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by

Elena Iñarra, Concepción Larrea and Elena Molis

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University of the Basque Country

The Stability of the Roommate Problem Revisited*

E. Inarra[†], C. Larrea[‡] and E. Molis[§]

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Abstract

The lack of stability in some matching problems suggests that alternative solution concepts to the core might be applied to find predictable matchings. We propose the absorbing sets as a solution for the class of roommate problems with strict preferences. This solution, which always exists, either gives the matchings in the core or predicts some other matchings when the core is empty. Furthermore, it satisfies an interesting property of outer stability. We also characterize the absorbing sets, determine their number and, in case of multiplicity, we find that they all share a similar structure.

KEYWORDS: Roommate problem, core, absorbing sets.

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[†] *Correspondence address:* Elena Inarra, Avda. L. Agirre 83, 48015 Bilbao, Spain. elena.inarra@ehu.es http://www.ehu.es/einarra.

[‡] Email: mariaconcepcion.larrea@ehu.es

[§] The research of this author is supported by UPV/EHU Grant PIFA010502. She also acknowledges the hospitality of the department of Economics of Maastricht University. Email: elena.molis@ehu.es

1 Introduction

Matching markets are of great interest in a variety of social and economic environments, ranging from marriages formation, through admission of students into colleges to matching firms with workers.[1] One of the aims pursued by the analysis of these markets is to find stable matchings. There are, however, some markets for which the set of stable matchings, *i.e.* the core, is empty. For these cases, we suggest that instead of using the common approach of restricting the preferences domain to deal with nonempty core matching markets, [2] other solution concepts may be applied to find predictable matchings. We argue that this alternative is a step in furthering our understanding of matching market performance.

The approach that we take consists of associating each matching market with an abstract system and then applying one of the existing solution concepts to solve it. The modeling of abstract systems deals with the problem of choosing a subset from a feasible set of alternatives. In these systems, a binary relation, which represents transitions between alternatives enforced by some agents, is defined. Various solution concepts have been proposed for solving abstract systems, such as the core, von Neumann-Morgenstern stable sets[3] (von Neumann-Morgenstern, 1947), subsolutions (Roth, 1976), admissible sets (Kalai, Pazner and Schmeidler, 1976), and absorbing sets. The notion of absorbing sets, which is the solution concept selected in our work, was first introduced by Schwartz (1970) and it coincides with the elementary dynamic solution (Shenoy, 1979).

We focus our attention on one-sided matching markets where each agent is allowed to form at most one partnership. These problems are known as *roommate problems* and are a generalization of the marriage problem, see Gale and Shapley (1962). In them each agent in a set ranks all others (including herself) according to her preferences. The abstract system associated with a roommate problem is the pair formed by the set of all matchings and a domination relation defined over this set which represents the existence of a blocking pair of agents that allows us to go from one matching to another. Matchings that are not blocked by any pair of agents are called stable. In this model the set of stable matchings equals the core. Roommate problems that do not admit any such

[1]See Roth and Sotomayor (1990) for a comprehensive survey of two-sided matching models.
[2]See for example, Roth (1985) and Kelso and Crawford (1982).
[3]Elhers (2007) studies von Neumann-Morgenstern stable sets in two-sided matching markets.

matchings are called *unsolvable*. Otherwise they are said to be *solvable*.

Core stability for solvable roommate problems has been studied by Gale and Shapley (1962), Irving (1985), Tan (1991), Abeledo and Isaak (1991), Chung (2000), Diamantoudi, Miyagawa and Xue (2004)) and Klaus and Klijn (2007) among others. With few exceptions, however, unsolvable roommate problems have not been so thoroughly studied. When there is no core stability, interest is rekindled in the application of other solution concepts to the class of roommate problems. Such interest is further enhanced from the empirical perspective in that as Pittel and Irving (1994) observe, when the number of agents increases, the probability of a roommate problem being solvable decreases fairly steeply.

Here we propose the *absorbing sets* as a solution for the class of roommate problems with strict preferences. In this context, an absorbing set is a set of matchings that satisfies the following two conditions: (i) any two distinct matchings inside the set dominate (directly or indirectly) each other and (ii) no matching in the set is dominated by a matching outside the set. We believe that the selection of this solution concept is well justified since for a solvable roommate problem it exactly provides the matchings in the core, and for an unsolvable roommate problem it gives a nonempty set of matchings. Furthermore, this solution has the property of outer stability in the sense that all matchings not in an absorbing set are (directly or indirectly) dominated by a matching that does belong to an absorbing set.[4] As a consequence of this property the matchings outside absorbing sets can be ruled out as reasonable matchings.

Among the scant literature on unsolvable roommate problems the papers by Tan (1990) and Abraham, Biró and Manlove (2005) are worthy of mention. The former investigates matchings with the maximum number of disjoint pairs of agents such that these pairs are stable among themselves and the latter looks at matchings with the smallest number of blocking pairs. Although for solvable roommate problems both proposals give the matchings in the core, for unsolvable ones it is easy to check that neither of them satisfies the outer stability property.

The notion of an absorbing set may perhaps be better understood if it is illustrated with the following metaphor:

Consider a point associated with each matching, and imagine that initially

[4]For marriage problems Roth and Vande Vate (1990) show that there exits a convergence domination path from any unstable matching to a stable one.

all these matching-points shine with light. Then the following process continues indefinitely: In each period each matching-point distributes its light equally among all its dominated matching-points. In this way, a matching-point remains lit either if it receives some light from another or if it is unable to transfer its light to any other matching-point. After an infinite number of periods only those matching-points belonging to a "stable constellation" (absorbing sets) will be permanently lit. Moreover, a stable constellation may be understood as an "energetically closed system" formed by a minimal set of self-lighting matching-points. In this terminology, a stable matching is precisely an energetically closed constellation which consists of a single matching-point.

The contribution of this paper to the analysis of the stability of the roommate problems can be summarized as follows:

First, we find that absorbing sets are determined by stable partitions. This notion was introduced by Tan (1991) as a structure generalizing the notion of a stable matching.[5] We also prove that if a roommate problem is solvable then an absorbing set is a singleton consisting of a stable matching and the union of all absorbing sets coincides with the core.

Second, we characterize the absorbing sets in terms of stable partitions. The characterization provided allows us to specify which and how many stable partitions determine absorbing sets. We also identify the matchings in these sets and give some of their features.

Third, we find that absorbing sets of an unsolvable roommate problem all share a similar structure. In terms of the metaphor described above they look like stable constellations forming "replicas" of one another.[6] Furthermore, we observe that all matchings in all absorbing sets have an interesting property of stability.

The rest of the paper is organized into the following sections. Section 2 contains the preliminaries. In Section 3 we study the absorbing sets of a roommate problem which are characterized in Section 4. We study the structure of these sets in Section 5 and Section 6 gathers some final comments. An appendix with the lemmas and their proofs concludes the paper.

[5]This author defines stable partitions to establish a necessary and sufficient condition for the existence of a stable matching in roommate problems with strict preferences.

[6]This is illustrated in Figure 2.

2 Preliminaries

A *roommate problem* is a pair $(N, (\succcurlyeq_x)_{x\in N})$ where N is a finite set of agents and for each agent $x \in N$, $\succcurlyeq_x$ is a complete, transitive preference relation defined over N. Let $\succ_x$ be the strict preference associated with $\succcurlyeq_x$. In this paper we only consider roommate problems with strict preferences, which we denote by $(N, (\succ_x)_{x\in N})$.

A *matching* μ is a one to one mapping from N onto itself such that for all $x \in N$ $\mu(\mu(x)) = x$, where $\mu(x)$ denotes the partner of agent x under the matching μ. If $\mu(x) = x$, then agent x is single under μ. Given $S \subseteq N$, $S = \emptyset$, let $\mu(S) = \{\mu(x) : x \in \mathcal{S}\}$. That is, $\mu(S)$ is the set of partners of the agents in S under μ. Let $\mu\mid_S$ be the mapping from S to N which denotes the restriction of μ to S. If $\mu(S) = S$ then $\mu\mid_S$ is a matching in $(S, (\succ_x)_{x\in S})$.

A pair of agents $\{x, y\} \subseteq N$ (possibly $x = y$) is a *blocking pair* of the matching μ if

$$y \succ_x \mu(x) \text{ and } x \succ_y \mu(y). \qquad [1]$$

That is, x and y prefer each other to their current partners at μ. If $x = y$, [1] means that agent x prefers being alone to being matched with $\mu(x)$. An agent $x \in N$ *blocks* a matching μ if this agent belongs to some blocking pair of μ. A matching is called *stable* if it is not blocked by any pair $\{x, y\}$. Let $\{x, y\}$ be a blocking pair of μ. A matching μ' is obtained from μ by *satisfying* $\{x, y\}$ if $\mu'(x) = y$ and for all $z \in N\backslash\{x, y\}$,

$$\mu'(z) = \begin{cases} z & if\ \mu(z) \in \{x, y\} \\ \mu(z) & \text{otherwise.} \end{cases}$$

That is, once $\{x, y\}$ is formed, their partners (if any) under μ are alone in μ', while the remaining agents are matched as in μ.

