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MAJORITIES WITH A QUORUM

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Majorities with a quorum*

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Abstract

Based on a general model of "quaternary" voting rule, sensitive to voters' choices between four different options (abstaining, voting "yes", voting "no" and staying home), we systematically study different types of majority and quorum. The model allows for a precise formulation of majority rules and quorum constraints. For such rules four types of majority can be defined. We also consider four types of quorum. Then we study the possible combinations of a majority system with a type of quorum and provide examples from rules actually used in parliaments.

1 Introduction

Most parliaments use 50% majorities to pass bills. If voters are only allowed vote yes or no, this majority requires that more than half the voters be in favor of the proposal or, equivalently, that the number of "yes"-voters be larger than the number of "no"-voters. When other options are available to voters (such as the possibility of abstaining or staying at home), these two requirements are no longer equivalent. In fact, they specify what is usually known as an "absolute" majority in the first case and a "simple" majority in the second. An intermediate majority is also used

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in some parliaments: a majority of the voters *present*. These majorities treat the different options differently (voting "yes", voting "no", abstaining and staying home), and induce some monotonicities that amount to a different partial order among these options with respect to their being more or less close to a "yes". One aim of this paper is to provide a systematic comparison of these majorities and provide examples of their use in different European parliaments.

Quorum constraints are also common in parliaments. The most frequent is the participation quorum, which requires a minimum number of voters to be present in order for a vote to be validly taken. Examples include the parliaments of Austria, Belgium, Spain, and the UK, among others. As is well-known the existence of a participation quorum introduces the possibility of strategic behavior for the simple majority or the majority of voters present: a voter against the proposal may have an incentive to stay at home rather than come and vote "no". Alternative forms of quorum can be found, such as the approval quorum, which requires a minimum number of votes in favor. This quorum is used for referendums in Germany (see Côrte-Real and Pereira, 2004) and in the Greek parliament. Up to 1890 the U.S. Congress used a quorum that required a minimum number of yes or no votes (see Vermeule, 2007). The comparison of these quorum constraints suggests the possibility of introducing new types of quorum. For instance, instead of imposing a maximum number of voters staying at home (as the participation quorum does), a maximum number of no votes can be imposed. Similarly one could impose a quorum requiring a minimum number of voters either in favor or abstaining. Other possible forms of quorum are discarded as less sensible.

Based on a general model of "quaternary" voting rule, sensitive to voters' choices between four different options (abstaining, voting "yes", voting "no" and staying home), this paper sets out to study systematically the different types of majority with and without quorums. We first examine different majorities and the monotonicities that they induce, study their properties and compare them. We then examine different types of quorum and finally we combine majorities and quorum constraints in all sensible ways. Particular attention is paid to an alternative majority that compares the number of votes in favor of the proposal with the votes of those who do not abstain. The rationale is that if indifferent voters are assumed to abstain, then the majority should be obtained on the basis of the remaining voters. This majority, which we refer to as the "majority of non abstaining voters", is also intermediate between the simple and the absolute majorities. As we show, it has interesting properties but has never been applied (to the best of our knowledge). Interestingly enough, this majority has an implicit participation quorum requirement that does not give rise to strategic voting.

The basic model is that of Laruelle and Valenciano's (2009) quaternary voting rules. This model is related to Freixas and Zwicker's (2003, 2009) voting rules with ordered levels of approval, but in the quaternary voting rules considered here, as fewer monotonicities are required, the options are not necessarily ordered. This permits us to consider rules that are excluded from Freixas and Zwicker's model, such as simple majorities with a participation quorum. We restrict our attention to anonymous rules (as those in most parliaments are) as in Freixas and Zwicker (2009) or Zwicker (2009). Other related papers include that of Côrte-Real and Pereira (2004), who study the possible simple majorities used in referendums with three options (no distinctions drawn between abstention and staying at home). They show that the participation quorum generates the "no-show" paradox, and compare these rules from the point of view of representation. Maniquet and Morelli (2009) also study referendum rules and compare the participation quorum and the approval quorum from the point of view of the preservation of the status quo. Dougherty and Edward (2010) consider all four options, and compare the simple majority with the absolute majority, using a welfarist criterion. We complement their approach by considering alternative majorities, and introducing different types of quorum.

The rest of the paper is organized as follows. Section 2 introduces the basic notation concerning dichotomous voting rules. The different majorities are presented in Section 3, and the different types of quorum in Section 4. The possible combinations of majorities with quorum constraints are systematically studied in Section 5. Finally, Section 6 summarizes the conclusions with a comparison between the different majorities and underlines the interest of the majority of non abstaining voters.

2 Dichotomous rules[1]

We assume that voters take part in a collective dichotomous decision where each of them may decide whether to participate or not in the vote. In they decide to turn out, they may vote yes, abstain or vote no. Four actions are thus possible. A vote profile keeps track of the actions chosen by each voter. If $\mathcal{N} = \{1, 2, ..., n\}$ denotes the set of voters, we represent a vote profile by a 4-partition $S = (S^Y, S^A, S^H, S^N)$ of $\mathcal{N}$, where S^Y the set of "yes"-voters, S^A is the set of abstaining voters, S^H is the set of those who stay at home and S^N is the set of "no"-voters. By $4^{\mathcal{N}}$ we denote the set of all such 4-partitions of $\mathcal{N}$. A voting rule $\mathcal{W}$ specifies which vote profiles lead to the acceptance of the proposal (otherwise the proposal is rejected[2]):

$$\mathcal{W} = \left\{(S^Y, S^A, S^H, S^N) \in 4^{\mathcal{N}} : S \text{ leads to the acceptance of the proposal}\right\}.$$

A vote configuration S is *winning if* $S \in \mathcal{W}$, and *losing* if $S \notin \mathcal{W}$. The following conditions are assumed for $\mathcal{W}$ to specify a sound voting rule:

Full support: If all voters vote "yes" the proposal is accepted:

$$(S^Y = \mathcal{N}) \Rightarrow (S \in \mathcal{W}).$$

[1] In this section we introduce the basic definitions, notation and terminology relative to dichotomous quaternary voting rules based on Laruelle and Valenciano (2009).

