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Veleza, Rodrigo, Thomson, William

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Rodrigo Velez[a] and William Thomson[b*]

[a]Department of Economics, Texas A&M University, College Station, TX 77843

[b]Department of Economics, University of Rochester, Rochester, NY 14627

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Abstract

We consider the problem of fairly allocating a bundle of privately appropriable and infinitely divisible goods among a group of agents with “classical” preferences. We propose to measure an agent’s “sacrifice” at an allocation by the size of the set of feasible bundles that the agent prefers to her consumption. As a solution, we select the allocations at which sacrifices are equal across agents and this common sacrifice is minimal. We then turn to the manipulability of this solution. In the tradition of Hurwicz (1972, Decision and Organization, U. Minnesota Press), we identify the equilibrium allocations of the manipulation game associated with this solution when all commodities are normal: (i) for each preference profile, each equal-division constrained Walrasian allocation is an equilibrium allocation; (ii) conversely, each equilibrium allocation is equal-division constrained Walrasian. (iii) Furthermore, we show that if normality of goods is dropped, then equilibrium allocations may not be efficient.

JEL Classification: C72, D63.

Key-words: equal-sacrifice solution; manipulation game; equal-division Walrasian solution.

1 Introduction

We consider the problem of fairly allocating a bundle of privately appropriable and infinitely divisible goods among a group of agents having equal rights on these goods. To make the objective of fairness operational, we propose to measure the sacrifice made by an agent at an allocation by the size of the set of feasible bundles that she prefers to her assignment, and to select the allocations at which sacrifices are equal across agents and this common sacrifice is minimal. We refer to the resulting solution as the "equal-sacrifice" solution.

First, we prove that the equal-sacrifice solution is well-defined under general assumptions on preferences, and that under some very mild additional monotonicity assumptions, equal-sacrifice allocations are also efficient.

Crawford (1980) advocates, for the two-agent case, the rule that selects, from the one of agent 1's indifference curves that divides the Edgeworth box into two regions of equal volumes, the allocation preferred by agent 2. This proposal suffers from treating the two agents asymmetrically, and it is not easily generalized to more than two agents. Nevertheless, basing the choice of an allocation on the size of upper contour sets is a natural assumption that we have retained. We have adapted it to handle arbitrary populations, and done so in a manner that delivers a symmetric treatment of agents. Our solution can also be thought of as a member of the following family. Specify for each agent a function that represents her preferences, a "welfare index" for her. Then, select the allocations at which these welfare indices take a common value, and this common value is maximal among all feasible allocations.

We then turn to the question of manipulability. To each rule can be associated a manipulation "game form" as follows: the strategy space of each agent is the space of preferences satisfying the properties that her relation is known to satisfy; the outcome function is the rule itself. If in this game, it is a dominant strategy for each agent to announce her true preferences, we say that the rule is "strategy-proof". It is well-known that on the domain on which we are operating, strategy-proofness is very restrictive. Indeed, no selection from the correspondence that associates with each economy its set of efficient allocations at which each agent finds her consumption at least as desirable as her endowment, is strategy-proof (Hurwicz, 1972, Serizawa, 2002). This conclusion also applies to all selections from the Pareto solution satisfying "equal treatment of equals" (Serizawa, 2002).[1] It follows from this latter result that the equal-sacrifice solution is not strategy-proof. It is of course easy to construct examples directly establishing this fact, and we will provide some.

[1] Barberà and Jackson (1995) drop efficiency and provide a characterization of the class of strategy-proof rules in the two-agent case, and under some additional property in the case of more than two agents.

The study of a solution should not stop with the observation that it is not strategy-proof, however. A violation of this property simply means that there are preference profiles such that, if all agents but possibly one tell the truth about their preferences, the last agent may benefit for not doing so. However, what should really concern us is not so much that agents may not be truthful, but rather that the allocations that the solution would specify for the true profile may not be reached. Thus, a determination of which allocations will be obtained is called for. Only knowing that an agent may benefit by behaving strategically, keeping fixed the announcements made by the others, does not suffice for that purpose. Several agents may be in that position, and any agent who is considering misrepresenting her preferences has to entertain the thought that others could do the same, and should take that fact into consideration when selecting her strategy. Consequently, we are led to associating with the solution a manipulation game, identifying its equilibria, and evaluating them in terms of the true preference profile.

Thus, the second objective of this paper is to characterize, for each preference profile, the equilibria of the manipulation game associated with the equal-sacrifice solution for that profile. We achieve this under the assumption that all goods are normal. Our main result is the following: the set of equilibrium outcomes of the manipulation game associated with the equal-sacrifice solution coincides with the set of equal-division constrained Walrasian allocations for the true preferences![2]

It is not uncommon that the equilibria of a manipulation game associated with an allocation rule contain the equal-division constrained Walrasian allocations (see below). Thus, the striking part of our results is the converse inclusion. No other allocation is reached at equilibrium. Typically, manipulation does not lead to such a relatively happy conclusion (these allocations, besides being efficient, have been a focal point in axiomatic studies of fair allocation). For instance, for each preference profile, the equilibrium allocations of a game similarly associated with the Walrasian solution itself are not necessarily constrained Walrasian for that profile. (In the two-agent case, they are all the allocations in the lens-shaped area defined by the true offer curves, as shown by Hurwicz, 1972.)

The following observations should provide some intuition for our result. First, the equal-sacrifice solution is very "sensitive". In particular, any change in any agent's indifference curve through her assigned bundle at some profile is likely to bring about a change

[2] A constrained Walrasian allocation is defined as a Walrasian allocation except that maximization of preferences takes place in "truncated" budget sets. Only consumption bundles that are part of a feasible allocation and meet the budget constraint are admissible. This variant of the Walrasian solution, introduced by Hurwicz (1979), only differs from it when some agents consume on the boundary of their consumption sets.

in the allocation the rule selects.[3] Moreover, if an agent's announced indifference curve through her assigned bundle is not linear, switching to a preference relation for which her indifference curve through her assigned bundle is flatter than it was originally (this is an "anti-monotonic" transformation of her preferences at that point, where "monotonicity" is understood in the sense of Maskin, 1999), increases the agent's apparent sacrifice. This calls for her sacrifice to be reduced. It is tempting at this point to conclude that equilibrium has to occur when all announcements are linear. One more step is needed however—and it turns out to be a technically delicate one—because a reduction in an agent's sacrifice in terms of preferences to which she may switch does not necessarily cause a parallel reduction in terms of her true preferences, which is what really matters to her. What is needed is an understanding of the circumstances under which this implication does hold. This is where the assumption of normality of goods comes into play. Under that assumption, the agent will indeed benefit. We also show that this assumption is necessary. Without it—and we give examples—equilibria exist involving non-linear preferences and whose outcomes are not constrained equal-division Walrasian for the true preferences.

2 Related literature

The manipulability of allocation rules on various domains has been the object of a number of studies.

In the context of exchange economies, early studies are Sobel (1981), for two agents, and Thomson (1984, 1987, 1988) for quasi-linear preferences. Constrained Walrasian allocations (or equal-income constrained Walrasian allocations) are shown to be equilibrium allocations, but there can be others. More recent papers have dealt with the manipulability of solutions to Nash's bargaining problem (Sobel, 2001, Kıbrıs, 2002) and have also derived equal-income constrained Walrasian allocations as equilibrium allocations.[4] This is why earlier we wrote that our conclusion that these allocations are equilibrium allocations of the game associated with the equal-sacrifice solution is not the surprising part. What is remarkable is that there are no other equilibrium allocations. We noted above that for two agents and for each specification of their preferences, the equilibrium allocations of the game associated with the Walrasian rule are delimited by the true offer curves. This region does contain the true Walrasian allocations but it also contains the endowment allocation,

[3]The sensitivity of our solution explains in part the difference between the conclusions we reach for it and what we know about the Walrasian solution. These issues are discussed by Thomson (1984), who shows the relevance of Maskin-monotonicity to the characterization of the set of equilibria.

[4]See also Crawford and Varian (1979) for an earlier study of the manipulation of utility functions in the context of bargaining.

and a continuum of allocations in between. Thus, Pareto efficiency is a possible outcome but it may also be that no gains from trade are achieved at equilibrium.

The context of the allocation of a single infinitely divisible good when agents have single-peaked preferences (Sprumont, 1983) is a rare one in which a strategy-proof rule exists. It is called the uniform rule. Other rules that have been proposed as providing fair outcomes for this class of problems. It so happens that for each preference profile, the manipulation game associated with a number of these rules has a unique equilibrium allocation, which is none other than the uniform allocation for that profile (Thomson, 1990, Bochet and Sakai, 2008.) Consequently, not only does manipulation not necessarily cause violations of efficiency, but it leads to a rule that has a number of other desirable properties (Thomson, 2006, provides an overview of these properties.) The similarity between these results and the main result of the current paper is due, in part, to the fact that the uniform rule can be thought of as a counterpart for the single-peaked model of the Walrasian concept when operated from equal division.

The problem of allocating an indivisible good when monetary compensations are feasible is studied by Tadenuma and Thomson (1995). Consider an allocation rule that selects envy-free allocations. Then, the equilibrium correspondence of its associated manipulation game is the entire no-envy solution. This conclusion can also be related to our main result. Indeed, for that model, the no-envy solution coincides with the equal-income Walrasian solution (Svensson, 1983). Nevertheless, no such coincidence takes place in either the classical model or the model with which we are concerned here.

In the context of matching (Roth and Sotomayor, 1990), no selection from the stable solution exists that is strategy-proof. However, for each preference profile, the set of undominated Nash equilibria of the manipulation game associated with either the "man-optimal" rule or the "woman-optimal" rule is the entire set of stable outcomes for that profile (Roth, 1984, Gale and Sotomayor, 1985.) Moreover, the set of Nash equilibria is the entire set of individually rational outcomes for the true preferences (Alcalde, 1996).

The manipulability of solutions has been studied under alternative behavioral assumptions. Crawford (1980) shows that for the rule that he had defined (see above), under maximin behavior, agent 1 would announce the one of her true indifference curves that divides the Edgeworth box into equal areas (volumes). The behavioral assumption under which he addresses the manipulation issue supposes an extreme form of risk aversion. We have found it more natural to assume Nash behavior.

Manipulation of preferences in economies with public goods is studied by Thomson (1979) and manipulation through endowments by Postlewaite (1979) and Thomson (1987). Manipulation of voting procedures is studied by Sanver and Zwicker (2004).

3 Model

3.1 The environment

We consider the problem of allocating a fixed social endowment $\Omega \equiv (\Omega^1, \ldots, \Omega^K) \in \mathbb{R}^K_{++}$, for some $K \in \mathbb{N}$, among a group of agents $N \equiv \{1, \ldots, n\}$. Their preferences are complete and transitive binary relations on $\mathbb{R}^K_+$. The generic preference is R_0. The symmetric and asymmetric parts of R_0 are I_0 and P_0 respectively. A preference R_0 is **monotone** if for each $\{b, b'\} \subset \mathbb{R}^K_+$, $b' \geq b$ implies $b' \, R_0 \, b$, and $b' \gg b$ implies $b' \, P_0 \, b$.[5] Let $\boldsymbol{\mathcal{U}}$ denote the domain of convex, continuous, and monotone preferences. A preference $R_0 \in \mathcal{U}$ is **semi-strictly monotone** if for each $\{b, b'\} \subset \mathbb{R}^K_+$ such that $b' \gneq b$, if $b \, P_0 \, 0$, then $b' \, P_0 \, b$. The sub-domain of $\mathcal{U}$ of semi-strictly monotone preferences is $\boldsymbol{\mathcal{R}}$.[6]

For each $R_0 \in \mathcal{U}$ and each $b \in \mathbb{R}^K_+$, the **indifference set of $\boldsymbol{R_0}$ at $\boldsymbol{b}$** is $I(R_i, b) \equiv \{b' \in \mathbb{R}^K_+ : b' \, I_0 \, b\}$ and the **constrained indifference set of $\boldsymbol{R_0}$ at $\boldsymbol{b}$** is $I^c(R_0, b) \equiv \{b' \in \mathbb{R}^K_+ : b' \, I_0 \, b,\ b' \leq \Omega\}$. The **upper contour set of $\boldsymbol{R_0}$ at $\boldsymbol{b}$** is $U(R_0, b) \equiv \{b' \in \mathbb{R}^K_+ : b' \, R_0 \, b\}$ and the **constrained upper contour set of $\boldsymbol{R_0}$ at $\boldsymbol{b}$** is $U^c(R_0, b) \equiv \{b' \in \mathbb{R}^K_+ : b' \, R_0 \, b,\ b' \leq \Omega\}$. The **strict upper contour set of $\boldsymbol{R_0}$ at $\boldsymbol{b}$** is $U(P_0, b) \equiv \{b' \in \mathbb{R}^K_+ : b' \, P_0 \, b\}$ and the **constrained strict upper contour set of $\boldsymbol{R_0}$ at $\boldsymbol{b}$** is $U^c(P_0, b) \equiv \{b' \in \mathbb{R}^K_+ : b' \, P_0 \, b,\ b' \leq \Omega\}$.

