The Uniform Rule in Economies with Single-Peaked Preferences, Endowments and Population-Monotonicity*

Bernardo Moreno**

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** Bernardo Moreno: University of Alicante.

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ABSTRACT

We consider the problem of fair allocate an infinitely divisible commodity among agents with single-peaked preferences and private endowments. First, we show that the adaptation of the *uniform rule* to this context is the only selection from the *no-envy in redistributions* and *Pareto efficient* solution to depend only on peaks and endowments. Second, we examine the implications of the requirement that a change in the population affect all agents that are present before and after the change in the same direction. We show there is no selection from the *no-envy* and *Pareto efficient* solution satisfying this requirement. However, if a mild additional restriction on the domain is imposed, there are selections from the *individually rational* and *Pareto efficient* solution satisfying it. Finally, we relax the *population-monotonicity* requirement by applying it only when the direction of the inequality between the sum of the endowments and the sum of the peaks does not change. Our main result is that there is only one selection from the *no-envy* and *Pareto efficient* solution satisfying this property. It is the *uniform rule*.

Keywords: Single-peaked preferences; Endowments; Population-Monotonicity; Uniform Rule.

1.- INTRODUCTION

The problem that we consider, for any finite set of agents with single-peaked preferences and with private endowments of an infinitely divisible commodity, is how to distribute equitably across agents the gains made possible by redistributions. By single-peaked preferences, we mean that each agent has an unique preferred consumption; he prefers more to less up to that point and less to more beyond it[1]. We search for desirable methods of reallocating the commodity.

The model in which there are no private endowments but there is a social endowment to be allocated has been analyzed, among others, by Sprumont (1991) and Thomson (1994a, 1994b, 1995). Recently, a slightly different model, in which preferences are defined over the whole real line is studied in Klaus, Peters and Storcken (1995). They show that the *uniform rule* is the unique rule satisfying *strategy-proofness*, *equal-treatment*[2] and *Pareto efficiency*. Our model can be interpreted in the following way. Consider the situation in which a commodity is initially divided. Since this division is not in general efficient, we can ask how to distribute across agents the gains made possible by redistributions. For instance, consider the pollution problem. Imagine a set of countries -may be, the european community-, each of one with a rate of pollution that

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[1] An example of a situation where such preferences arise is the following. A team of workers has been assigned a task, each one of them has to work a determinated number of hours and they are paid an hourly wage, if their disutility of labor is concave, then their induced preferences over the labor they supply are single-peaked.

[2] **Strategy-proofness** says that, if preferences are private information, in the game where each agent reports his preference it is a (weakly) dominant strategy to reveal one's true preference. **Equal-treatment** says if the individual endowments and preferences of two agents are equal up to a translation, then each agent should be indifferent between his own allocation and the translated allocation of the other agent.

depends of the development of this country. In general, countries more developments -and, for instance richer- than others has more pollution. The more development countries are willing to give this pollution to other country and pay money for this. On the other hand, the less development countries are willing to receive this pollution and an amount of money. Note that the pollution can not be through away. Their preferences over pollution are single-peaked.

As in these earlier studies, our approach is axiomatic. Our main concern is to study the behavior of solutions when the number of agents and, since agents have their own endowments, possibly the sum of endowments change. We also look for solutions that depend only on preferred consumptions and endowments. We first consider the requirement of *endowment-peak-only*, which says that if a redistribution is chosen, then if the preferences change but the preferred consumptions and the private endowments are the same, the initial redistribution still has to be chosen. We are also interested in solutions satisfying *Pareto efficiency* and desirable distributional requirements. Since *no-envy* in final consumptions is in general, incompatible with *individual rationality*, we will ask for allocations being obtained through an *envy-free redistribution*: an allocation is obtained through an *envy-free redistribution* if no agent would prefer someone else's redistribution to his own. When we ask whether *endowment-peak-only* is compatible with *no-envy in redistributions* and *Pareto efficiency*, the answer is yes: the adaptation of the *uniform rule* to this context is the only selection from the *no-envy in redistributions* and *Pareto efficient* solution satisfying *endowment-peak-only*. Moreover, the *uniform rule* is *individually rational* (Theorem 1 below).

Next, we consider the requirement of *population-monotonicity* which says that a change in the population should affect all agents that are present before and after the change in the same direction. We first ask whether *population-monotonic* selections from the *no-envy in redistribution* and *Pareto efficient* solution exist. The answer is that there is no such selection. However, we show that there exist selections from the *individually rational* and *Pareto efficient* solution satisfying *population-monotonicity*. We provide a solution, the *proportion-sacrifice* solution, which satisfies these three requirements under a minor additional restriction on preferences.

If we examine the proof of the negative result, we realize that the difficulty occurs whenever the change in population reverses the direction of the inequality between the sum of preferred consumptions and the sum of the endowments. This suggests weakening *population-monotonicity* by applying it only when such changes do not occur. Suppose that initially the sum of endowments is smaller than the sum of the preferred consumptions; then if new agents whose endowments are smaller than their preferred consumptions come in, the direction of the inequality does not change. Conversely, suppose that initially the sum of endowment is greater than the sum of the preferred consumptions; then if new agents whose endowments are greater than their preferred consumptions come in, again the direction of the inequality does not change. Note that it is not clear what happen in the first case, with the arrival of a agent with his endowment greater than his preferred consumption, or in the second case, with the arrival of a agent with his endowment greater than his preferred consumption. We formulate a weaker requirement that applies only when the direction of the inequality between the sum of the endowments and the sum of preferred consumptions does not change. When combined with efficiency, the requirement says that the arrival (departure) of agents whose endowments are smaller than their preferred consumptions when initially more of commodity would be socially desirable, all agents initially present are made worse (better) off. The arrival (departure) of this kind of agents is a bad (good) news. In contrast, the arrival (departure) of agents whose endowments are greater than their preferred consumptions when initially more of commodity would be socially desirable, means that all agents initially present are made better (worse) off. The arrival (departure) of this kind of agents is a good (bad) news. We refer to this property as *one-sided population-monotonicity*.

Finally, we ask whether there are *one-sided population-monotonic* selections from the *no-envy in redistributions* and *Pareto efficient* solution. The answer is yes: the *uniform rule* is the only selection from the *no-envy in redistributions* and *Pareto efficient* solution satisfying *one-sided population-monotonicity* (Theorem 2 below)

The paper proceeds as follows. In section 2, we introduce the model. In section 3, we show that the adaptation of the *uniform rule* to this context is the only selection from the *no-envy* *in redistributions* and *Pareto efficient* solution to depend only on peaks and endowments. In section 4, we examine the implications of the requirement that a change in the population affect all agents that are present before and after the change in the same direction. We show there is no selection from the *no-envy* and *Pareto efficient* solution satisfying this requirement. However, if a mild additional restriction on the domain is imposed, there are selections from the *individually rational* and *Pareto efficient* solution satisfying it. In section 5, we relax the *population-monotonicity* requirement by applying it only when the direction of the inequality between the sum of the endowments and the sum of the peaks does not change. Our main result is that there is only one selection from the *no-envy* and *Pareto efficient* solution satisfying this property. It is the *uniform rule*. Finally, in section 6, we conclude.