[2] Here we focus on dichotomous voting rules, where there are only two possible final outcomes, acceptance or rejection. For rules with more than two outcomes, see Freixas and Zwicker (2003).

Null support: If no voter votes "yes" then the proposal is rejected:

$$(S^Y = \varnothing) \Rightarrow (S \notin \mathcal{W}).$$

In addition to these conditions, the following monotonicities (i.e. transfers of votes that keep a winning profile winning) are assumed.

***AY*-monotonicity:** If $S \in \mathcal{W}$, then $T \in \mathcal{W}$ for any T such that $S^Y \subseteq T^Y$, $S^N = T^N$ and $S^H = T^H$.

***HY*-monotonicity:** If $S \in \mathcal{W}$, then $T \in \mathcal{W}$ for any T such that $S^Y \subseteq T^Y$, $S^N = T^N$ and $S^A = T^A$.

***NA*-monotonicity:** If $S \in \mathcal{W}$, then $T \in \mathcal{W}$ for any T such that $S^A \subseteq T^A$, $S^Y = T^Y$ and $S^H = T^H$.

The monotonicity conditions can be summarized by the following diagram indicating the transition of votes that do not change the winning character of a winning vote configuration or, in other terms, the (partial) order between the different options with respect to their relative proximity to a "yes":

$$\begin{array}{ccc} & Y & \\ \nearrow & & \nwarrow \\ A & & H \\ \uparrow & & \\ N & & \end{array}$$

As can be seen immediately, AY-monotonicity together with NA-monotonicity imply what can be referred to as NY-monotonicity with obvious meaning (if the set of 'yes' voters increases at the expense of the set of "no"-voters a winning configuration cannot become losing). Note that this monotonicity ($N \to Y$), implied by the other two, is omitted in the diagram for the sake of simplicity. These diagrams are often used later to represent the monotonicities of different classes of QVRs. Then we have the following definition[3] (Laruelle and Valenciano, 2009):

Definition 1 *An n-voter "quaternary dichotomous voting rule" (QVR) is a set $\mathcal{W}$ of 4-partitions of $\mathcal{N}$ that satisfies full-support, null-support, NA-monotonicity, AY-monotonicity and HY-monotonicity.*

Here we restrict our attention to *anonymous* voting rules, where only the number of voters who have chosen each of the different options matter, not their identities. In other words, anonymous rules can be specified in terms of the number of voters that

[3] In Laruelle and Valenciano (2009) a more general definition is given, nevertheless, as pointed out there, all the real-world rules admitting abstention and no voting fit into this more restricted notion.

chose each option, i.e., whether a vote configuration is winning or not depends only on vector (s^Y, s^A, s^H, s^N), where s^Y is the number of "yes"-voters, s^A is the number of abstaining voters, s^H is the number of voters who stay at home and s^N is the number of "no"-voters (of course, $s^Y + s^A + s^H + s^N = n$).

As to the relation between the voters' actions and their preferences, what can be said in general is this:

Proposition 2 *In any quaternary voting rule, voting "yes" is always a weakly dominant strategy for voters in favor of the proposal.*

That is, a voter in favor of the proposal will never regret picking the action ranked first (Y). As to voters against the proposal the most that can be said in general is that they should always pick some of the actions furthest from a "yes". As the option of staying at home cannot in general be compared with abstention or the "no"-option, a voter against the proposal may be better off voting "no", or staying at home, depending on the vote configuration. Note the difference with Freixas and Zwicker's (2003) (4,2)-rules, where options (i.e. "levels of support" in their terminology) are linearly ordered, so that any two options can always be pairwise compared, and a voter against the proposal should always vote "no".

Note also that if the option of staying at home is eliminated, we obtain the ternary rules introduced by Felsenthal and Machover (1997), but if we eliminate instead the option of abstaining we obtain ternary rules not covered by their model as options are not necessarily ordered. An example of such rule is the following, where abstention is not allowed:

Example 3 *The Austrian parliament uses the following ternary voting rule:*

$$\mathcal{W} = \left\{ (S^Y, S^H, S^N) \in 3^{\mathcal{N}} : s^Y + s^N > \tfrac{1}{2}n \quad \& \quad s^Y > s^N \right\}.$$

Finally, if we drop the options of staying at home and abstaining, we get the binary rules. An example of such a rule is provided by the Norwegian parliament where, as put by Rasch (1995, p. 491): "*The norm [..] is that each legislator should be present when the plenary votes are taken, and everybody has to vote either in favour or against the motion(s) under discussion.*"

Example 4 *The Norwegian Storting uses a binary 1/2-majority:*

$$\mathcal{W} = \left\{ (S^Y, S^N) \in 2^{\mathcal{N}} : s^Y > \tfrac{1}{2}n \right\}.$$

Freixas and Zwicker (2003) generalize the notion of weighted rule for (j, k)-rules. This can be adapted to anonymous dichotomous quaternary voting rules as follows.