Let $R_0 \in \mathcal{R}$, $b \in \mathbb{R}^K_+$, and $p \in \mathbb{R}^K_+$. The **set of prices that support $\boldsymbol{U(R_0, b)}$ at $\boldsymbol{b}$** is **$\mathrm{Supp}(\boldsymbol{R_0, b})$**. An **income expansion path for $\boldsymbol{R_0}$ at prices $\boldsymbol{p}$** is a function $V : \mathbb{R}_+ \to \mathbb{R}^K_+$, such that for each $w \in \mathbb{R}_+$, $w = p \cdot V(w)$ and $p \in \mathrm{Supp}(R_0, V(w))$; V is **quasi-strictly increasing** if its component functions $\{V_k\}_{k=1}^K$ are strictly increasing up to a possible "flat part" containing the origin. Formally, V is quasi-strictly increasing if for each each $k \in \{1, \ldots, K\}$, and each $\{w, w'\} \subset R_+$ such that $w < w'$, $0 < V_k(w)$ implies $V_k(w) < V_k(w')$.

We consider four additional preference domains.

- A preference R_0 has **quasi-strictly increasing income expansion paths** (i.e., all goods are normal) if for each $b \in \mathbb{R}^K_+ \setminus \{0\}$ and each $p \in \mathrm{Supp}(R_0, b)$, there is a quasi-strictly increasing income expansion path for R_0 at prices p that passes through b.[7] Let $\boldsymbol{\mathcal{I}}$ be the domain of preferences in $\mathcal{R}$ with quasi-strictly increasing

[5] We use the following vector inequalities. For each $L \in \mathbb{N}$ and each $\{x, x'\} \subset \mathbb{R}^L$: $x' \geq x$ if for each $l \in \{1, \ldots, L\}$, $x'_l \geq x_l$; $x' \gneq x$ if $x' \geq x$ and $x' \neq x$; and $x' \gg x$ if for each $l \in \{1, \ldots, L\}$, $x'_l > x_l$.

[6] Semi-strict monotonicity of R_0 is weaker than strict monotonicity of R_0, which says that for each $\{x, y\} \subset \mathbb{R}^K_+$ such that $x \gneq y$, $x \, P_0 \, y$. For instance, Cobb-Douglas preferences are semi-strictly monotone, but not strictly monotone (violations of the strict form of the property occur on the boundary.)

[7] Observe that if $p \in \mathbb{R}^K_{++}$ and $R_0 \in \mathcal{R}$, then each income expansion path relative to prices p that passes through b starts at $0 \in \mathbb{R}^K_+$.

income expansion paths.

- A preference R_0 is **smooth** if for each $b \in \mathbb{R}^K_{++}$, there is a unique $p \in \mathbb{R}^K_{++}$ that supports $U(R_0, b)$ at b, i.e., $|\text{Supp}(R_0, b)| = 1$. Let $\boldsymbol{\mathcal{S}}$ be the domain of smooth preferences in $\mathcal{R}$.

- A preference R_0 is **homothetic** if for each $b \in \mathbb{R}^K_+ \setminus \{0\}$ and each $p \in \text{Supp}(R_0, b)$, the ray passing through b is an income expansion path for R_0 at prices p.[8] Let $\boldsymbol{\mathcal{H}}$ be the domain of homothetic preferences in $\mathcal{R}$.

- A preference R_0 is a **positively oriented linear preference**, if there is $p \in \mathbb{R}^K_{++}$ such that for each $\{b, b'\} \subset R^K_+$, $b \, R_0 \, b'$ if and only if $p \cdot b \geq p \cdot b'$. For each $p \in \mathbb{R}^K_{++}$, the linear preference associated with p is L^p. Let $\boldsymbol{\mathcal{L}}$ be the domain of positively oriented linear preferences.[9]

Lemma 9 in Appendix states that $\mathcal{H} \subsetneq \mathcal{I}$, and thus the following inclusion relations hold among the preference domains above: $\mathcal{L} \subsetneq \mathcal{H} \subsetneq \mathcal{I} \subsetneq \mathcal{R}$ and $\mathcal{L} \subsetneq \mathcal{S} \subsetneq \mathcal{R}$.

Agent i's generic preference is $R_i \in \mathcal{U}$, and the generic preference profile is $R \equiv (R_i)_{i \in N} \in \mathcal{U}^N$. For each $R \in \mathcal{U}^N$, each $i \in N$, and each $R'_i \in \mathcal{U}$, the profile $(R_{-i}, R'_i) \in \mathcal{U}^N$ is obtained from R by replacing R_i by R'_i.

The **set of feasible allocations** is $Z \equiv \{z \equiv (z_i)_{i \in N} \in \prod_N \mathbb{R}^K_+ : \sum_{i \in N} z_i \leq \Omega\}$. Agent i's allotment at $z \in Z$ is $z_i \equiv (z_i^k)_{k=1}^K \in \mathbb{R}^K_+$. A **solution** associates to each preference profile a non-empty subset of Z. The generic solution is F. A **selector** f from a solution F is a function that associates with each $R \in \mathcal{R}^N$ an element of $F(R)$. We write $f \in F$.

A solution F is **essentially single-valued** if for each $R \in \mathcal{R}^N$, each $\{z, z'\} \subseteq F(R)$, and each $i \in N$, $z \, I_i \, z'$.

3.2 Manipulation of a solution

When a solution F recommends for a particular economy a set of allocations, as opposed to a singleton, one has to ask the question: how will agents manipulate F? Consider some profile of announced preferences. Suppose that a given F-optimal allocation, say z, is chosen for that profile. If an agent unilaterally deviates, and the set of F-optimal allocations for the new profile is not a singleton, then there could be some allocations in

[8] Since indifference sets with linear pieces are allowed for preferences in $\mathcal{R}$, then for a given price vector, income expansion paths are not necessarily unique.

[9] Observe that linear preferences with indifference sets parallel to some coordinate subspaces are not included in $\mathcal{L}$. Moreover, $\mathcal{L} \subset \mathcal{R}$.

the set at which she is better off, some at which she is worse off, and some at which her welfare is unaffected.[10]

Several behavioral assumptions can be formulated to deal with this indeterminacy. For instance, one can suppose that the agent is optimistic, i.e., given the others' reports, if by reporting different preferences, there is at least one F-optimal allocation for the new profile that she prefers to z, then she will not stick with her initial announcement. Alternatively, one can suppose that she will switch only if she prefers all of the F-optimal allocations for the new profile to z. Of course, one can think of yet other behavioral assumptions.

We solve this issue in a way that bypasses any speculation about which behavioral assumption is the most appropriate. We complete the allocation process by assuming that each agent is asked to report not only her preferences, but also a bundle, which we interpret as the one she would like to receive. Moreover, we add to the specification of the outcome function a "selector" to break ties between the F-optimal outcomes for the reported preferences when the F-optimal set is not a singleton and the profile of reported bundles is not in this set. Now the outcome function is complete: if the profile of reported bundles is an F-optimal allocation for the profile of reported preferences, then it is the outcome of the allocation process; otherwise, the selector determines this outcome.

Adding a selector allows us to understand the strength of the behavioral assumptions in the strategic analysis of solutions. If the outcomes that obtain at equilibrium are independent of this selector, then these outcomes will result under any of the aforementioned behavioral assumptions. In Section 5 we show that this is the case for the equal-sacrifice solution in the domain of preferences with quasi-strictly increasing income expansion paths.

For a given solution and one of its selectors, we now formally define the game form associated with them for some domain of preferences. Let $\mathcal{D} \subseteq \mathcal{U}$ be a domain, F a solution, and $f \in F$. The **game form** $\boldsymbol{\langle S(\mathcal{D})^N, F^f \rangle}$ is defined as follows: (i) each agent's strategy space is $\boldsymbol{S(\mathcal{D})} \equiv \mathcal{D} \times \mathbb{R}^K_+$; and (ii) given the strategy profile $(R, z) \equiv (R_i, z_i)_{i \in N} \in S(\mathcal{D})^N$, the outcome is

$$\boldsymbol{F^f(R, z)} \equiv \begin{cases} z & \text{if } z \in F(R) \\ f(R) & \text{otherwise.} \end{cases}$$

For each $R^0 \in \mathcal{U}^N$, the **game** $\boldsymbol{\langle S(\mathcal{D})^N, F^f, R^0 \rangle}$ is obtained by augmenting the game form $\langle S(\mathcal{D})^N, F^f \rangle$ by the preference profile R^0.

A **Nash equilibrium of** $\boldsymbol{\langle S(\mathcal{D})^N, F^f, R^0 \rangle}$ is a strategy profile $(R, z) \in S(\mathcal{D})^N$, such

[10] This issue arises even for an *essentially single-valued* solution. For such a solution, agents are seemingly indifferent among the recommended allocations under the reported preferences, but this could not be the case for their true preferences.

that for each $i \in N$ and each $(R'_i, z'_i) \in S(\mathcal{D})$,

$$F_i^f(R,z)\, R_i^0\, F_i^f(R_{-i}, R'_i, z_{-i}, z'_i).$$

For each game $\langle S(\mathcal{D})^N, F^f, R^0\rangle$, the set of **Nash equilibria** is $\boldsymbol{\mathcal{N}\langle S(\mathcal{D})^N, F^f, R^0\rangle}$ and the set of **Nash equilibrium outcomes** is $\boldsymbol{\mathcal{O}\langle S(\mathcal{D})^N, F^f, R^0\rangle}$.

If for each pair of selectors of F, f and g, $\mathcal{O}\langle S(\mathcal{D})^N, F^f, R^0\rangle = \mathcal{O}\langle S(\mathcal{D})^N, F^g, R^0\rangle$, we denote this common set by $\boldsymbol{\mathcal{O}\langle S(\mathcal{D})^N, F, R^0\rangle}$.

3.3 Additional notation

For each pair $\{x^1, x^2\} \subset \mathbb{R}_+^K$, let **seg**$[\boldsymbol{x^1, x^2}]$ be the segment connecting x^1 and x^2. For each list $\{x^1, x^2, \ldots, x^l\} \subset \mathbb{R}_+^K$, let **bro.seg**$[\boldsymbol{x^1, x^2, \ldots, x^l}]$ be the broken segment connecting these points in that order, and let **con.hull**$\boldsymbol{\{x^1, x^2, \ldots, x^l\}}$ be the convex hull of $\{x^1, x^2, \ldots, x^l\}$. Finally, for each pair of vectors $\{x^1, x^2\} \subset \mathbb{R}_+^K$ such that $x^1 \leq x^2$, let **rec**$\boldsymbol{\{x^1, x^2\}}$ be the rectangle $\{y \in \mathbb{R}_+^K : x^1 \leq y \leq x^2\}$. For each $b \in \mathbb{R}_+^2$ and each $m \in \mathbb{R}_+$, let **ray**$\boldsymbol{\{b, m\}}$ be the ray emanating from b with slope m.

4 The equal-sacrifice solution

The **Pareto solution**, $\boldsymbol{P}$, and the **weak Pareto solution**, $\boldsymbol{P^w}$, are defined as usual: for each $R \in \mathcal{U}^N$ and each $z \in Z$, $z \in P(R)$ if and only if there is no $z' \in Z$ such that (i) for each $i \in N$, $z'_i\, R_i\, z_i$, and (ii) there is $j \in N$ such that $z'_j\, P_j\, z_j$; $z \in P^w(R)$ if and only if there is no $z' \in Z$ such that for each $i \in N$, $z'_i\, P_i\, z_i$.

Let μ be the Lebesgue measure on $\mathbb{R}^K$. For each $R_0 \in \mathcal{U}$ and each $b \in \mathbb{R}_+^K$, let $\boldsymbol{a(R_0, b)} \equiv \mu(U^c(R_0, b))$, i.e., the size of the constrained upper contour set of R_0 at b. Since resources are owned collectively in our model, then when an agent with preferences R_0 consumes b, she "sacrifices" her option of consuming the bundles in $U^c(R_0, b)$. Thus, $a(R_0, b)$ is a reasonable measure of the sacrifice of R_0 at b, the measure that assigns equal weights to all bundles.

For each $R \in \mathcal{U}^N$, let $\Psi(R)$ be the set of feasible allocations at which sacrifices are equal across agents, i.e., $\Psi(R) \equiv \{z \in Z : \text{for each } \{i,j\} \subseteq N,\ a(R_i, z_i) = a(R_j, z_j)\}$. The **equal-sacrifice solution**, $\boldsymbol{E}$, associates with each $R \in \mathcal{U}^N$ the set of allocations at which sacrifices are equal across agents, and this common sacrifice is minimal:

$$E(R) \equiv \left\{z \in \Psi(R) : \text{for each } \{i,j\} \subseteq N \text{ and each } z' \in \Psi(R),\ a(R_i, z_i) \leq a(R_j, z'_j)\right\}.$$

The following theorem states that E is a well-defined and *essentially single-valued* solution. We present the proof in Appendix.

Theorem 1. For each $R \in \mathcal{U}^N$, $E(R) \neq \emptyset$. Moreover, E is essentially single-valued.

The following lemma concerns efficiency properties of the equal-sacrifice allocations. First, they are weakly Pareto efficient. Second, the lemma identifies two wide classes of economies in which they are in fact Pareto efficient.[11]

Lemma 1. Let $R \in \mathcal{U}^N$. Then, (i) $E(R) \subseteq P^w(R)$, and (ii) If $|N| = 2$ or $R \in \mathcal{R}^N$, then $E(R) \subseteq P(R)$.