Tan (1991) establishes a necessary and sufficient condition for the solvability of roommate problems with strict preferences in terms of stable partitions. This notion, considered by this author as a generalization of the notion of a stable matching, can be formally defined as follows:[7]

Let $A = \{a_1, ..., a_k\} \subseteq N$ be an ordered set of agents. The set A is a *ring* if $k \geq 3$ and for all $i \in \{1, ..., k\}$, $a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i$ (subscript modulo k).

[7] See Biró *et al.* (2007) for a clarifying interpretation of this notion.

The set A is a pair of mutually acceptable agents if $k = 2$ and for all $i \in \{1, 2\}$, $a_{i-1} \succ_{a_i} a_i$ (subscript modulo 2)[8]. The set A is a singleton if $k = 1$.

A *stable partition* is a partition P of N such that:
(i) For all $A \in P$, the set A is a ring, a mutually acceptable pair of agents or a singleton, and
(ii) For any sets $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_l\}$ of P (possibly $A = B$), the following condition holds:

$$\text{if} \quad b_j \succ_{a_i} a_{i-1} \text{ then } b_{j-1} \succ_{b_j} a_i,$$

for all $i \in \{1, ..., k\}$ and $j \in \{1, ..., l\}$ such that $b_j \neq a_{i+1}$.

Condition (ii) may be interpreted as a notion of stability over the partitions satisfying Condition (i).

The following assertion is proven by Tan (1991).

Remark 1 *A roommate problem $(N, (\succ_x)_{x \in N})$ has no stable matchings if and only if there exists a stable partition with some odd ring. Moreover, any two stable partitions have exactly the same odd rings.* [9]

Using the notion of a stable partition Inarra *et al.* (2007) introduce some specific matchings, called P-stable matchings, defined as follows:

Definition 1 *Let P be a stable partition. A P-stable matching is a matching μ such that for each $A = \{a_1, ..., a_k\} \in P$, $\mu(a_i) \in \{a_{i+1}, a_{i-1}\}$ for all $i \in \{1, ..., k\}$ except for a unique j where $\mu(a_j) = a_j$ if A is odd.*

Given the use made of the notions of a P-stable matching and a stable partition in deriving our results, it may be helpful to illustrate them with the numerical example given in Inarra *et al.* (2007).

[8] Hereafter we omit subscript modulo k.
[9] A ring is odd (even) if its cardinal is odd (even).

EXAMPLE 1 Consider the following 6-agent roommate problem:

$$2 \succ_1 3 \succ_1 1 \succ_1 4 \succ_1 5 \succ_1 6$$

$$3 \succ_2 1 \succ_2 2 \succ_2 4 \succ_2 5 \succ_2 6$$

$$1 \succ_3 2 \succ_3 3 \succ_3 4 \succ_3 5 \succ_3 6$$

$$5 \succ_4 4 \succ_4 1 \succ_4 2 \succ_4 3 \succ_4 6$$

$$4 \succ_5 5 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 6$$

$$6 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 4 \succ_6 5$$

It is easy to verify that $P = \{\{1,2,3\},\{4,5\},\{6\}\}$ is a stable partition where $A_1 = \{1,2,3\}$ is an odd ring, $A_2 = \{4,5\}$ is a pair of mutually acceptable agents and $A_3 = \{6\}$ is a singleton. This partition can be represented graphically as follows:

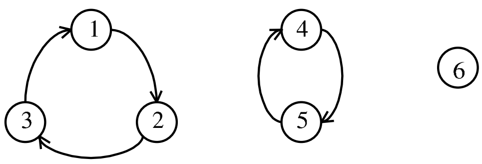

Figure 1.- A stable partition P.

The P-stable matchings associated with the stable partition P are: $\mu_1 = [\{1\},\{2,3\},\{4,5\},\{6\}]$, $\mu_2 = [\{2\},\{1,3\},\{4,5\},\{6\}]$ and $\mu_3 = [\{3\},\{1,2\},\{4,5\},\{6\}]$.

3 Absorbing sets for the roommate problem

We study in this section the absorbing sets of the class of roommate problems with strict preferences and find that every of these sets is determined by some stable partition. We also show that if a roommate problem is solvable then every absorbing set contains only one matching which is stable. Furthermore, the union of all of them coincides with the core.

An *abstract system* is a pair $(X, \mathcal{R})$ where X is a finite set of alternatives and $\mathcal{R}$ is a binary relation on X. Two of the solution concepts proposed to solve an abstract system are the core and absorbing sets. In what follows, we associate a roommate problem with strict preferences with an abstract system and define these two solution concepts in this particular setting. Let $\mathcal{M}$ denote the set of all matchings. Set $X = \mathcal{M}$ and define a binary relation R on $\mathcal{M}$ as follows: Given two matchings μ, $\mu' \in \mathcal{M}$, $\mu' R \mu$ if and only if μ' is obtained from μ by satisfying a blocking pair of μ. We say that μ' *directly dominates* μ if $\mu' R \mu$. Hereafter the system associated with the roommate problem $(N, (\succ_x)_{x \in N})$ is the pair $(\mathcal{M}, R)$. Let R^T denote the transitive closure of R. Then $\mu' R^T \mu$ if and only if there exists a finite sequence of matchings $\mu = \mu_0, \mu_1, ..., \mu_m = \mu'$ such that, for all $i \in \{1, ..., m\}$, $\mu_i R \mu_{i-1}$. We say that μ' *dominates* μ if $\mu' R^T \mu$.

As it has been mentioned in the introduction, the conventional solution considered in matching problems is the *core*. In roommate problems, however, the core may be empty and *absorbing sets* stand out as a good candidate for an alternative solution concept. For these problems an absorbing set can be formally defined as follows:

Definition 2 *A nonempty subset $\mathcal{A}$ of $\mathcal{M}$ is an absorbing set of $(\mathcal{M}, R)$ if the following conditions hold:*
(i) For any two distinct $\mu, \mu' \in \mathcal{A}$, $\mu' R^T \mu$.
(ii) For any $\mu \in \mathcal{A}$ there is no exists $\mu' \notin \mathcal{A}$ such that $\mu' R \mu$.

Condition (i) means that matchings of $\mathcal{A}$ are symmetrically connected by the relation R^T. That is, every matching in an absorbing set is dominated by any other matching in the same set. Condition (ii) means that the set $\mathcal{A}$ is R-closed. That is, no matching in an absorbing set is directly dominated by a matching outside the set.

A nice property of this solution is that it always exists, although, in general, it may be not unique. Theorem 1 in Kalai *et al.* (1977) states that if a set of alternatives, X, is finite then the admissible set (the union of absorbing sets) is nonempty (see also Theorem 2.5 in Shenoy (1979)). Thus either of these two results allows us to conclude that any $(\mathcal{M}, R)$ has at least one absorbing set. Absorbing sets also satisfy the property of *outer stability* which says that every matching not belonging to an absorbing set is dominated by a matching that does belong to an absorbing set.[10]

We are interested in determining which matchings form absorbing sets and to that end we use the notion of a stable partition. Throughout the paper,

[10]This is shown in Kalai *et al.* (1976).

however, we only consider stable partitions which do not contain even rings. This does not imply a loss of generality since Proposition 3.2 in Tan (1991) states that every even ring in a stable partition can be broken into pairs of mutually acceptable agents preserving stability.

Let P be a stable partition. We denote by $\mathcal{A}_P$ the set formed by all the P-stable matchings and those matchings that dominate them. Our first theorem establishes that an absorbing set is one of these sets $\mathcal{A}_P$. As it is shown in Example 2, however, not every set $\mathcal{A}_P$ need to be an absorbing set.

Theorem 1 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. If $\mathcal{A}$ is an absorbing set then $\mathcal{A} = \mathcal{A}_P$ for some stable partition P.*

Proof. First, we prove that there exists a P-stable matching $\overline{\mu}$ such that $\overline{\mu} \in \mathcal{A}$. Let μ be an arbitrary matching of $\mathcal{A}$. If μ is a P-stable matching for some stable partition P then $\overline{\mu} = \mu$ and we are done. Otherwise, by Theorem 1 in Inarra *et al.* (2007), there exists a P-stable matching $\overline{\mu}$ such that $\overline{\mu} R^T \mu$ and by Condition (ii) of Definition 2 we have $\overline{\mu} \in \mathcal{A}$.

Now, we prove that $\mathcal{A} = \mathcal{A}_P$. By Lemma 2, we have $\mathcal{A}_P = \{\overline{\mu}\} \cup \{\mu \in \mathcal{M}\colon \mu R^T \overline{\mu}\}$.

($\subseteq$): Let $\mu \in \mathcal{A}$. We must show that $\mu \in \mathcal{A}_P$. If $\mu = \overline{\mu}$ and given that $\overline{\mu} \in \mathcal{A}_P$ we are done. Suppose that $\mu \neq \overline{\mu}$. Since $\overline{\mu} \in \mathcal{A}$, by Condition (i) of Definition 2, we have $\mu R^T \overline{\mu}$. Hence $\mu \in \mathcal{A}_P$ as desired.