Definition 5 *An n-voter weighted anonymous quaternary voting rule, denoted by*

$$\mathcal{Q}(Q; w^Y, w^A, w^H, w^N),$$

is specified by a system of weights $w = (w^Y, w^A, w^H, w^N)$ *such that* $w^Y \geq w^A \geq w^N$ *and* $w^Y \geq w^H$*, and a quota* Q*, so that a vote configuration* $(S^Y, S^A, S^H, S^N) \in 4^{\mathcal{N}}$ *is winning if*

$$s^Y w^Y + s^A w^A + s^H w^H + s^N w^N > Q.$$

The inequalities between weights correspond to the monotonicities assumed for all QVRs. As shown below, not all voting rules in parliaments can be expressed as anonymous weighted voting rules. Majorities with quorum are usually *double* weighted, that is, the intersection of two weighted rules[4].

3 Quaternary majorities

When there are only two options, voting "yes" or voting "no", the requirement for a proposal to be adopted by a majority can be stated in two equivalent ways: by requiring the number of "yes"-voters to be larger than half the total number of voters, or by requiring the number of "yes"-voters to be larger than the number of "no"-voters. Evidently the two conditions are equivalent when only voting "yes" or "no" is possible (i.e. $s^Y + s^N = n$):

$$s^Y > \tfrac{1}{2}n \;\Leftrightarrow\; s^Y > s^N.$$

Once there is at least a third option, these two requirements are no longer equivalent. For ternary and quaternary voting rules these two conditions define two different majorities, namely the *simple majority* ($s^Y > s^N$) and the *absolute majority* ($s^Y > \frac{1}{2}n$). For quaternary voting rules, though, a third type of majority is found in real-world examples which is intermediate between the absolute majority and the simple majority: the *majority of present voters,* which requires

$$s^Y > \tfrac{1}{2}(s^Y + s^A + s^N).$$

But observe that an alternative intermediate requirement is also possible:

$$s^Y > \tfrac{1}{2}(s^Y + s^H + s^N).$$

To the best of our knowledge this majority has never been proposed, although a rationale for it can be found. Assuming that indifferent voters abstain, it makes sense to require a majority of the non indifferent voters. We refer to this majority as *majority of non abstaining voters.*

[4] In Laruelle and Valenciano (2009) it is proved that any QVR can be expressed as the intersection of a finite number of weighted QVRs (non anonymous in general).

Observe that in each of these four rules at least two options are equivalent: abstaining (A) and staying at home (H) in the simple majority, A and voting "no" (N) in the majority of present voters, H and N in the majority of non abstaining voters. In the absolute majority, three options are undistinguishable: A, H and N.

Each of these majorities is in fact the particular case $q = 1/2$ within a particular family of q-majority rules. Moreover, each of these families of rules is a particular class of anonymous *weighted* quaternary voting rules that we systematically define, indicating the monotonicities satisfied by each family, checking in every case that it is a family of weighted rules and providing examples from the real world whenever we have found them.

Definition 6 *The "simple q-majority", where* $1/2 \leq q < 1$*, is the rule given by*

$$\mathcal{W}^{qSM} := \left\{ (S^Y, S^A, S^H, S^N) \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^N) \right\}.$$

Comment: These rules satisfy the monotonicities indicated in the diagram (where "$A \equiv H$" means that abstaining and staying at home are equivalent options), and $\mathcal{W}^{qSM} = \mathcal{Q}(Q; w^Y, w^A, w^H, w^N)$ with quota $Q = 0$ and the system of weights given.

$$\begin{array}{cc} Y & w^Y = 1 - q \\ \uparrow & \\ A \equiv H & w^A = w^H = 0 \\ \uparrow & \\ N & w^N = -q \end{array}$$

The simple majority is the particular case $q = 1/2$.

Example 7 *The Swedish Riksdag uses a* $1/2$*-simple majority.*

Definition 8 *The "q-majority of present voters", where* $1/2 \leq q < 1$*, is the rule given by*

$$\mathcal{W}^{qPM} := \left\{ (S^Y, S^A, S^H, S^N) \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^A + s^N) \right\}.$$

Comment: These rules satisfy the monotonicities indicated in the diagram, and $\mathcal{W}^{qPM} = \mathcal{Q}(Q; w^Y, w^A, w^H, w^N)$ with quota $Q = 0$ and the system of weights given.

$$\begin{array}{cc} Y & w^Y = 1 - q \\ \uparrow & \\ H & w^H = 0 \\ \uparrow & \\ A \equiv N & w^A = w^N = -q \end{array}$$

The majority of present voters is the particular case $q = 1/2$.

Example 9 *The Finish parliament uses a* $1/2$*-majority of present voters.*

Definition 10 *The "q-majority of non abstaining voters", where $1/2 \leq q < 1$, is the rule given by*

$$\mathcal{W}^{qEM} := \left\{ (S^Y, S^A, S^H, S^N) \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^H + s^N) \right\}.$$

Comment: These rules satisfy the monotonicities indicated in the diagram, and $\mathcal{W}^{qEM} = \mathcal{Q}(Q; w^Y, w^A, w^H, w^N)$ with quota $Q = 0$ and the system of weights given.

$$\begin{array}{cc} Y & w^Y = 1 - q \\ \uparrow & \\ A & w^A = 0 \\ \uparrow & \\ H \equiv N & w^H = w^N = -q \end{array}$$

The atorementioned example of majority of non abstaining voters is the particular case $q = 1/2$.

