CONSISTENT BELIEFS, LEARNING AND DIFFERENT EQUILIBRIA IN OLIGOPOLISTIC MARKETS*

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A B S T R A C T

The aim of this paper is to show the relation among equilibria in models with different levels of rationality. The different levels of rationality are defined in terms of the number of iterations that a player can perform with a simple rule for updating beliefs. It is shown that the rule converges to the Nash equilibrium without any increase in the complexity of players.

KEYWORDS: Consistent Beliefs; Learning; Rationality Level.

1 Introduction

This paper proposes a rule for updating beliefs in a simple symmetric oligopoly model with a linear demand function and constant marginal costs.

The crucial features of this rule are the following:

First, each firm i has a belief about how other firms respond to $i's$ production level, and this belief is correct. Beliefs of firms form a consistent self-reference system.

Second, beliefs about beliefs are seen as iterations of the beliefs formation process, and those iterations are identified with a rationality level.

Third, it is assumed that these beliefs are elements of a one parameter family, and by the assumption of symmetry, firm $i's$ beliefs about any other firm can be summarized by that parameter.

The main result of this paper is that for a number of firms larger than three, beliefs updating leads to the competitive equilibrium. In contrast, when the number of firms is two, it leads to the Cournot-Nash equilibrium. If we allow for asymmetric levels of rationality, a Stackelberg solution can also be an equilibrium.

A motivation of this rule for updating beliefs is in the next section. In section 3 I present the model with an example. The stability properties of the beliefs updating rule are investigated in sections 4 and 5, where the main results are stated.

Most of the recent literature on learning in games concentrates in developing learning algorithms where past play of the opponent is used to estimate future play. The approach I propose is that players should be able to learn the rule opponents use to select their strategies. This is important because it is more profitable for a player to learn opponents reasoning processes rather than opponents frequencies of play.

2 Motivation

A pervasive criticism of the game theoretic models, as suggested by Fudenberg and Kreps in their CORE lectures in May 1990 has emerged concerning the application of the Nash equilibrium. Where do these equilibria come from? How does one choose in the case of multiple equilibria in order to make predictions? In the CORE lectures, learning was presented as a justification for the extended use of the Nash equilibrium in economics. Not only did learning yield a Nash equilibrium but was also a way how to reach it. Unfortunately a lot of the underlying assumptions of the learning process were made in order to reach the desired result thereby introducing an ad hoc element into the analysis. This is the case of fictitious play.

The fictitious play (Brown [2]) justification has become, after a degree of reinterpretation (Fudenberg and Kreps [9]), a model of learning. In order to illustrate how fictitious play works, the following example is useful. Imagine that two players repeatedly play the game that is described in the table below. Imagine also that they have some prior beliefs about the strategy that their opponent will choose. Let bR and bC be the beliefs held by the Row player and column player respectively. Set arbitrarily $bR = (0, 1)$ and $bC = (1, 0)$, suggesting, therefore, that the Row player will believe that his counterpart will play his second strategy (column 2), with probability 1, while the Column player will believe that the Row player will play his first strategy (row 1) with probability 1. They observe the pure strategy chosen by the opponent. If a strategy is mixed, however, they will not be able to observe the 'impurity' of the strategy that is inherent in the 'mixture'.

	$Column1$	$Column2$
$Row1$	1, 0	3, 2
$Row2$	2, 1	4, 0

The history of play and how beliefs are updated is represented in the next table. In the first column and first row the beliefs about each rival have been set arbitrarily as explained above. In the next column the expected payoffs for the two strategies of each player are displayed and, finally, a comparison between these expected payoffs indicates the best or optimal choice for each player in each round. In the second line the updating of beliefs is achieved by adding 1 to the observed strategy and, after a process of normalisation, the relative likelihood about the rival is obtained. The process is then repeated to show how the system converges.

Table 1.

Round 1	Beliefs about the rival	Expected payoffs	Choice
Player Row	0,1	3,4	row 2
Player Column	1,0	0,2	column 2
Round 2	Beliefs about the rival	Expected payoffs	Choice
Player Row	0,1	3,4	row 2
Player Column	1/2,1/2	1/2,1	column 2
Round 3	Beliefs about the rival	Expected payoffs	Choice
Player Row	0,1	3,4	row 2
Player Column	1/4,3/4	3/4,1/2	column 2
Round 4	Beliefs about the rival	Expected payoffs	Choice
Player Row	0,1	3,4	row 2
Player Column	1/8,7/8	7/8,1/4	column 1
Round 5	Beliefs about the rival	Expected payoffs	Choice
Player Row	1/2,1/2	2,3	row 2
Player Column	1/16,15/16	15/16,1/8	column 1

As this table shows, fictitious play possesses some useful properties. *In any history generated by fictitious play, if a strategy profile is played that is a strict Nash equilibrium, then all subsequent play will be part of that strategy profile.* And *if a strategy profile is played for all but a finite number of periods, then that strategic profile is a Nash equilibrium.* Fudenberg[6]

But the fictitious play mechanism itself raises some issues when interpreted as a learning process. A real model of learning should first answer the basic question. What does a player learn? In the model, learning does not take place at any moment. The only thing it shows is that if the players update their beliefs using that particular updating rule, then they will choose their part in the Nash equilibrium profile. But they do not learn about how the other player reason, because if they are able to learn that feature of the opponent then they will not update their beliefs according to the fictitious play rule.

The use of fictitious play as an example illustrates this point clearly. If player Row thinks that player Column is using fictitious play to compute his strategy then, maintaining from the beginning an adherence to row 1 will doubtless force player two to play column 2. Once this situation has been attained player Row will not deviate from the strategy because he has seen through the counter strategy, namely that player Column was revising his beliefs according to fictitious play model. As a result, any surprising deviation by player Row to row 2 will then be followed by a revision by player Column of his beliefs that, sooner or later, will provoke the undesirable payoff of 2 for the Row player. At the same time, player Column will inevitably be pleased with the reasoning that is followed by player Row. This observation opens the possibility of learning so as to avoid any Nash equilibrium inefficient outcomes.

3 Example

Consider n firms facing the inverse demand function $P = A - X$, where A is a positive real number and X stands for the total supply obtained by adding up the individual quantities, $X = x_1 + \ldots + x_n$. Firms are indexed by the elements of a subset of the natural numbers. The firms produce a homogeneous good with no costs (it is easy to show that this assumption on the cost function can be extended to all degree 2 polynomial cost functions).