2.- THE MODEL

The model is as in Thomson (1995). There is an infinite population of "potential agents", indexed by the positive integers, $\mathbb{N}$. Let $\tilde{\mathbb{N}}$ denote the class of finite subsets of $\mathbb{N}$. Each agent $i \in \mathbb{N}$ is equipped with a continuous preference relation $\mathbf{R_i}$ defined over $\mathbb{R}_+$ and **an endowment** $\omega_i \in \mathbb{R}_+$. Let $\mathbf{P_i}$ denote the strict preference relation associated with R_i, and $\mathbf{I_i}$ the indifference relation. These preference relations are **single-peaked**: for each R_i, there is a number, denoted by $\mathbf{p(R_i)} \in \mathbb{R}_+$, such that for all z_i, $z'_i \in \mathbb{R}_+$ if $z'_i < z_i \leq p(R_i)$, or $p(R_i) \leq z_i < z'_i$, then $z_i P_i z'_i$. The preference relation R_i can be described in terms of the function $r_i: \mathbb{R}_+ \cup \{\infty\} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ defined as follows: given $z_i \leq p(R_i)$, $r_i(z_i) \geq p(R_i)$ and $z_i I_i r_i(z_i)$ if such a number exists, and $r_i(z_i) = \infty$ otherwise; given $z_i \geq p(R_i)$, $r_i(z_i) \leq p(R_i)$ and $z_i I_i r_i(z_i)$ if such a number exists, and $r_i(z_i) = 0$ otherwise. (The number $r_i(z_i)$ is the consumption "on the other side" of agent i's preferred consumption that he finds indifferent to z_i, if such a consumption exists; it is 0 or ∞ otherwise.) Let $r_i(\infty) = \lim_{z_i \rightarrow \infty} r_i(z_i)$. Let $\Re$ be the class of single-peaked preference relations on $\mathbb{R}_+$. For all $N \in \tilde{\mathbb{N}}$, let $\Re^N$ denote the cartesian product of $|N|$ copies of $\Re_+$, indexed by the members of N. Similarly, let $\mathbb{R}^N_+$ denote the cartesian product of $|N|$ copies of $\mathbb{R}$. We write $R_N = (R_i)_{i \in N}$, $p(R_N) = (p(R_i))_{i \in N}$, and $\omega_N = (\omega_i)_{i \in N}$. **An economy** is a pair $\mathbf{e = (R_N, \omega_N)} \in \Re^N \times \mathbb{R}^N_+$.

For all $N \in \tilde{\mathbb{N}}$ and $e = (R_N, \omega_N) \in \Re^N \times \mathbb{R}^N_+$, let $\mathbf{Z(e)} = \{(z_i)_{i \in N} \in \mathbb{R}^N_+ : \Sigma_N z_i = \Sigma_N \omega_i\}$ denote **the set of feasible allocations of e.**

A **solution** is a mapping φ which associates with every $N \in \tilde{\mathbb{N}}$ and $e = (R_N, \omega_N) \in \Re^N \times \mathbb{R}^N_+$, a non-empty subset of Z(e). The intersection of two solutions φ and φ' is denoted $\varphi\varphi'$.

For all $N \in \tilde{\mathbb{N}}$ and $e = (R_N, \omega_N) \in \Re^N \times \mathbb{R}^N_+$, **the allocation $z \in Z(e)$ is Pareto efficient for e** if $z \in Z(e)$ and there is no $z' \in Z(e)$ with $z'_i R_i z_i$ for all $i \in N$, and $z'_i P_i z_i$ for some $i \in N$. Let **P(e)** be the set of these allocations. It is **individually rational for e** if $z \in Z(e)$, and $z_i R_i \omega_i$ for all $i \in N$. Let **I(e)** be the set of these allocations. **The allocation $z \in Z(e)$ is envy-free in final consumptions for e** if $z \in Z(e)$ if for all $i,j \in N$, $z_i R_i z_j$ (Foley (1967)).

Note that when all agents' endowments are greater than their peaks or when all agents' endowments are smaller than their peaks, there is a unique *individually rational* allocation which is the profile of endowments.

Note that, in our context, it is not meaningful to ask for *no-envy* in final consumptions since the intersection of the set of *envy-free* and the set of *individually rational* allocations may be empty. However, we can ask how to distribute equitably across agents the gains made possible by redistributions. We will therefore be looking for notions of **equitable redistributions**. A permutation of order n is a bijection Π from the set of agents to itself. Let $\prod$ be the collection of all permutations of order n. For all $N \in \tilde{N}$, let $T \subset \mathbb{R}^N$ be the set of **feasible net redistributions:** $T=\{t \in \mathbb{R}^N: \Sigma_N t_i = 0\}$.

Definition. For all $N \in \tilde{N}$ and $e=(R_N,\omega_N) \in \mathfrak{R}^N \times \mathbb{R}^N_+$, the net redistribution **$t \in T$ is envy-free for e**[3] if $\omega + t \in Z(e)$ and, for no $i \in N$, for no permutation, $\Pi \in \prod^n$, such that $\omega_i + \Pi_i(t) \in \mathbb{R}_+$, we have $(\omega_i + \Pi_i(t)) P_i (\omega_i + t_i)$. Let **F(e)** be the set of allocations obtained through envy-free redistributions.

For all $N \in \tilde{N}$ and $e=(R_N,\omega_N) \in \mathfrak{R}^N \times \mathbb{R}^N_+$, the set of *Pareto efficient* allocations obtained through *envy-free redistributions*[4] is non-empty. Indeed, the **uniform allocation**, introduced next, always exists, and satisfies *no-envy* and *Pareto efficiency*.

Definition. For all $N \in \tilde{N}$ and $e=(R_N,\omega_N) \in \mathfrak{R}^N \times \mathbb{R}^N_+$, the allocation $z \in Z(e)$ **is the uniform allocation of e** if there is $\lambda \in \mathbb{R}_+$ such that

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[3] Note that the concept of an *envy-free redistribution* is the same as the concept of an *envy-free trade* (Kolm (1972); Schmeidler and Vind (1972)), but we have preferred to use the word redistribution since it captures better the normative side of the model.

[4] In the sequel, when we talk about envy-free allocations, we mean allocations obtained through envy-free redistributions.

(i) when $\Sigma_N p(R_i) \geq \Sigma_N \omega_i$, then for all $i \in N$, $z_i = \min\{\lambda + \omega_i, p(R_i)\}$

(ii) when $\Sigma_N p(R_i) \leq \Sigma_N \omega_i$ then for all $i \in N$, $z_i = \max\{\omega_i - \lambda, p(R_i)\}$.

Let **U(e)** denote the uniform allocation of e.

The idea behind the *uniform rule* is the following. Suppose that the sum of the peaks is greater than the sum of the endowments. All agents whose endowments are greater than their peaks receive their peaks. Starting from their endowments, all agents whose endowments are smaller than their peaks receive the same amount until all the commodity is allocated or until the agent with the smaller distance between his endowment and his peak, reaches his peak. In the latter case, the agent with the smaller distance between his endowment and his peak does not receive anything more. This process continues until all the commodity is allocated. Conversely, suppose that the sum of the peaks is smaller than the sum of the endowments. All agents whose endowments are smaller than their peaks receive their peaks. Starting from their endowments, all agents whose endowments are greater than their peaks give the same amount until all the commodity is reallocated or until the agent with the smaller distance between his endowment and his peak, reaches his peak. In the latter case, the agent with the smaller distance between his endowment and his peak does not give anything more. This process continues until all the commodity is reallocated.