Definition 11 *The "absolute q-majority", where $1/2 \leq q < 1$, is the rule given by*

$$\mathcal{W}^{qAM} := \left\{ (S^Y, S^A, S^H, S^N) \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^A + s^H + s^N) \right\}.$$

Comment: These rules satisfy the monotonicities indicated in the diagram, and $\mathcal{W}^{qAM} = \mathcal{Q}(Q; w^Y, w^A, w^H, w^N)$ for the system of weights given and quota $Q = 0$.

$$\begin{array}{cc} Y & w^Y = 1 - q \\ \uparrow & \\ A \equiv H \equiv N & w^A = w^H = w^N = -q \end{array}$$

When $q = 1/2$ we have the absolute majority rule. These rules, with different quotas, are often used in parliaments as the following examples show:

Example 12 *The Estonian parliament, the French National Assembly, the Hungarian National Assembly, the National Council of the Slovak Republic and the Spanish Congress of Deputies use a absolute 1/2-majority*[5]*. The Estonian parliament, French National Assembly, the Greek parliament, the Polish Sjem, the National Council of the Slovak Republic, and the Great National Assembly of Turkey parliaments use an absolute 3/5-majority*[6]*. The French parliament ("Haute Cour"), the Italian Chamber*

[5]It is used in the Estonian parliament for special legislation, in the French National Assembly and in the Spanish Congress of Deputies for "organic legislation", in the Hungarian National Assembly for the election of the Prime Minister, and in the National Council of the Slovac Republic for votes of no confidence.

[6]It is used in the Estonian parliament for the amendment of the constitution, in the French National Assembly for the ratification of EU treaties, in the Greek parliament for national questions of crucial importance, in the Polish Sjem to reject the veto of the President, and in the National Council of the Slovak Republic for the election of the President.

of Deputies and the Polish Sjem use an absolute $2/3$*-majority*[7]*. The Danish Folketing uses an absolute* $5/6$*-majority for the delegation of powers.*

As all majorities are QVR weighted rules, and QVR weighted rules are "linear rules" (see Laruelle and Valenciano, 2009), that is, the options (once equivalent ones are identified) are linearly ordered, we can deduce the voters' actions from their preferences. We already know that a voter in favor of the proposal should always vote in favor of the proposal. A voter against the proposal should always pick one of the "lowest" actions (equivalent if more than one!).

Proposition 13 *For voters against the proposal,*

(i) Voting "no" is a weakly dominant strategy in any simple q-majority.

(ii) Voting "no" and abstaining are weakly dominant indifferent strategies in any q-majority of present voters.

(iii) Voting "no" and staying at home are weakly dominant indifferent strategies in any q-majority of non abstaining voters.

(iv) Voting "no", staying at home and abstaining are weakly dominant indifferent strategies in any absolute q-majority.

Therefore, an indifferent voter who does not want to favor any option should either stay home or abstain in any simple majority, stay home in a present majority, and abstain in a majority of non abstaining voters, but must necessarily favour one option or the other in an absolute majority. Thus no strategic considerations enter when voters have strict preferences. As shown below, this is no longer the case when quorum conditions enter into the definition of the rule.

In general, for a given q, the requirement for adopting a proposal is weakest under the simple q-majority, and strongest under the absolute q-majority, while the q-majority of non abstaining voters and the present q-majority are intermediate. More precisely, we have:

Proposition 14 *For any q, we have*

$$\mathcal{W}^{qAM} \subseteq \mathcal{W}^{qPM} \subseteq \mathcal{W}^{qSM}$$

$$\mathcal{W}^{qAM} \subseteq \mathcal{W}^{qEM} \subseteq \mathcal{W}^{qSM}.$$

[7]It is used in the French parliament ("Haute Cour") for the removal fron office of the President, in the Italian Chamber of Deputies for the election of the President and in the Polish Sjem to change the constitution.

Thus the absolute majority is the most demanding majority. This means that the chances of adopting a proposal will be largest under the simple majority, smallest under the absolute majority, and intermediate for the majority of non abstaining voters or the majority of present voters.

Remarks: (i) As it is straightforward to check, all the majority rules considered fit into Definition 1. They all satisfy full support and null support conditions and the basic monotonicities assumed for all QVR. As is obvious from the diagrams, all classes of majority rules present further monotonicities *in addition to* the basic ones.
(ii) The restriction of the quota, $q \geq 1/2$, ensures that all majority rules are *proper* in the following sense. A set of voters $R \subseteq \mathcal{N}$ is "strong winning" for a QVR $\mathcal{W} \subseteq 4^{\mathcal{N}}$ if for all $S \in 4^{\mathcal{N}}$ such that $S^Y = R$, $S \in \mathcal{W}$. That is, if all members of a strong winning set of voters are in favor of the proposal they will win a vote whatever the others do. Then it can easily be checked that for any q-majority rule if $q \geq 1/2$ no pair of disjoint strong winning sets of voters does exist. This condition extends that of properness for binary rules with the same motivation: it prevents two disjoint sets of individuals with opposed preferences from passing contradictory bills.[8]

4 Quorum constraints

Under the simple majority a proposal may be adopted with an extremely low number of votes in favor. For instance, if just one voter votes "yes" and all others either abstain or stay home the proposal is accepted under the simple majority. When this is considered undesirable, one form or other of quorum is adopted. In the rules actually used by parliaments we have found three types of quorum. The first one, which we refer to as "participation quorum", requires a minimum number of voters to be present (or, equivalently, that a given number of absentees is not exceeded). This is the most commonly used form. A second type is used in German and Hungarian referendums (See Côrte-Real and Pereira, 2004). It requires a minimum number of votes in favor of the proposal. We refer to it as an "approval quorum"[9]. Up to 1890 the U.S. Congress[10] used another quorum, as mentioned in Vermeule (2007), which required that a minimum of voters vote yes or not.