A typical profit function is

$$\pi_i = (A - x_i - \sum_{j \neq i} x_j)x_i \ all \ i = 1 \ldots n.$$

Taking the F.O.C. we get the following:

$$\frac{d\pi_i}{dx_i} = A - 2x_i - \sum_{j \neq i} x_j - x_i \frac{d \sum_{j \neq i} x_j}{dx_i} = 0 \ all \ i = 1 \ldots n.$$

Arranging terms, we get the following:

$$x_i(2 + \sum_{j \neq i} \frac{dx_j}{dx_i}) + \sum_{j \neq I} x_j = A \ all \ i = 1 \ldots n.$$

Defining $k_i = 1 + \sum_{j \neq i} \frac{dx_j}{dx_i}$ we end up with a system of equations. According to this definition of k, it is interesting to differentiate the following cases:

1 $if \quad k = 0 \quad or \quad 1 + \sum \frac{\delta x_i}{\delta x_j} = 0$

2 $if \quad k = 1 \quad or \quad \sum \frac{\delta x_i}{\delta x_j} = 0$

3 $if \quad k = n \quad or \quad \sum \frac{\delta x_i}{\delta x_j} = n - 1.$

Case 1 is the competitive equilibrium. Case 2 is the Nash-Cournot equilibrium. Case 3 is the collusive solution. To verify this relations it is enough to solve the maximization programs and to plug on them these values of k. The system of equations is:

$$A = x_i(1 + k_i) + \sum_{j \neq i} x_j$$

$$\vdots$$

$$\vdots$$

$$A = x_n(1 + k_n) + \sum_{j \neq n} x_j.$$

This system can be written as $DX = A$ where D is the nxn coefficients matrix, X is the nx1 column vector of x_s and A is a nx1 column vector of independent terms.

The coefficient matrix D is:

$$\begin{pmatrix} 1+k_1 & 1 & \dots & 1 \\ 1 & 1+k_2 & \dots & 1 \\ \vdots & & & \\ 1 & \dots & & 1+k_n \end{pmatrix}$$

A simple repeated operation of substraction -first line minus second line. Second line minus third line, and so on, leaving the n_{th} line untouched- gives the transformed equivalent of matrix D.

$$\begin{pmatrix} k_1 & -k_2 & 0 & \dots & 0 \\ 0 & k_2 & -k_3 & \dots & 0 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \dots & 1+k_n \end{pmatrix}$$

The column vector of independent terms takes the form of

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ A \end{pmatrix}$$

From each of the equations of the system we can verify

$$x_1 = \frac{k_2}{k_1} x_2, \dots, x_j = \frac{k_{j+1}}{k_j} x_{j+1} \; 1 \leq j \leq n-1,$$

and

$$x_n = \frac{1}{A + k_n} \left(A - \sum_{i=1}^{n-1} x_i \right).$$

Therefore, through a chain of substitutions,

$$x_1 = \frac{k_2}{k_1}\frac{k_3}{k_2}, \ldots, \frac{k_n}{k_{n-1}} x_n,$$

so that

$$x_1 = \frac{k_n}{k_1} x_n.$$

Similarly,

$$x_j = \frac{k_n}{k_j} x_n \; for \; all \; 1 \leq j \leq n-1$$

and

$$x_n = \frac{1}{1+k_n}\left(A - k_n x_n \sum_{j=1}^{n-1} \frac{1}{k_j}\right).$$

After some additional manipulation we have

$$x_n = \frac{A}{1 + k_n\left(1 + \sum_{j=1}^{n-1} \frac{1}{k_j}\right)}. \tag{1}$$

To obtain an expression for the profit function, we have to follow this sequence of equalities:

$$X = \sum_{j=1}^{n} x_j = x_n + \sum_{j=1}^{n-1} x_j = A - (1+k_n)x_n + x_n = A - k_n x_n = A\left(1 - \frac{k_n}{1 + k_n\left(1 + \sum_{j=1}^{n-1} \frac{1}{k_j}\right)}\right).$$

To simplify matters let

$$\xi = 1 + k_n\left(1 + \sum_{j=1}^{n-1} \frac{1}{k_j}\right).$$

From here it is easy to see that

$$\pi_i = (A - X)x_i = A\left(1 - \left(1 - \frac{k_n}{\xi}\right)\right) x_i.$$

Simplifying the last expression, taking into account that $x_i = \frac{k_n}{k_i} x_n$, we obtain

$$\pi_i = \left(\frac{Ak_n}{\xi}\right)^2 \frac{1}{k_i}. \tag{2}$$

Taking the first order condition yields

$$\frac{d\pi_i}{dk_i} = \frac{2}{k_i}\left(\frac{Ak_n}{\xi}\right)\left(\frac{-Ak_n \frac{d\xi}{dk_i}}{\xi^2}\right) - \left(\frac{Ak_n}{\xi}\right)^2 \frac{1}{k_i^2}.$$

But

$$\frac{d\xi}{dk_i} = k_n \frac{-1}{k_i^2}.$$

Making the substitution, and taking the expression of the profit function in to account, we finally obtain

$$\frac{d\pi_i}{dk_i} = \pi_i \left(\frac{2k_n}{\xi k_i^3} - \frac{1}{k_i^2} \right)$$

Therefore the condition for a maximum is given by

$$\left(\frac{2k_n}{\xi k_i^3} - \frac{1}{k_i^2} \right) = 0,$$

or equivalently $\xi k_i - 2k_n = 0$. Taking into account the definition of ξ, we can write

$$k_i \left(1 + k_n + k_n \sum_{j \neq i}^{n-1} \frac{1}{k_j} + \frac{k_n}{k_i} \right) - 2k_n = 0.$$

Solving for k_i, we get the main expression

$$k_i^* = \frac{1}{1 + \sum_{j \neq i} \frac{1}{k_j}} \; for \; all \; i = 1 \ldots n \tag{3}$$

Then we are ready to state the following

Proposition 1 *If k_i is bounded from above, $k_i \leq C$ and $k_i \geq 0$ for all $i = 1 \ldots n$, the optimal belief tends to the competitive equilibrium optimal belief when the number of firms increases. In other words $k_i \to 0$ when $n \to \infty$ $\forall i$.*

Proof

Let k_j take any arbitrary value $j = 1, 2 \ldots n$. Increase the number of firms and let them have any arbitrary belief. Assume that k_i^* does not converge to zero. That happens only if

$$\frac{1}{k_{n+1}} + \frac{1}{k_{n+2}} + \ldots + \frac{1}{k_{n+m}} < \infty$$

but because beliefs are bounded by C, the following inequality holds:

$$\frac{1}{k_{n+1}} + \frac{1}{k_{n+2}} + \ldots + \frac{1}{k_{n+m}} \geq 1/C + 1/C + \ldots 1/C = m/C$$

Clearly m/C does not converge when $m \to \infty$, and therefore a contradiction. Q.E.D.

