-1}(1,m_w^{q-1}(e'_{q+1-k})),$$

and $m_w^{q-1}(e'_{q+1-k}) \geq m_w^q(e'_2)$. Then, $m_w^{q-1}(e_2) \geq m_w^q(e'_2)$.■

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