($\supseteq$): Let $\mu \in \mathcal{A}_P$. We must show that $\mu \in \mathcal{A}$. If $\mu = \overline{\mu}$ since $\overline{\mu} \in \mathcal{A}$ we are done. If $\mu \neq \overline{\mu}$ then $\mu R^T \overline{\mu}$. As $\overline{\mu} \in \mathcal{A}$, by Condition (ii) of Definition 2 it follows that $\mu \in \mathcal{A}$. ■

We have seen that P-stable matchings are derived from stable partitions. On the other hand, every absorbing set contains some P-stable matching. Thus, informally, stable partitions may be interpreted as those structures which determine absorbing sets.

Using the previous theorem we derive the following interesting result.

Corollary 2 *If the roommate problem $(N, (\succ_x)_{x \in N})$ is solvable then $\mathcal{A}$ is an absorbing set if and only if $\mathcal{A} = \{\mu\}$ for some stable matching μ.*

Proof. If $\mathcal{A}$ is an absorbing set then, by Theorem 1, $\mathcal{A} = \mathcal{A}_P$ for some stable partition P. Now, as the roommate problem is solvable, by Remark 1 the stable partition P does not contain any odd ring. Hence there exists a unique P-stable matching μ which is stable by stability of P. Then $\mathcal{A}_P = \{\mu\}$ and therefore $\mathcal{A} = \{\mu\}$. Conversely, if $\mathcal{A} = \{\mu\}$ for some stable matching μ, then $\mathcal{A}$ satisfies Conditions (i) and (ii) of Definition 2. Hence $\mathcal{A}$ is an absorbing set. ■

As a result of the corollary above we have that the union of all absorbing sets coincides with the core. Thus absorbing sets may be considered as a generalization of this solution concept in roommate problems with strict preferences.

To clarify the notion of absorbing sets we consider the following numerical example which will also be used elsewhere in the paper to illustrate other results.

EXAMPLE 2 Consider the following 10-agent roommate problem:

$$2 \succ_1 3 \succ_1 4 \succ_1 5 \succ_1 6 \succ_1 7 \succ_1 8 \succ_1 9 \succ_1 1 \succ_1 10$$
$$3 \succ_2 1 \succ_2 4 \succ_2 5 \succ_2 6 \succ_2 7 \succ_2 8 \succ_2 9 \succ_2 2 \succ_2 10$$
$$1 \succ_3 2 \succ_3 4 \succ_3 5 \succ_3 6 \succ_3 7 \succ_3 8 \succ_3 9 \succ_3 3 \succ_3 10$$
$$7 \succ_4 8 \succ_4 9 \succ_4 5 \succ_4 6 \succ_4 1 \succ_4 2 \succ_4 3 \succ_4 4 \succ_4 10$$
$$8 \succ_5 9 \succ_5 7 \succ_5 4 \succ_5 6 \succ_5 5 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 10$$
$$9 \succ_6 7 \succ_6 8 \succ_6 4 \succ_6 5 \succ_6 6 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 10$$
$$5 \succ_7 6 \succ_7 1 \succ_7 4 \succ_7 9 \succ_7 8 \succ_7 7 \succ_7 2 \succ_7 3 \succ_7 10$$
$$6 \succ_8 4 \succ_8 5 \succ_8 7 \succ_8 9 \succ_8 8 \succ_8 1 \succ_8 2 \succ_8 3 \succ_8 10$$
$$4 \succ_9 5 \succ_9 6 \succ_9 7 \succ_9 8 \succ_9 9 \succ_9 1 \succ_9 2 \succ_9 3 \succ_9 10$$
$$10 \succ_{10} 1 \succ_{10} \ldots$$

In this example there are three stable partitions: $P_1 = \{\{1,2,3\},\{4,7\},\{5,8\},\{6,9\},\{10\}\}$, $P_2 = \{\{1,2,3\},\{4,8\},\{5,9\},\{6,7\},\{10\}\}$ and $P_3 = \{\{1,2,3\},\{4,9\},\{5,7\},\{6,8\},\{10\}\}$. Consider stable partition P_2. The associated P_2-stable matchings are: $\mu_1 =[\{1\},\{2,3\},\{4,8\},\{5,9\},\{6,7\},\{10\}]$, $\mu_2 =[\{2\},\{1,3\},\{4,8\},\{5,9\},\{6,7\},\{10\}]$ and $\mu_3=[\{3\},\{1,2\},\{4,8\},\{5,9\},\{6,7\},\{10\}]$ and the set $\mathcal{A}_{P_2} = \{\mu_1,\mu_2,\mu_3\}$. Notice that any of these matchings dominates each other but they are not directly dominated by any matching outside $\mathcal{A}_{P_2}$. Therefore $\mathcal{A}_{P_2}$ is an absorbing set. In addition, matching $\mu_1 = [\{1\},\{2,3\},\{4,8\},\{5,9\},\{6,7\},\{10\}]$ can be derived from the P_1-stable matching $\mu = [\{1\},\{2,3\},\{4,7\},\{5,8\},\{6,9\},\{10\}]$ by satisfying the following sequence of blocking pairs: $\{1,7\}$, $\{4,8\}$, $\{5,9\}$, $\{6,7\}$. Hence μ_1 belongs to $\mathcal{A}_{P_1}$. It is easy to verify, however, that μ does not dominate μ_1. Thus $\mathcal{A}_{P_1}$ is not an absorbing set since it does not satisfy Condition (i) of Definition 2.

4 Characterization of absorbing sets

Given a roommate problem with strict preferences, the main purpose of this section is to characterize its absorbing sets, which, as we have already asserted, are determined by some stable partitions. To do this we introduce an iterative process which proves to be efficient for this task. In particular, given a stable partition P (hence the set $\mathcal{A}_P$ is immediately defined) the process determines the set D_P, which is formed by those agents that block some matching in $\mathcal{A}_P$. The set S_P formed by those agents that do not block any matching in $\mathcal{A}_P$ happens to play a crucial role in the characterization of absorbing sets. Specifically, considering the set of stable partitions restricted to these sets of non-blocking agents, only the maximal ones determine the absorbing sets. As a consequence of this result, the number of these sets is obtained immediately. Furthermore, the characterization provided allows us to put forward an interesting feature of the matchings of an absorbing set. All these results are illustrated by using Example 2.

Formally, given a stable partition P, let D_P denote the set of agents that block some matching in $\mathcal{A}_P$ and let $S_P = N \backslash D_P$. The set D_P can be determined by an *iterative process* in a finite number of steps. In order to do it, we define inductively a sequence of sets $\langle D_t \rangle_{t=0}^{\infty}$ as follows:

(i) for $t = 0$, D_0 is the union of all odd rings of P.

(ii) for $t \geq 1$, $D_t = D_{t-1} \cup B_t$ where $B_t = \{b_1(t), ..., b_{l_t}(t)\} \in P$ ($l_t =$ 1 or 2), $B_t \nsubseteq D_{t-1}$, and there is a set $A_t = \{a_1(t), ..., a_{k_t}(t)\} \in P$ such that $A_t \subseteq D_{t-1}$ and

$$b_j(t) \succ_{a_i(t)} a_i(t) \text{ and } a_i(t) \succ_{b_j(t)} b_{j-1}(t), \qquad [2]$$

for some $i \in \{1, ..., k_t\}$ and $j \in \{1, ..., l_t\}$.[11]

Given that P contains a finite number of sets, then $D_t = D_{t-1}$ for some t. Let r be the minimum number such that $D_{r+1} = D_r$. Then, by Lemma 3, $D_r = D_P$. Note that, for any set $A \in P$, either $A \subseteq D_P$ or $A \subseteq S_P$. Let

[11] If such set does not exist then $D_t = D_{t-1}$.

$P\mid_{D_P}=\{A\in P : A\subseteq D_P\}$ and $P\mid_{S_P}=\{A\in P : A\subseteq S_P\}$. Then $P\mid_{D_P}$ and $P\mid_{S_P}$ are stable partitions of the sets D_P and S_P respectively.

Before proceeding, we need an additional definition. Let $\mathcal{P}=\{P\mid_{S_P}: P$ is a stable partition$\}$. Given a stable partition P, we say that $P\mid_{S_P}$ is maximal in $\mathcal{P}$ if there is not a stable partition P' such that $P\mid_{S_P}\subset P'\mid_{S_{P'}}$. Our next result gives a characterization of the absorbing sets in terms of stable partitions.

