IKERLANAK

REPLICATOR DYNAMICS AND EVOLUTIONARY STABLE STRATEGIES IN HETEROGENEOUS GAMES

by

André Barreira da Silva Rocha, Annick Laruelle and Peio Zuazo

2011
Working Paper Series: IL. 57/11

Departamento de Fundamentos del Análisis Económico I

Ekonomi Analisiaren Oinarriak I Saila

eman ta zabal zazu

UPV EHU

University of the Basque Country

Replicator Dynamics and Evolutionary Stable Strategies in Heterogeneous Games[*]

André Barreira da Silva Rocha[†], Annick Laruelle[‡§] and Peio Zuazo[¶]

December 20, 2011

Abstract

We generalise and extend the work of Iñarra and Laruelle (2011) by studying two person symmetric evolutionary games with two strategies, a heterogeneous population with two possible types of individuals and incomplete information. Comparing such games with their classic homogeneous version with complete information found in the literature, we show that for the class of anti-coordination games the only evolutionarily stable strategy vanishes. Instead, we find infinite neutrally stable strategies. We also model the evolutionary process using two different replicator dynamics setups, each with a different inheritance rule, and we show that both lead to the same results with respect to stability.

1 Introduction

Evolutionary game theory (EGT) applied to 2 person static games with complete information and homogeneous population has been widely discussed in the literature since Maynard Smith and Price (1973) introduced the key concept of evolutionary stable strategy (ESS). Although the latter is a more robust concept than the Nash equilibrium (NE), the main drawback is that, like the NE, ESS is a static concept and game theorists do not provide an answer on how the ESS is achieved or which one is selected when, like in the classic coordination game, two ESS persist as outcomes of the game. These two limitations were overcome when Taylor and Jonker (1978) introduced the replicator dynamics (RD) and its connection between the static ESS and the dynamic concept of evolutionary equilibrium (EE). In symmetric two-player games, every population state that is an ESS is asymptotically stable (i.e., an EE) in the RD. Also, Bomze and Weibull (1995) showed that in symmetric two-player games, every population state that is a neutrally stable strategy (NSS) is neutrally stable in the RD.

In this paper, we analyze all classes of symmetric games with two players and two strategies allowing for two important differences when compared to the general literature on both deterministic and stochastic EGT found in several branches of science such as biology, physics, economics and sociology. We assume both incomplete information and the existence of a heterogeneous population. Regarding the latter, in line with standard models on EGT, we consider a single population which is split into subgroups of individuals programmed to play one pure strategy. But, additionally, we assume that individuals in this population are randomly assigned one of two possible types.

These ideas are the same originally proposed in Iñarra and Laruelle (2011) where they study a static hawk-dove game with finite population and incomplete information in order to provide theoretical support to the empirical results of Dennis et al. (2008). The latter carries out an experiment with domestic fowls where the animals are randomly marked at the back of their heads. Such a marking results in the individuals being able to identify the type of their opponent (marked or unmarked) during a contest but they are not able to identify their own type. We extend Iñarra and Laruelle (2011) by not restricting our attention to the hawk-dove game. Instead we generalise the analysis to all classes of static games. We consider a very large population and we extend the static analysis of the problem by modeling it as a dynamic game using RD.

The importance of studying all classes of symmetric games assuming a heterogeneous population stems from the fact that a vast literature on homogeneous population exists. In many papers it is recognized the importance of extending the models to a heterogeneous

population in further research. Examples of recent papers with homogeneous population can be found in Ji and Xian-Jia (2011) where they study the evolution of a single homogeneous finite population according to a Moran process considering also the possibility of mutations during reproduction. They focus their study on the hawk-dove game and the coordination game assuming different intensities of selection and different mutation rates. Xu et al. (2011) study the effect of punishment on the emergence of cooperation in the homogeneous population snowdrift game. In addition to the strategies to cooperate and to defeat, they include a strategy to punish. A punisher is simply a cooperator who is willing to pay a cost and receive a bit less payoff in exchange to make a defector get even less. Platkowski and Zakrzewski (2011) study the homogeneous population Rock-Paper-Scissors game using imitation dynamics and assuming that strategies with higher attractiveness have a large likelihood to be imitated. The attractiveness of a strategy follows a Cobb-Douglas utility function depending on the payoff of the strategy in the normal form game and its popularity, measured by its proportion of adoption, among the population. We could bring several more examples without getting close to exhaust the list of papers found in the literature.