3.- ENDOWMENT-PEAK-ONLY

A noteworthy feature of the *uniform rule* is that it depends only on peaks and endowments. In fact, the *uniform rule* is the only selection from the *no-envy* and *Pareto efficient* solution to depend only on peaks and endowments. The extension to this context of *peak-only*[5] (Sprumont (1991); and Thomson (1990)) is what we call **endowment-peak-only**.

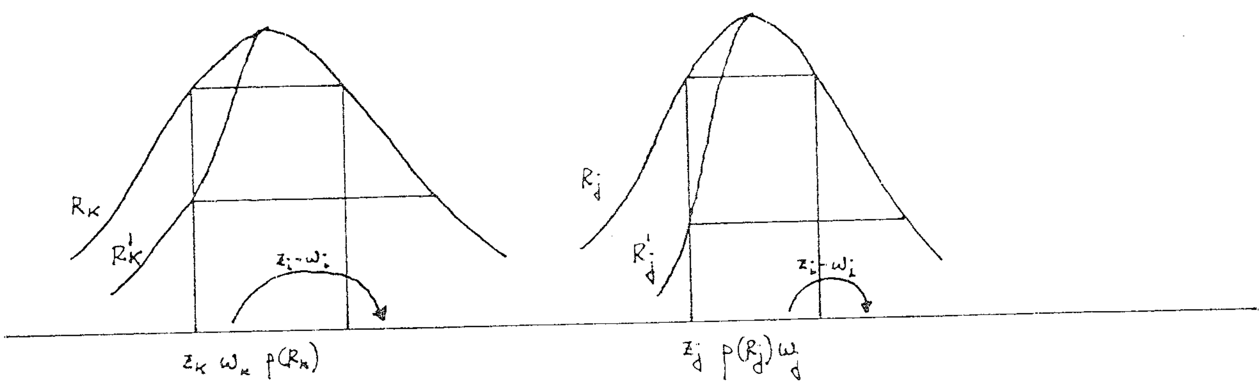

Figure 1. Characterization of the *uniform rule* on the basis of *endowment-peak-only* (Theorem 1, Claims 1 and 2).

Endowment-peak-only: A solution φ satisfies the *endowment-peak-only* property if for all $N \in \tilde{N}$ and $e=(R_N,\omega_N)$, $e'=(R'_N,\omega'_N) \in \Re^N \times \mathbb{R}^N_+$, if $p(R^N)=p(R'_N)$ and $\omega_N=\omega'_N$, then $\varphi(e)=\varphi(e')$.

When we impose *endowment-peak-only* on selections from the *no-envy* and *Pareto efficient* solution, we obtain the first characterization of the *uniform rule*:

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[5] **Peak-only:** A solution φ satisfies the *peak-only* property if for all $N \in \tilde{N}$ and $e=(R_N,\omega_N)$, $e'=(R'_N,\omega'_N) \in \Re^N \times \mathbb{R}^N_+$, if $p(R_N)=p(R'_N)$, then $\varphi(e)=\varphi(e')$.

Theorem 1. The *uniform rule* is the only selection from the *no-envy* and *Pareto efficient* solution satisfying *endowment-peak-only*.

Proof: Let $\varphi \subseteq FP$ satisfy *endowment-peak-only*. Let $N \in \tilde{N}$ and $e=(R_N,\omega_N) \in \Re^N \times \mathbb{R}^N_+$ be such that $\Sigma_N p(R_i) \geq \Sigma_N \omega_i$. The proof for the case $\Sigma_N p(R_i) \leq \Sigma_N \omega_i$ is similar and therefore it is omitted. Suppose, to get a contradiction, that $\varphi(e) \neq U(e)$. Let $z=\varphi(e)$. Since $\varphi \subseteq P$ this implies that for all $i \in N$, $z_i \leq p(R_i)$. The proof of the Theorem follows from the four claims below:

Claim 1. (Figure 1) **For all $j \in N$ such that $\omega_j \geq p(R_j)$, $z_j = p(R_j)$.** Suppose, to get a contradiction, that for some $j \in N$ such that $\omega_j \geq p(R_j)$, $z_j < p(R_j)$. By feasibility, for some $i \in N$ such that $\omega_i \leq p(R_i)$, $z_i > \omega_i$. Since $\varphi \subseteq F$, $z_j R_j (\omega_j + (z_i - \omega_i))$. Let now $e'=(R'_N,\omega'_N) \in \Re^N \times \mathbb{R}^N_+$ be such that $p(R_N)=p(R'_N)$, $\omega_N = \omega'_N$, and $(\omega_j+(z_i-\omega_i))P'_j z_j$. Since φ is *endowment-peak-only*

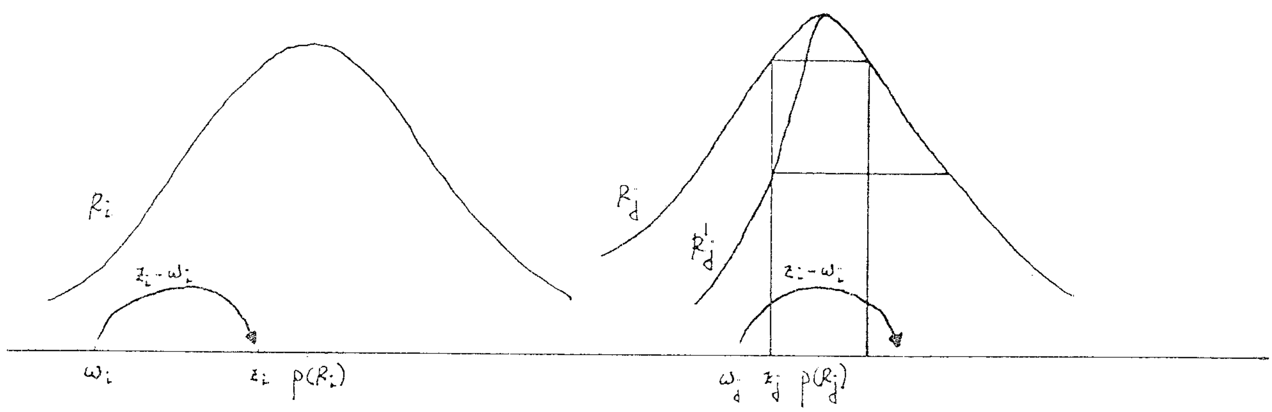

Figure 2. Characterization of the *uniform rule* on the basis of *endowment-peak-only* (theorem 1, claim 4).

$\varphi(e')=\varphi(e)$, so that $z \in \varphi(e')$. But since $(\omega_j+(z_i-\omega_i))P'_j z_j$, we obtain a contradiction to $\varphi \subseteq F$.

Claim 2. (Figure 1) **For all $k \in N$ such that $\omega_k \leq p(R_k)$, $z_k \geq \omega_k$.** Suppose, to get a contradiction, that for some $k \in N$ such that $\omega_k \leq p(R_k)$, $z_k < \omega_k$. By feasibility, for some $i \in N \setminus \{k\}$ such that $\omega_i \leq p(R_i)$, $z_i > \omega_i$. Since $\varphi \subseteq F$, $z_k R_k (\omega_k + (z_i - \omega_i))$. Note that this is possible only if $(\omega_k+(z_i-\omega_i)) \geq r(z_k)$. Let $e'=(R'_N,\omega'_N) \in \Re^N \times \mathbb{R}^N_+$ be such that $p(R_N)=p(R'_N)$, $\omega_N=\omega'_N$, and $(\omega_k+(z_i-$

$\omega_i))P'_k z_k$. Since φ is *endowment-peak-only* $\varphi(e')=\varphi(e)$, so that $z\in\varphi(e')$. But since $(\omega_k+(z_i-\omega_i))P'_k z_k$ we obtain a contradiction to $\varphi\subseteq F$.