Theorem 3 *Let $(N,(\succ_x)_{x\in N})$ be a roommate problem. $\mathcal{A}$ is an absorbing set if and only if $\mathcal{A}=\mathcal{A}_P$ for some stable partition P such that $P\mid_{S_P}$ is maximal in $\mathcal{P}$.*

Proof. ($\Longrightarrow$): Let $\mathcal{A}$ be an absorbing set. Then, by Theorem 1, $\mathcal{A}=\mathcal{A}_P$ for some stable partition P. We prove that $P\mid_{S_P}$ is maximal in $\mathcal{P}$. Assume that $P\mid_{S_P}$ is not maximal, *i.e.*, there exists a stable partition P' such that $P\mid_{S_P}\subset P'\mid_{S_{P'}}$. Let μ and μ' be a P-stable matching and a P'-stable matching respectively. Thus, by Lemma 6, $\mu' R^T\mu$. Now, since $\mu\in\mathcal{A}_P$ and $\mathcal{A}=\mathcal{A}_P$ we have $\mu\in\mathcal{A}$. Hence, by Condition (ii) of Definition 2 $\mu'\in\mathcal{A}$. But then, by Condition (i), $\mu R^T\mu'$ and therefore, by Lemma 6, $P'\mid_{S_{P'}}\subseteq P\mid_{S_P}$, contradicting that $P\mid_{S_P}\subset P'\mid_{S_{P'}}$.
($\Longleftarrow$): Let P be a stable partition such that $P\mid_{S_P}$ is maximal in $\mathcal{P}$. We prove that $\mathcal{A}_P$ is an absorbing set, *i.e.*, $\mathcal{A}_P$ satisfies Conditions (i) and (ii) of Definition 2. By Lemma 2, $\mathcal{A}_P=\{\overline{\mu}\}\cup\left\{\mu\in\mathcal{M} : \mu R^T\overline{\mu}\right\}$ where $\overline{\mu}$ is a P-stable matching. Let $\mu\in\mathcal{A}_P$. If there exists $\mu'\in\mathcal{M}$ such that $\mu' R\mu$ then $\mu' R^T\overline{\mu}$. Hence $\mu'\in\mathcal{A}_P$ and Condition (ii) follows.
Now we show that $\mathcal{A}_P$ satisfies Condition (i). It suffices to prove that $\overline{\mu} R^T\mu$ for all $\mu\in\mathcal{A}_P$ such that $\mu\neq\overline{\mu}$. If μ is not a P'-stable matching for any stable partition P', by Theorem 1 in Inarra *et al.* (2007), there exists a P'-stable matching μ' such that $\mu' R^T\mu$. Since $\mu R^T\overline{\mu}$ we have $\mu' R^T\overline{\mu}$ (if μ is a P'-stable matching for some stable partition P' then $\mu'=\mu$ can be considered.) Thus, by Lemma 6, $P\mid_{S_P}\subseteq P'\mid_{S_{P'}}$ and since $P\mid_{S_P}$ is maximal in $\mathcal{P}$, it follows that $P\mid_{S_P}=P'\mid_{S_{P'}}$. But then $\overline{\mu} R^T\mu'$ and since $\mu' R^T\mu$ we conclude that $\overline{\mu} R^T\mu$ as desired. ∎

As an immediate consequence of the previous theorem and Lemma 7, the number of absorbing sets in a roommate problem is straightforward to determine. This corollary implies a necessary and sufficient condition for the uniqueness of the solution.

Corollary 4 *Let* $(N, (\succ_x)_{x\in N})$ *be a roommate problem the number of absorbing sets is equal to the number of distinct maximal partitions of* $\mathcal{P}$.

Let $\mathcal{A}$ be an absorbing set. $D_{\mathcal{A}}$ denotes the set of agents blocking some matching in $\mathcal{A}$, while the set $S_{\mathcal{A}} = N\backslash D_{\mathcal{A}}$ is formed by those agents that do not block any matching in $\mathcal{A}$. The following corollary, derived easily from Theorem 3 and Lemma 5, shows that all matchings in an absorbing set $\mathcal{A}$ have some identical pairings formed by the agents in $S_{\mathcal{A}}$ which, in addition, are stable among them.

Corollary 5 *Let* $(N, (\succ_x)_{x\in N})$ *be a roommate problem. For any absorbing set* $\mathcal{A}$ *such that* $S_{\mathcal{A}} \neq \emptyset$ *the following conditions hold:*
(i) For any $\mu \in \mathcal{A}$, $\mu(S_{\mathcal{A}}) = S_{\mathcal{A}}$ *and* $\mu\mid_{S_{\mathcal{A}}}$ *is stable for* $(S_{\mathcal{A}}, (\succ_x)_{x\in S_{\mathcal{A}}})$.
(ii) For any μ, $\mu' \in \mathcal{A}$, $\mu\mid_{S_{\mathcal{A}}} = \mu'\mid_{S_{\mathcal{A}}}$.

To illustrate the iterative process and results described above, consider again Example 2. We apply the process to the following stable partition $P_1 = \{\{1,2,3\},\{4,7\},\{5,8\},\{6,9\},\{10\}\}$. Note that P_1 contains a unique odd ring. Then $D_0 = \{1,2,3\}$. Let $B_1 = \{4,7\}$ and $A_1 = \{1,2,3\}$. Since $7 \succ_1 1$ and $1 \succ_7 4$, then $D_1 = D_0 \cup B_1 = \{1,2,3,4,7\}$. Consider now the sets $B_2 = \{5,8\}$ and $A_2 = \{4,7\}$. As $8 \succ_4 4$ and $4 \succ_8 5$, then $D_2 = D_1 \cup B_2 = \{1,2,3,4,7,5,8\}$. Finally, let $B_3 = \{6,9\}$ and $A_3 = \{5,8\}$. Since $9 \succ_5 5$ and $5 \succ_9 6$, then $D_3 = D_2 \cup B_3 = \{1,2,3,4,7,5,8,6,9\}$ and the process finishes. Hence $D_{P_1} = D_3$. By repeating the same process to the remaining stable partitions, we obtain $D_{P_2} = D_{P_3} = \{1,2,3\}$. Hence $P_1\mid_{D_{P_1}} = \{\{1,2,3\},\{4,7\},\{5,8\},\{6,9\}\}$ and $P_2\mid_{D_{P_2}} = P_3\mid_{D_{P_3}} = \{\{1,2,3\}\}$, and $P_1\mid_{S_{P_1}} = \{\{10\}\}$, $P_2\mid_{S_{P_2}} = \{\{4,8\},\{5,9\},\{6,7\},\{10\}\}$ and $P_3\mid_{S_{P_3}} = \{\{4,9\},\{5,7\},\{6,8\},\{10\}\}$.

Notice that $P_2\mid_{S_{P_2}}$ and $P_3\mid_{S_{P_3}}$ are the maximal partitions of $\mathcal{P}$. Therefore, by Theorem 3 and Corollary 4, this roommate problem has exactly two absorbing sets $\mathcal{A}$ and $\mathcal{A}'$ where $\mathcal{A} = \mathcal{A}_{P_2}$ and $\mathcal{A}' = \mathcal{A}_{P_3}$.

Regarding Corollary 5, consider the absorbing set $\mathcal{A} = \mathcal{A}_{P_2} = \{\mu_1, \mu_2, \mu_3\}$, formed by the following matchings: $\mu_1 = [\{1\},\{2,3\},\{4,8\},\{5,9\},\{6,7\},\{10\}]$, $\mu_2 = [\{2\},\{1,3\},\{4,8\},\{5,9\},\{6,7\},\{10\}]$ and $\mu_3 = [\{3\},\{1,2\},\{4,8\},\{5,9\},\{6,7\},\{10\}]$. In this case, $D_{\mathcal{A}} = \{1,2,3\}$ and $S_{\mathcal{A}} = \{4,5,6,7,8,9,10\}$. We observe that $\mu_1\mid_{S_{\mathcal{A}}} = \mu_2\mid_{S_{\mathcal{A}}} = \mu_3\mid_{S_{\mathcal{A}}}$ being the matching $[\{4,8\},\{5,9\},\{6,7\},\{10\}]$ stable for the roommate problem $(S_{\mathcal{A}}, (\succ_x)_{x\in S_{\mathcal{A}}})$.

5 Structure of absorbing sets

In this section we investigate the structure of the absorbing sets in case of multiplicity. We also observe a property of stability verified by all matchings of the absorbing sets.

Given an absorbing set $\mathcal{A}$ such that $D_{\mathcal{A}} \neq \emptyset$, let $\mathcal{A} \mid_{D_{\mathcal{A}}} = \{\mu \mid_{D_{\mathcal{A}}} : \mu \in \mathcal{A}\}$. Analogously, if $S_{\mathcal{A}} \neq \emptyset$, let $\mathcal{A} \mid_{S_{\mathcal{A}}} = \{\mu \mid_{S_{\mathcal{A}}} : \mu \in \mathcal{A}\}$

Theorem 6 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. For any two absorbing sets $\mathcal{A}$ and $\mathcal{A}'$, the following conditions hold:*
(i) $D_{\mathcal{A}} = D_{\mathcal{A}'}$ and $S_{\mathcal{A}} = S_{\mathcal{A}'}$.
(ii) $\mathcal{A} \mid_{D_{\mathcal{A}}} = \mathcal{A}' \mid_{D_{\mathcal{A}'}}$.
(iii) $\mathcal{A} \mid_{S_{\mathcal{A}}}$ and $\mathcal{A}' \mid_{S_{\mathcal{A}'}}$ are singletons consisting of a stable matching in $(S, (\succ_x)_{x \in S})$, where $S = S_{\mathcal{A}} = S_{\mathcal{A}'}$.