Notice that this result is an extension to the well known convergence of Nash equilibrium to competitive equilibrium when the number of firms increases. Here we state that this convergence is achieved from any point.

Proposition 2 *For any given n, for any j, if $k_j \to 0$ then $k_i^* \to 0$*

Proof

$$\lim_{k_j \to 0} \frac{1}{k_j} = \infty$$

$$\lim_{k_j \to 0} \sum_{l \neq i}^{n} \frac{1}{k_l} = \infty$$

And therefore $k_i^* \to 0$. Q.E.D.

This simple proposition states that one competitive firm can induce total competition in the whole industry just showing a competitive behaviour.

Proposition 3 *Any profile $\chi = (x_1 \ldots x_n)$ is symmetric $(x_1 = \ldots = x_n)$ if and only if $(k_1 = \ldots = k_n)$.*

Proof

It is clear that if $k_i = k_l \; \forall i, l$ then $x_i = x_l \; \forall i, l$.

To prove the sufficiency we need the generalized to the n firms case expressions for x_i. See in appendix I the derivation to reach the required expressions:

$$x_i = \frac{A}{1 + k_i \left(1 + \sum_{j \neq i}^{n} \frac{1}{k_j}\right)} = \frac{A}{1 + k_l \left(1 + \sum_{j \neq l}^{n} \frac{1}{k_j}\right)} = x_l$$

Then

$$k_i \left(1 + \sum_{j \neq i}^{n} \frac{1}{k_j}\right) = k_l \left(1 + \sum_{j \neq l}^{n} \frac{1}{k_l}\right)$$

If we call $S = \sum_{j=1}^{n} \frac{1}{k_j}$ then we can write the last expression as

$$k_i + k_i(S - \frac{1}{k_i}) = k_l + k_l(S - \frac{1}{k_l}),$$

which simplifies to $k_i - k_l = S(k_l - k_i)$. This can only be true if $S = -1$ which is impossible because $k_i > 0 \forall i$ or $k_l = k_i$. This is true $\forall i, l$. Q.E.D.

In fact this proposition can be proved because no costs (different among firms) were assumed. With equal costs the proposition is still valid. In the case that differing marginal costs are assumed the proposition no longer holds. What is always true is that in any symmetric equilibrium, beliefs have to be symmetric. In the next proposition we state some properties about the equilibrium. The proof is constructed in a way that it also shows that the equilibrium exists.

Proposition 4 *The system of equations described by*

$$k_i^* = \frac{1}{1 + \sum_{j \neq i}^{n} \frac{1}{k_j}} \quad i = 1 \dots n$$

has a unique symmetric equilibrium. Furthermore, the equilibrium is the competitive equilibrium. In equilibrium, beliefs are symmetric.

Proof

Because the equilibrium is symmetric assume $k_1 = k_2 = \ldots = k_n = k^0 \neq 0$. Then $k_i^* = \frac{1}{1+(n-1)\frac{1}{k^0}}$, which implies that $k_i^* \leq k^0$. Otherwise, $k_i^* = \frac{1}{1+(n+1)\frac{1}{k^0}} > k^0$ implies that $k^0 < 2 - n \leq 0$ impossible by assumption. Therefore must happen that $k_i^* \leq k^0$. If $k_i^* = k^0$ then $k^0 = \frac{1}{1+(n-1)\frac{1}{k^0}}$ which in turn implies $k^0 = 2 - n$ and as long as $k_i \geq 0$ and $n \geq 2$, the solution is $n = 2$ and $k^0 = 0$ and then the proposition is proved. If $k_i^* < k^0$ then call $k_i^* = \frac{1}{1+(n-1)\frac{1}{k^0}} = k^1$ and assume now that $k_1 = k_2 = \ldots = k_n = k^1$. Then

$$k_i^* = \frac{1}{1 + (n-1)\frac{1}{k^1}} = \frac{1}{1 + (n-1) + (n-1)^2 \frac{1}{k^0}}$$

and call this new number k^2. Repeating the process we find a sequence of numbers $k^0, k^1, k^2 \ldots$. To complete the proof, we have to show that this sequence is convergent to zero. But this is obvious if we look at the general term of the sequence. It is

$$k^m = \frac{1}{\sum_{p=0}^{m-1} (n-1)^p + (n-1)^{p+1} \frac{1}{k^0}}$$

and it converges to zero with the number of iterations. Q.E.D.

Notice that even in the case that marginal costs are different but constant, the system does not change at all.

Customising the model for a duopolistic relationship, in equation 3, and indexing the two firms with a and p results in the following:

$$k_a = 1 + \frac{dx_p}{dx_a} \text{ and } k_p = 1 + \frac{dx_a}{dx_p}.$$

Notice that, in general, k_i can be anything between 0 and n. In this duopolistic case k_i is anything between 0 and 2, because two is the number of firms. The equivalent of equation 3 for a duopolistic relation gives the optimal levels of production depending on the beliefs. The solution is

$$x_a^* = \frac{k_p A}{(1 + k_a)(1 + k_p) - 1} \text{ and } x_p^* = \frac{k_a A}{(1 + k_p)(1 + k_a) - 1}. \quad (4)$$

Plugging (4) into the profit function, we get the final result

$$\pi_a^*(k_a, k_p) = A^2 \frac{k_p^2 k_a}{(k_a + k_p + k_a k_p)^2} \text{ and } \pi_p^*(k_a, k_p) = A^2 \frac{k_a^2 k_p}{(k_a + k_p + k_a k_p)^2}. \qquad (5)$$

The beauty of the concept is that the profits of each firm depend on what they think about the opponent and what the opponent thinks about them, i.e. profits depend only on beliefs. So in order to maximize their profits firms only have to choose their optimal belief.

The optimal belief that a should have about p given the belief p has about a can be computed from these two profit functions. The same can be done for p.

$$\frac{d\pi_a(k_a,k_p)}{dk_a} = 0 \text{ yields to } k_a = \frac{k_p}{1+k_p} \qquad (6)$$

$$\frac{d\pi_p(k_a,k_p)}{dk_p} = 0 \text{ yields to } k_p = \frac{k_a}{1+k_a} \qquad (7)$$

These two equations are a particular case of the idea presented in section 2. Each player, instead of having a prior about the actions of the other player, the players observe a certain action and the he asks to himself: "which one is the belief the other player has about me in order to make his action optimal". Once he has this information he find his best response which in turn determines uniquely a belief about the other player. The other player will reason in the same way. The function describing this process is called the beliefs formation process. Learning means the ability of each player to disentangle how the other player form his beliefs.