Claim 3. For all i, $j\in N$ such that $\omega_i\leq p(R_i)$, $\omega_j\leq p(R_j)$ and $p(R_i)-\omega_i\geq p(R_j)-\omega_j$, we have $z_i-\omega_i\geq z_j-\omega_j$. Suppose, to get a contradiction, that for some i, $j\in N$ such that $\omega_i\leq p(R_i)$, $\omega_j\leq p(R_j)$, and $p(R_i)-\omega_i\geq p(R_j)-\omega_j$, we have $z_i-\omega_i<z_j-\omega_j$. Since $\varphi\subseteq P$, it is clear that $(\omega_i+(z_j-\omega_j))P_i z_i$, in violation to $\varphi\subseteq F$.

Claim 4. (Figure 2) **For all i, $j\in N$ such that $\omega_i\leq p(R_i)$, $\omega_j\leq p(R_j)$ and $p(R_i)-\omega_i\geq p(R_j)-\omega_j$, we have $z_i-\omega_i>z_j-\omega_j$ if and only if $z_j=p(R_j)$.** Suppose, to get a contradiction, that for some i, $j\in N$ such that $\omega_i\leq p(R_i)$, $\omega_j\leq p(R_j)$, and $p(R_i)-\omega_i\geq p(R_j)-\omega_j$, we have $z_i-\omega_i>z_j-\omega_j$ and $z_j<p(R_j)$. Since $\varphi\subseteq F$, $z_j R_j(\omega_j+(z_i-\omega_i))$. Note that this is possible only if $(\omega_j+(z_i-\omega_i))\geq r(z_j)$. Let now $e'=(R'_N,\omega'_N)\in\Re^N\times\mathbb{R}^N_+$ be such that $p(R_N)=p(R'_N)$, $\omega_N=\omega'_N$, and $(\omega_j+(z_i-\omega_i))P'_j z_j$. Since φ is *endowment-peak-only*, $\varphi(e')=\varphi(e)$, so that $z\in\varphi(e')$. But since $(\omega_j+(z_i-\omega_i))P'_j z_j$, we obtain a contradiction to $\varphi\subseteq F$. **Q.E.D.**

4.- POPULATION-MONOTONICITY

We now consider changes in the number of agents and, since newcomers may arrive with their own endowments, possibly in the amount of the commodity to be allocated. We will demand that all agents initially present be affected in the same direction by the arrival of new agents. A general version of this requirement was proposed and studied by Thomson (1983a, 1983b). Chun (1986) considered a quasi-linear model of cost allocation and proposed the condition that all agents be affected in the same direction by the arrival of additional agents. The next requirement is an adaptation of Thomson (1995) to the context in which agents are entitled to endowments.

Population-monotonicity: For all $N, N' \in \tilde{N}$ with $N' \subset N$, for all $e=(R_N, \omega_N) \in \Re^N \times \mathbb{R}^N_+$, either $\varphi_i(e) R_i \varphi_i(e')$ for all $i \in N'$, where $e'=(R_{N'}, w_{N'})$ or $\varphi_i(e') R_i \varphi_i(e)$ for all $i \in N'$.

The *uniform rule* is an appealing solution since it satisfies several desirable properties, as we have seen. But, unfortunately, it is not *population-monotonic*, as we now show by means of an example:

Example 1. Let $N=\{1,2,3\}$ and $e=(R_N, \omega_N) \in \Re^N \times \mathbb{R}^N_+$ be such that $p(R_N)=(1,2,2)$ and $\omega_N=(0.6, 2.3, 2.3)$. Note that $\Sigma_N p(R_i)=5<5.2=\Sigma_N \omega_i$. We have $U(e)=(1, 2.1, 2.1)$. Let $N'=\{1,2\}$, and $e'=(R_{N'}, \omega_{N'})$. Note that $\Sigma_{N'} p(R_i)=3>2.9=\Sigma_{N'} \omega_i$. We have $U(e')=(0.9, 2)$. Thus, in the change from e to e', agent 1 loses and agent 2 gains, in violation of *population-monotonicity*.

But the situation is worse, since this is not just a problem with the *uniform rule*. *Population-monotonicity* is a strong requirement when combined with *no-envy* and *Pareto efficiency*. In fact, there is no solution satisfying these three requirements, as we show in the next proposition.

Proposition 1. There is no selection from the *no-envy* and *Pareto efficient* solution satisfying *population-monotonicity*.

Proof: Let $N=\{1,2,3,4\}$ and $e=(R_N,\omega_N)\in\Re^N\times\mathbb{R}^N_+$ be such that $p(R_1)=5$, with $5.35P_1 4.75$, $p(R_2)=5$, with $0.6I_2 5.35$, R_2 be representable by a function that is linear on each of the intervals $[0,5]$ and $[5,36]$, $p(R_3)=10$, with $9.75P_3 10.1$, $p(R_4)=10$, with $14.25I_4 9.75$, R_4 be representable by a function that is linear on each of the intervals $[0,10]$ and $[10,36]$, and $\omega_N=(2.5,2.5,12,12)$. Let $\varphi\subseteq FP$ be *population-monotonic* and $z=\varphi(e)$. Note that $\Sigma_N p(R_i)=30>29=\Sigma_N\omega_i$, and since $\varphi\subseteq P$, $z_i\leq p(R_i)$ for all $i\in N$. Since $\varphi\subseteq F$, $\omega_1=\omega_2$ and $p(R_1)=p(R_2)$, we have $z_1=z_2$. Since $\varphi\subseteq F$, $\omega_3=\omega_4$ and $p(R_3)=p(R_4)$, we have $z_3=z_4$. Since $\varphi\subseteq F$, $z_1=z_2$, $z_3=z_4$, $14.25I_4 9.75$, and given the definition of R_4, we have $z_2\in[4.5,4.75]$ and $z_4\in[9.75,10]$.

Let us now introduce an additional agent, agent 5. Let $R_5\in\Re$ be such that $p(R_5)=5$ and let $\omega_5=7$. Let $N'=\{1,2,3,4,5\}$ and $e'=(R_{N'},\omega_{N'})\in\Re^{N'}\times\mathbb{R}^{N'}_+$ be the resulting economy. Let $z'=\varphi(e')$. Note that $\Sigma_{N'}p(R_i)=35<36=\Sigma_{N'}\omega_i$, and since $\varphi\subseteq P$, $z'_i\geq p(R_i)$ for all $i\in N'$. Since $\varphi\subseteq F$, $\omega_1=\omega_2$ and $p(R_1)=p(R_2)$, we have $z'_1=z'_2$. Since $\varphi\subseteq F$, $\omega_3=\omega_4$ and $p(R_3)=p(R_4)$, we have $z'_3=z'_4$. Since $\varphi\subseteq F$, $z'_1=z'_2$, $z'_3=z'_4$, $0.6I_2 5.35$, and given the definition of R_2, we have $z'_2\in[5,5.35]$, $z'_4\in[10.1,31/3]$ and $z'_5\in[5.1,16/3]$. Since $5.35P_1 4.75$ and $9.75P_3 10.1$, in the change from e to e', agent 1 gains and agent 3 loses, in violation of *population monotonicity*.