Proof. Let $\mathcal{A}$ and $\mathcal{A}'$ be two absorbing sets. Then, by Theorem 3, there exist some stable partitions P and P' such that $\mathcal{A} = \mathcal{A}_P$, $\mathcal{A}' = \mathcal{A}_{P'}$ where $P \mid_{S_P}$ and $P' \mid_{S_{P'}}$ are maximal in $\mathcal{P}$.
(i) Since $S_{\mathcal{A}} = S_P$ and $S_{\mathcal{A}'} = S_{P'}$ and, by Lemma 9, $S_P = S_{P'}$, then $S_{\mathcal{A}} = S_{\mathcal{A}'}$. Therefore $D_{\mathcal{A}} = D_{\mathcal{A}'}$.
(ii) It is very easy to verify that $\mathcal{A} \mid_{D_{\mathcal{A}}}$ and $\mathcal{A}' \mid_{D_{\mathcal{A}'}}$ are absorbing sets in $(D, (\succ_x)_{x \in D})$ where $D = D_{\mathcal{A}} = D_{\mathcal{A}'}$ such that $\mathcal{A} \mid_{D_{\mathcal{A}}} = \mathcal{A}_{P|_{D_P}}$ and $\mathcal{A}' \mid_{D_{\mathcal{A}'}} = \mathcal{A}_{P'|_{D_{P'}}}$. Since $S_{P|_{D_P}} = S_{P'|_{D_{P'}}} = \emptyset$, from Lemma 7, we conclude that $\mathcal{A} \mid_{D_{\mathcal{A}}} = \mathcal{A}' \mid_{D_{\mathcal{A}'}}$.
(iii) It follows directly from Corollary 5. ■

The three conditions of the previous theorem provide all absorbing sets of a roommate problem with strict preferences with a similar structure. Following the metaphor described in the introduction we can say that absorbing sets resemble identical stable constellations, as illustrated in Figure 2.

To explain numerically this last result, consider the two absorbing sets of Example 2: $\mathcal{A} = \mathcal{A}_{P_2}$ and $\mathcal{A}' = \mathcal{A}_{P_3}$. Since $D_{\mathcal{A}} = D_{\mathcal{A}'} = \{1, 2, 3\}$ we have $\mathcal{A} = \{\mu_1, \mu_2, \mu_3\}$ and $\mathcal{A}' = \{\mu_1', \mu_2', \mu_3'\}$ where μ_1, μ_2, μ_3 are the P_2-stable matchings and μ_1', μ_2', μ_3' are the P_3-stable matchings (see Figure 2). Additionally, $\mathcal{A} \mid_{D_{\mathcal{A}}} = \{\mu_1 \mid_{D_{\mathcal{A}}}, \mu_2 \mid_{D_{\mathcal{A}}}, \mu_3 \mid_{D_{\mathcal{A}}}\}$ and $\mathcal{A}' \mid_{D_{\mathcal{A}'}} = \left\{\mu_1' \mid_{D_{\mathcal{A}'}}, \mu_2' \mid_{D_{\mathcal{A}'}}, \mu_3' \mid_{D_{\mathcal{A}'}}\right\}$ where $\mu_1 \mid_{D_{\mathcal{A}}} = \mu_1' \mid_{D_{\mathcal{A}'}} = [\{1\}, \{2, 3\}]$, $\mu_2 \mid_{D_{\mathcal{A}}} = \mu_2' \mid_{D_{\mathcal{A}'}} = [\{1, 3\}, \{2\}]$ and

$\mu_3 \mid_{D_{\mathcal{A}}} = \mu_3' \mid_{D_{\mathcal{A}'}} = [\{1,2\}, \{3\}]$. Furthermore, $\mathcal{A} \mid_{S_{\mathcal{A}}}$ and $\mathcal{A}' \mid_{S_{\mathcal{A}'}}$ are respectively singletons consisting of the stable matchings $\mu = [\{4,8\}, \{5,9\}, \{6,7\}, \{10\}]$ and $\mu' = [\{4,9\}, \{5,7\}, \{6,8\}, \{10\}]$ in $(S, (\succ_x)_{x \in S})$ where $S = \{4,5,6,7,8,9,10\}$.

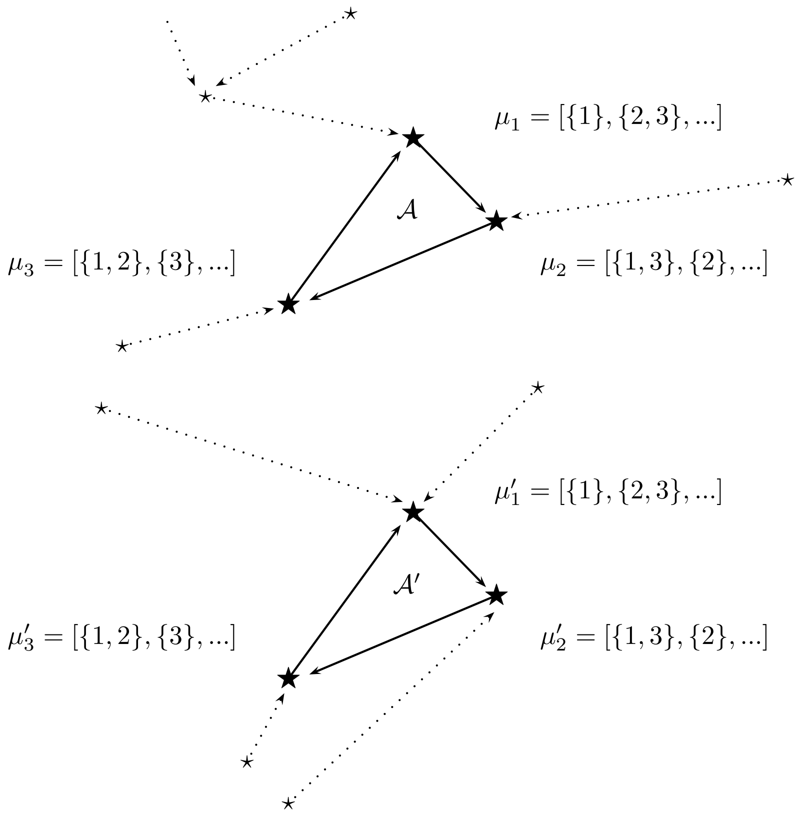

Figure 2.- The two absorbing sets of the roommate problem in Example 2.

To conclude this section, let us introduce one more definition to explain an interesting property of stability verified by the matchings in absorbing sets. Given a matching μ we say that a pair of agents $\{x, y\}$ matched under μ is *strongly stable* if, for any matching $\mu' \in \mathcal{M}$ such that $\mu' R^T \mu$, agents x and y are also matched under μ'. In particular, from Corollary 5, we know that for every matching μ in an absorbing set $\mathcal{A}$ those pairs matched under $\mu \mid_{S_{\mathcal{A}}}$ are strongly stable. Furthermore, Theorem 6 (iii) shows that all matchings in all absorbing sets have the same number of strongly stable pairs. These two results jointly with the outer stability property guarantee that all matchings in all absorbing sets have the greatest number of strongly stable pairs among the matchings in $\mathcal{M}$.

6 Some concluding comments

In this paper we have claimed that the matchings of absorbing sets are predictable in the class of roommate problems with strict preferences. Our results, however, leave three interesting questions opened for analysis.

1. Although the solution proposed seems to be very good at ruling out some matchings with confidence, it may not be so efficient at selecting the most "reasonable" ones. Thus, when an absorbing set has multiple matchings, a discriminating criterion should be used if a more refined outcome is desired. Let us discuss this briefly with the following numerical example.

EXAMPLE 3 Consider the following 4-agent roommate problem:

$$2 \succ_1 3 \succ_1 4 \succ_1 1$$
$$3 \succ_2 1 \succ_2 4 \succ_2 2$$
$$1 \succ_3 2 \succ_3 4 \succ_3 3$$
$$1 \succ_4 2 \succ_4 3 \succ_4 4$$

In this instance, the unique absorbing set is formed by the following matchings: $\mu_1 = [\{1,2\},\{3,4\}]$, $\mu_2 = [\{1\},\{2,3\},\{4\}]$, $\mu_3 = [\{1,4\},\{2,3\}]$, $\mu_4 = [\{2\},\{1,3\},\{4\}]$, $\mu_5 = [\{1,3\},\{2,4\}]$ and $\mu_6 = [\{3\},\{1,2\},\{4\}]$. The application of a standard criterion such as Pareto optimality[12] over the unique absorbing set gives the matchings μ_1, μ_3 and μ_5.

2. A natural extension of the approach followed in this paper is the study of absorbing sets as solution for roommate problems with weak preferences. In this case, by contrast with roommate problems with strict preferences, when the core is nonempty our solution concept can select more matchings than just those in the core. The following example given by Chung (2000) shows this.

EXAMPLE 4 Consider the following 4-agent roommate problem:

$$4 \succ_1 2 \succ_1 3 \succ_1 1$$
$$3 \succ_2 1 \succ_2 2 \succ_2 4$$
$$1 \succ_3 2 \succ_3 4 \succ_3 3$$
$$1 \sim_4 4 \succ_4 \ldots$$

[12]Informally, a matching μ is said to be Pareto optimal if there is no other matching μ' such that some agent is better off in μ' than in μ and no agent is worse off in μ' than in μ.

In this instance, there are two absorbing sets: $\mathcal{A} = \{[\{1,2\},\{3,4\}]\}$ and $\mathcal{A}' = \{[\{1\},\{2,3\},\{4\}], [\{2\},\{1,3\},\{4\}], [\{3\},\{1,2\},\{4\}]\}$.