All three above-mentioned types of quorum impose a condition on the minimum number of "yes"-votes, possibly added to other votes, as a proportion k, with $0 < k < 1$, of the total number of votes. If we examine all combinatorially possible conditions of this type, we obtain the following cases, enumerated with the corresponding monotonicities:

[8] In Laruelle and Valenciano (2009) the notion of proper rule is extended in this way to quaternary voting rules.

[9] Côrte-Real and Pereira (2004) call these types of quorum "voting threshold" and "majority threshold". Here we take the terms used by Maniquet and Morelli (2008).

[10] Then Speaker Thomas Reed replaced it by a participation quorum.

$$\begin{array}{cccc} s^Y > kn & s^Y + s^A > kn & s^Y + s^H > kn & s^Y + s^N > kn \\ Y & Y \equiv A & Y \equiv H & Y \equiv N \\ \uparrow & \uparrow & \uparrow & \uparrow \\ A \equiv H \equiv N & H \equiv N & A \equiv N & A \equiv H \end{array}$$

$$\begin{array}{ccc} s^Y + s^H + s^N > kn & s^Y + s^A + s^N > kn & s^Y + s^A + s^H > kn \\ Y \equiv H \equiv N & Y \equiv A \equiv N & Y \equiv A \equiv H \\ \uparrow & \uparrow & \uparrow \\ A & H & N \end{array}$$

Some of these conditions are difficult to justify, other make sense. Condition $s^Y > kn$ is the above-mentioned *approval quorum*. Condition $s^Y + s^A > kn$, which we call the *weak approval quorum,* can be seen as an extension of the approval quorum to those who abstain (and are possibly indifferent). It imposes there that should be a sufficient number of voters who are in favor or indifferent. Condition $s^Y + s^H > kn$ would be the extension of the approval quorum to those who stay at home, without counting those who abstain. We have found no rationale for this condition that would encourage supporters to stay home, so we do not take this form of quorum into consideration. We do not consider a quorum of the form $s^Y + s^N > kn$, nor of the form $s^Y + s^H + s^N > kn$ either, because these conditions violate *NA*-monotonicity[11]. Condition $s^Y + s^A + s^N > kn$ is the above-mentioned *participation quorum that* sets an upper bound on the number of voters staying at home. Finally, we refer to condition $s^Y + s^A + s^H > kn$ as the *rejection quorum.* It sets an upper bound on the number of "no"-voters: $s^N < (1-k)n$, and can be seen as a condition symmetric to that of approval quorum, which sets a lower bound on the number of "yes"-voters.

Thus, given any voting rule, four different rules can be defined imposing in addition any of the four types of quorum that we consider.

Definition 15 *Let $\mathcal{W} \subseteq 4^N$ be a QVR, by adding a k-approval quorum a new rule, denoted by $\mathcal{W}_{kY}$, is specified:*

$$\mathcal{W}_{kY} := \left\{ S \in \mathcal{W} : s^Y > kn \right\};$$

by adding a weak k-approval quorum a new rule, denoted by $\mathcal{W}_{kYA}$, is specified:

$$\mathcal{W}_{kYA} := \left\{ S \in \mathcal{W} : \ s^Y + s^A > kn \right\};$$

by adding a k-participation quorum a new rule, denoted by $\mathcal{W}_{kYAN}$, is specified:

$$\mathcal{W}_{kYAN} := \left\{ S \in \mathcal{W} : s^Y + s^A + s^N > kn \right\};$$

[11] Under these conditions, abstaining would be considered as strictly worse than the "no" option for the acceptance of a proposal. Indeed, condition $s^Y + s^N > kn$ was used in the U.S. Congress, but it was eliminated and replced by a participation quorum in 1890.

by adding a k-rejection quorum a new rule, denoted by $\mathcal{W}_{kYAH}$*, is specified:*

$$\mathcal{W}_{kYAH} := \left\{ S \in \mathcal{W} : s^Y + s^A + s^H > kn \right\} .$$

Observe that, for a fixed k, the approval quorum is the strongest requirement, followed by the weak approval quorum. The weakest requirements are the participation quorum and the rejection quorum, which cannot be compared.

Proposition 16 *For any quaternary voting rule* $\mathcal{W}$ *and any given* k *we have*

$$\mathcal{W}_{kY} \subseteq \mathcal{W}_{kYA} \subseteq \mathcal{W}_{kYAN} \subseteq \mathcal{W},$$

$$\mathcal{W}_{kY} \subseteq \mathcal{W}_{kYA} \subseteq \mathcal{W}_{kYAH} \subseteq \mathcal{W}.$$

Thus, the weaker the quorum the greater the chances of adopting a proposal.

5 Majorities with quorum

We now systematically combine the different q-majorities with the different types of k-quorum. In every case the resulting rules satisfy the monotonicities displayed by *both* the original rule and the quorum rule, therefore resulting in less monotonicities than either of them in general. For each matching majority/quorum we give the resulting diagram of monotonicities of such rules and a double system of weights that, with quotas $Q = 0$ and $K = 0$, specify them in the form

$$\mathcal{Q}(Q; w^Y, w^A, w^H, w^N) \cap \mathcal{Q}(K; v^Y, v^A, v^H, v^N) \qquad (1)$$

We also provide examples from parliaments whenever found and available. In each case a relation between k and q should be imposed for the combination to make sense and to ensure that neither of the rules is contained in the other (i.e. the condition that defines one is implied by the condition that specifies the other). First, note that

$$\mathcal{W}^{qAM} = \mathcal{W}_{qY},$$

that is, for any q, a q-approval quorum, the strongest q-quorum, is equivalent to a q-absolute majority, the strongest of the q-majorities. Thus it does not make sense to add any quorum constraint to a q-absolute majority. Likewise, to require a k-approval quorum together with any q-majority only makes sense if $k < q$, otherwise it would amount to merely assuming a k-absolute majority.