The interpretation for equations (6) and (7) is that, given $a's$ belief about p and given that p is following the same reasoning that a is following, $a's$ optimal belief is given by equation (6). Furthermore, $a's$ belief is consistent, as shall be shown, with the belief formation that p has about a and that is given by equation (7). These two equations have been plotted in figure 1.

A modification in $a's$ beliefs makes p revise his beliefs, which in turn induces a revision of $a's$ beliefs and so on. Interestingly the final outcome is the competitive equilibrium and not the Nash-Cournot equilibrium, i.e. $k_i = 1$ is not contained in the belief equilibrium that is given by the dynamics of the plotted system. Assume that the system given by equations (6) and (7) is at a disequilibrium point given by $k_a = k_p = d$ ($d \geq 0$). In the resulting dynamic one player reacts by matching his best response function, given the beliefs the other has. Then the other player reacts also by matching his best response function given the observed reaction of the first, and so on and so forth until the equilibrium point where $k_a = k_p = 0$ is reached. A possible sequence is provided by the following table when $d = 1$.

Figure 1: Beliefs

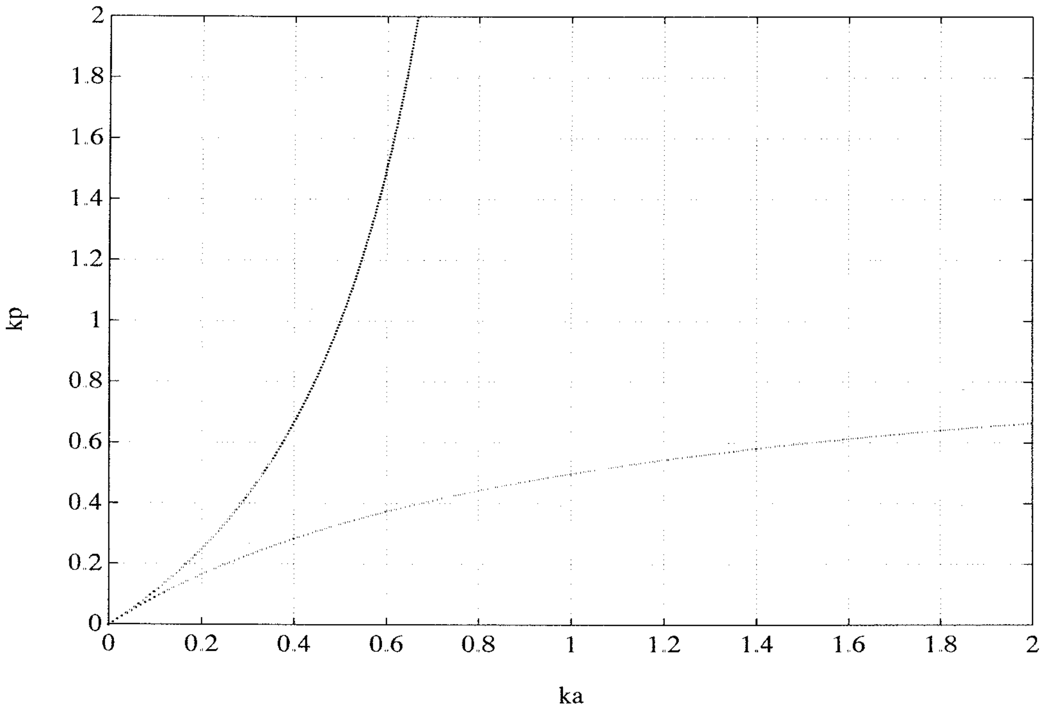


Table2.

k_a	k_p	$x_a A^{-1}$	$x_p A^{-1}$	$\pi_a A^2$	$\pi_p A^2$	$\frac{dx_p}{dx_a}$	$\frac{dx_p}{dx_a}$
1	1	1/3	1/3	1/9	1/9	—	—
1/2	1	1/2	1/4	1/8	1/16	−1/2	−2
1/2	1/3	1/3	1/2	1/18	1/12	−3/2	−2/3
1/4	1/3	1/2	3/8	1/16	3/64	−3/4	−4/3
...	...	...	...	...	...	...	...
0	0	0.5	0.5	0	0	−1	−1

Observe the main characteristic of the process. The revision of beliefs is made consistent with the variation of the others' output. Therefore when $k_a = 1/2$ the value for $\frac{dx_p}{dx_a} = -1/2$, and recalling the definition of k_i it is clear that they match.

Several festures of the equilibrium should be noted here:

It can be easily shown that the worst outcome for p is that a thinks that p is competitive, i.e. the equilibrium belief, because it will force p to be competitive too, with the result that both firms loose any possibility

As the final outcome of the above belief revision process is the worst overall outcome for firms and if firms are able to forsee this, firms should not be expected to follow such a reasoning process.

4 Is cooperation possible?

The above argument shows clearly that players are willing to manipulate the belief formation process when improvements are to be expected. In the model presented in section 3 there is an explicit equation for player $i's$ belief formation process.

In order to make the argument that follows as clear as possible let us make use of the duopolistic case in the next subsection. Once the argument is clear we will provide a further generalisation in the second subsection.

4.1 A reasoning that leads to cooperation

Someone could argue that if firms know the belief formation process of the other firm, they should forsee the competitive outcome and hence refuse to follow such a belief formation process. They will notice that their beliefs are correlated because $a's$ optimal belief affects $p's$ optimal belief and vice versa. They should take this correlation into account when computing their optimal belief.

Let's formalise these ideas. Assume that there are two markets. In each of them there are two firms. One is firm a and the other is firm p. The demand curve is the same as before. The only difference is that the products, being still homogeneous in each of the markets, differ from one another. In market 1 firms behave as described in the model in section 3. Market 2 opens one period later. Therefore firms a and p in the second market have observed the whole process followed by firms a and p in market 1. This process is described by equations (4) and made particular for a duopoly

Firms a and p in market 2 will realise that beliefs were correlated. Therefore when computing their optimal beliefs they will take account of this fact.