3. Another potential extension may be the application of our approach to more general choice problems such as hedonic games[13](See Dreze and Greenberg (1980)), or network formation models (see, for instance, Jackson and Wolinsky (1996)). An arbitrary hedonic game can be associated with an abstract system where the set of alternatives is the set of all coalitional partitions that can be formed by the agents involved in the problem. Analogously, for network formation models, Page, Wooders and Kamat (2005) define abstract systems associated with these problems where the set of alternatives is formed by a set of networks. In these two specific systems the binary relation represents transitions from one alternative to another and, as in our paper, absorbing sets could be proposed as a solution for them whenever their corresponding cores are empty.

[13]Diamantoudi and Xue (2003) and Barberá and Gerber (2003) have pointed out that roommate problems can be considered as a special case of hedonic games.

7 Appendix

Lemma 1 *Given a stable partition P. For any two distinct P-stable matchings μ and μ', $\mu' R^T \mu$.*

Proof. If P does not contain any odd ring then there exists a unique P-stable matching and we are done. Suppose that P contains some odd ring. Let $A_1, ..., A_r$ be the odd rings of P and $T = \bigcup\limits_{i=1}^{r} A_i$.

Set $A_1 = \{a_1, ..., a_k\}$. As A_1 is a ring then

$$a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i, \qquad [3]$$

for all $i = \{1, ..., k\}$. By Definition 1, since μ and μ' are P-stable matchings, there are two agents a_l, $a_s \in A_1$ such that $\mu(a_l) = a_l$ and $\mu'(a_s) = a_s$. Now, since $\mu(a_l) = a_l$ and $\mu(a_{l-1}) = a_{l-2}$, by condition [3], $\{a_l, a_{l-1}\}$ blocks μ, inducing a P-stable matching μ_1 for which $\mu(a_{l-2}) = a_{l-2}$. Repeating the process, we obtain a sequence of P-stable matchings μ_0, μ_1,..., μ_i, ... as follows:
(i) $\mu_0 = \mu$.
(ii) For $i \geq 1$, μ_i is the P-stable matching obtained from μ_{i-1} by satisfying the blocking pair $\{a_{l-2(i-1)}, a_{l-2(i-1)-1}\}$.

Let $m_1 \in \{1, ..., k\}$ such that $a_{l-2m_1} = a_s$. Then $\mu = \mu_0, \mu_1, ..., \mu_{m_1}$ is a finite sequence of P-stable matchings such that, for all $i \in \{1, ..., m_1\}$, $\mu_i R \mu_{i-1}$ and $\mu_{m_1} \mid_{A_1} = \mu' \mid_{A_1}$.
Consider now the ring A_2. Reasoning in the same way as before, for μ_{m_1} and μ' we obtain a finite sequence of P-stable matchings μ_{m_1}, μ_{m_1+1}, ..., $\mu_{m_1+m_2}$ such that, for all $i \in \{m_1+1, ..., m_1+m_2\}$, $\mu_i R \mu_{i-1}$ and $\mu_{m_1+m_2} \mid_{(A_1 \cup A_2)} = \mu' \mid_{(A_1 \cup A_2)}$.
Repeating the same procedure to the remaining odd rings, eventually we obtain a finite sequence of P-stable matchings $\mu = \mu_0, \mu_1, ..., \mu_m$, where $m = \sum\limits_{i=1}^{r} m_i$, and such that, for all $i \in \{1, ..., m\}$, $\mu_i R \mu_{i-1}$ and $\mu_m \mid_T = \mu' \mid_T$. Now, since $\mu_m \mid_{(N \setminus T)} = \mu' \mid_{(N \setminus T)}$, then $\mu_m = \mu'$ and the proof is complete. ■

Lemma 2 *Let P be a stable partition and $\overline{\mu}$ be a P-stable matching. Then, $\mathcal{A}_P = \{\overline{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\}$.*

Proof. It follows directly from the definition of $\mathcal{A}_P$ and Lemma 1. ■

Lemma 3 $Dr = D_P$

Proof. ($\subseteq$): First we prove that $D_0 \subseteq D_P$. Let $A = \{a_1, ..., a_k\}$ be an odd ring of P. We must show that $a_i \in D_P$ for all $i \in \{1, ..., k\}$. Consider the P-stable matching μ such that $\mu(a_i) = a_i$. As $\mu(a_{i-1}) = a_{i-2}$ and $a_i \succ_{a_{i-1}} a_{i-2}$ and $a_{i-1} \succ_{a_i} a_i$ then $\{a_i, a_{i-1}\}$ is a blocking pair of μ and therefore $a_i \in D_P$.
Now we prove that, for each $t \in \{1, ..., r\}$, the following conditions hold:
a) $B_t \subseteq D_P$.
b) There exists a matching $\mu_t \in \mathcal{A}_P$ such that

$$\mu_t(x) = \begin{cases} x & \text{if } x \in B_t \\ \\ \overline{\mu}_t(x) & \text{otherwise,} \end{cases}$$

where $\overline{\mu}_t$ is a P-stable matching.

We argue by induction on t.
If $t = 1$, we have $A_1 = \{a_1(1), ..., a_{k_1}(1)\}$ and $B_1 = \{b_1(1), ..., b_{l_1}(1)\}$.[14] Since $A_1 \subseteq D_0$ then A_1 is an odd ring of P. Consider the P-stable matching μ such that $\mu(a_i) = a_i$. Since $\mu(b_j) = b_{j-1}$, by [2], we have $b_j \succ_{a_i} \mu(a_i)$ and $a_i \succ_{b_j} \mu(b_j)$. Hence $\{a_i, b_j\}$ is a blocking pair of μ and therefore $b_j \in D_P$. Let μ' be the matching obtained from μ by satisfying this blocking pair. Now, since $a_i \succ_{b_j} b_{j-1}$, by the stability of P, $a_{i-1} \succ_{a_i} b_j$. As $\mu'(a_{i-1}) = a_{i-2}$ and $a_i \succ_{a_{i-1}} a_{i-2}$, then $\{a_i, a_{i-1}\}$ is a blocking pair of μ' which induces a matching $\widetilde{\mu} \in \mathcal{A}_P$ such that $\widetilde{\mu}(x) = x$ if $x \in B_1$ and $\widetilde{\mu}(x) = \overline{\mu}(x)$ otherwise, where $\overline{\mu}$ is the P-stable matching such that $\overline{\mu}(a_{i-2}) = a_{i-2}$. Let $\mu_1 = \widetilde{\mu}$ and $\overline{\mu}_1 = \overline{\mu}$. Then, if $l = 1$ we are done. If $l = 2$, to complete the proof we need to show that $b_{j-1} \in D_P$. But this is trivial because as agents b_j and b_{j-1} are alone under μ_1, $\{b_j, b_{j-1}\}$ is a blocking pair of μ_1 and therefore $b_{j-1} \in D_P$.
Now assume that $t \geq 2$. We consider two cases:
Case 1. A_t is an odd ring. Reasoning in the same way as before for the sets A_t and B_t, the result follows.
Case 2. A_t is not an odd ring. Then $A_t = B_s$ for some $s < t$. By the inductive hypothesis, there exists $\mu_s \in \mathcal{A}_P$ such that $\mu_s(x) = x$ if $x \in B_s$ and $\mu_s(x) = \overline{\mu}_s(x)$ otherwise, where $\overline{\mu}_s$ is a P-stable matching. As $\mu_s(a_i) = a_i$ and $\mu_s(b_j) = \overline{\mu}_s(b_j) = b_{j-1}$, by [2], we have $b_j \succ_{a_i} \mu_s(a_i)$ and $a_i \succ_{b_j} \mu_s(b_j)$. Hence $\{a_i, b_j\}$ is a blocking pair of μ_s and therefore $b_j \in D_P$. Let μ'_s be the matching obtained from μ_s by satisfying this blocking pair. Since $a_i \succ_{b_j} b_{j-1}$, by the

[14] Abusing notation, we write a_i and b_j instead of $a_i(t)$ and $b_j(t)$ for all t.

stability of P, $a_{i-1} \succ_{a_i} b_j$ and as $\mu'_s(a_{i-1}) = a_{i-1}$ then $\{a_i, a_{i-1}\}$ is a blocking pair of μ'_s, which induces a matching $\widetilde{\mu}_s \in \mathcal{A}_P$ such that $\widetilde{\mu}_s(x) = x$ if $x \in B_t$ and $\widetilde{\mu}_s(x) = \overline{\mu}_s(x)$ otherwise. Then, choosing $\mu_t = \widetilde{\mu}_s$ and $\overline{\mu}_t = \overline{\mu}_s$ and reasoning in the same way as before, the result follows.

Finally, as $D_0 \subseteq D_P$ and, for each $t \in \{1, ..., r\}$, $B_t \subseteq D_P$ we conclude that $D_r \subseteq D_P$.

$(\supseteq)$: We prove that D_r contains all the blocking pairs of the matchings in $\mathcal{A}_P$. Now, by Lemma 2, $\mathcal{A}_P = \{\overline{\mu}\} \cup \left\{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\right\}$ where $\overline{\mu}$ is a P-stable matching. Hence it suffices to show that, for any finite sequence of matchings $\overline{\mu} = \mu_0, \mu_1, ..., \mu_m$ such that, for all $s \in \{1, ..., m\}$, μ_s is obtained from μ_{s-1} by satisfying the blocking pair $\{x_s, y_s\}$, therefore $\{x_s, y_s\} \subseteq D_r$.