The following lemma is an application of the Second Fundamental Theorem of Welfare Economics. We present the proof in Appendix.

Lemma 2. Let $R \in \mathcal{R}^N$ and $z \in E(R)$. Then, (i) for each $i \in N$, $\Omega\, P_i\, z_i\, P_i\, 0$, (ii) $\sum_{i \in N} z_i = \Omega$, and (iii) there is $p \in \mathbb{R}^K_{++}$ such that for each $i \in N$, $p \in \text{Supp}(R_i, z_i)$.

The **equal-division constrained Walrasian solution**, $\boldsymbol{W^c_{ed}}$, associates with each $R \in \mathcal{U}^N$ the set of allocations

$$W^c_{ed}(R) \equiv \{z \in Z : \exists p \in \mathbb{R}^K_+ \text{ s.t. for each } i \in N \text{ and each } z'_i \in U^c(P_i, z_i),\ p \cdot z'_i > p \cdot \tfrac{\Omega}{n}\}.$$

The following lemma is a consequence of the monotonicity properties of preferences in the domain $\mathcal{R}$. We omit the straightforward proof.

Lemma 3. Let $R \in \mathcal{R}^N$ and $z \in W^c_{ed}(R)$. Then, (i) $\sum_{i \in N} z_i = \Omega$, and (ii) if $p \in \mathbb{R}^K_+$ supports z as a member of $W^c_{ed}(R)$, then $p \in \mathbb{R}^K_{++}$.

5 The manipulability of the equal-sacrifice solution

Our main theorem characterizes the equilibrium correspondence of the manipulation game associated with the equal-sacrifice solution on the domain $\mathcal{I}$. For each preference profile in $\mathcal{I}^N$, each equal-division constrained Walrasian allocation is an equilibrium allocation. Conversely, for each selector $e \in E$, each equilibrium allocation of the game associated with the equal-sacrifice solution and e and the profile is an equal-division constrained Walrasian allocation for that profile.

[11] The equal-sacrifice solution may select weakly Pareto efficient allocations that are not Pareto efficient when there are at least three agents and preferences are in $\mathcal{U} \setminus \mathcal{R}$. Let $(R_i)_{i \in N} \in \mathcal{U}^N$ and for each $i \in N$, let u_i be the function defined by: for each $z_i \in \mathbb{R}^K_+$, $u_i(z_i) \equiv \mu(\text{rec}\{0, \Omega\}) - a(R_i, z_i)$. It is easily seen that $E(R)$ contains the allocations whose image under $(u_i)_{i \in N}$ is the Kalai-Smorodinsky (K-S) bargaining solution for the comprehensive hull of the convex problem $u(Z)$. It is well known that K-S may select weakly efficient allocations that are not efficient for more than three agents on the convex domain. An example showing this fact is easily adapted to prove the parallel statement for E.

Theorem 2. For each $R^0 \in \mathcal{I}^N$ and each $e \in E$,

$$\mathcal{O}\langle S(\mathcal{I})^N, E^e, R^0\rangle = W^c_{ed}(R^0).$$

Since the characterization above is independent of the particular selector from E, then we conclude that these are the outcomes that result from the manipulation of this solution. The following corollary states this result.

Corollary 1. For each $R^0 \in \mathcal{I}^N$,

$$\mathcal{O}\langle S(\mathcal{I})^N, E, R^0\rangle = W^c_{ed}(R^0).$$

The proof of Theorem 2 follows from four lemmas.

Let $e \in E$ and $R^0 \in \mathcal{I}^N$. Our first lemma identifies profiles of actions that do not constitute equilibria of game $\langle S(\mathcal{I})^N, E^e, R^0\rangle$. It states conditions under which at least one agent can benefit by changing her action.

Lemma 4. Let $\mathcal{D} \subseteq \mathcal{R}$ be such that $\mathcal{H} \subseteq \mathcal{D}$ and $e \in E$. Let $R^0 \in \mathcal{D}^N$ and $(R, z) \in S(\mathcal{D})^N$. If there are $i \in N$, $z' \in Z$, $p \in \mathbb{R}^K_{++}$, and $a \in \mathbb{R}_{++}$ such that:

(1) for each $j \in N \setminus \{i\}$, p supports $U(R_j, z'_j)$ at z'_j,

(2) for each $j \in N \setminus \{i\}$, $a(R_j, z'_j) = a$,

(3) $z'_i \, P^0_i \, E^e_i(R, z)$,

(4) $\mu(\text{rec}\{z'_i, \Omega\}) < a < a(L^p, z'_i)$,

then $(R, z) \notin \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0\rangle$.

Proof. We construct a preference $R'_i \in \mathcal{D}$ such that

$$E^e_i(R_{-i}, R'_i, z_{-i}, z'_i) \, P^0_i \, E^e_i(R, z).$$

Let $\alpha \in [0, 1)$ and A be the set obtained by translating the boundary of $\mathbb{R}^K_+$ so that the origin is translated to z'_i, i.e., $A \equiv \{x_i \in \mathbb{R}^K_+ : z'_i \leq x_i\} \setminus \{x_i \in \mathbb{R}^K_+ : z'_i \ll x_i\}$. For each ray in the direction of a strictly positive vector, r, the **α-convex combination of A and $I(L^p, z'_i)$ through r** is the point in $\mathbb{R}^K_+$ obtained as a convex combination of $r \cap A$ with weight α and $r \cap I(L^p, z'_i)$ with weight $1 - \alpha$. Let A_α be the set obtained by taking α-convex combinations of A and $I(L^p, z'_i)$ through all the strictly positive rays. Let $R_i(\alpha)$ be the preference whose indifference set through the origin is the boundary of $\mathbb{R}^K_+$ and whose indifference sets in the interior of $\mathbb{R}^K_+$ are obtained as homothetic images of A_α. Clearly

$R_i(\alpha)$ is homothetic and such that $I(R_i(\alpha), z_i') = A_\alpha$. The function $\alpha \mapsto a(R_i(\alpha), z_i')$ is continuous and as $\alpha \to 1$, $a(R_i(\alpha), z_i') \to \mu(\text{rec}\{z_i', \Omega\})$. By hypothesis (4), $\mu(\text{rec}\{z_i', \Omega\}) < a < a(L^p, z_i') = a(R_i(0), z_i')$. Thus, by the Intermediate Value Theorem, there is $\beta \in (0, 1)$ such that $a(R_i(\beta), z_i') = a$. Let $R_i' \equiv R_i(\beta)$.[12] Since $\mathcal{H} \subseteq \mathcal{D}$, then $R_i' \in \mathcal{D}$.

Since $U(R_i', z_i') \subseteq U(L^p, z_i')$, then $p \in \text{Supp}(R_i', z_i')$. Thus, by hypothesis (1), $z' \in P(R_{-i}, R_i')$. Since $a(R_i', z_i') = a$, then by hypothesis (2), $z' \in E_i(R_{-i}, R_i')$. By construction, the upper contour set of R_i' at z_i' intersects the hyperplane with normal p passing through z_i' only at z_i', i.e., $\{z_i'' : z_i''\, R_i'\, z_i',\ p \cdot z_i'' = p \cdot z_i'\} = \{z_i'\}$. Let $z'' \in E(R_{-i}, R_i')$. We prove that $z_i'' = z_i'$. Since for each $j \in N \setminus \{i\}$, $z_j''\, I_j\, z_j'$, then for each $j \in N \setminus \{i\}$, $p \cdot z_j'' \geq p \cdot z_j'$ and thus, $p \cdot z_i'' \leq p \cdot z_i'$. Since $p \in \text{Supp}(R_i', z_i')$ and $z_i''\, I_i'\, z_i'$, then $p \cdot z_i' \leq p \cdot z_i''$. Thus, $p \cdot z_i' = p \cdot z_i''$. Consequently, $z_i'' = z_i'$. By hypothesis (3), $E_i^e(R_{-i}, R_i', z_{-i}, z_i') = z_i'\, P_i^0\, E_i^e(R, z)$. Thus, $(R, z) \notin \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0\rangle$. $\square$

The next lemma states that if a domain $\mathcal{D} \subseteq \mathcal{I}$ is such that $\mathcal{H} \subseteq \mathcal{D}$, then, for each $e \in E$, at each equilibrium of $\langle S(\mathcal{D})^N, E^e, R^0\rangle$, agents report parallel linear indifference sets through their consumption bundles within the "feasible box," i.e., the set $\text{rec}\{0, \Omega\}$.

Lemma 5. Let $\mathcal{D} \subseteq \mathcal{I}$ be such that $\mathcal{H} \subseteq \mathcal{D}$. Let $e \in E$ and $R^0 \in \mathcal{R}^N$. If $z \in \mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0\rangle$, then there is $p \in \mathbb{R}_{++}$ such that for each $i \in N$,

$$I^c(R_i, z_i) = I^c(L^p, z_i).$$

Proof. Let $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0\rangle$. Let $z \equiv E^e(R, z^*)$. By Lemma 2, for each $i \in N$, $z_i \gneq 0$ and there is $p \in \mathbb{R}^K_{++}$ such that for each $i \in N$, $p \in \text{Supp}(R_i, z_i)$. We claim that for each $i \in N$, $I^c(R_i, z_i) = I^c(L^p, z_i)$. Suppose by contradiction that there is $i \in N$ for whom $I^c(R_i, z_i) \neq I^c(L^p, z_i)$. By Lemma 2, $\Omega\, P_i\, z_i\, P_i\, 0$. Since preferences are semi-strictly monotone, then $\mu(\text{rec}\{z_i, \Omega\}) < a(R_i, z_i)$. Moreover, since preferences are continuous, then $a(R_i, z_i) < a(L^p, z_i)$. Let $\delta \equiv a(L^p, z_i) - a(R_i, z_i) > 0$.

For each $j \in N \setminus \{i\}$, let V_j be a quasi-strictly increasing income expansion path for R_j at prices p that passes through z_j.[13] Since preferences are continuous, then for each $j \in N \setminus \{i\}$, the function $w \in \mathbb{R}_+ \mapsto a(R_j, V_j(w))$ is continuous.[14] Let $\eta \in \mathbb{R}_+$ be such that $\eta < \frac{\delta}{2}$. Since $\delta < \mu(\text{rec}\{0, \Omega\}) - a(R_i, z_i)$, then by the Intermediate Value Theorem, there are $(w_j^\eta)_{j \in N \setminus \{i\}} \ll (p \cdot z_j)_{j \in N \setminus \{i\}}$ such that for each $j \in N \setminus \{i\}$, $a(R_j, V_j(w_j^\eta)) = a(R_j, z_j) + \eta$.

[12] R_i' is semi-strictly monotone, but it is not strictly monotone. It is easy to construct a preference R_i'' with the same properties as R_i' but strictly monotone.

[13] These income expansion paths exist because $R \in \mathcal{I}^N$.

[14] Since preferences are continuous, then the function $x_j \in \mathbb{R}^K_+ \mapsto a(R_j, x_j)$ is continuous; since V_j is quasi-strictly increasing, then it is continuous. Thus, $a(R_i, V_j(\cdot_w))$ is the composition of two continuous functions.

Let $z_i^\eta \equiv \Omega - \sum_{j\in N\setminus\{i\}} V_j(w_j^\eta)$. Since each V_j is quasi-strictly increasing, then for each $\eta > 0$, $z_i^\eta \gneq z_i$. Since preferences are semi-strictly monotone, then $z_i^\eta \, P_i^0 \, z_i$. Observe that as $\eta \to 0$, $z_i^\eta \to z_i$. Let $\nu > 0$ be such that $a(L^p, z_i) - a(L^p, z_i^\nu) < \frac{\delta}{2}$. Let $z_i' \equiv z_i^\nu$, and for each $j \in N \setminus \{i\}$, let $z_j' \equiv V_j(w_j^\nu)$. Clearly, $z' \in Z$. Let $a \equiv a(R_i, z_i) + \nu$. By construction, for each $j \in N \setminus \{i\}$, $a(R_j, z_j') = a$.

Agent i and the objects $z' \in Z$, $p \in \mathbb{R}^K_{++}$, and $a \in \mathbb{R}_{++}$ satisfy the first three properties in Lemma 4. We claim that they also satisfy the fourth property. By construction of z' and a, $a(L^p, z_i) - \frac{\delta}{2} < a(L^p, z_i')$. Recall that $\delta \equiv a(L^p, z_i) - a(R_i, z_i)$. Thus, $a(R_i, z_i) + \frac{\delta}{2} < a(L^p, z_i')$. Since $\nu < \frac{\delta}{2}$, then $a \equiv a(R_i, z_i) + \nu < a(L^p, z_i')$. Finally, since $z_i' \geq z_i$, then $\mu(\text{rec}\{z_i', \Omega\}) \leq \mu(\text{rec}\{z_i, \Omega\}) < a(R_i, z_i) < a$. Thus, by Lemma 4, $(R, z^*) \notin \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0\rangle$. This is a contradiction. $\square$

Let $e \in E$. The following lemma states that if a domain $\mathcal{D} \subseteq \mathcal{I}$ is such that $\mathcal{H} \subseteq \mathcal{D}$, then all equilibrium allocations of $\langle S(\mathcal{D})^N, E^e, R^0\rangle$ are equal-division constrained Walrasian for R^0.