The correct way to proceed for firm a in the second market is to take a total derivative in his profit function, equation 8. For simplicity I have set $A = 1$.

$$\frac{d\pi_a}{dk_a} = \frac{\left(k_p^2 + 2k_a k_p \frac{\partial k_p}{\partial k_a}\right)(k_a + k_p + k_a k_p)^2 - 2k_a k_p^2 (k_a + k_p + k_a k_p)\left(1 + \frac{\partial k_p}{\partial k_a} + k_p + k_a \frac{\partial k_p}{\partial k_a}\right)}{(k_a + k_p + k_a k_p)^4} = 0,$$

which simplifies to

$$-k_a k_p - k_a k_p^2 + k_p^2 + 2k_a^2 \frac{\partial k_p}{\partial k_a} = 0, \tag{8}$$

which is a relation that beliefs and *sophisticated beliefs* must optimally satisfy.[1]

Sophisticated beliefs cannot be arbitrary. To have a sophisticated belief logically implies that a belief already exists and that it is related with others' beliefs. A minimum requirement of consistency is, therefore, that the sophisticated belief refers to the observed beliefs formation process.

If firm a in the second market thinks that firm p in market 2 will behave like firm p in market 1, then firm a in market 2 has to solve the following system of equations:

$$-k_a k_p - k_a k_p^2 + k_p^2 + 2k_a^2 \frac{\partial k_p}{\partial k_a} = 0,$$

$$k_p = \frac{k_a}{1 + k_a}.$$

From the second equation in the above system it is possible to compute:

$$\frac{dk_p}{dk_a} = \frac{1}{(1 + k_a)^2}. \tag{9}$$

In fact, if firm p in market 2 behaves like firm p in market 1, firm a in market 2 has to take this variations as true.

Plugging equation (9) into equation (8), two results are reached. First, a new behavioral equation is obtained for firm a, namely

$$k_a = \frac{3k_p}{1 + k_p}. \tag{10}$$

Second, the solution of the system is $k_a = 1$ and $k_p = 1/2$. Following the same reasoning, now for firm p in market 2 we obtain a similar system of equations with the following results:

$$k_p = \frac{3k_a}{1 + k_a}. \tag{11}$$

The solution of the system is $k_p = 1$ and $k_a = 1/2$

These two equations form a system. This system has a solution in $k_a = k_p = 2$, which is the outcome that was desired,i.e. $k = n$ or cooperation. The next table illustrates how the outcome is reached over time if players are to start with the Nash conjectures.

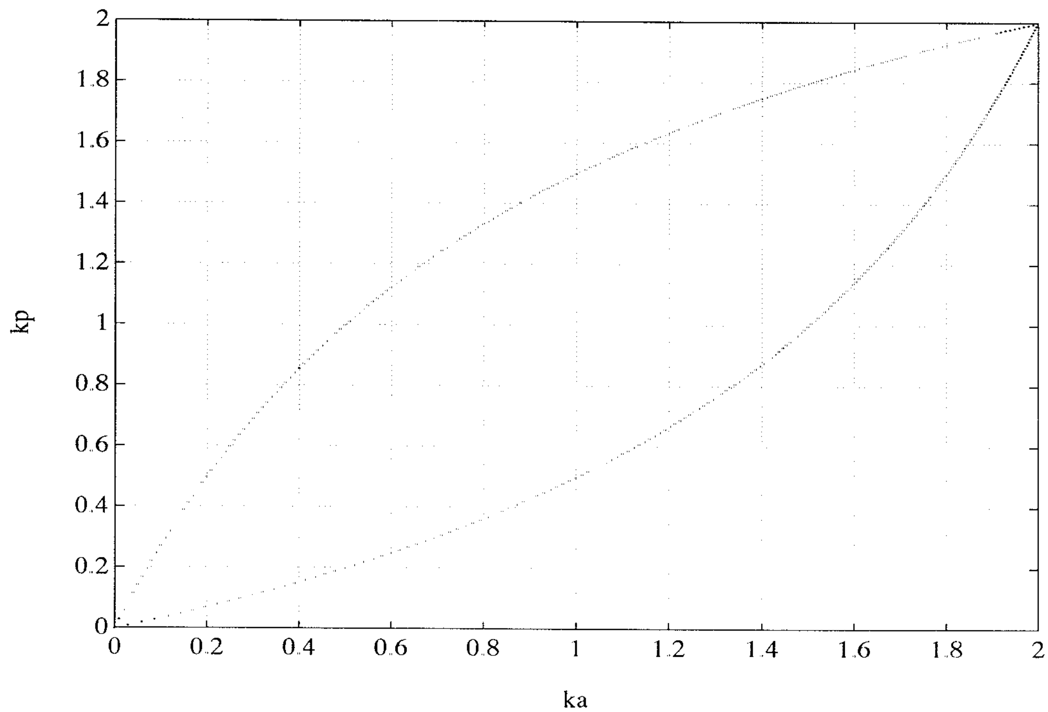
[1] I am calling *sophisticated belief* the resulting belief from the knowledge that i's beliefs and j's beliefs are mutually dependent.

Table3.

k_a	k_p	x_a	x_p	π_a	π_p
1	1	0.333	0.333	0.111	0.111
3/2	1	0.25	0.375	0.0938	0.1406
3/2	9/5	0.333	0.25	0.1350	0.1125
27/14	9/5	0.25	0.2647	0.1205	0.1291
...	...	...	...	...	...
$6561/3281 \approx 2$	$6561/3281 \approx 2$	0.25	0.25	0.125	0.125

A plot of these two best-belief-best-reply functions is given in figure 2

Figure 2: Sophisticated beliefs



Further points are to be noted:

First, the functions are continuous and therefore, more robust than results which sustain cooperation as a Nash equilibrium with the use of, for example grim,[2] players.

[2]A grim player is a nasty player. When he is playing a repeated Prisoner's Dilemma, he defects forever as soon as he perceives a defection from other player.