We argue by induction on s.

If $s = 1$, then $\{x_1, y_1\}$ is a blocking pair of $\overline{\mu}$. Let $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_l\}$ be the sets of P such that $x_1 \in A$ and $y_1 \in B$. Then $x_1 = a_i$ and $y_1 = b_j$ for some i and j. As $\{x_1, y_1\}$ blocks $\overline{\mu}$ we have $y_1 \succ_{x_1} \overline{\mu}(x_1)$ and $x_1 \succ_{y_1} \overline{\mu}(y_1)$, *i.e.*, $b_j \succ_{a_i} \overline{\mu}(a_i)$ and $a_i \succ_{b_j} \overline{\mu}(b_j)$. Suppose, by contradiction, that $\{a_i, b_j\} \nsubseteq D_r$. If $\{a_i, b_j\} \cap D_r = \emptyset$ then A and B are not odd rings hence, by Definition 1, we have $\overline{\mu}(a_i) = a_{i-1}$ and $\overline{\mu}(b_j) = b_{j-1}$. But then $b_j \succ_{a_i} a_{i-1}$ and $a_i \succ_{b_j} b_{j-1}$, contradicting the stability of P. If $a_i \in D_r$ and $b_j \notin D_r$ since $\overline{\mu}(b_j) = b_{j-1}$ we have $b_j \succ_{a_i} a_i$ and $a_i \succ_{b_j} b_{j-1}$. Hence, by [2], $b_j \in D_r$, and we reach a contradiction. If we assume that $a_i \notin D_r$ and $b_j \in D_r$, a similar contradiction is reached.

Suppose now that $s \geq 2$. Then $\{x_s, y_s\}$ blocks μ_{s-1}. Consider the sets $A' = \{a'_1, ..., a'_{k'}\}$ and $B' = \{b'_1, ..., b'_{l'}\}$ of P such that $x_s = a'_i$ and $y_s = b'_j$ for some i and j. First we prove that if $x_s \notin D_r$ then $\mu_{s-1}(x_s) = \overline{\mu}(x_s)$. We argue by contradiction. If $\mu_{s-1}(x_s) \neq \overline{\mu}(x_s)$ we have $\{x_s, \overline{\mu}(x_s)\} \cap \{x_i, y_i\} \neq \emptyset$ for some $i < s$. By the inductive hypothesis, $\{x_i, y_i\} \subseteq D_r$ hence $\{x_s, \overline{\mu}(x_s)\} \cap D_r \neq \emptyset$ and since $x_s \notin D_r$, it follows that $\overline{\mu}(x_s) \in D_r$. But $\overline{\mu}(x_s) \in A'$ so $A' \subseteq D_r$ and therefore $x_s \in D_r$, which contradicts that $x_s \notin D_r$. In a similar manner, the reader may see that if $y_s \notin D_r$ then $\mu_{s-1}(y_s) = \overline{\mu}(y_s)$. Now, reasoning in the same way as before for $\{x_s, y_s\}$ the result follows. ■

Lemma 4 *Let P be a stable partition. Then, there exists $\mu^* \in \mathcal{A}_P$ such that*

$$\mu^*(x) = \begin{cases} x & \text{if } x \in D_P \backslash D_0 \\ \\ \overline{\mu}(x) & \text{otherwise,} \end{cases}$$

where $\overline{\mu}$ is a P-stable matching.

Proof. By Lemma 3 we have $D_r = D_P$. We argue by induction on r.
If $r = 0$, consider $\mu^* = \overline{\mu}$, where $\overline{\mu}$ is any P-stable matching.
For $r \geq 1$, by Lemma 3 (see its proof), there exists $\mu_r \in \mathcal{A}_P$ such that $\mu_r(x) = x$ if $x \in B_r$ and $\mu_r(x) = \overline{\mu}_r(x)$ otherwise, where $\overline{\mu}_r$ is a P-stable matching. Let $N' = N \backslash B_r$. Then $P' = P \backslash \{B_r\}$ is a stable partition of N' for which $D_{P'} = D_{r-1}$. Therefore, by the inductive hypothesis, there exists $\mu' \in \mathcal{A}_{P'}$ such that $\mu'(x) = x$ if $x \in D_{P'} \backslash D_0$ and $\mu'(x) = \overline{\mu}'(x)$ otherwise, where $\overline{\mu}'$ is a P'-stable matching. Let μ^* and $\overline{\mu}$ be such that $\mu^* |_{N'} = \mu'$, $\mu^* |_{B_r} = \mu_r |_{B_r}$, $\overline{\mu} |_{N'} = \overline{\mu}'$ and $\overline{\mu} |_{B_r} = \overline{\mu}_r |_{B_r}$. Clearly, $\overline{\mu}$ is a P-stable matching. Now, we show that $\mu^* \in \mathcal{A}_P$. If $\mu^* = \mu_r$ since $\mu_r \in \mathcal{A}_P$ we are done. Otherwise, as $\mu^* |_{N'} \in \mathcal{A}_{P'}$ and $\mu_r |_{N'}$ is a P'-stable matching we have $\mu^* |_{N'} R^T \mu_r |_{N'}$. Hence $\mu^* R^T \mu_r$ and since $\mu_r \in \mathcal{A}_P$ then $\mu^* \in \mathcal{A}_P$. Obviously μ^* satisfies the assertion in this lemma. ■

Lemma 5 *Let P be a stable partition such that $S_P \neq \emptyset$. The following conditions hold:*
(i) For any $\mu \in \mathcal{A}_P$, $\mu(S_P) = S_P$ and $\mu |_{S_P}$ is stable for $(S_P, (\succ_x)_{x \in S_P})$.
(ii) For any μ, $\mu' \in \mathcal{A}_P$, $\mu |_{S_P} = \mu' |_{S_P}$.

Proof. By Lemma 2, $\mathcal{A}_P = \{\overline{\mu}\} \cup \left\{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\right\}$ where $\overline{\mu}$ is a P-stable matching.
(i) Let $\mu \in \mathcal{A}_P$. We prove that, for each $x \in S_P$, $\mu(x) \in S_P$. Let $x \in S_P$ and $A \in P$ such that $x \in A$. Then $A \subseteq S_P$. If $\mu = \overline{\mu}$, as $\overline{\mu}$ is a P-stable matching, by Definition 1, $\mu(x) \in A$ and since $A \subseteq S_P$ we have $\mu(x) \in S_P$. If $\mu \neq \overline{\mu}$ then $\mu R^T \overline{\mu}$ and since $\{x, \overline{\mu}(x)\} \subseteq S_P$ it follows that $\mu(x) = \overline{\mu}(x)$ and therefore $\mu(x) \in S_P$. Clearly $\mu |_{S_P}$ is stable.
(ii) Since $\mu |_{S_P} = \overline{\mu} |_{S_P}$ for all $\mu \in \mathcal{A}_P$, the result follows directly. ■

Lemma 6 *Let P and P' be two distinct stable partitions and let μ and μ' be a P-stable matching and a P'-stable matchings respectively. Then, $\mu' R^T \mu$ if and only if $P |_{S_P} \subseteq P' |_{S_{P'}}$.*

Proof. ($\Longrightarrow$): It is trivial if $P |_{S_P} = \emptyset$. Suppose that $P |_{S_P} \neq \emptyset$. Let $A \in P$ such that $A \subseteq S_P$. We must prove that $A \in P'$ and $A \subseteq S_{P'}$. As $\mu' R^T \mu$ then

$\mu' \in \mathcal{A}_P$ hence $\mathcal{A}_{P'} \subseteq \mathcal{A}_P$. Therefore $S_P \subseteq S_{P'}$. Now, by Lemma 5, we have $\mu' \mid_{S_P} = \mu \mid_{S_P}$ and since $\mu(A) = A$, it follows that $\mu'(A) = A$. Hence $A \in P'$. Moreover, as $A \subseteq S_P$ and $S_P \subseteq S_{P'}$ then $A \subseteq S_{P'}$.