5.1 Simple majorities with quorum

If we add the different quorum constraints to a simple q-majority, we obtain the following double-weighted rules.

Simple q-majority with k-approval quorum ($0 < k < q$):

$$\mathcal{W}_{kY}^{qSM} := \left\{ S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^N) \ \&\ s^Y > kn \right\}.$$

The monotonicities of such rules and a double system of weights that, with quotas $Q = 0$ and $K = 0$, specify them in the form (1) are the following:

$$\begin{array}{ccc}
Y & w^Y = 1 - q & v^Y = 1 - k \\
\uparrow & & \\
A \equiv H & w^A = w^H = 0 & v^A = v^H = -k \\
\uparrow & & \\
N & w^N = -q & v^N = -k
\end{array}$$

Example 17 *The rule used for referendum in Germany and Hungary is a 1/2-simple majority with an 1/4-approval quorum (source: Côrte-Real and Pereira, 2004).*

Simple q-majority with weak k-approval quorum:

$$\mathcal{W}_{kYA}^{qSM} := \left\{ S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^N) \ \&\ s^Y + s^A > kn \right\}.$$

The resulting monotonicities of such rules and a double system of weights to specify them in the form (1) with quotas $Q = K = 0$ are:

$$\begin{array}{ccc}
Y & w^Y = 1 - q & v^Y = 1 - k \\
\uparrow & & \\
A & w^A = 0 & v^A = 1 - k \\
\uparrow & & \\
H & w^H = 0 & v^H = -k \\
\uparrow & & \\
N & w^N = -q & v^N = -k
\end{array}$$

Simple q-majority with k-participation quorum:

$$\mathcal{W}_{kYAN}^{qSM} := \left\{ S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^N) \ \&\ s^Y + s^A + s^N > kn \right\}.$$

Their monotonicities and a double system of weights (with quotas $Q = K = 0$) are:

$$\begin{array}{ccccc}
 & Y & & w^Y = 1 - q & v^Y = 1 - k \\
 & \uparrow & & & \\
 & A & & w^A = 0 & v^A = 1 - k \\
\nearrow & & \nwarrow & & \\
N & & H & w^N = -q,\ w^H = 0 & v^N = 1 - k,\ v^H = -k
\end{array}$$

Example 18 *The Belgian Chamber of Representatives, the Polish Sjem and the Italian Chamber of Deputies use a* $1/2$*-simple majority with a* $1/2$*-participation quorum. The UK House of Commons uses a* $1/2$*-simple majority with a* $40/650$*-participation quorum.*

Simple q-majority with k-rejection quorum:

$$\mathcal{W}_{kYAH}^{qSM} := \left\{S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^N) \ \& \ s^Y + s^A + s^H > kn\right\}.$$

Their monotonicities and a double system of weights with quotas $Q = 0$ and $K = 0$ are:

$$\begin{array}{ccc} Y & w^Y = 1-q & v^Y = 1-k \\ \uparrow & & \\ A \equiv H & w^A = w^H = 0 & v^A = v^H = 1-k \\ \uparrow & & \\ N & w^N = -q & v^N = -k \end{array}$$

From the diagram, we can see that the participation quorum breaks the linear order of the options and thus introduces the possibility of strategic behavior (it may be better for a voter against the proposal to stay at home rather than go and vote "no"). But for the simple majority with the other types of quorum we have:

Proposition 19 *For voters against the proposal, voting "no" is a weakly dominant strategy in any simple* q*-majority rule with either an approval quorum, a weak approval quorum or a rejection quorum.*

5.2 Majorities of voters present with quorum

We now combine the different types of quorum with q-majorities of present voters, and obtain the following rules.

q-Majority of voters present with k-approval quorum ($0 < k < q$):

$$\mathcal{W}_{kY}^{qPM} := \left\{S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^A + s^N) \ \& \ s^Y > kn\right\}.$$

Their monotonicities and a double system of weights with quotas $Q = 0$ and $K = 0$ are:

$$\begin{array}{ccc} Y & w^Y = 1-q & v^Y = 1-k \\ \uparrow & & \\ H & w^H = 0 & v^H = -k \\ \uparrow & & \\ A \equiv N & w^A = w^N = -q & v^A = v^N = -k \end{array}$$

Example 20 *The rule used in the Greek parliament is a* $1/2$*-present majority with a* $1/4$*-approval quorum.*

q-Majority of voters present with weak k-approval quorum:

$$\mathcal{W}^{qPM}_{kYA} := \left\{ S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^A + s^N) \ \& \ s^Y + s^A > kn \right\},$$

The monotonicities and an example of a system of weights with quotas $Q = 0$ and $K = 0$ are:

$$\begin{array}{ccccc} & Y & & w^Y = 1-q & v^Y = 1-k \\ \nearrow & & \nwarrow & & \\ A & & H & w^A = -q,\ w^H = 0 & v^A = 1-k,\ v^H = -k \\ \searrow & & \swarrow & & \\ & N & & w^N = -q & v^N = -k \end{array}$$

q-Majority of voters present with k-participation quorum:

$$\mathcal{W}^{qPM}_{kYAN} =: \left\{ S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^A + s^N) \ \& \ s^Y + s^A + s^N > kn \right\}.$$