Lemma 6. Let $\mathcal{D} \subseteq \mathcal{I}$ be such that $\mathcal{H} \subseteq \mathcal{D}$. Let $e \in E$. Then, for each $R^0 \in \mathcal{R}^N$,

$$\mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0\rangle \subseteq W^c_{ed}(R^0).$$

Proof. Let $(R, z^*) \in \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0\rangle$ and $z \equiv E^e(R, z^*)$. From Lemma 5, there is $p \in \mathbb{R}_{++}$ such that for each $i \in N$, $I^c(R_i, z_i) = I^c(L^p, z_i)$. Since $z \in E(R)$, then for each $\{i, j\} \subseteq N$, $a(L^p, z_i) = a(L^p, z_j)$. Thus, for each $i \in N$, $p \cdot z_i = p \cdot \frac{\Omega}{n}$ and then $z \in W^c_{ed}(R)$. We claim that $z \in W^c_{ed}(R^0)$ and p supports z as a member of $W^c_{ed}(R^0)$. That is, we claim that for each $i \in N$ and each $x_i \in U^c(P_i^0, z_i)$, $p \cdot x_i > p \cdot \frac{\Omega}{n}$. Suppose by contradiction that there is $i \in N$ and $x_i \in U^c(P_i^0, z_i)$ such that $p \cdot x_i \leq p \cdot \frac{\Omega}{n}$. Since preferences are continuous, then we can suppose w.l.o.g. that $x_i \gg 0$. For each $\alpha \in [0,1]$, let $z_i^\alpha \equiv \alpha x_i + (1-\alpha)\Omega$. Since R_i^0 is semi-strictly monotone and $x_i \ll \Omega$, then for each $\alpha \in [0,1]$, $z_i^\alpha \, P_i^0 \, z_i$. Also, since $p \cdot x_i \leq p \cdot \frac{\Omega}{n}$, then by the Intermediate Value Theorem, there is $\beta \in (0,1]$ such that $0 \ll z_i^\beta \in U^c(P_i^0, z_i)$ and $p \cdot z_i^\beta = p \cdot \frac{\Omega}{n}$. For each $j \in N \setminus \{i\}$, let $z_j^\beta \equiv (\Omega - z_i^\beta)/(n-1)$. Thus, $z^\beta \in Z$. Moreover, for each $j \in N \setminus \{i\}$, $p \cdot z_j^\beta = p \cdot \frac{\Omega}{n}$ and $p \in \text{Supp}(R_j, z_j^\beta)$.

For each $j \in N \setminus \{i\}$, let V_j be a quasi-strictly increasing income expansion path for R_j at prices p that passes through z_j^β.[15] For each $j \in N \setminus \{i\}$, the function $w \in \mathbb{R}_+ \mapsto a(R_j, V_j(w))$, is continuous.[16] Let $\eta \in (0, a(L^p, \frac{\Omega}{n}))$. By the Intermediate Value Theorem, there are $(w_j^\eta)_{j\in N\setminus\{i\}} \gg (p \cdot z_j)_{j\in N\setminus\{i\}}$ such that for each $j \in N \setminus \{i\}$, $a(R_i, V_j(w_j^\eta)) = a(R_j, z_j) - \eta$. Let $y_i^\eta \equiv \Omega - \sum_{j\in N\setminus\{i\}} V_j(w_j^\eta) \lneq z_i^\beta$. Since each V_j is quasi-strictly increasing,

[15] These income expansion paths exist because $R \in \mathcal{I}^N$.

[16] See Footnote 14.

and thus, continuous, then as $\eta \to 0$, $y_i^\eta \to z_i^\beta$. Thus, as $\eta \to 0$, $\mu(\text{rec}\{y_i^\eta, \Omega\}) \to \mu(\text{rec}\{z_i^\beta, \Omega\})$. Moreover, for each $j \in N \setminus \{i\}$, as $\eta \to 0$, $V_j(w_j^\eta) \to z_j^\beta$. Let $\nu > 0$ be small enough such that: (i) for each $j \in N \setminus \{i\}$, $a(R_j, V_j(w_j^\nu)) > \mu(\text{rec}\{y_i^\nu, \Omega\})$, (ii) $y_i^\nu \gg 0$, and (iii) $y_i^\nu \, P_i^0 \, z_i$. Let $z_i' \equiv y_i^\nu$ and for each $j \in N \setminus \{i\}$, let $z_j' \equiv V_j(w_j^\nu)$. Clearly, $z' \in Z$. Let $a \in \mathbb{R}_{++}$ be the common value of $a(R_j, z_j') = a(R_j, z_j) - \nu = a(L^p, z_j) - \nu$ for $j \in N$. Since $a < a(L^p, z_i)$ and $a(L^p, z_i) < a(L^p, z_i')$, then $a < a(L^p, z_i')$. Thus, $\mu(\text{rec}\{z_i', \Omega\}) < a < a(L^p, z_i')$.

Agent i and the objects $z' \in Z$, $p \in \mathbb{R}^K_{++}$, and $a \in \mathbb{R}_{++}$ satisfy the four properties in Lemma 4. Thus, $(R, z^*) \notin \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0\rangle$. This is a contradiction. $\square$

The following lemma states that if agents can report linear preferences, then all equal-division constrained Walrasian allocations are equilibrium outcomes.

Lemma 7. Let $\mathcal{D} \subseteq \mathcal{R}$ be such that $\mathcal{L} \subseteq \mathcal{D}$ and $e \in E$. Then, for each $R^0 \in \mathcal{D}^N$,

$$\mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0\rangle \supseteq W^c_{ed}(R^0).$$

Proof. Let $R^0 \in \mathcal{D}^N$ and $z \in W^c_{ed}(R^0)$. We show that there is $R \in \mathcal{L}^N$ such that $(R, z) \in \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0\rangle$ and $E^e(R, z) = z$. Since $z \in W^c_{ed}(R^0)$, then by Lemma 3, there is $p \in \mathbb{R}^K_{++}$ such that for each $i \in N$ and each $z_i' \in U^c(P_i^0, z_i)$, $p \cdot z_i > p \cdot \frac{\Omega}{n}$. Let $R \equiv (L_i^p)_{i \in N}$. Let $i \in N$, $R_i' \in \mathcal{D}$, and $z' \in E(R_{-i}, R_i')$. Observe that for each $\{j, k\} \subseteq N \setminus \{i\}$, $p \cdot z_j' = p \cdot z_k'$. We claim that $p \cdot z_i' \leq p \cdot \frac{\Omega}{n}$ and thus $z_i \, R_i^0 \, z_i'$. Suppose by contradiction that $p \cdot z_i' > p \cdot \frac{\Omega}{n}$. Thus, for each $j \in N \setminus \{i\}$, $a(L^p, \frac{\Omega}{n}) < a(R_j, z_j')$. Efficiency of E implies p supports $U(R_i', z_i')$, i.e., for each $x_i \in U^c(R_i', z_i')$, $p \cdot x_i \geq p \cdot z_i'$. Thus, $a(R_i', z_i') \leq a(L^p, z_i') < a(L^p, \frac{\Omega}{n})$. Consequently, for each $j \in N \setminus \{i\}$, $a(R_i', z_i') < a(R_j, z_j')$. This contradicts $z' \in E(R_{-i}, R_i')$. Thus, for each $i \in N$ and each $(R_i', z_i') \in S(\mathcal{D})$, $E_i^e(R, z) \, R_i^0 \, E_i^e(R_{-i}, R_i', z_{-i}, z_i')$. Thus, $(R, z) \in \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0\rangle$. Finally, since $z \in E(R)$, then $E^e(R, z) = z$. $\square$

Now we can complete the proof of our main characterization.

Proof of Theorem 2. Let $e \in E$ and $\mathcal{D} \equiv \mathcal{I}$. Thus, $\mathcal{I} \supseteq \mathcal{D} \supseteq \mathcal{H} \supseteq \mathcal{L}$, and then by Lemmas 6 and 7, $\mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0\rangle \subseteq W^c_{ed}(R^0) \subseteq \mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0\rangle$. Thus, $\mathcal{O}\langle S(\mathcal{I})^N, E^e, R^0\rangle = W^c_{ed}(R^0)$. $\square$

The following corollary characterizes the equilibrium outcomes of the manipulation game associated with the equal-sacrifice solution on the domain of economies with homothetic preferences.

Corollary 2. For each $R^0 \in \mathcal{H}^N$ and each $e \in E$, $\mathcal{O}\langle S(\mathcal{H})^N, E, R^0\rangle = W^c_{ed}(R^0)$

Proof. Let $e \in E$ and $\mathcal{D} \equiv \mathcal{H}$. Thus, $\mathcal{I} \supseteq \mathcal{D} \supseteq \mathcal{H} \supseteq \mathcal{L}$, and then by Lemmas 6 and 7, $\mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0\rangle \subseteq W^c_{ed}(R^0) \subseteq \mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0\rangle$. Thus, $\mathcal{O}\langle S(\mathcal{H})^N, E^e, R^0\rangle = W^c_{ed}(R^0)$. $\square$

6 Discussion

Let $e \in E$. In this section we show that if our assumptions on preferences of semi-strict monotonicity and quasi-strictly increasing income expansion paths are dropped, then equilibrium outcomes of $\langle S(\mathcal{D}), E^e, R^0\rangle$ are not necessarily equal-division constrained Walrasian for R^0. We also show that this is so even if smoothness of preferences is required. We do this by means of three examples.

The next lemma provides conditions that characterize the Nash equilibria in the two-agent case when preferences are not necessarily semi-strictly monotone. It facilitates the presentation of the examples that follow.

Lemma 8. Assume $N \equiv \{1,2\}$. Let $\mathcal{D} \subseteq \mathcal{U}$, $R^0 \in \mathcal{U}^N$, $R \in \mathcal{D}^N$, and $z \in E(R)$ be such that $z_1 + z_2 = \Omega$. Suppose that for each $z_1' \in U^c(P_1^0, z_1)$, and each $p \in \mathbb{R}^K_+$ supporting $U^c(R_2, \Omega - z_1')$ at $\Omega - z_1'$, $a(L^p, z_1') < a(R_2, \Omega - z_1')$, and that the parallel statement obtained by exchanging the roles of the two agents holds. Then for each $e \in E$, $(R, z) \in \mathcal{N}\langle S(\mathcal{D})^N, E^e, R^0\rangle$ and $z \in \mathcal{O}\langle S(\mathcal{D})^N, E^e, R^0\rangle$.

Proof. Let N, $\mathcal{D}$, R^0, R, and z be as in the statement of the lemma. We claim that for each $e \in E$, each $i \in N$, and each $(R_i', z_i') \in S(\mathcal{D})$, $E_i^e(R, z)\, R_i^0\, E_i^e(R_{-i}, R_i', z_{-i}, z_i')$.

Suppose by contradiction that the above statement is false for say, $e \in E$ and agent 1: there are $(R_1', z_1') \in S(\mathcal{D})$ such that $E_1^e(R_1', R_2, z_1', z_2)\, P_1^0\, E_1^e(R, z)$.

Thus, $E_1^e(R_1', R_2, z_1', z_2) \neq E_1^e(R, z)$. We claim that $E^e(R_1', R_2, z_1', z_2) \neq (z_1', z_2)$. Suppose by contradiction that $E^e(R_1', R_2, z_1', z_2) = (z_1', z_2)$. Since $z \in Z$ and $z_1 + z_2 = \Omega$, then $z_1' \leq z_1$. Thus, $z_1 = E_1^e(R, z)\, R_1^0\, E_1^e(R_1', R_2, z_1', z_2)$. This is a contradiction.

Since $E^e(R_1', R_2, z_1', , z_2) \neq (z_1', z_2)$, then $E^e(R_1', R_2, z_1', z_2) = e(R_1', R_2)$. Let $\hat{z} \equiv e(R_1', R_2)$. By Lemma 1, $\hat{z} \in P(R_1', R_2)$. Thus, there is $p \in \mathbb{R}^K_+$ that supports $U^c(R_2, \hat{z}_2)$ at $\hat{z}_2$ and also supports $U^c(R_1', \hat{z}_1)$ at $\hat{z}_1$. Thus, $a(R_1', \hat{z}_1) \leq a(L^p, \hat{z}_1)$. Moreover, by hypothesis, $a(L^p, \hat{z}_1) < a(R_2, \hat{z}_2)$. Thus, $a(R_1', \hat{z}_1) < a(R_2, \hat{z}_2)$ and $\hat{z} \notin E(R_1', R_2)$. This is a contradiction. ☐

The following example shows that in Theorem 2, semi-strict monotonicity of preferences ($\mathcal{D} \subseteq \mathcal{R}$) cannot be replaced by monotonicity of preferences ($\mathcal{D} \subseteq \mathcal{U}$) .

Example 1. Let $N \equiv \{1,2\}$ and $\Omega \equiv (1,1) \in \mathbb{R}^2_{++}$. We specify $R^0 \in \mathcal{U}^N$ and construct a profile $(R, z) \in (\mathcal{D} \times \mathbb{R}^2_+)^N$ such that preferences R are homothetic but not semi-strictly increasing, i.e., $R \in (\mathcal{U} \setminus \mathcal{R})^N$, $(R, z) \in \mathcal{N}\langle S(\mathcal{U})^N, E, R^0\rangle$, $z \in \mathcal{O}\langle S(\mathcal{U})^N, E, R^0\rangle$, $I^c(R_2, z_2)$ is not linear, and $z \notin W^c_{ed}(R^0)$.