Q.E.D.

For the problem of fair division, Thomson (1995) stated that there is no selection from the *no-envy* (in final consumptions) solution satisfying *population-monotonicity*. The counterpart of this proposition is not true in our context, since the solution that always selects the profile of endowments satisfies *no-envy in redistributions* and *population monotonicity*.

These impossibilities are disappointing and since we do not want to drop the Pareto efficient requirement, we can try to relax either the distributional requirement or the monotonicity requirement. Let us start with the distributional requirement. Note that when combined with *efficiency*, *no-envy* is a strong requirement since as soon as two agents have the same endowments and the same peaks, they have to consume the same amount. If we examine the proof of the negative result stated in Proposition 1, we realize that one of the features of the proof is that we deal with economies in which there are agents with the same endowments and peaks, but whose preferences are otherwise, very different. Thomson (1995) showed for the problem of fair division that there are not *population-monotonic* selections from the *individual rationality from equal division* solution. The question then is whether, in our context, there exist selections from the *individually rational* and *Pareto efficient* solution satisfying *population-monotonicity*. The answer is yes. There are such solutions, if a minor restriction is imposed on the domain. The solution defined next is in the spirit of the *equal-sacrifice* solution (Thomson, 1995). It involves evaluating an allocation $z \in Z(e)$ on the basis of $c_i(z_i)=|r_i(z_i)-z_i|$ and $c_i(\omega_i)=|r_i(\omega_i)-\omega_i|$. The number $c_i(z_i)$ -respectively $c_i(\omega_i)$-, which is the size of agent i's upper contour set at z_i -respectively at ω_i-, can be interpreted as a measure of his "sacrifice at z -respectively ω-" (Thomson, 1995). The idea is to compare the sacrifice at z_i with the sacrifice at ω_i and select an efficient allocation at which the ratio $c_i(z_i)/c_i(\omega_i)$ be equal across agents. In order to avoid the difficulties that occur with economies for which the sacrifices are infinite for some agents at the endowment, we restrict to the domain of economies for which for all $i \in \mathbb{N}$, $r_i(\omega_i) < \infty$.

Definition. For all $N \in \tilde{N}$ and $e=(R_N,\omega_N) \in \Re^N \times \mathbb{R}^N_+$, the allocation $\mathbf{z} \in \mathbf{Z(e)}$ **is the proportional-sacrifice allocation of e**, if there exist $\lambda \in [0,1]$ such that

(i) when $\Sigma_N p(R_i) \geq \Sigma_N \omega_i$ then for all $i \in N$ such that $\omega_i \neq p(R_i)$, $\lambda=(r_i(z_i)-z_i)/|r_i(\omega_i)-\omega_i|$ and $z_i=p(R_i)$ otherwise

(ii) when $\Sigma_N p(R_i) \leq \Sigma_N \omega_i$ then for all $i \in N$ such that $\omega_i \neq p(R_i)$, $\lambda=(z_i-r_i(z_i))/|r_i(\omega_i)-\omega_i|$ and $z_i=p(R_i)$ otherwise.

Let **Prosac(e)** denote the *proportional-sacrifice* allocation of e.

On the domain of economices for which for all $i \in \mathbb{N}$, $r_i(\omega_i) < \infty$, the *proportional-sacrifice* solution is a selection from the *individually rational* and *Pareto efficient* solution satisfying *population-monotonicity*. By Proposition 1, we know that the *proportional-sacrifice* solution does not necessarily select allocations obtained through an *envy-free redistribution*.

5.- ONE-SIDED POPULATION-MONOTONICITY

We now return to the negative result stated in Proposition 1, and instead of dropping the *no-envy in redistributions* requirement, we ask whether there are ways of relaxing the monotonicity requirement. If we examine again the proof of the negative result stated in Proposition 1, we realize that another feature of the proof is that the change in population and the accompanying change in the endowments reverses the direction of the inequality between the sum of the peaks and the sum of the endowments. This fact suggests weakening the *population-monotonicity* condition by applying it only when the change in the population produces no such reversals. The following requirement is an adaptation of Thomson (1995) to our context.

One-sided population-monotonicity: For all $N, N' \in \mathbb{N}$ with $N' \subset N$, for all $e=(R,\omega) \in \Re^N \times \mathbb{R}^N_+$ if either (i) $\Sigma_N p(R_i) \leq \Sigma_N \omega_i$ and $\Sigma_{N'} p(R_i) \leq \Sigma_{N'} \omega_i$ or (ii) $\Sigma_N p(R_i) \geq \Sigma_N \omega_i$ and $\Sigma_N p(R_i) \geq \Sigma_{N'} \omega_i$, then for all $i \in N'$, $\varphi_i(e) R_i \varphi_i(e')$, where $e' = (R_{N'}, \omega_{N'})$, or for all $i \in N'$, $\varphi_i(e') R_i \varphi_i(e)$.

We find that several solutions are *one-sided population-monotonic*. Our first example is one of the possible extensions of the *proportional* solution (Thomson, 1995) to our model. Note that we always can define an allocation proportional to the agents' endowments or to the agents' peaks. However, when the sum of the peaks is greater than the sum of the endowments, an allocation proportional to the agents' endowments may not be well-defined if one of the endowments is equal to zero. On the other hand, when the sum of the peaks is smaller than the sum of the endowments, an allocation proportional to the agents' peaks may not be well-defined. In order to avoid these difficulties, we define a solution that it is proportional to the agents' peaks when the sum of the peaks is greater than the sum of the endowments, and it is proportional to the agents' endowments when the sum of the peaks is smaller than the sum of the endowments.

Definition. For all $N \in \tilde{N}$ and $e=(R_N, \omega_N) \in \Re^N \times \mathbb{R}^N_+$, the allocation $z \in Z(e)$ **is the asymmetrically-proportional allocation of e**, if there exists $\lambda \in \mathbb{R}_+$ such that

(i) when $\Sigma_N p(R_i) \geq \Sigma_N \omega_i$, then for all $i \in N$ such that $\omega_i \leq p(R_i)$, $z_i = \max\{\lambda p(R_i), \omega_i\}$ or $z_i = p(R_i)$ otherwise

(ii) when $\Sigma_N p(R_i) \leq \Sigma_N \omega_i$, then for all $i \in N$ such that $\omega_i \geq p(R_i)$, $z_i = \max\{\lambda \omega_i, p(R_i)\}$ or $z_i = p(R_i)$ otherwise.

Let **Apro(e)** denote the *asymmetrically-proportional* allocation of e.

Note that λ in the definition always exists, even if some agent's endowment is zero or some agent's peak is zero. Clearly, the *asymmetrically-proportional* allocation is *individually rational* and *efficient*. Another solution that satisfies *one-sided population-monotonicity* is the adaptation of the *equal-distance* solution (Thomson, 1994a) to our context.