($\Longleftarrow$): By Lemma 4, there exists $\mu^* \in \mathcal{A}_P$ such that $\mu^*(x) = x$ if $x \in D_P \backslash D_0$ and $\mu^*(x) = \overline{\mu}(x)$ otherwise, where $\overline{\mu}$ is a P-stable matching. First we prove that there exists a P'-stable matching $\widetilde{\mu}$ such that $\widetilde{\mu} R^T \mu^*$. Consider the P'-stable matching $\widetilde{\mu}$ such that $\widetilde{\mu} \mid_{D_0} = \overline{\mu} \mid_{D_0}$. As $\mu^*(x) = \overline{\mu}(x)$ for all $x \in D_0$ then $\widetilde{\mu} \mid_{D_0} = \mu^* \mid_{D_0}$. Furthermore, if $S_P \neq \emptyset$ since $P \mid_{S_P} \subseteq P' \mid_{S_{P'}}$ we have $\widetilde{\mu} \mid_{S_P} = \overline{\mu} \mid_{S_P}$ and as $\mu^*(x) = \overline{\mu}(x)$ for all $x \in S_P$, it follows that $\widetilde{\mu} \mid_{S_P} = \mu^* \mid_{S_P}$. Then, for each $x \in D_P \backslash D_0$, we have $\widetilde{\mu}(x) \in D_P \backslash D_0$ (otherwise, $\widetilde{\mu}(x) = \mu^*(x) = x$ hence $x \notin D_P \backslash D_0$). Let $(D_P \backslash D_0)' = \{x \in D_P \backslash D_0 : \widetilde{\mu}(x) \neq x\}$. First of all, note that $(D_P \backslash D_0)' \neq \emptyset$ (if $(D_P \backslash D_0)' = \emptyset$ then $\mu^* = \widetilde{\mu} = \overline{\mu}$ and therefore $P = P'$). Now we can write $(D_P \backslash D_0)' = \cup_{i=1}^{s} \{x_i, y_i\}$ where $y_i = \widetilde{\mu}(x_i)$. Since agents x_i and y_i are alone under μ^* we can consider the finite sequence of matchings $\mu^* = \mu_0, \mu_1, ..., \mu_s$ where, for all $i \in \{1, ..., s\}$, μ_i is obtained from μ_{i-1} by satisfying the blocking pair $\{x_i, y_i\}$. Then we have $\mu_s = \widetilde{\mu}$. Therefore $\widetilde{\mu} R^T \mu^*$ and since $\mu^* R^T \overline{\mu}$ we conclude that $\widetilde{\mu} R^T \overline{\mu}$. Finally, the result follows directly by Lemma 1. ■

Lemma 7 *Let P and P' be two stable partitions. $\mathcal{A}_P = \mathcal{A}_{P'}$ if and only if $P \mid_{S_P} = P' \mid_{S_{P'}}$.*

Proof. Suppose that $\mathcal{A}_P = \mathcal{A}_{P'}$. Let $\overline{\mu}$ and $\widetilde{\mu}$ be a P-stable matching and a P'-stable matching respectively. By Lemma 2, we have $\mathcal{A}_P = \{\overline{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\}$ and $\mathcal{A}_{P'} = \{\widetilde{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \widetilde{\mu}\}$. As $\mathcal{A}_P = \mathcal{A}_{P'}$ then $\widetilde{\mu} \in \mathcal{A}_P$ and $\overline{\mu} \in \mathcal{A}_{P'}$. If $\widetilde{\mu} = \overline{\mu}$ then $P = P'$ and we are done. If $\widetilde{\mu} \neq \overline{\mu}$ we have $\widetilde{\mu} R^T \overline{\mu}$ and $\overline{\mu} R^T \widetilde{\mu}$. Hence, by Lemma 6, $P \mid_{S_P} = P' \mid_{S_{P'}}$.

The converse is analogous. ■

Lemma 8 *Let P and P' be two stable partitions. Then for each $A \in P$ either $A \subseteq D_{P'}$ or $A \subseteq S_{P'}$.*

Proof. Let $A \in P$. If A is an odd ring then $A \subseteq D_{P'}$. If A is a singleton the result is trivial. Assume, therefore, that A is a pair of mutually acceptable agents. Let $A = \{x, y\}$. Suppose, by contradiction, and without loss of generality, that $x \in S_{P'}$ and $y \in D_{P'}$. By Lemma 4, we know that there exists a matching $\mu' \in \mathcal{A}_{P'}$ such that $\mu'(x) = x$ if $x \in D_{P'} \backslash D_0$ and $\mu'(x) = \widetilde{\mu}(x)$ otherwise, where $\widetilde{\mu}$ is a P'-stable matching. To reach a contradiction we prove that $\{x, y\}$ blocks μ' by using a proposal-rejection procedure intuitively described as follows. Let $y_0 = y$. Let x_1 denote the predecessor of y_0 in P[15] and $y_1 = \mu'(x_1)$. As agent y_0 prefers x_1 to being alone, y_0 proposes x_1. If x_1 accepts the proposal (that is, x_1 prefers y_0 to his partner under μ') the pair $\{x_1, y_0\}$ blocks μ' and the procedure concludes. Otherwise, let x_2 be the predecessor of y_1 in P and $y_2 = \mu'(x_2)$. Since agent x_1 prefers y_1 to y_0, then, by the stability of P, agent y_1 prefers x_2 to x_1. So y_1 becomes a new proposer in the procedure and offers x_2 the possibility of forming a new pair. Then if x_2 accepts the proposal, the pair $\{x_2, y_1\}$ blocks μ' and the procedure concludes. Otherwise, it may continue iteratively in this manner.

Formally, we define inductively a sequence of pairs $\langle \{x_n, y_n\} \rangle_{n=0}^{\infty}$, that are matched under μ' as follows:

(i) $x_0 = \mu'(y)$ and $y_0 = y$.

(ii) For $n \geq 1$, x_n is the predecessor of y_{n-1} in P and $y_n = \mu'(x_n)$.

Given that N is finite there exists $r \in \mathbb{N}$ such that $y_n \succ_{x_n} y_{n-1}$ for all $n = 1, ..., r-1$ and $y_{r-1} \succ_{x_r} y_r$. Thus the procedure generates the blocking pair $\{x_r, y_{r-1}\}$ of μ' and therefore agents x_r and y_{r-1} are in $D_{P'}$. We now show that $r = 1$. If, on the contrary, $r \geq 2$ since $y_{r-1} \in D_{P'} \backslash D_0$ then agent y_{r-1} is single under μ'. Hence $x_{r-1} = y_{r-1}$. But then $y_{r-2} \succ_{x_{r-1}} y_{r-1}$, contradicts the choice of r (x_{r-1} would accept the proposal of y_{r-2}). So, $r = 1$ and since $x_1 = x$ and $y_0 = y$ we have $\{x, y\}$ blocks μ'. Hence $x \in D_{P'}$ and we have reached a contradiction. ■

Lemma 9 *If P and P' are two stable partitions such that $P\mid_{S_P}$ and $P'\mid_{S_{P'}}$ are maximal in $\mathcal{P}$, then $S_P = S_{P'}$.*

Proof. Suppose, by contradiction, that $S_P \neq S_{P'}$. Then $S_P \cap D_{P'} \neq \emptyset$ or $S_{P'} \cap D_P \neq \emptyset$. We assume, without loss of generality, that $S_P \cap D_{P'} \neq \emptyset$ (otherwise, the argument will be identical except for that the roles of P and P' which are interchanged). By Lemma 8, for each $A \in P$ either $A \subseteq D_{P'}$ or $A \subseteq S_{P'}$. Let $P^* = \{A \in P : A \subseteq D_{P'}\} \cup \{A' \in P' : A' \subseteq S_{P'}\}$ be a partition of N. It is easy to verify that P^* is stable. Now we prove that $D_{P^*} \subseteq D_P \cap D_{P'}$.

[15] Given $x \in N$, we say that y is the predecessor of x in P if y is the immediate predecessor of x in A, where $A \in P$ such that $x \in A$.

By the iterative process described in Section 4, there exists a finite sequence of sets $\langle D_t^* \rangle_{t=0}^{r^*}$ such that:

(i) D_0^* is the union of all odd rings of P^*.

(ii) For $t \geq 1$, $D_t^* = D_{t-1}^* \cup D_t^*$ where $B_t^* = \{b_1^*(t), ..., b_{l_t^*}^*(t)\} \in P^*$ ($l_t^* = 1$ or 2), $B_t^* \nsubseteq D_{t-1}^*$, for which there is a set $A_t^* = \{a_1^*(t), ..., a_{k_t^*}^*(t)\} \in P^*$, $A_t^* \subseteq D_{t-1}^*$ and

$$b_j^*(t) \succ_{a_i^*(t)} a_i^*(t) \text{ and } a_i^*(t) \succ_{b_j^*(t)} b_{j-1}^*(t), \quad [4]$$

for some $i \in \{1, ..., k_t^*\}$ and $j \in \{1, ..., l_t^*\}$.

Then, by Lemma 3, $D_{P^*} = D_{r^*}^*$. We prove by induction on t that, for each $t = 0, ..., r^*$, $D_t^* \subseteq D_P \cap D_{P'}$. If $t = 0$, this is trivial. Assume that $t \geq 1$. It is suffices to prove that $B_t^* \subseteq D_P \cap D_{P'}$. By Lemma 8, we only need show that $b_j^*(t) \in D_P \cap D_{P'}$. Since $A_t^* \subseteq D_{t-1}^*$, by the inductive hypothesis, $a_i^*(t) \in D_P \cap D_{P'}$. Clearly $b_j^*(t) \in D_{P'}$ (otherwise, $B_t^* \in P'$ and since $a_i^*(t) \in D_{P'}$, by [4], $b_j^*(t) \in D_{P'}$). So $B_t^* \in P$ and since $a_i^*(t) \in D_P$, from [4] it follows that $b_j^*(t) \in D_P$, as desired.

Finally, since $D_{P^*} \subseteq D_P \cap D_{P'}$ we have $S_{P'} \cup (S_P \cap D_{P'}) \subseteq S_{P^*}$ and therefore $P' \mid_{S_{P'}} \subset P^* \mid_{S_{P^*}}$, contradicting the maximality of $P' \mid_{S_{P'}}$. ■

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