• Specifying true preferences (Figure 1 (a)). Let $q \in \mathbb{R}^2_{++}$ be such that $q_1 > q_2$ and $R_1^0 \equiv L^{(q_1+1, q_2)}$. Let $R_2^0 \in \mathcal{U}$ be the homothetic preference for which $I(R_2^0, (0,1)) \equiv \{(x, 1) \in \mathbb{R}^K_+ : x \in \mathbb{R}_+\}$.

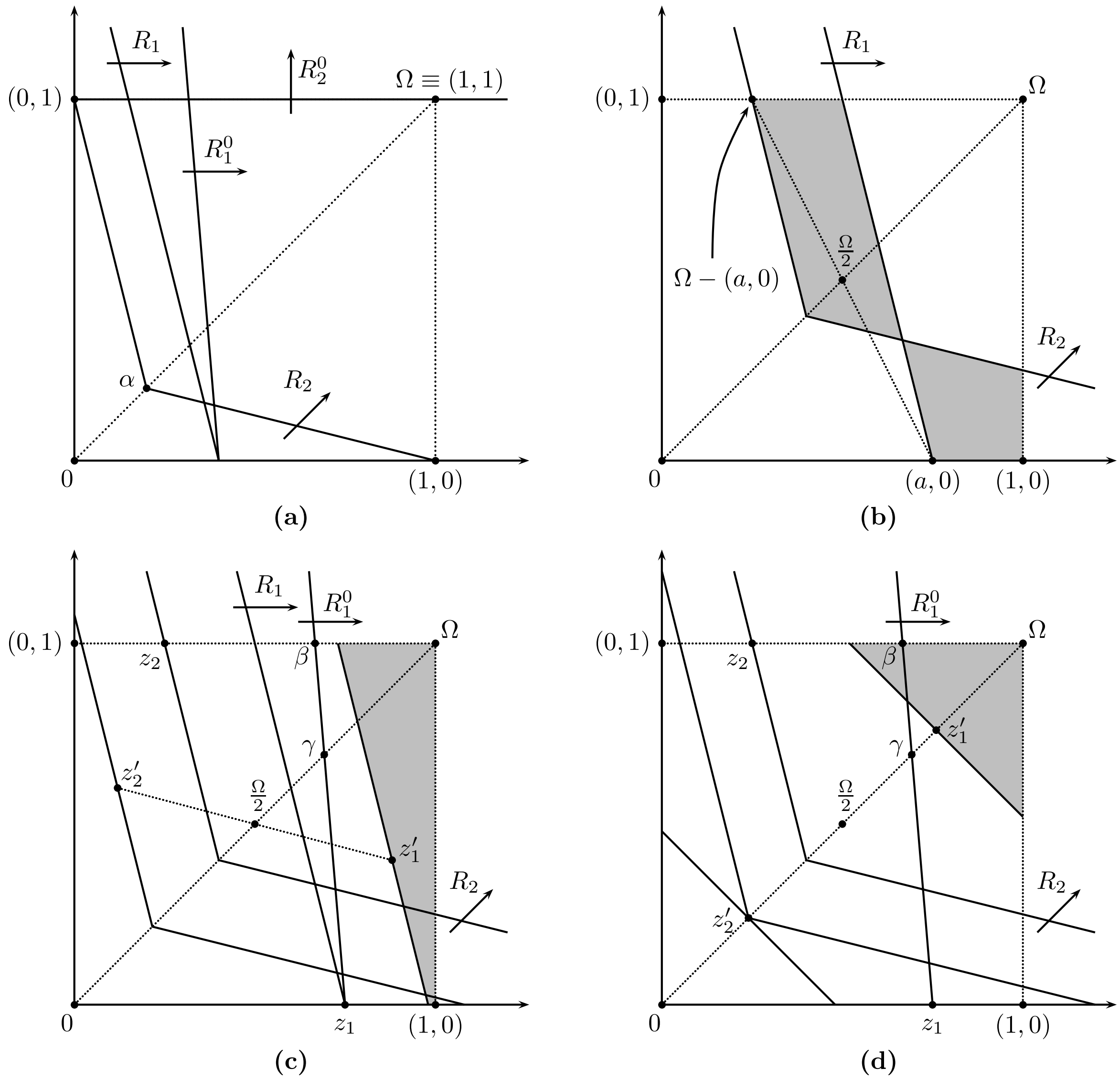

Figure 1: Example 1.

• Specifying reported preferences. Let $R_1 \equiv L^q$. Let $\alpha \equiv \text{seg}[0, \Omega] \cap I(R_1, (0, 1))$. Let $R_2 \in \mathcal{U}$ be the homothetic preference such that $I(R_2, (1, 0)) \equiv \text{bro.seg}[(0, 1), \alpha, (1, 0)]$.

• Identifying an equilibrium outcome, z (Figure 1 (b)). The function $x \in \mathbb{R}_+ \mapsto a(R_1, (x, 0))$ is continuous and strictly decreasing on $[\frac{1}{2}, 1]$; also, the function $x \in \mathbb{R}_+ \mapsto a(R_2, \Omega - (x, 0))$ is continuous and strictly increasing on $[\frac{1}{2}, 1]$. Moreover, $a(R_1, (\frac{1}{2}, 0)) > a(R_2, \Omega - (\frac{1}{2}, 0))$ and $a(R_1, (1, 0)) < a(R_2, \Omega - (1, 0))$. Thus, by the Intermediate Value Theorem, there is $a \in [\frac{1}{2}, 1]$ such that $a(R_1, (a, 0)) = a(R_2, \Omega - (a, 0))$. In Figure 1 (b), this is equivalent to the equality of the measures of the two shaded areas forming a bow-tie. Let $z_1 \equiv (a, 0)$ and $z_2 \equiv \Omega - z_1$.

We claim that for each $e \in E$, $(R, z) \in \mathcal{N}\langle S(\mathcal{U})^N, E^e, R^0\rangle$. Observe that $z \in E(R)$ and that for each $z' \in Z$, $z_2 \, R_2^0 \, z_2'$. Thus, R_2 is a best response to R_1. By Lemma 8, it is enough to show that for each $z' \in Z$ such that $z_1' \, P_1^0 \, z_1$, and each $p \in \mathbb{R}^2_+$ supporting $U^c(R_2, z_2')$ at z_2', we have $a(L^p, z_1') < a(R_2, z_2')$. Let $\beta \equiv \text{seg}[(0,1), \Omega] \cap I(R_1^0, z_1)$ and $\gamma \equiv \text{seg}[z_1, \beta] \cap \text{seg}[0, \Omega]$ (Figure 1 (c)). There are five cases.

Case 1: $z_1' \in \text{con.hull}\{z_1, \gamma, \Omega, (1,0)\}$ but $z_1' \notin \text{bro.seg}[(1,0), \Omega, \gamma]$ (Figure 1 (c)). Then, the unique $p \in \mathbb{R}^2_+$ (up to a positive scalar multiplication) supporting $U^c(R_2, z_2')$ at z_2', is $p = q$. Since $a(L^q, z_1') < a(L^q, z_1) = a(R_1, z_1) = a(R_2, z_2) < a(R_2, z_2')$, then $a(L^p, z_1') < a(R_2, z_2')$.

Case 2: $z_1' \in \text{seg}[\Omega, (1,0)]$. Then, for each $p \in \mathbb{R}^2_+$ supporting $U^c(R_2, z_2')$ at z_2', $\frac{p_2}{p_1} \leq \frac{q_2}{q_1}$. Since $z_1' \in \text{seg}[\Omega, (1,0)]$, then $a(L^p, z_1') \leq a(L^q, z_1')$. Now, since $a(L^q, z_1') < a(L^q, z_1) = a(R_1, z_1) = a(R_2, z_2) < a(R_2, z_2')$, then $a(L^p, z_1') < a(R_2, z_2')$.

Case 3: $z_1' \in \text{seg}[\gamma, \Omega]$ (Figure 1 (d)). Then, for each $p \in \mathbb{R}^2_+$ supporting $U^c(R_2, z_2')$ at z_2', $\frac{q_2}{q_1} \leq \frac{p_2}{p_1} \leq \frac{q_1}{q_2}$. A simple geometric argument shows that for each $p \in \mathbb{R}^2_+$ satisfying this inequality, $a(L^p, z_1') \leq a(L^q, z_1')$. Thus, $a(L^p, z_1') < a(L^q, z_1) = a(R_1, z_1) = a(R_2, z_2) < a(R_2, z_2')$.

Case 4: $z_1' \in \text{con.hull}\{\gamma, \Omega, \beta\}$ but $z_1' \notin \text{bro.seg}[\gamma, \Omega, \beta]$. A symmetric argument to the one in Case 1 applies.

Case 5: $z_1' \in \text{seg}[\beta, \Omega]$. A symmetric argument to the one in Case 2 applies.

Concluding: Observe that $z \notin P(R^0)$. Thus, $z \notin W^c_{ed}(R^0)$. $\square$

The following example shows that in Lemma 5, the assumption that income expansion paths be quasi-strictly increasing ($\mathcal{D} \subseteq \mathcal{I}$) cannot be dropped (cannot be replaced by $\mathcal{D} \subseteq \mathcal{R}$.)

Example 2. Let $N \equiv \{1, 2\}$ and $\Omega \equiv (1,1) \in \mathbb{R}^2_{++}$. We specify $R^0 \in \mathcal{R}^{\{1,2\}}$ and construct $(R, z) \in (\mathcal{R} \setminus \mathcal{I} \times \mathbb{R}^2_+)^{\{1,2\}}$ such that $(R, z) \in \mathcal{N}\langle S(\mathcal{R})^N, E, R^0\rangle$, $z \in \mathcal{O}\langle S(\mathcal{R})^N, E, R^0\rangle$, $I^c(R_1, z_1)$ and $I^c(R_2, z_2)$ are not linear, and $z \notin W^c_{ed}(R^0)$.

• Specifying true preferences. Let $R_1^0 \in \mathcal{H}$ be the homothetic preference whose upper contour set at $(\frac{1}{5}, \frac{4}{5})$ is the intersection of the upper contour sets of $L^{(8,1)}$ and $L^{(4,3)}$ (Figure 2 (a)). Let $R_2^0 \in \mathcal{H}$ be the symmetric image of R_1^0 with respect to the 45^o line, i.e., for each $\{x, y\} \subset \mathbb{R}^2_+$, $x \, R_2^0 \, y$ if and only if $(x_2, x_1) \, R_1^0 \, (y_2, y_1)$.

• Specifying reported preferences. Let us specify R_1 (Figure 2 (b)). Let Q be the union of seg $\left[\left(\frac{2}{5}, 0\right), \left(\frac{1}{5}, \frac{4}{5}\right)\right]$ and the half line with slope 4 starting at $\left(\frac{1}{5}, \frac{4}{5}\right)$.[17] The indifference

[17] The ray with slope 4 starting at $\left(\frac{1}{5}, \frac{4}{5}\right)$ is the set of points $\{(\frac{1}{5}\lambda, \frac{4}{5}\lambda) \in \mathbb{R}^2_+ : \lambda \geq 1\}$.

sets of R_1 to the left of Q are linear with normal $(2,1)$. The indifference sets of R_1 to the right of Q are linear with normal $(4,3)$.

The indifference sets of R_1 can be alternatively described as follows. For each $a \in [0, \frac{4}{5}]$, $I(R_1,(0,a)) = I(L^{(\frac{4}{5},\frac{2}{5})},(0,a))$. For each $a \in]\frac{4}{5}, \frac{6}{5}]$,

$$I(R_1,(0,a)) = \text{bro.seg}\left[(0,a), \left(\frac{4}{5} - \frac{1}{2}a, 2a - \frac{8}{5}\right), \left(\frac{13}{6}a - \frac{4}{3}, 0\right)\right].$$

Let $\hat{R}_1 \in \mathcal{H}$ be the preference for which $I(\hat{R}_1,(0,\frac{6}{5})) = I(R_1,(0,\frac{6}{5}))$. For each $a \in]\frac{6}{5}, +\infty[$, $I(R_1,(0,a)) = I(\hat{R}_1,(0,a))$.

Now, R_2 is the symmetric image of R_1 with respect to the 45^o line.

Claim 1: $R \notin \mathcal{I}$. Indeed, $(1,1) \in \text{Supp}(R_1,(\frac{1}{5},\frac{4}{5}))$, and for each $0 \lneq \Delta \in \mathbb{R}^K_+$ such that $(\frac{1}{5},\frac{4}{5}) - \Delta \gneq 0$, $(1,1) \notin \text{Supp}(R_i,(0,1)-\Delta)$. Thus, no quasi-strictly increasing path of maximizers of R_1 at prices $(1,1)$ passes through $(\frac{1}{5},\frac{4}{5})$ (Figure 2 (b)).

• Identifying an equilibrium outcome, z. Let $z_1 \equiv (\frac{1}{5},\frac{4}{5})$ and $z_2 \equiv (\frac{4}{5},\frac{1}{5})$.