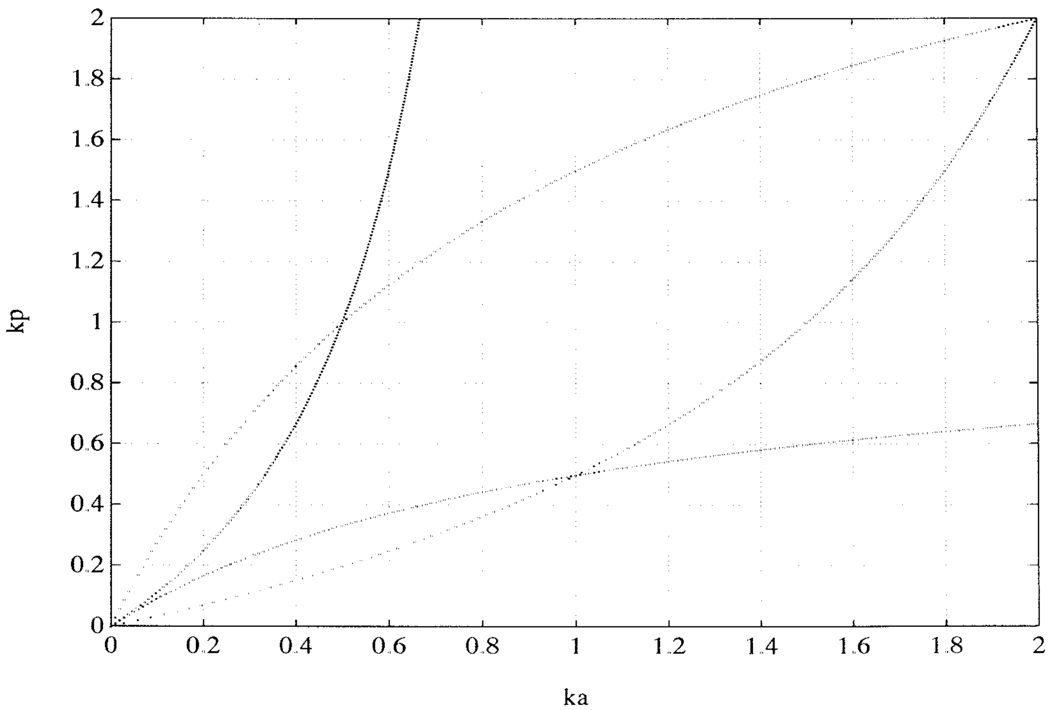
Second, the system converges to the solution from any starting point. Therefore the stability of the system is guaranteed.

Third, as was shown at the beginning, the beliefs must be symmetric if the efficient solution is to be obtained.

Fourth, the process has two equilibria: the cooperative solution and the Bertrand case. Only the former should be expected, however, because the Bertrand case is unstable as will be proven in proposition 6.

In figure 3 the two best responses are plotted for the two duopolists. Two of them correspond to the first order beliefs and two to the sophisticated beliefs.

Figure 3: Two best beliefs



Notice that it could be the case that, say player a, has a sophisticated belief and player p has a simple consistent belief. In this case the solution of the system is $k_a = 1$ and $k_p = 1/2$. This solution correspond to the Stakelberg equilibrium in which player a is the follower and player p is the leader. The reason for it is clear. Player p is too naive to foresee the outcome in case that both players behave according to equation 3, but player a can anticipate this outcome. We have seen therefore that the competitive oucome is reached when both players are consistent (Bresnahan [1]), the collusive outcome when both are smart, and Stakelberg in the case that one player has a sophisticated belief and the other player has not. In figure 3 all these options appear as an equilibrium point for each situation.

4.2 The generalization to the N firm case

As we have shown in the duopoly case, if firms realise that their beliefs are correlated (sophisticated beliefs), they can manipulate the process of belief formation. Let us develop the generalization for the sophisticated beliefs.

The profit function shown in equation (5) can be re-written as

$$\pi_i(k_1 \dots k_n) = \left(\frac{A}{1+\sum_{j=1}^{n}\frac{1}{k_j}}\right)^2 \frac{1}{k_i}.$$

Now we take the first derivative, taking into account that, firm i realizes the link between its own beliefs and others' beliefs.

$$\frac{d\pi_i(k_1 \dots k_n)}{dk_i} = \frac{-\left(1+\sum_{j=1}^{n}\frac{1}{k_j}\right)^2 - 2k_i\left(1+\sum_{j=1}^{n}\frac{1}{k_j}\right)\left(\frac{d\sum(1/k_j)}{dk_i}\right)}{\left(\left(1+\sum_{j=1}^{n}\frac{1}{k_j}\right)^2 k_i\right)^2} = 0,$$

or

$$-\left(1+\sum_{j=1}^{n}\frac{1}{k_j}\right) - 2k_i\left(\frac{d\sum(1/k_j)}{dk_i}\right) = 0.$$

Taking the derivative inside the parenthesis

$$-\left(1+\sum_{j=1}^{n}\frac{1}{k_j}\right) - 2k_i\sum_{j=1}^{n}\frac{-dk_j/dk_i}{k_j^2} = 0.$$

Taking into account that

$$\frac{dk_j}{dk_i} = \frac{1}{k_i^2\left(1+\sum_{l\neq j}^{n}\frac{1}{k_l}\right)^2}$$

and doing the substitution, we end up with

$$-\left(1+\sum_{j=1}^{n}\frac{1}{k_j}\right) + 2k_i\sum_{j=1}^{n}\frac{1}{k_i^2k_j^2\left(1+\sum_{l\neq j}^{n}1/k_l\right)^2} = 0.$$

The last expression simplifies when we see that, in the denominator, k_j appears with its inverse, canceling one another. So

$$-\left(1+\sum_{j=1}^{n}\frac{1}{k_j}\right) + 2k_i\sum_{j=1}^{n}\frac{1}{k_i^2} = 0.$$

Solving for k_i we finally obtain

$$k_i^* = \frac{2n-1}{1+\sum_{j\neq i}^{n}\frac{1}{k_j}} \quad \forall i = 1\ldots n. \tag{12}$$

Equation (12) generates a system of n equations.

Proposition 5 *The system generated by equation (12) possesses two symmetric equilibria. One is the competitive equilibrium and the other is the collusive equilibrium.*

Proof

Define the function

$$G(k) = k - \frac{2n-1}{1+(n-1)\frac{1}{k}}.$$

Because the equilibrium is symmetric, $k_1 = \ldots = k_n = k$, those values of k such that $G(k^*) = 0$, are a solution of the system. But $G(k^*) = 0$ implies $k(k-n) = 0$, satisfied by $k = 0$ and $k = n$, which are, respectively, the competitive and collusive beliefs. Q.E.D.

To give a proposition about the stability of the system generated by equation (12) let us induce a particular dynamic. Making time explicit in equations (3), for the duopolistic case, (6) and (7) we write,

$$k_t^a = \frac{k_{t-1}^p}{1+k_{t-1}^p},$$

and

$$k_t^p = \frac{k_{t-1}^a}{1+k_{t-1}^a}.$$

From these two equations substituting one into the other finding a recursive formula we get equation (13) in which any value of k_a and k_t can be obtained at any moment of time τ.

$$k_{t+\tau} = \frac{k_t}{1+\tau k_t} \quad all\ \tau = 0, 1, 2, \ldots \tag{13}$$

Just a simple look to equation (13) is enough to see that as $\tau \to \infty$, $k_{t+\tau} \to 0$ which in turn is a critical point of the system, because once the system reaches $k_t = 0$ then $k_{t+\tau} = 0$ for all subsequent periods. This was stated in proposition 4.