Definition. For all $N \in \tilde{N}$ and $e = (R_N, \omega_N) \in \Re^N \times \mathbb{R}^N_+$, the allocation **$z \in Z(e)$ is the equal-distance allocation of e**, if there exists $d \geq 0$ such that

(i) when $\Sigma_N p(R_i) \geq \Sigma_N \omega_i$, then for all $i \in N$ such that $\omega_i \leq p(R_i)$, $z_i = \max\{\omega_i, p(R_i) - d\}$ and $z_i = p(R_i)$ otherwise

(ii) when $\Sigma_N p(R_i) \leq \Sigma_N \omega_i$, then for all $i \in N$ such that $\omega_i \geq p(R_i)$, $z_i = \min\{\omega_i, p(R_i) + d\}$ and $z_i = p(R_i)$ otherwise.

Let **Dis(e)** denote the *equal-distance* allocation of e.

The *equal-distance* solution is single-valued and produces *individually rational* and *efficient* allocations. It follows from Example 1 that neither the *asymmetrically-proportional* solution nor the *equal-distance* solution are *population-monotonic*. Moreover, neither solution necessarily select allocations obtained through *envy-free redistributions*, as we show by means of the following examples:

Example 2. Let $N = \{1,2,3\}$ and $e = (R, \omega) \in \Re^N \times \mathbb{R}^N_+$ be such that $p(R_N) = (5,5,10)$, $11/2 P_2 9/2$ and $\omega_N = (2,3,14)$. Note that $\Sigma_N p(R_i) = 20 > 19 = \Sigma_N \omega_i$. Let $z = Apro(e)$. Then, we have $z = (9/5, 9/5, 10)$. Since $(\omega_2 + (z_1 - \omega_1)) = 11/2 P_2 9/2$, *Apro(e)* is not an allocation obtained through an *envy-free redistribution*.

Example 3. Let $N=\{1,2,3\}$ and $e=(R,\omega)\in\Re^N\times\mathbb{R}^N_+$ be such that $p(R_N)=(5,5,10)$ and $\omega_N=(1,2,14)$. Note that $\Sigma_N p(R_i)=20>17=\Sigma_N\omega_i$. Let $z=$*Dis(e)*. Then, we have $z=(7/2,7/2,10)$. Since $(\omega_2+(z_1-\omega_1))=9/2P_2 7/2$, *Dis(e)* is not an allocation obtained through an *envy-free redistribution*[6].

All the solutions presented before, the *uniform rule*, the *asymmetrically-proportional* solution and the *equal-distance* solution, satisfy *one-sided population-monotonicity*. However, when we impose *one-sided population-monotonicity* on selections from the *no-envy* and *Pareto efficient* solution, only the *uniform rule* remains acceptable, as we show in the next proposition.

Theorem 2. The *uniform rule* is the only selection from the *no-envy* and *Pareto efficient* solution satisfying *one-sided population-monotonicity*.

Proof: Let $\varphi\subseteq FP$ be a *one-sided population-monotonic* solution. Let $N\in\tilde{N}$ and $e=(R_N,\omega_N)\in\Re^N\times\mathbb{R}^N_+$ be such that $\Sigma_N p(R_i)\geq\Sigma_N\omega_i$. The proof for the case $\Sigma_N p(R_i)\leq\Sigma_N\omega_i$ is similar and therefore it is omitted. Suppose, to get a contradicition, that $\varphi(e)\neq U(e)$. Let $z=\varphi(e)$. Since $\varphi\subseteq P$ this implies that for all $i\in N$, $z_i\leq p(R_i)$. The proof of the Theorem follows from the four claim below:

Claim 1. (Figure 3) **For all $j\in N$ such that $\omega_j\geq p(R_j)$, $z_j=p(R_j)$.** Suppose, by contradiction, that for some $j\in N$ such that $\omega_j\geq p(R_j)$, $z_j<p(R_j)$. By feasibility and efficiency, for some $i\in N$, $\omega_i<z_i\leq p(R_i)$. Since $\varphi\subseteq F$, $z_j R_j (w_j+(z_i-\omega_i))$. Let $N'\in\tilde{N}$ be a set of agents such that $|N'|=|N|$

[6] Note that we need at least three agents to proof that the *equal-distance* allocation is not obtained through an *envy-free redistribution*.

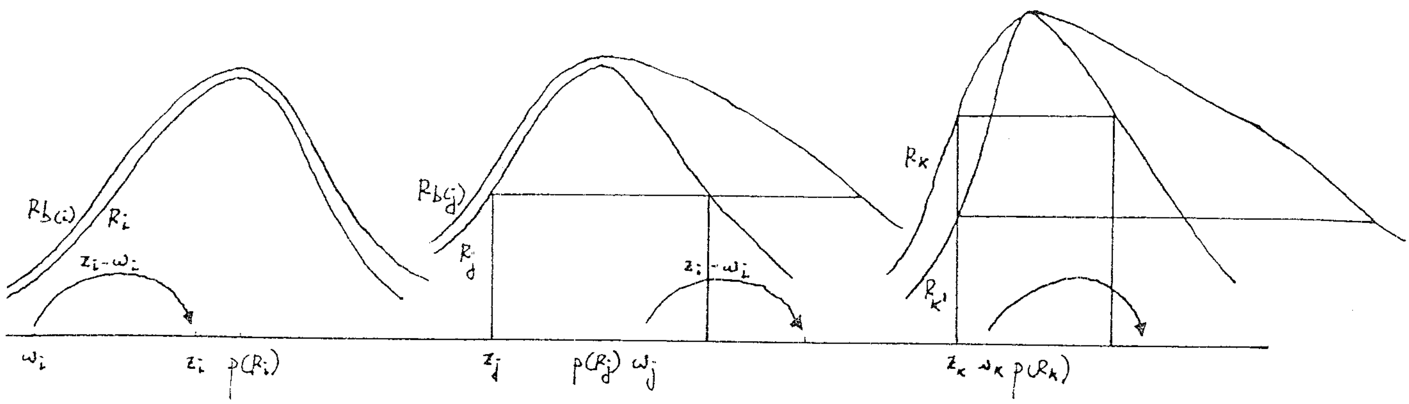

Figure 3. Characterization of the *uniform rule* on the basis of *one-sided population-monotonicity* (Theorem 2, Claims 1 and 2).

and $N \cap N' = \varnothing$. Let $b{:}N \rightarrow N'$ be a bijection. Let $R_{b(i)} = R_i$ for all $i \in N \backslash \{j\}$, let $R_{b(j)} \in \Re$ be such that $p(R_{b(j)}) = p(R_j)$, and $(\omega_{b(j)} + (z_i - \omega_i)) P_{b(j)} z_j$, and let $\omega_{b(i)} = \omega_i$, for all $i \in N$. Let $N'' = N \cup N'$. Let $e'' = (R_{N''}, \omega_{N''}) \in \Re^{N''} \times \mathbb{R}^{N''}_{+}$. Let $z'' = \varphi(e'')$. Note that $\Sigma_N p(R_i) + \Sigma_{N'} p(R_{b(i)}) \geq \Sigma_N \omega_i + \Sigma_{N'} \omega_{b(i)}$, and since $\varphi \subseteq P$, for all $i \in N$ $z''_i \leq p(R_i)$, and $z''_{b(i)} \leq p(R_{b(i)})$. Since $\varphi \subseteq F$, $\omega_{b(i)} = \omega_i$ for all $i \in N \backslash \{j\}$, and $p(R_{b(i)}) = p(R_i)$ for all $i \in N \backslash \{j\}$, we have $z''_{b(i)} = z''_i$ for all $i \in N \backslash \{j\}$. Since $\varphi \subseteq FP$, $\omega_{b(j)} = \omega_j$, and $p(R_{b(j)}) = p(R_j)$, we have $z''_{b(j)} = z''_j$. Since $\varphi \subseteq F$, $z''_{b(j)} = z''_j$, and $(\omega_{b(j)} + (z_i - \omega_i)) P_{b(j)} z_j$, we have $z''_{b(j)} \neq z_j$. Thus, in the change from e to e'', if $z''_{b(j)} > z_j$ ($z''_{b(j)} < z_j$) agent j gains (loses) and at least one $i \in N \backslash \{j\}$ loses (gains) in violation of *one-sided population-monotonicity*.