Claim 2: $(R,z) \in \mathcal{N}\langle S(\mathcal{R})^N, E^e, R^0\rangle$. Observe that $z \in E(R)$. Thus, by Lemma 8 and the symmetry of our construction, it is enough to show that for each $R'_2 \in \mathcal{R}$ such that $e_2(R_1,R'_2)\, P^0_2\, z_2$, and each $p \in \mathbb{R}^K_+$ supporting $U^c(R_1, e_1(R_1,R'_2))$ at $e_1(R_1,R'_2)$, we have $a(L^p, e_2(R_1,R'_2)) < a(R_1, e_1(R_1,R'_2))$. Let $R'_2 \in \mathcal{R}$, $z' \equiv e(R_1,R'_2)$, and A the "residual complement" of the constrained upper contour set of R^0_2 at z_2, i.e., $A \equiv \{z'_1 \leq \Omega : \Omega - z'_1\, P^0_2, z_2\}$. It is easy to see that

$$A = \text{con.hull}\left\{0, \left(\frac{3}{10},0\right), z_1, \left(\frac{1}{20},1\right), (0,1)\right\} \setminus \text{bro.seg}\left[\left(\frac{3}{10},0\right), z_1, \left(\frac{1}{20},1\right)\right].$$

There are three cases.

Case 1: $z'_1 \in A \setminus \text{bro.seg}[(0,1),0,\left(\frac{3}{10},0\right)]$ (Figure 2 (d)). Then, the unique $p \in \mathbb{R}^2_+$ (up to positive scale multiplication) supporting $U^c(R_1,z'_1)$ at z'_1 is $(2,1)$. Observe that $a(L^{(2,1)},z_2) < a(R_2,z_2)$, i.e., of the two shaded sets forming a bow-tie in Figure 2 (c), the measure of the upper set is greater than the measure of the lower one. Since $a(L^{(2,1)},z'_2) \leq a(L^{(2,1)},z_2) < a(R_2,z_2) = a(R_1,z_1) \leq a(R_1,z'_1)$, then $a(L^{(2,1)},z'_2) < a(R_1,z'_1)$.

Case 2: $z'_1 \in \text{seg}[(0,1),0]$. Then, for each $p \in \mathbb{R}^2_+$ supporting $U^c(R_1,z'_1)$ at z'_1, $\frac{p_2}{p_1} \leq \frac{1}{2}$. Since $z'_2 \in \text{seg}[(1,0),\Omega]$, then $a(L^p,z'_2) \leq a(L^{(2,1)},z'_1)$ and the argument in Case 1 shows that $a(L^p,z'_2) < a(R_1,z'_1)$.

Case 3: $z'_1 \in \text{seg}[0,\left(\frac{3}{10},0\right)]$. Then, for each $p \in \mathbb{R}^2_+$ supporting $U^c(R_1,z'_1)$ at z'_1, $\frac{p_2}{p_1} \geq \frac{1}{2}$. Since $z'_j \in \text{seg}[(\frac{7}{10},1),\Omega]$, then $a(L^p,z'_2) < a(R_1,z_1)$. Now, since $a(R_1,z_1) < a(R_1,z'_1)$, then $a(L^p,z'_2) < a(R_1,z'_1)$.

Concluding: Since $(0,1)\, P^0_1\, z_1$ and $(1,0)\, P^0_2\, z_2$, then $z \notin P(R^0)$. Thus, $z \notin W^c_{ed}(R^0)$.
□

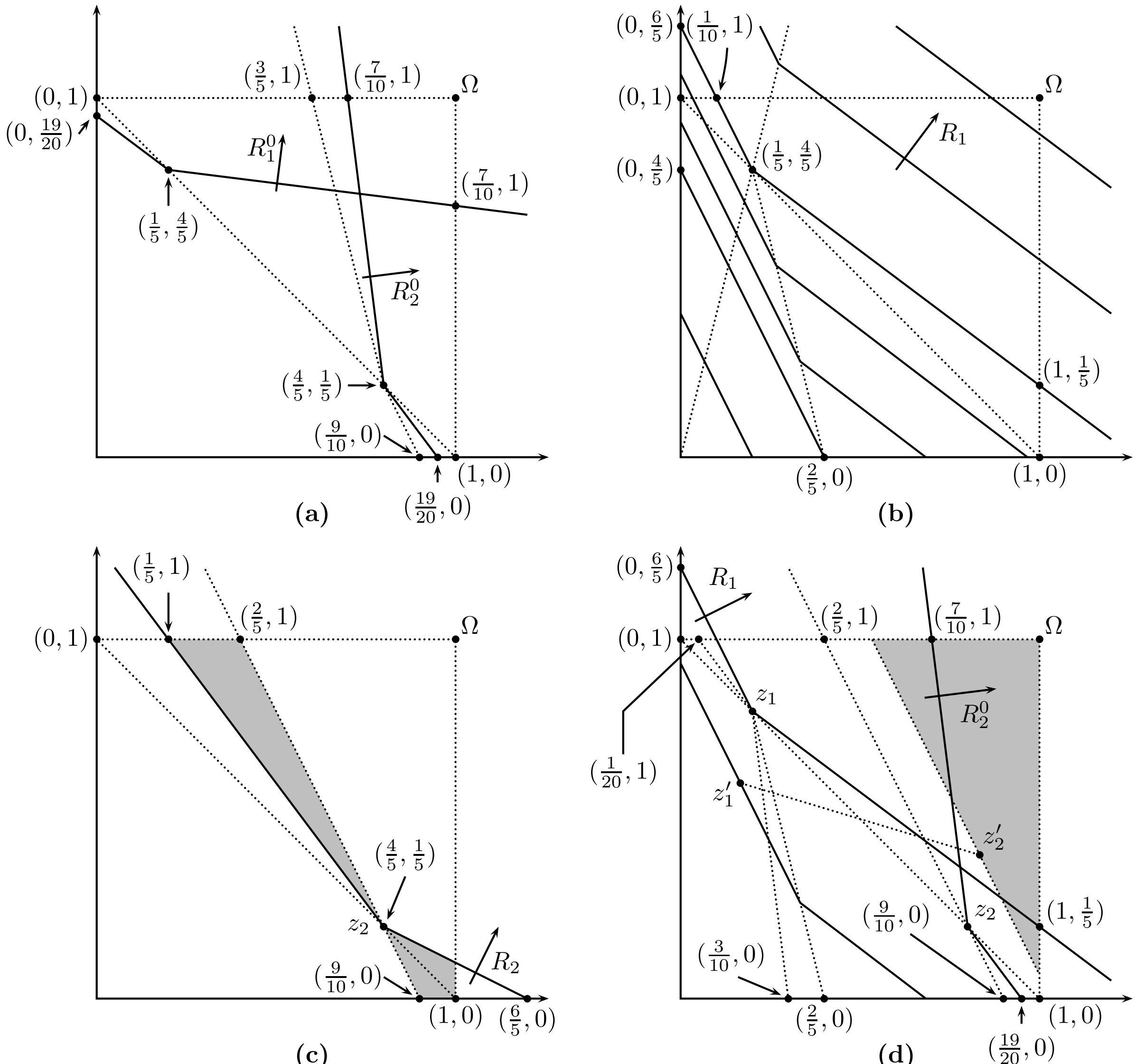

Figure 2: Example 2.

Let F be a solution. A selector $f \in F$ is an **equal-division selector**, if for each $R \in \mathcal{R}^N$ such that $z_{ed} \equiv \frac{1}{|N|}(\Omega, \ldots, \Omega) \in F(R)$, we have $f(R) = z_{ed}$.

The following example shows that in Lemma 5, the assumption of quasi-strictly increasing expansion path ($\mathcal{D} \subseteq \mathcal{I}$) cannot be replaced by the assumption of smoothness of preferences ($\mathcal{D} \subseteq \mathcal{S}$).

Example 3. Let $N \equiv \{1, 2\}$, $\lambda > 4\sqrt{2}$, and $\Omega \equiv (\lambda, \lambda) \in \mathbb{R}^2_{++}$. We specify $R^0 \in \mathcal{S}^N$ and construct $(R, z) \in (\mathcal{S} \setminus \mathcal{I} \times \mathbb{R}^2_+)^{\{1,2\}}$ such that for each equal-division selector $e \in E$, $(R, z) \in \mathcal{N}\langle S(\mathcal{S})^N, E^e, R^0\rangle$, $I^c(R_1, z_1)$ and $I^c(R_2, z_2)$ are linear, and $z \notin W^c_{ed}(R^0)$.

• Specifying true preferences (Figure 3 (a)). Let $R_1^0 \in (\mathcal{H} \cap \mathcal{S})^N$ be such that $U(R_1^0, (\lambda, 0)) \cap$

$\{(x_1,x_2)\in\mathbb{R}^K_+ : x_1+x_2<\lambda\}\neq\emptyset$ and for each $x\in U(R^0_1,(\lambda,0))\cap\{(x_1,x_2)\in\mathbb{R}^K_+ : x_1+x_2\leq\lambda\}$, $||(\lambda,0)-x||<\frac{1}{2}$. Let $R^0_2\equiv L^{(1,1)}\in(\mathcal{H}\cap\mathcal{S})^N$.

• Specifying reported preferences (Figure 3 (b)). We construct $\hat{R}\in\mathcal{R}^N$ and $z\in Z$, such that $(\hat{R},z)=\mathcal{N}\langle S(\mathcal{R})^N,E^e,R^0\rangle$ and $z=\mathcal{O}\langle S(\mathcal{R})^N,E^e,R^0\rangle$; later we smooth out $\hat{R}$ and construct $R\in\mathcal{S}^N$ such that $(R,z)=\mathcal{N}\langle S(\mathcal{S})^N,E^e,R^0\rangle$ and $z\in\mathcal{O}\langle S(\mathcal{S})^N,E^e,R^0\rangle$.

We first specify $\hat{R}_1$ (Figure 3 (b)).

(i) For each $a\in[0,\lambda]\subset\mathbb{R}_+$, $I(\hat{R}_1,(0,a))\equiv I(L^{(1,1)},(0,1))$. Let $\alpha\equiv(\frac{\sqrt{2}}{2},\lambda-\frac{\sqrt{2}}{2})$ and $\beta\equiv(\sqrt{2},\lambda-\sqrt{2})$; note that $||(0,\lambda)-\alpha||=1$ and $||\alpha-\beta||=1$.

(ii) For each $a\in]\lambda,\frac{3}{2}\lambda]\subset\mathbb{R}_+$ let $t(a)\equiv\text{seg}[(0,a),\beta]\cap\text{ray}\{\alpha,1\}$ and $I(\hat{R}_1,(0,a))\equiv\text{bro.seg}[(0,a),t(a),(t_2(a)-\frac{\sqrt{2}}{2},0)]$. Observe that $\text{seg}[t(a),(t_2(a)-\frac{\sqrt{2}}{2},0)]$ has slope -1 and for each $a\in]\lambda,\frac{3}{2}\lambda]$, the line that passes through $(0,a)$ and whose slope is that of $\text{seg}[(0,a),t(a)]$, namely, $-\frac{a-t_2(a)}{t_1}$, intersects at β the line that passes through $(0,\lambda)$ with slope -1.

Let $\tilde{R}_1\in\mathcal{H}$ be the homothetic preference for which $I(\tilde{R}_1,(0,\frac{3}{2}\lambda))\equiv I(\hat{R}_1,(0,\frac{3}{2}\lambda))$. For each $a\in]\frac{3}{2}\lambda,+\infty[\subset\mathbb{R}_+$, let $I(\hat{R}_1,(0,a))\equiv I(\tilde{R}_1,(0,a))$. Finally, let $\hat{R}_2\equiv\hat{R}_1$.

Claim 1: $\hat{R}_1\notin\mathcal{I}$. Indeed, $(0,\lambda)$ is a maximizer for $\hat{R}_1$ at prices $(1,1)$, and for each $0\lneq\Delta\in\mathbb{R}^K_+$ such that $||(0,\lambda)-\Delta||<\frac{1}{2}$, $(0,\lambda)+\Delta$ is not a maximizer for $\hat{R}_1$ at prices $(1,1)$. Thus, no quasi-strictly increasing path of maximizers of $\hat{R}_1$ at prices $(1,1)$ passes through $(0,\lambda)$ (Figure 3 (b)).

Let $z_1\equiv(\lambda,0)$ and $z_2\equiv(0,\lambda)$.

Claim 2: $(\hat{R},z)\in\mathcal{N}\langle S(\mathcal{R})^N,E^e,R^0\rangle$. Note that $z\in E(\hat{R})$. Thus, by Lemma 8, it is enough to show that for each $R'_1\in\mathcal{R}$ such that $e_1(R'_1,\hat{R}_2)\,P^0_1\,z_1$, and each $p\in\mathbb{R}^K_+$ supporting $U^c(\hat{R}_2,e_2(R'_1,\hat{R}_2))$ at $e_2(R'_1,\hat{R}_2)$, $a(L^p,e_1(R'_1,\hat{R}_2))<a(R_2,e_2(R'_1,\hat{R}_2))$, and the parallel statement obtained by exchanging the roles of the two agents hold.

We prove the first statement of Claim 2. Let $R'_1\in\mathcal{R}$ and $p\in\mathbb{R}^K_+$ be as specified above. Let $\tilde{z}\equiv e(R'_1,\hat{R}_2)$. We prove that $a(L^p,e_1(R'_1,\hat{R}_2))<a(R_2,e_2(R'_1,\hat{R}_2))$. There are five cases.