Their monotonicities and a system of weights with quotas $Q = 0$ and $K = 0$ are:

$$\begin{array}{ccccc} & Y & & w^Y = 1-q & v^Y = 1-k \\ \nearrow & & \nwarrow & & \\ A \equiv N & & H & w^A = w^N = -q,\ w^H = 0 & v^A = v^N = 1-k,\ v^H = -k \end{array}$$

Example 21 *The rule used in the German Budestag, the Italian Senate the Spanish Congress of Deputies, the Czech Chamber of Deputies, the Latvian Saeima, and the Polish Sjem is a 1/2-majority of present with a 1/2-participation quorum. The rule used in the Latvian Saeima to amend the Constitution is a 2/3-present majority with a 2/3-participation quorum.*

q-Majority of voters present with k-rejection quorum:

$$\mathcal{W}^{qPM}_{kYAH} := \left\{ S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^A + s^N) \ \& \ s^Y + s^A + s^H > kn \right\},$$

Their monotonicities and an example of a system of weights with quotas $Q = 0$ and $K = 0$ are:

$$\begin{array}{ccc} Y & w^Y = 1-q & v^Y = 1-k \\ \uparrow & & \\ H & w^H = -q & v^H = 1-k \\ \uparrow & & \\ A & w^A = 0 & v^A = 1-k \\ \uparrow & & \\ N & w^N = -q & v^N = -k \end{array}$$

Again the introduction of a participation quorum induces strategic behavior for voters against the proposal. Also note that in this case voting "no" and abstaining are equivalent options. With the other types of quorum there is at least one weakly dominant strategy for voters against the proposal.

Proposition 22 *For voters against the proposal:*

(i) Voting "no" and abstaining are equivalent and weakly dominant strategies in any q-majority of voters present with an approval quorum ($\mathcal{W}_{kY}^{qPM}$)

(ii) Voting "no" is a weakly dominant strategy in any q-majority of voters present with an approval quorum ($\mathcal{W}_{kYA}^{qPM}$) or a rejection quorum ($\mathcal{W}_{kYAH}^{qPM}$).

In most cases the rules used in parliaments, as in the examples provided, consist of one type of majority with (or without) a form of quorum. Nevertheless, a double quorum is feasible, and the Grand National Assemby of Turkey provides an interesting example, which gives rise to a *triple*-weighted majority rule.

Example 23 *The rule used in the Grand National Assemby of Turkey (ordinary motions) is a* $1/2$*-majority of voters present with* $1/3$*-participation quorum and a* $1/4$*-approval quorum, namely*

$$\mathcal{W}^{TP} = \left\{ S \in 4^{\mathcal{N}} : s^Y > \tfrac{1}{2}(s^Y + s^A + s^N),\ s^Y + s^A + s^N > \tfrac{1}{3}n,\ s^Y > \tfrac{1}{4}n \right\}.$$

The monotonicities can be summarized as follows:

$$\begin{array}{ccc} & Y & \\ \nearrow & & \nwarrow \\ A \equiv N & & H \end{array}$$

and a triple system of weights with quotas 0 *is the following:*

$w^Y = \frac{1}{2}$	$v^Y = \frac{2}{3}$	$u^Y = \frac{3}{4}$
$w^A = w^N = -\frac{1}{2},\ w^H = 0$	$v^A = v^N = \frac{2}{3},\ v^H = -\frac{1}{3}$	$u^A = u^N = u^H = -\frac{1}{4}$

5.3 Majorities of non abstaining voters with a quorum

Interestingly enough, the majority of non abstaining voters entails an implicit weak approval quorum (and thus a participation quorum and a rejection quorum).

Proposition 24 *For any* n*, if* $s^Y > q(s^Y + s^H + s^N)$ *then* $s^Y + s^A > qn$*.*

Proof. We can rewrite $s^Y > q(s^Y + s^H + s^N)$ using $s^Y + s^A + s^H + s^N = n$ in order to obtain $(1-q)(n - s^H - s^N - s^A) > q(s^H + s^N)$ and $(1-q)(n - s^A) > s^H + s^N$. Thus $(1-q)n > s^H + s^N$ or $s^Y + s^A > qn$. ■

We then have

$$\mathcal{W}^{qEM} = \mathcal{W}_{qYA}^{qEM} = \mathcal{W}_{qYAN}^{qEM} = \mathcal{W}_{qYAH}^{qEM}.$$

Thus, if $k \leq q$ weak k-approval quorum, k-participation quorum and k-rejection quorum are not binding for a q-majority of non abstaining voters. Therefore, if $k < q$ only the introduction of an k-approval quorum leads to a new rule: in the other cases it must be assumed that $q < k$.

q-Majority of non abstaining voters with k-approval quorum ($0 < k < q$):

$$\mathcal{W}_{kY}^{qEM} := \left\{S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^H + s^N) \ \& \ s^Y > kn\right\}.$$

Their monotonicities and a system of weights with quotas $Q = 0$ and $K = 0$ are:

$$\begin{array}{ccc} Y & w^Y = 1-q & v^Y = 1-k \\ \uparrow & & \\ A & w^A = 0 & v^A = -k \\ \uparrow & & \\ H \equiv N & w^H = w^N = -q & v^H = w^N = -q \end{array}$$

q-Majority of non abstaining voters with weak k-approval quorum:

$$\mathcal{W}_{kYA}^{qEM} := \left\{S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^H + s^N) \ \& \ s^Y + s^A > kn\right\},$$

where $1/2 \leq q < k < 1$.