In addition, there are also papers assuming two populations. Cressman et al. (1998) study the evolutionary dynamics of crime assuming two types of individuals. But the framework is different from ours because they consider two homogeneous populations, one of owners and another of criminals. Individuals of one population only interact with individuals drawn from the opponent population, ruling out own-population effects (see Friedman, 1998), that is, the fitness function of an individual depends only on the state of the opponent population.

In terms of results, our first main contribution to the literature regards the static version of the game. We show that, when both a heterogeneous population and incomplete information exist, the only class of games in which the results differ from their classic homogeneous versions with complete information is the anti-coordination game. In such class of games, the only ESS vanishes when heterogeneity is added to the population. Instead we find infinite NSS. For the classes of prisoner dilemma and coordination games, the number of ESS matches in both the homogeneous and heterogeneous games. Also, any ESS in these two classes of games consists of both players adopting the same pure action against any type of opponent and this pure action matches with the ESS in the homogeneous game.

Our second contribution is related to the dynamic analysis. We show that in games with heterogeneous populations and incomplete information the natural selection process can be modeled using two different RD. Both dynamics lead to the same results. The first RD follows the standard literature and relies solely on pure strategies while the alternative RD relies on both pure and mixed strategies. The strategies define the two actions that an individual will

adopt when contesting against the two different types of opponents. Throughout the paper, the lack of complete information does not change the well established result that asymptotic stability corresponds to evolutionary stable strategies while neutral stability is related to neutrally stable strategies.

The rest of the paper is organized as follows. Section 2 reviews the homogeneous games with two players and two strategies with complete information and introduces the heterogeneous game with incomplete information. Section 3 gives the ESS. Section 4 reviews the RD for homogeneous games and introduces two RD approaches for the heterogeneous games, comparing their results. Section 5 concludes.

2   Homogeneous game versus heterogeneous game

We study symmetric games with two players and two strategies, H or D. The payoff matrix is given by

$$
\begin{array}{ccc}
 & H & D \\
H & (a_{11}, a_{11}) & (a_{12}, a_{21}) \\
D & (a_{21}, a_{12}) & (a_{22}, a_{22})
\end{array}
$$

We consider generic games, i.e., those such that $a_{11} \neq a_{21}$ and $a_{22} \neq a_{12}$. We can represent a strategy by the probability of playing the first strategy H. Let $u(\alpha, \beta)$ (respectively $v(\alpha, \beta)$) be the expected payoff of the row player (respectively the column player) when the row player plays H with probability α (and D with probability $1 - \alpha$) and the column player plays H with probability β (and D with probability $1 - \beta$). We obtain

$$
\begin{aligned}
u(\alpha, \beta) &= \alpha\beta a_{11} + \alpha(1 - \beta)a_{12} + (1 - \alpha)\beta a_{21} + (1 - \alpha)(1 - \beta)a_{22} \\
v(\alpha, \beta) &= u(\beta, \alpha).
\end{aligned}
$$

As the game is symmetric we need to keep track only of the payoff function of one player (say the row player). It can be rewritten as

$$
u(\alpha, \beta) = \beta a_{21} + (1 - \beta)a_{22} + \left[(a_1 + a_2)\beta - a_2 \right]\alpha \tag{1}
$$

with $a_1 = a_{11} - a_{21}$ and $a_2 = a_{22} - a_{12}$. As will be seen in the following, the solution concepts only depend on the coefficients a_1 and a_2. In other words the matrix payoff is equivalent to $\mathcal{M}$, where

$$
\mathcal{M} = \left(\begin{array}{cc} (a_1, a_1) & (0, 0) \\ (0, 0) & (a_2, a_2) \end{array} \right).
$$

4

Definition 1 *A matrix $\mathcal{M}$ is (i) dom when $a_1 a_2 < 0$, (ii) coord when $a_1 > 0$ and $a_2 > 0$, or (iii) anti when $a_1 < 0$ and $a_2 < 0$.*

This is a complete classification of all 2×2 symmetric games as for generic games $a_1 \neq 0$ and $a_2 \neq 0$. For coord and anti matrices (for which $a_1 + a_2 \neq 0$) we can define $a = \frac{a_2}{a_1 + a_2}$. A well-known example of game with a dom matrix is the prisoner dilemma, an example with a coord matrix is the stag-hare game and an example with an anti matrix is the hawk-dove game.