Case 1: $\tilde{z}^1_1+\tilde{z}^2_1>\lambda$ and $\tilde{z}_1\ll\Omega$. Then, the unique p (up to a positive scale transformation) supporting $U^c(\hat{R}_2,\tilde{z}_2)$ at $\tilde{z}_2$ is $(1,1)$. Since $\tilde{z}^1_2+\tilde{z}^2_2<\lambda$, then $a(L^{(1,1)},\tilde{z}_1)<a(L^{(1,1)},\tilde{z}_2)=a(R_2,\tilde{z}_2)$.

Case 2: $\tilde{z}^1_1+\tilde{z}^2_1>\lambda$ and $\tilde{z}^1_1=\lambda$. The claim is clearly true if $\tilde{z}_1=\Omega$. Suppose now that $\tilde{z}_1\neq\Omega$. Thus, $\tilde{z}^2_1<\lambda$ and for each p supporting $U^c(\hat{R}_2,\tilde{z}_2)$ at $\tilde{z}_2$, we have $p_1\leq p_2$. Thus, $U^c(L^p,\tilde{z}_1)\subseteq U^c(L^{(1,1)},\tilde{z}_1)$ and $a(L^p,\tilde{z}_1)<a(L^{(1,1)},\tilde{z}_2)=a(R_2,\tilde{z}_2)$.

Case 3: $\tilde{z}^1_1+\tilde{z}^2_1>\lambda$, $\tilde{z}^2_1=\lambda$, and $\tilde{z}_1\neq\Omega$. A symmetric argument to the one in Case 2 shows that for each p supporting $U^c(\hat{R}_2,\tilde{z}_2)$ at $\tilde{z}_2$, $a(L^p,\tilde{z}_1)<a(L^{(1,1)},\tilde{z}_2)=a(R_2,\tilde{z}_2)$.

Case 4: $\tilde{z}^1_1+\tilde{z}^2_1=\lambda$. Then, $\tilde{z}^1_2+\tilde{z}^1_2=\lambda$. Moreover, since $\tilde{z}_1\,P^0_1\,z_1$, then $\tilde{z}^2_2<\lambda$. Thus, the unique p (up to positive scale transformations) supporting $U^c(\hat{R}_2,\tilde{z}_2)$ at $\tilde{z}_2$ is $(1,1)$.

Since $z_1 \, P_1^0 \, \frac{\Omega}{2}$ and e is an equal-division selector, then $(\frac{\Omega}{2}, \frac{\Omega}{2}) \notin E(R_1', \hat{R}_2)$, for otherwise $e(R_1', \hat{R}_2) = (\frac{\Omega}{2}, \frac{\Omega}{2})$. Thus, p supports $U^c(R_1', \tilde{z}_1)$ at $\tilde{z}_1$, $a(R_1', \tilde{z}_1) < a(L^p, \tilde{z}_1) = a(\hat{R}_2, \tilde{z}_2)$, and $\tilde{z} \notin E(R_1', \hat{R}_2)$. Thus, this case cannot occur.

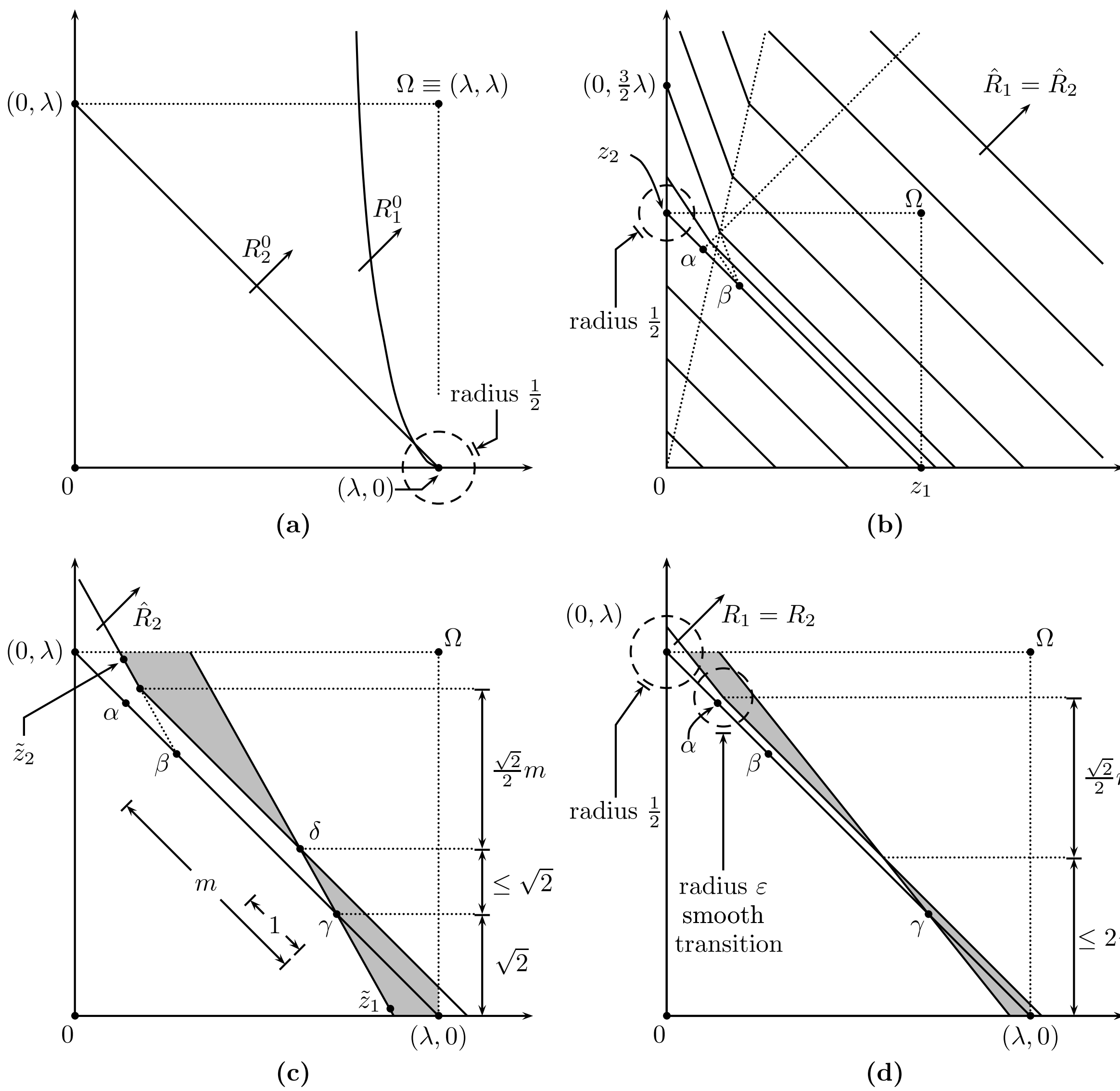

Figure 3: Example 3. In order to help visualize our geometrical argument, we have exaggerated the distance between $\tilde{z}_1$ and $(\lambda, 0)$ in Panel (c) with respect to preferences shown in Panel (a).

Case 5: $\tilde{z}_1^1 + \tilde{z}_1^2 < \lambda$ (Figure 3 (c)). Then, $||\tilde{z}_1 - z_1|| < \frac{1}{2}$ and the unique p (up to positive scale transformations) supporting $U^c(R_2, \tilde{z}_2)$ at $\tilde{z}_2$ is (p_1, p_2) where $p_1 = \tilde{z}_2^2 - \lambda - \sqrt{2}$ and

$p_2 = \tilde{z}_2^1 - \sqrt{2}$, i.e., the normal vector to the line that pases through $\tilde{z}_2$ and β (Figure 3 (b)). Let $\gamma \equiv I(L^p, \tilde{z}_1) \cap \mathrm{seg}[(0,\lambda),(\lambda,0)]$ and $\delta = (\delta^1, \delta^2) \equiv I(L^p, \tilde{z}_1) \cap I(R_2, \tilde{z}_2)$. Note that $||\gamma - (\lambda, 0)|| = 2$ and $\delta^2 \leq 2\sqrt{2}$. We claim that $a(L^p, \tilde{z}_1) < a(R_2, \tilde{z}_2)$. This is equivalent to $\mu(U^c(R_2, \tilde{z}_2) \setminus U^c(L^p, \tilde{z}_1)) > \mu(U^c(L^p, \tilde{z}_1) \setminus U^c(R_2, \tilde{z}_2))$, i.e., from the measures of the two shaded sets forming a bow-tie in Figure 3 (c), the measure of the upper set is greater than the measure of the lower one. Indeed, let $m \equiv \sqrt{2}\lambda - 4$. Then, the upper set is a proper superset of a congruent set to the lower one if $\frac{\sqrt{2}}{2}m > 2\sqrt{2}$. Since $\lambda > 4\sqrt{2}$, this last inequality holds.

We now prove the second statement of Claim 2, i.e., for each $R_2' \in \mathcal{R}$ such that $e_2(\hat{R}_1, R_2')\, P_2^0\, z_2$, and each $p \in \mathbb{R}_+^2$ supporting $U^c(R_1, e_1(\hat{R}_1, R_2'))$ at $e_1(\hat{R}_1, R_2')$, $a(L^p, e_2(\hat{R}_1, R_2')) < a(R_1, e_2(\hat{R}_1, R_2'))$. A symmetric argument to Cases 1, 2, and 3 above proves this statement (there is no counterpart of Cases 4 and 5.)

Now, we smooth out $\hat{R}$ and construct $R \in \mathcal{S}^N$ such that $(R, z) = \mathcal{N}\langle S(\mathcal{S})^N, E^e, R^0\rangle$ and $z = \mathcal{O}\langle S(\mathcal{S})^N, E^e, R^0\rangle$. The only points at which the indifference sets of $\hat{R}_1$ and $\hat{R}_2$ have multiple supporting lines are $\{t(a) : \lambda < a < \frac{3}{2}\lambda\}$ and the half line starting at $t(\frac{3}{2}\lambda)$ with direction $t(\frac{3}{2}\lambda)$. For a small $\varepsilon > 0$ there are smooth preferences, say R_1 and R_2, whose indifference sets "coincide" with the indifference sets of $\hat{R}_1$ and $\hat{R}_2$ outside each open ball with radius ε and centered at each of these "kinks" (Figure 3 (d)). Moreover, if ε is small enough, then the same argument that shows that $(\hat{R}, z) = \mathcal{N}\langle S(\mathcal{R})^N, E^e, R^0\rangle$ and $z = \mathcal{O}\langle S(\mathcal{R})^N, E^e, R^0\rangle$ can be used to show that $(R, z) = \mathcal{N}\langle S(\mathcal{S})^N, E^e, R^0\rangle$ and $z = \mathcal{O}\langle S(\mathcal{S})^N, E^e, R^0\rangle$.[18]

Concluding: Observe that $z \notin P(R^0)$. Thus, $z \notin W_{ed}^c(R^0)$.□

7 Concluding comment and a conjecture

Our results have implications for what is called "implementation" – for introductions and surveys, see Corchón (1996), Jackson (2001), and Maskin and Sjöström (2002). A "game form" consists of a profile of strategy spaces, one for each agent, and an outcome function, a function that associates with each preference profile an allocation. Once a preference profile is given, we have a "game." A game form "implements a solution" if for each preference profile, the set of equilibrium allocations of the resulting game coincides with the set of allocations that the solution would select for this profile. Thus, our main result

[18]Formally, since $\lambda > 4\sqrt{2}$, there is $R_2 \in \mathcal{S}$ such that: (i) for each $a \in [0, \lambda] \subset \mathbb{R}_+$, $I(R_2, (0, a)) \equiv I(\hat{R}_2, (0, a))$; (ii) there is $0 < \varepsilon < \frac{\sqrt{2}}{2}m - 2\sqrt{2}$ such that for each $a \in]\lambda, \frac{3}{2}\lambda] \subset \mathbb{R}_+$, $I(R_2, (0, a)) \cap \{y \in \mathbb{R}_+^K : ||y - t(a)|| < \varepsilon\} \equiv I(\hat{R}_2, (0, a)) \cap \{y \in \mathbb{R}_+^K : ||y - t(a)|| < \varepsilon\}$; and (iii) for each $\tilde{z}_2 \in \mathbb{R}_+^K$ such that $||(0, \lambda) - \tilde{z}_2|| < \frac{1}{2}$ and $\tilde{z}_2^1 + \tilde{z}_2^2 > \lambda$, the positive normal vector to the the line that passes through $\tilde{z}_2$ and β supports $U^c(R_2, \tilde{z}_2)$ at $\tilde{z}_2$.

can be seen as providing an implementation of the equal-division constrained Walrasian solution.

Since the pioneering work of Hurwicz, a variety of game forms have been defined achieving this objective, under a range of requirements on the game form and on the domain of admissible preferences. For some of these game forms, strategy spaces are finite dimensional Euclidean spaces. One can argue that from the viewpoint of simplicity, such game forms are preferable to the game form associated with the equal-sacrifice rule. Consequently, we do not emphasize the fact that our game form provides an implementation of this solution.