Proceeding in the same way with equations (10) and (11) we obtain

$$k_t^a = \frac{3k_{t-1}^p}{1+k_{t-1}^p},$$

and

$$k_t^p = \frac{3k_{t-1}^a}{1+k_{t-1}^a}$$

After some manipulations we arrive at the equation

$$k_{t+\tau} = \frac{9^{\tau/2}k_t}{1 + 4k_t\,(1 + 9 + 9^2 + \ldots + 9^{\tau/2-1})}. \tag{14}$$

From this equation we can prove the dynamic version of proposition 5 where we stated that the system had two critical points, one in zero and the other one in n (in this case $n = 2$). In addition the global stability of the system around the cooperative equilibrium can be proven.

Proposition 6 *The system described by equation (14) has two critical points. One competitive and the other collusive. Under this dynamics the competitive equilibrium is unstable and the collusive equilibrium is stable.*

Proof

Equation (14) contains in the denominator the sum of a finite sequence. This sum S exists and is equal to $\frac{9^{\tau/2}-1}{8}$. Substituting this value into equation (14) we get

$$k_{t+\tau} = \frac{9^{\tau/2}k_t}{\frac{1}{2}9^{\tau/2}k_t - \frac{1}{2}k_t + 1}.\ \forall\tau = 1, 2, \ldots$$

And

$$\lim_{\tau\to\infty} \frac{9^{\tau/2}k_t}{\frac{1}{2}9^{\tau/2}k_t - \frac{1}{2}k_t + 1} = 2.$$

This happens for any starting value of $k_t \neq 0$. It is clear that for any τ if $k_t = 0$ then $k_{t+\tau} = 0$. Q.E.D.

5 Rationality of a Higher Order

We have seen, in the last section, what happens when to smart players play. This section is divided in two subsections. In the first subsection I introduce the idea of rationality of a higher order, and its consequences when the number of players is two. There we will see that the learning process converges to the Cournot-Nash equilibrium. In the second subsection I will extend the analysis to the N firms case. There I show that for four or more players the process converges to competition.

5.1 Rationality of a higher order when the number of firms is two

Let me briefly summarize the whole argument. First I have given the beliefs formation process of a player. To do this we have applied the idea that any observed production decision for an oligopolist must be optimal in accord to some belief about the response of the opponents. Let us call this operation to be a result of rationality of order 1, and denote it by r_1. Then, in section 3, we have argued, as we did in the discussion of fictitious play, that this can be anticipated by the other players. We saw that the resulting reaction function was different and a best response. Let us call this operation to be a result of rationality of order 2, and denote it by r_2. We have already shown that two players with this level of rationality can achieve cooperation.

Continuing in this fashion we can define rationality of higher order. An r_3 player will optimize taking account of the fact that he has observed that the opponent was r_2. Notice that the beliefs are uniquely determined and therefore the players can know the rationality level of the opponent.

Then, as we did before, firm a has to solve the following system of equations:

$$-k_a k_p - k_a k_p^2 + k_p^2 + 2k_a^2 \frac{\partial k_p}{\partial k_a} = 0,$$

$$k_p = \frac{3k_a}{1 + k_a}.$$

The resulting behavioral equation, now hold by a r_3 player will be:

$$k_a = \frac{5/3k_p}{1 + k_p}.$$

For player p will be:

$$k_p = \frac{5/3k_a}{1 + k_a}.$$

Notice that there is no substantial increase in the difficulty to shift from r_2 to r_3. The procedure has been to iterate the same profit maximization formula. Similarly we can go to r_4 and so forth.

The sequence of behavioral equations can be written as

$$k_j = \frac{\phi_t k_i}{1 + k_i}, \ j, i = a, p, \ j \neq i.$$

Where ϕ_t is a constant, and the subindex t indicates the iteration or the rationality level. It can be easily verified that this coefficient evolves according to this sequence:

$$\phi_t = 1 + \frac{2}{\phi_{t-1}}. \tag{15}$$

This sequence is decreasing and bounded below. It has a limit and this limit is $l = 2$. Therefore the limiting behavioral equations are the following:

$$k_a = \frac{2k_p}{1 + k_p},$$

$$k_p = \frac{2k_a}{1+k_a},$$

They intersect each other at $k_a = k_p = 1$, i.e. in the Cournot-Nash equilibrium. They are also the best replay as reaction functions. This is the main result of this paper: the learning process converges to the Cournot-Nash equilibrium.

In the next payoff matrix I have computed the profits for each duopolist when different levels of rationality are matched.

Table4.

	r_1	r_2	r_3	r_4	r_5
r_1	$(0,0)$	$(0.125, 0.0625)$	$(0.0625, 0.0468)$	$(0.0937, 0.0585)$	$(0.0781, 0.0537)$
r_2	$(0.0625, 0.125)$	$(0.125, 0.125)$	$(0.0973, 0.14062)$	$(0.1093, 0.1367)$	$(0.1015, 0.1396)$
r_3	$(0.0468, 0.0625)$	$(0.14062, 0.0937)$	$(0.09375, 0.09375)$	$(0.1171, 0.09765)$	$(0.1054, 0.09667)$
r_4	$(0.0585, 0.0937)$	$(0.1367, 0.1093)$	$(0.09765, 0.1171)$	$(0.11718, 0.11718)$	$(0.1074, 0.1181)$
r_5	$(0.0537, 0.0781)$	$(0.1396, 0.1015)$	$(0.09667, 0.1054)$	$(0.1181, 0.1074)$	$(0.10742, 0.10742)$

The unique Nash equilibrium of this game is in r_∞, r_∞. Two things are to be noted here. First, the main feature of this process is that there is no fictitious play here. Once a player has observed the rationality level of the opponent he can react optimally increasing, if he can, his level of smartness. Second, if we assume that player 1 is r_3 and player 2 is r_6, the second player will stop in r_4. The obvious reason is that r_6, his maximum level of smartness, is not a best response to r_3. Notice, however, that there is no reason to arbitrarily bound the smartness level. The reason is that there is no substantial difference in the computational capabilities. More intelligent players have to compute more but not more complex computations.