Claim 2. (Figure 3) **For all $k \in N$ such that $\omega_k \leq p(R_k)$, $z_k \geq \omega_k$.** Suppose, to get a contradiction, that for some $k \in N$ such that $\omega_k \leq p(R_k)$, $z_k < \omega_k$. By feasibility and efficiency, for some $i \in N \backslash \{k\}$, $\omega_i < z_i \leq p(R_i)$. Since $\varphi \subseteq F$, $z_k R_k (\omega_k + (z_i - \omega_i))$. Note that this is possible only if $(\omega_k + (z_i - \omega_i)) \geq r_k(z_k)$. Let us now introduce an additional agent, agent k'. Let $R_{k'} \in \Re$ be such that $p(R_k) = p(R_{k'})$, $r_{k'}(\omega_{k'}) > (\omega_{k'} + (z_i - \omega_i) + (\omega_k - z_k))$ and let $\omega_k = \omega_{k'}$. Let $e' = (R_N, R_{k'}, \omega_N, \omega_{k'}) \in \Re^{N \cup \{k'\}} \times \mathbb{R}^{N \cup \{k'\}}_{+}$ be the resulting economy and $z' = \varphi(e')$. Since $\varphi \subseteq FP$, $\omega_k = \omega_{k'}$ and $p(R_k) = p(R_{k'})$, we have $z'_k = z'_{k'}$. Note that $z'_j = z_j$ for all $j \in N \backslash \{i\}$, $z'_{k'} = z_k$ and $z'_i = z_i + (w_{k'} - z'_{k'})$ is a feasible allocation. Since $\varphi \subseteq F$ and $r_{k'}(\omega_{k'}) > (\omega_{k'} + (z_i - \omega_i) + (\omega_k - z_k))$, the above allocation is not obtained through an *envy-free redistribution*. Indeed, if $\omega_{k'} > z'_{k'} > z_k$, agent k gains and by *one-sided population-monotonicity*, all agents gain. Therefore, $z'_i \geq z_i$. By

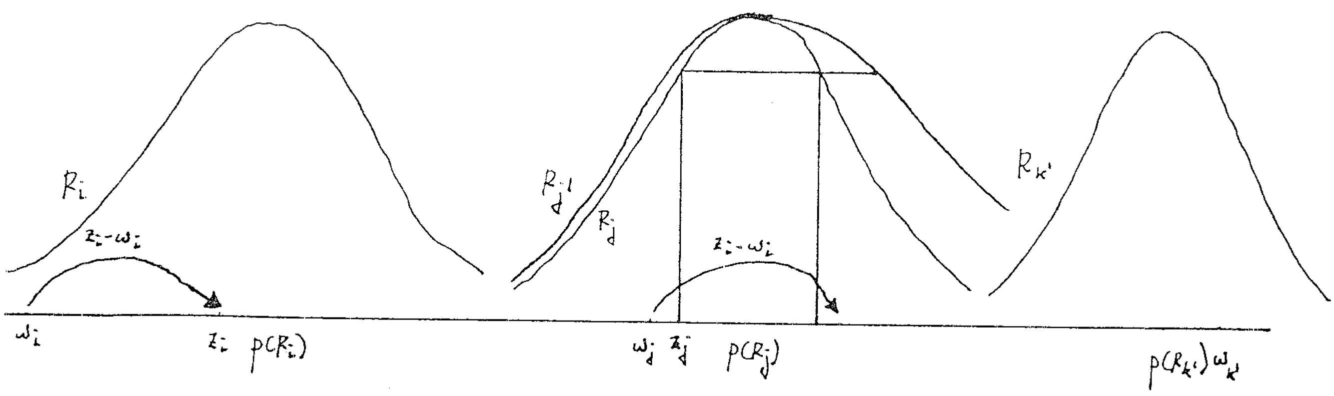

Figure 4. Characterization of the *uniform rule* on the basis of *one-sided population-monotonicity* (Theorem 2, Claim 4).

feasibility, $z'_i<(z_i+(\omega_{k'}-z_k))$ and since $r_{k'}(\omega_{k'})>(\omega_{k'}+(z_i-\omega_i)+(\omega_k-z_k))$, agent k' envies the redistribution of agent i, in violation to $\varphi\subseteq F$. Thus, in the change from e to e', if $z'_{k'}\geq\omega_{k'}$ ($z'_{k'}<z_k$) agent k gains (loses) and at least one $i\in N\backslash\{k\}$ loses (gains), in violation of *one-sided population-monotonicity*.

Claim 3. For all i, $j\in N$ such that $\omega_i\leq p(R_i)$, $\omega_j\leq p(R_j)$ and $p(R_i)-\omega_i\geq p(R_j)-\omega_j$, we have $z_i-\omega_i\geq z_j-\omega_j$. Suppose, to get a contradiction, that for some i, $j\in N$ such that $\omega_i\leq p(R_i)$, $\omega_j\leq p(R_j)$, and $p(R_i)-\omega_i\geq p(R_j)-\omega_j$, we have $z_i-\omega_i<z_j-\omega_j$. Since $\varphi\subseteq P$, it is clear that $(w_i+(z_j-\omega_j))P_iz_i$, in violation of $\varphi\subseteq F$.

Claim 4. (Figure 4) **For all i, $j\in N$ such that $\omega_i\leq p(R_i)$, $\omega_j\leq p(R_j)$ and $p(R_i)-\omega_i\geq p(R_j)-\omega_j$, we have $z_i-\omega_i>z_j-\omega_j$ if and only if $z_j=p(R_j)$.** Suppose, to get a contradiction, that for some i, $j\in N$ such that $\omega_i\leq p(R_i)$, $\omega_j\leq p(R_j)$, and $p(R_i)-\omega_i\geq p(R_j)-\omega_j$, we have $z_i-\omega_i>z_j-\omega_j$ and $z_j<p(R_j)$. Since $\varphi\subseteq F$, $z_jR_j(\omega_j+(z_i-\omega_i))$. Note that this is possible only if $(\omega_j+(z_i-\omega_i))\geq r_j(z_j)$. Let us now introduce two additional agents, agents j' and k'. Let $R_{j'}\in\Re$ be such that $p(R_j)=p(R_{j'})$, $(\omega_{j'}+(z_i-\omega_i))P_{j'}z_j$ and let $\omega_j=\omega_{j'}$. Let $R_{k'}\in\Re$ and $\omega_{k'}-p(R_{k'})=z_j-\omega_j$. Let $e'=(R_N,R_{j'},R_{k'},\omega_N,\omega_{j'},\omega_{k'})\in\Re^{N\cup\{j'\}\cup\{k'\}}\times\mathbb{R}^{N\cup\{j'\}\cup\{k'\}}_+$ be the resulting economy. Note that $\Sigma_Np(R_i)+p(R_{j'})+p(R_{k'})\geq\Sigma_N\omega_i+\omega_{j'}+\omega_{k'}$ and let $z'=\varphi(e')$. Since $\varphi\subseteq P$, we have $z'_i\leq p(R_i)$ for all $i\in N$, $z'_{j'}\leq p(R_{j'})$, and $z'_{k'}\leq p(R_{k'})$. Since $\varphi\subseteq F$, $\omega_j=\omega_{j'}$ and $p(R_j)=p(R_{j'})$, we have $z'_j=z'_{j'}$. By