An extension of our results to situations where agents have individual endowments is possible, although adapting the definition of the equal-sacrifice solution itself is not entirely straightforward. The following appears the most natural to us. Consider an allocation that each agent finds at least as desirable as her endowment. Then, for each agent, identify the consumption bundles that she could receive at an allocation that each agent finds at least as desirable as her endowment. Then, calculate the ratio of the size of the subset of bundles that she finds at least as desirable as her assigned bundle to the size of the subset of these bundles that she finds at least as desirable as her endowment. Finally, select the allocations at which these ratios are all equal. We conjecture that under the same assumptions on the domain, and for each preference profile, the set of equilibrium allocations of the manipulation game associated with this rule is the set of constrained Walrasian allocations for that profile.

Appendix

Lemma 9. $\mathcal{H} \subsetneq \mathcal{I}$.

Proof. We show that $\mathcal{H} \subsetneq \mathcal{I}$ by means of an example in $\mathbb{R}^2_+$. We construct $R_0 \in \mathcal{I}$ such that $R_0 \notin \mathcal{H}$. Let us define $R_0 \in \mathcal{I}$. For each $a \in [0,1] \subset \mathbb{R}_+$, let $I(R_0,(0,a)) \equiv I(L^{(1,1)},(0,a))$. For each $a \in]1,\infty[$, be the set

$$\text{bro.seg}\left[(0,a), I(L^{(3,1)},(0,a)) \cap \text{ray}\{(0,1),1\}, I(L^{(1,3)},(a,0)) \cap \text{ray}\{(1,0),1\}, (a,0)\right].$$

Claim: $R_0 \in \mathcal{I}$, i.e., for each $x_0 \in \mathbb{R}^2_+$ and each $p \in \mathbb{R}^K_{++}$ such that x_0 is a maximizer for R_0 at prices p, there is a quasi-strictly increasing path of maximizers of R_0 at prices p that passes through x_0. There are three cases.

Case 1: $x_0^1 - 1 \geq x_0^2$. Then, we consider the path whose graph is $\text{seg}[(0,0),(1,0),x_0] \cup \text{ray}\{x_0,1\}$ (Figure 4).

Case 2: $x_0^1+1 > x_0^2 > x_0^1-1$. Then, we consider the path whose graph is seg$[(0,0), x_0] \cup$ ray$\{x_0, 1\}$ (Figure 4).

Case 3: $x_0^2 \geq x_0^1+1$. Then, we consider the path whose graph is seg$[(0,0),(0,1), x_0] \cup$ ray$\{x_0, 1\}$.

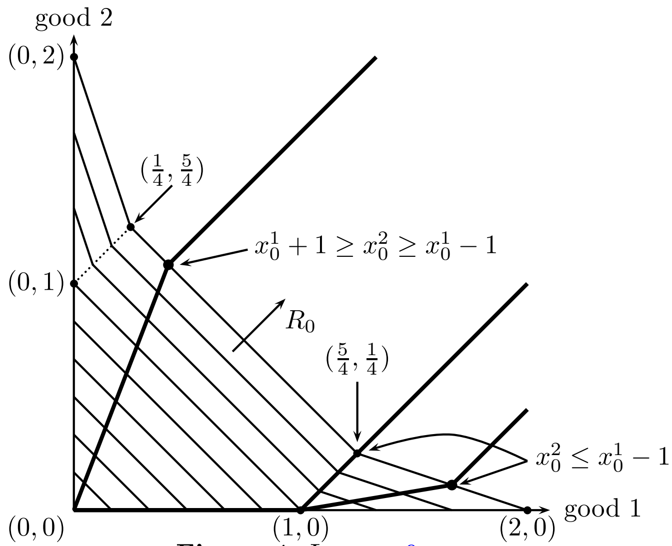

Figure 4: Lemma 9.

$\square$

Proof of Theorem 1. Let $R \in \mathcal{U}$ and $\Theta \equiv \prod_{t=1}^{K} \Omega^t$. Observe that for each $i \in N$, the function u_i, defined by $x_i \in \mathbb{R}_+^K \mapsto \Theta - a(R_i, x_i)$, is a continuous representation of R_i. Let $\Phi(R) \equiv \{\nu \in \mathbb{R}_+ :$ there is $z \in Z$ s.t. for each $i \in N$, $a(R_i, z_i) = \nu\}$. Since $\Theta \in \Phi(R)$, then $\Phi(R)$ is non-empty. Clearly, $\Phi(R) \equiv \{\nu \in \mathbb{R}_+ :$ there is $z \in Z$ s.t. for each $i \in N$, $\Theta - u(z_i) = \nu\}$. Continuity of u implies that $\Phi(R)$ is closed. Since $\Phi(R)$ is bounded, then it is compact. Let $\nu^* \equiv \min \Phi(R)$. Since $\nu^* \in \Phi(R)$, then there is $z \in Z$ such that for each $i \in N$, $a(R_i, z_i) = \nu^*$. Moreover, $z \in \Psi(R)$. Let $z' \in \Psi(R)$. Then, the common value of $a(R_i, z'_i)$ for $i \in N$, is in $\Phi(R)$. Thus, for each $\{i,j\} \subseteq N$, $a(R_i, z_i) = v^* \leq a(R_j, z'_j)$. Thus, $z \in E(R)$.

Let $\{z, z'\} \subseteq E(R)$. Let $i \in N$. By definition of E, $a(R_i, z_i) = a(R_i, z'_i)$. Thus, $u_i(z_i) = u_i(z'_i)$. Since u_i represents R_i, then $z_i \, I_i \, z'_i$. Thus, E is *essentially single-valued*. $\square$

Proof of Lemma 1. Let $R \in \mathcal{U}^N$ and $z \in E(R)$. We proceed in three steps.

(i) $\boldsymbol{z \in P^w(R)}$. Suppose by contradiction that there is $z' \in Z$ such that for each $i \in N$, $z'_i \, P_i \, z_i$. Let $\nu \equiv \max_{j \in N} a(R_j, z'_j)$. Thus, for each $i \in N$, $a(R_i, z'_i) \leq \nu < a(R_i, z_i)$.

Since for each $i \in N$, $a(R_i, 0) \geq \nu$ and preferences are monotone and continuous, then by the Intermediate Value Theorem, there is $z'' \in Z$ such that for each $i \in N$, $a(R_i, z''_i) = \nu$. Thus, $z \notin E(R)$. This is a contradiction.

(iia) If $|N| = 2$, then, $E(R) \subseteq P(R)$. Suppose by contradiction that there is $z \in E(R)$ such that $z \notin P(R)$. Then, there are $z' \in Z$, $i \in N$ and $j \in N$ such that $z'_i \, R_i \, z_i$ and $z'_j \, P_j \, z_j$. Since preferences are monotone, we can assume w.l.o.g. that $z'_i + z'_j = \Omega$. We claim that $\Omega \, P_i \, z'_i$. Since $z \in P^w(R)$, then $z'_i \, I_i \, z_i$. Suppose by contradiction that $z'_i \, I_i \, \Omega$. Then, $a(R_i, z'_i) = 0$ and thus, $a(R_i, z_i) = 0$. Since $z \in E(R)$, then $a(R_j, z_j) = 0$. But, since $z'_j \in U^c(P_j, z_j)$, then $a(R_j, z_j) > 0$. This is a contradiction. Now, since preferences are continuous, then there is $\alpha \in (0, 1)$ such that $(1 - \alpha)z'_j \, P_j \, z_j$. Since $z'_i + z'_j = \Omega$, then $z'_i + \alpha z'_j = (1 - \alpha)z'_i + \alpha\Omega$ (we are able to make this claim because $|N| = 2$). Thus, $z'_i + \alpha z'_j \gg z'_i$. Since preferences are monotone, then $(z'_i + \alpha z'_j) \, P_i \, z'_i$. Let z'' be the allocation defined by: $z''_i \equiv z'_i + \alpha z'_j$ and $z''_j \equiv (1 - \alpha)z'_j$. Since $z' \in Z$, then $z'' \in Z$. Since $z''_i \, P_i \, z_i$ and $z''_j \, P_j \, z_j$, then $z \notin P^w(R)$. This is a contradiction.

(iib) If $R \in \mathcal{R}^N$, then, $E(R) \subseteq P(R)$.

We first prove that for each $i \in N$, $\Omega \, P_i \, z_i \, P_i \, 0$. Let $i \in N$. Since preferences are semi-strictly monotone and $z_i \leq \Omega$, then $z_i \, I_i \, \Omega$ implies $z_i = \Omega$. Thus $\Omega \, P_i \, z_i$, for otherwise for each $j \in N \setminus \{i\}$, $z_j = 0$. We prove now that $z_i \, P_i \, 0$. Suppose by contradiction that $z_i \, I_i \, 0$. Since preferences are continuous and monotone and $z \in E(R)$, then for each $j \in N \setminus \{i\}$, $z_j \, I_j \, 0$. Since for each $j \in N$, $\frac{\Omega}{n} \, P_j \, 0$, then $z \notin P^w(R)$. This is a contradiction.

Now, suppose by contradiction that there is $z \in E(R)$ such that $z \notin P(R)$. Then, there is $z' \in Z$ such that for each $i \in N$, $z'_i \, R_i \, z_i$ and for some $j \in N$, $z'_j \, P_j \, z_j$. Now, since preferences are continuous, then there is $\alpha \in (0, 1)$ such that $(1 - \alpha)z'_j \, P_j \, z_j$. Since $z'_j \, P_i \, z_j \, P_j \, 0$, then $z'_j \neq 0$. Let $i \in N \setminus \{j\}$. Since $R \in \mathcal{R}^N$ and $z'_i \, R_i \, z_i \, P_i \, 0$, then $(z'_i + \frac{\alpha}{|N|-1} z'_j) \, P_i \, z_i$. Let z'' be the allocation defined by: $z''_j \equiv (1 - \alpha)z'_j$, and for each $i \in N \setminus \{j\}$, $z''_i \equiv z'_i + \frac{\alpha}{|N|-1} z'_j$. Since $z' \in Z$, then $z'' \in Z$. Since for each $i \in N$, $z''_i \, P_i \, z_i$, then $z \notin P^w(R)$. This is a contradiction. □

Proof of Lemma 2. Let $R \in \mathcal{R}^N$ and $z \in E(R)$. Since $R \in \mathcal{R}^N$, then by Lemma 1, $z \in P(R)$.

- **(i) For each $i \in N$, $\Omega \, P_i \, z_i \, P_i \, 0$.** See (iib) in the proof of Lemma 1.
- **(ii) $\sum_{i \in N} z_i = \Omega$.** Since $z \in P(R)$, preferences are semi-strictly monotone, and for each $i \in N$, $z_i \, P_i \, 0$, then $\sum_{i \in N} z_i = \Omega$.
- **(iii) There is $p \in \mathbb{R}^K_{++}$ such that for each $i \in N$, $p \in \text{Supp}(R_i, z_i)$.** Since $z \in P(R)$, then by the Second Theorem of Welfare Economics (Mas-Colell et al., 1995, Proposition 16.D.1), there is $p \in \mathbb{R}^K \setminus \{0\}$ and a vector of wealth levels $(w_i)_{i \in N} \in \mathbb{R}^N$, such that: (i) $\sum_{i \in N} w_i = p \cdot \Omega$ and (ii) for each $i \in N$ and each $x \in \mathbb{R}^K_+$, if $x \, P_i \, z_i$, then $p \cdot x \geq w_i$.

Since preferences are monotone and for each $i \in N$, $z_i \, P_i \, 0$, then it is not the case that $p \leq 0$. Thus, by semi-strict monotonicity of preferences, for each $i \in N$, $p \cdot z_i \geq w_i$. Since $\sum_{i \in N} z_i = \Omega$ and $\sum_{i \in N} w_i = p \cdot \Omega$, then for each $i \in N$, $p \cdot z_i = w_i$. We claim that $p \geq 0$. Suppose by contradiction that there is $k \in \{1, \ldots, K\}$ such that $p_k < 0$. Let $i \in N$. For each $\delta \in \mathbb{R}_{++}$, $p \cdot (z_i + \delta 1^k) < w_i$. Moreover, since preferences are semi-strictly monotone and $z_i \, P_i \, 0$, then $(z_i + \delta 1^k) \, P_i \, z_i$. This is a contradiction. We claim that $p \gg 0$. Suppose by contradiction that there is $l \in \{1, \ldots, K\}$ such that $p_l = 0$. Recall that $p \neq 0$. Since $\sum_{i \in N} z_i = \Omega$, then there are $k \in \{1, \ldots, K\}$ and $i \in N$ such that $p_k > 0$ and $z_i^k > 0$. Let $\delta \in \mathbb{R}_{++}$. Thus, $p \cdot (z_i + \delta 1^l) = w_i$. Since preferences are semi-strictly monotone and $z_i \, P_i \, 0$, then $(z_i + \delta 1^l) \, P_i \, z_i$. Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < z_i^k$. Thus, $(z_i + \delta 1^l - \varepsilon 1^k) \in \mathbb{R}_+^K$. By continuity if preferences, if ε is small enough, then $(z_i + \delta 1^l - \varepsilon 1^k) \, P_i \, z_i$. Since $p_k > 0$, then $p \cdot (z_i + \delta 1^l - \varepsilon 1^k) < w_i$. This is a contradiction.

Let $i \in N$. Since $p \gg 0$ and $z_i \, P_i \, 0$, then $p_i \cdot z_i = w_i > 0$. A simple argument (Mas-Colell et al. (1995), Proposition 16.D.2) shows that since $w_i > 0$, then $p \in \text{Supp}(R_i, z_i)$. $\square$

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