5.2 The N firm case

In section 3 the argument about consistent beliefs gave as a result equation (3). Further, in section 4.2 I got equation (12). If we continue iterating we get a recursive formula for the constant in the next expression:

$$k_i = \frac{\phi_t}{1+\sum_{j\neq i}^{n} 1/k_j}, \quad i = 1\ldots n.$$

This recursive formula is:

$$\phi_t = 1 + 2(n-1)1/\phi_{t-1}. \tag{16}$$

It is easy to verify that for $n = 2$ we get equation (15).

Proposition 7 *The learning process converges to the Cournot-Nash equilibrium if and only if $n = 2$.*

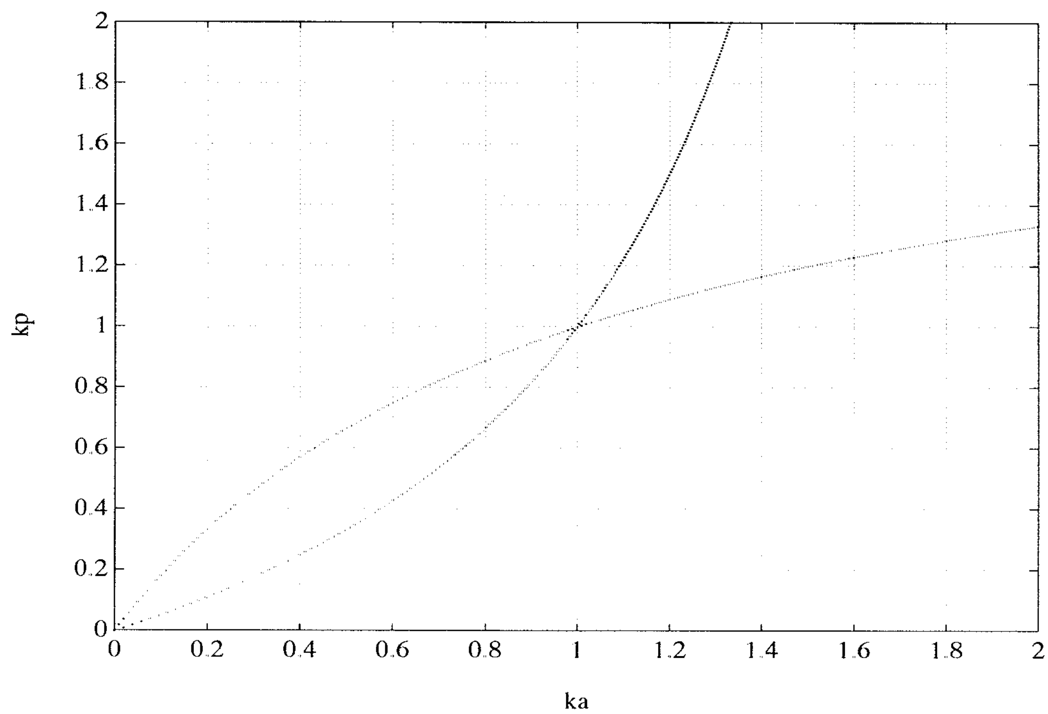
Proof

The sequence described by equation (16) has a limit. This limit is $l = 1 + 2(n-1)1/l$. The solution for this equation is: $l = (1 + \sqrt{1 + 8(n-1)}/2$. This is the constant when the iterative process has reached the limit. Therefore, the expression for k_i would be:

$$k_i^* = \frac{l}{1 + \sum_{j \neq i}^{n} \frac{1}{k_j}} \; for \; all \; i = 1 \ldots n.$$

Recall that the Cournot-Nash equilibrium is characterized by $k_1 = k_2 = \ldots = k_n = 1$. In the denominator we have, in the Cournot-Nash equilibrium, $1 + (n-1)$. Therefore the process will reach the Cournot-Nash equilibrium as a symmetric equilibrium iff we find the constant n in the numerator. But this is to say that $l = n$. After minor manipulations we find $n = 2$. The converse has been proved in subsection 5.1. Q.E.D.

Figure 4: Convergence to Cournot-Nash when r_∞, r_∞ are matched



Proposition 8 *The learning process converges to competition for any $n \geq 4$*

Proof. The symmetric equlibrium can be easily computed from:

$$k_i^* = \frac{l}{1 + \sum_{j \neq i}^{n} \frac{1}{k_j}} \quad for\ all\ i = 1 \ldots n.$$

as $k = \frac{l(n)}{1+(n-1)\frac{1}{k}}$, or $k = l - (n-1)$. When $n = 4$, $l = 3$, so that the symmetric equilibrium is $k = 0$, or competition. It is clear that for $n > 4$, $k < 0$, but this is impossible because with this solution profits are negative for all firms, and in addition $k = 0$, and zero profits, is always a solution for any ϕ_t. Q.E.D.

In case that $n = 3$, we can find an equilibrium point in the midway from Cournot-Nash to competition. It is surprising the speed of convergence to competition when the number of firms increases. With four firms it takes a litle while, but with seven firms competition is reached at the third iteration. However as shown in proposition five, collusion is always possible. From here it is easy to see that those markets with very smart players will perform very bad in relation with those markets whose players are bounded to r_2.

6 Conclusions

In this paper I have tried to analyze the behaviour of firms in an oligopolistic market when the principal aim of a firm is to learn about how the other players compute their best responses. The presented procedure have at least the following adventages when compared to fictitious play:

1. It is a real model of learning. It is possible for each player to find out the precise level of smartness of the opponent.

2. Players do not need to have any prior about the opponent. They simply see what the opponent has done, then they find out the belief the opponent must have about him in order to make optimal the decision taken. Therefore there is no a priori suposition about the procedure that firms should follow. Each firm only apply a priciple of rationality: what has been observed must be optimal for some belief. Once a firm has tested the rationality level of the opponent she maximizes profits with the new information.

3. Convergency is guaranteed from any starting point in the beliefs space if we use the Cournot "tatonnement" process over any pair of reaction functions.

This paper is a generalisation of Bresnahan duopoly model. Here his argument is extended to any number of firms. The beliefs of firms are consistent with those observed responses regardless how far they are from competition, whereas in Bresnahan model the conjectures are lovally consistent. This paper also characterize different equilibria in oligopolistic markets as a result of the interaction of different rationality levels. It also shows that competition is associated with the lowest rationality level. This result is of particular interest when boundedly rational agents are considered. As stated in proposition 8, competition is also reached when players are infinitely smart but the number of firms is larger than 4. The conclusion is that infinitely smart players can perform as bad as firms bounded to r_1 rationality level.

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