Claim 1, we have $z_{k'}=p(R_{k'})$. Note that $z'_i=z_i$ for all $i\in N$, $z'_{j'}=z_j$ and $z_{k'}=p(R_{k'})$ is a feasible allocation. Since $\varphi\subseteq F$, $z'_{j'}=z'_j$, and $(\omega_{j'}+(z_i-\omega_i))P_{j'}z_j$, we have $z'_{j'}\neq z_j$. Thus, in the change from e to e', if $z'_{j'}>z_j$ ($z'_{j'}<z_j$) agent j gains (loses) and at least one $i\in N\backslash\{j\}$ loses (gains), in violation of *one-sided population-monotonicity*. **Q.E.D.**

Note that here, in contrast with the problem of fair allocation (Thomson, 1995), we have not used *replication-invariance*[7] to characterize the *uniform rule* as the only *one-sided population-monotonic* selection from the *no-envy* and *Pareto efficient* solution. In fact, although in Claim 1, we duplicate the number of agents, we only need to introduce agents with the same distance between the endowment and the peak than an old one, in order to maintein the direction of the inequality between the sum of preferred consumptions and the sum of the endowments[8].

The following requirement is an adaptation of Thomson (1995) to our context:

Replication-invariance: For all N, $N'\in\tilde{N}$, for all $k\in\mathbb{N}$, for all $e=(R_N,\omega_N)\in\Re^N\times\mathbb{R}^N_+$ and $e'=(R'_{N'},\omega'_{N'})\in\Re^{N'}\times\mathbb{R}^{N'}_+$, for all $z\in\varphi(e)$ and $z'\in Z(e')$, if $|N'|=k|N|$, $R'_{N'}$ is a k-replica of R_N, $\omega'_{N'}$ is a k-replica of ω_N, and z'is a k-replica of z, then $z'=\varphi(e')$.

Since the *uniform rule* is *replication-invariant*, it is clear that if a subsolution of the *no-envy* and *Pareto efficient* solution satisfies *one-sided population-monotonicity*, then it is *replication-invariant*. In the problem of fair division (Thomson, 1995), since one-sided population-monotonicity can not be applied when the economy is replicated, the counterpart of the above statement is not necessarily true. Finally, Theorem 2 is a tight result: If we drop any

[7] **Replication-invariance** says that if a solution selects an allocation for an economy, the replica of this allocation has to be selected by this solution in the replicated economy.

[8] When the sum of the peaks is smaller than the sum of the endowments, in order to proof a counterpart of Claim 1, we introduce an agent with the same distance between the endowment and the peak but may be placed in a different position on the real line and with different preferences.

one of the three requirements, new solutions emerge satisfying the remaining two, i.e. the axioms are independent. Theorem 2 is indeed a full axiomatic characterization.

1. **Drop no-envy.** The *equal-distance* solution, the *proportional* solution and the *proportional-sacrifice* solution satisfy *Pareto efficiency* and *one-sided population-monotonicity*.

2. **Drop Pareto-efficiency.** The solution that always selects the profile of endowments satisfies *no-envy* and *one-sided population monotonicity*.

3. **Drop one-sided population-monotonicity.** The solution S, defined next, satisfies *no-envy* and *Pareto efficiency*. For all $N \in \tilde{N}$ and $e=(R_N,\omega_N) \in \Re^N \times \mathbb{R}^N_+$, let $N'=\{i \in N: \omega_i \leq p(R_i)\}$ and $N''=\{i \in N: \omega_i > p(R_i)\}$. Let $d'=\min_{i \in N'}\{r_i(\omega_i)-\omega_i\}$ and $d''=\min_{i \in N''}\{\omega_i - r_i(\omega_i)\}$. Then the allocation **$z \in Z(e)$ is such that** **$z \in S(e)$** if there is $\lambda \in \mathbb{R}_+$ such that
(i) when $\Sigma_N p(R_i) \geq \Sigma_N \omega_i$, then for all $i \in N$, $z_i=\min\{\hat{\omega}i+\lambda, p(R_i)\}$ where $\hat{\omega}_i=\omega_i$ for all $i \in N'$ and $\hat{\omega}_i=\min\{\omega_i-d', p(R_i)\}$ for all $i \in N''$
(ii) when $\Sigma_N p(R_i) \leq \Sigma_N \omega_i$ then for all $i \in N$, $z_i=\max\{\hat{\omega}_i-\lambda, p(R_i)\}$ where $\hat{\omega}_i=\omega_i$ for all $i \in N''$ and $\hat{\omega}_i=\max\{\omega_i+d'', p(R_i)\}$ for all $i \in N'$.

6.- CONCLUSIONS

We considered the problem of fair division of a commodity when agents have single-peaked preferences and endowments. First, we studied minimal informational requirements, and characterized the *uniform rule* as the only selection from the *no-envy in redistribution* and *Pareto efficient* solution satisfying *endowment-peak-only*.

We also study two properties of solutions dealing with possible changes in the number of agents. The *uniform rule* does not satisfy the property of *population-monotonicity*, but no selection from the *no-envy in redistributions* and *Pareto efficient* solution does. However, we can find selections from the *individually rational* and *Pareto efficient* solution satisfying *population-monotonicity*. Indeed, the *proportional-sacrifice* solution does. Finally, we showed that the *uniform rule* is the only selection from the *no-envy in redistribution* and *Pareto efficient* solution satisfying a weaker requirement of monotonicity in the population, *one-sided population-monotonicity*.

Our results are summarized in the following table. A "yes" in row i and column j means that the solution in row i satisfies the condition in column j. A "no" means the opposite.

	Individually rational	No-envy	Endowment-peak-only	Population-monotonicity	One-sided population-monotonicity
Uniform rule	Yes	Yes	Yes	No	Yes
Asymmetrically-proportional solution	Yes	No	Yes	No	Yes
Equal-distance solution	Yes	No	Yes	No	Yes
Proportional sacrifice solution	Yes	No	No	Yes*	Yes*

* A domain restriction is needed for this positive result.

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Our results are summarized in the following table. A "yes" in row i and column j means that the solution in row i satisfies the condition in column j. A "no" means the opposite.

	Individually rational	No-envy	Endowment-peak-only	Population-monotonicity	One-sided population-monotonicity
Uniform rule	Yes	Yes	Yes	No	Yes
Asymmetrically-proportional solution	Yes	No	Yes	No	Yes
Equal-distance solution	Yes	No	Yes	No	Yes
Proportional sacrifice solution	Yes	No	No	Yes*	Yes*

* A domain restriction is needed for this positive result.

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