Their monotonicities and a system of weights with quotas $Q = 0$ and $K = 0$ are:

$$\begin{array}{ccc} Y & w^Y = 1-q & v^Y = 1-k \\ \uparrow & & \\ A & w^A = 0 & v^A = 1-k \\ \uparrow & & \\ H \equiv N & w^H = w^N = -q & v^H = w^N = -k \end{array}$$

We discard a q-majority of non abstaining voters with a k-participation quorum because it yields an undesired monotonicity $H \to N$, which places not turning out further away than "no" from "yes". Thus, the only remaining combination is

q-Majority of non abstaining voters with k-rejection quorum:

$$\mathcal{W}_{kY}^{qEM} := \left\{S \in 4^{\mathcal{N}} : s^Y > q(s^Y + s^H + s^N) \ \& \ s^Y + s^A + s^H > kn\right\},$$

where $1/2 \leq q < k < 1$.

Their monotonicities and a system of weights with quotas $Q = 0$ and $K = 0$ are:

$$\begin{array}{ccc} Y & w^Y = 1-q & v^Y = 1-k \\ \uparrow & & \\ A & w^A = 0 & v^H = 1-k \\ \uparrow & & \\ H & w^H = -q & v^A = 1-k \\ \uparrow & & \\ N & w^N = -q & v^N = -k \end{array}$$

Note that in all three cases the resulting rule entails a linear order of the options (up to identification of equivalent ones), so that no possibility of strategic behavior emerges.

Proposition 25 *For voters against the proposal, voting "no" and staying at home are equivalent and weakly dominant strategies in any q-majority of non abstaining voters rule with an approval quorum or a weak approval quorum, while voting "no" is a weakly dominant strategy in any q-majority of non abstaining voters rule with a rejection quorum.*

Remarks: (i) Note that all the rules combining majorities of different types with different forms of quorum considered in this section fit into Definition 1, and they are all proper in the sense of remark-(ii) at the end of Section 3.
(ii) Only the simple q-majority and the q-majority of voters present with a k-participation quorum, of which many examples are to be found in parliaments, give rise to the possibility of strategic behavior (along with the special case of the Grand National Assembly of Turkey).

6 Conclusion

Majority rules are used in many parliaments. They all require the number of votes in favor of the proposal to be larger than a certain proportion of a group of voters always containing the "yes"-voters, and the "no"-voters. This group may or may not include the voters who abstain and/or the voters who stay home. We have systematically examined the possible majorities, which include the majority of non abstaining voters. Our analysis shows that majorities are ordered rules (i.e. options are linearly ordered) with two or three levels of approval. The "yes"-option is always ranked on the first level of approval (and is the only option on this level), while the "no"-option is always ranked on the lowest level. The differences between the majorities lie in the number of levels and in how the two remaining options are ranked. The absolute majority has only two levels, while the other majorities have three. The absolute majority does not distinguish between abstention, staying at home and the "no"-option. The simple majority does not distinguish between staying at home and abstaining: these two options are ranked on the second level. The majority of voters present does not distinguish between the "no"-option and abstention, which are ranked on the third level, while staying at home is ranked in the second level. An alternative majority is what we have called the "majority of non abstaining voters". In this case the "no"-option and staying home are undistinguishable and are ranked on the third level, while abstention lies on the second and intermediate level. As majorities are ordered, there exists at least one weakly dominant action for non indifferent voters: the "yes"-option for the voters in favor and an option ranked on the lowest level for the voters against the proposal. If indifferent voters do not wish to favour any option there exists the possibility of picking an option on the second intermediate level, except in the absolute majority, where only two levels exist.

We have also studied quorum constraints. Each quorum leads to a two-level ranking of the options. The addition of a quorum to a majority may maintain the number of levels, increase it, or make two options incomparable, with the resulting rule no longer being ordered, which may give rise to strategic behavior. The approval quorum ranks the “yes”-option first, and the remaining option are considered as equivalent on the second level. Therefore adding the approval quorum to any majority (or even to any rule) does not modify the ranking of the options. The weak approval quorum ranks the “yes”-option and abstention on the first level. This approval quorum introduces a fourth level in the simple majority: all options become ordered, with the “yes”-option first, abstention second, staying home third and the “no”-option last. Adding such a quorum to the majority of voters present makes two options incomparable, as abstention is ranked below staying at home in the majority while the opposite holds for the quorum. The addition of the participation quorum to the simple majority or the majority of voters present has a similar effect: the participation quorum ranks the staying home option below the “no”-option, while the opposite holds in the two majorities. As a result, adding a participation quorum makes the staying home option and the “no”-option incomparable. As the rejection quorum ranks the “no”-option last and does not distinguish between the remaining options, introducing the quorum into the majorities keeps the resulting rules ordered. A fourth level is however introduced in the case of the majority of voters present, where the addition of the quorum leads to a ranking with the “yes”-option first, abstention second, staying home third and the “no”-option last.

Finally we want to stress that the majority of non abstaining voters appears as natural as the other majorities but nevertheless does not seem to have been used to date in any parliament. This majority has the advantage of having an implicit participation quorum, which neither the simple majority nor the majority of present voters has. As a result it is immune to the strategic behavior of the voters, while a certain number of voters present is necessary to adopt a proposal, and is less demanding than the absolute majority.

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