$$
\begin{array}{ccc}
\begin{array}{c}
\quad\ \text{H} \qquad \text{D} \\
\begin{array}{c|cc}
\text{H} & (2,2) & (0,3) \\
\text{D} & (3,0) & (1,1)
\end{array} \\
\text{prisoner dilemma}
\end{array}
&
\begin{array}{c}
\quad\ \text{H} \qquad \text{D} \\
\begin{array}{c|cc}
\text{H} & (1,1) & (1,0) \\
\text{D} & (0,1) & (2,2)
\end{array} \\
\text{stag-hare game}
\end{array}
&
\begin{array}{c}
\qquad\ \text{H} \qquad\qquad \text{D} \\
\begin{array}{c|cc}
\text{H} & (\frac{v-c}{2}, \frac{v-c}{2}) & (v,0) \\
\text{D} & (0,v) & (\frac{v}{2}, \frac{v}{2})
\end{array} \\
\text{hawk-dove } (v < c)
\end{array}
\end{array}
$$

We denote a game by $\Gamma(\mathcal{M})$, and refer to it as the *homogeneous* game in the sense that it is played among a homogeneous population.

By contrast we consider heterogeneous population, where the division of the population into two types is done artificially, and where individuals fail to recognize their own type but do recognize their opponent's type. We extend Iñarra and Laruelle (2011)'s study of the hawk-dove game in this context and see how it affects the evolutionarily stable strategies and the replicator dynamics.

The population is divided into two types of individuals, I and II, where x $(0 < x < 1)$ is the proportion of individuals of type I. We assume that x is constant.[1] Assuming that the population is very large, at any bilateral encounter between two individuals the probabilities of the four possible encounters, which we denote respectively by $p(I,I)$, $p(I,II)$, $p(II,I)$ and $p(II,II)$, are given by

$$p(I,I) = x^2,\ p(II,II) = (1-x)^2,\ p(I,II) = p(II,I) = x(1-x). \tag{2}$$

Any individual in the population can say "I know who you are but I do not know who I am". She finds herself in a position of choosing a probability of playing H for each type of opponent. But the strategy cannot depend on her own type. A *strategy* can thus be represented by $\boldsymbol{\alpha} = (\alpha_I, \alpha_{II})$ where α_I gives the probability of choosing H when facing an opponent of type I, and α_{II} gives the probability of choosing H when facing an opponent of type II. There are four pure strategies, HH where $\alpha_I = \alpha_{II} = 1$, DD where $\alpha_I = \alpha_{II} = 0$,

---

[1] In the dynamic process this corresponds to assuming that each new offspring is randomly assigned a type in such a way that x does not change. The allocation of type is completely independent of the strategy the offspring is programmed to play.

DH where $\alpha_I = 0$ and $\alpha_{II} = 1$, and HD where $\alpha_I = 1$ and $\alpha_{II} = 0$. We denote by $\Delta(\boldsymbol{\alpha})$ the frequency of playing action H:

$$\Delta(\boldsymbol{\alpha}) = x\alpha_I + (1 - x)\alpha_{II}. \tag{3}$$

Of course, $0 \leq \Delta(\boldsymbol{\alpha}) \leq 1$, the extreme values are reached for two of the pure strategies, $\Delta(HH) = 1$ and $\Delta(DD) = 0$, while $\Delta(HD) = x$ and $\Delta(DH) = 1 - x$.

The expected payoff of an individual playing $\boldsymbol{\alpha} = (\alpha_I, \alpha_{II})$ while the opponent plays $\boldsymbol{\beta} = (\beta_I, \beta_{II})$ is the sum of the expected payoffs she would obtain in every possible encounter weighted by its probability of occurrence. Therefore the expected payoff of an individual playing $\boldsymbol{\alpha}$ against an opponent playing $\boldsymbol{\beta}$, that we denote by $U(\boldsymbol{\alpha}, \boldsymbol{\beta})$, is given by

$$U(\boldsymbol{\alpha}, \boldsymbol{\beta}) = p(I, I)u(\alpha_I, \beta_I) + p(II, I)u(\alpha_I, \beta_{II}) + p(I, II)u(\alpha_{II}, \beta_I) + p(II, II)u(\alpha_{II}, \beta_{II}).$$

Using (1), (2) and (3) we can rewrite

$$U(\boldsymbol{\alpha}, \boldsymbol{\beta}) = [(a_1 + a_2)\Delta(\boldsymbol{\beta}) - a_2]\,\Delta(\boldsymbol{\alpha}) + g(\boldsymbol{\beta}) \tag{4}$$

where

$$g(\boldsymbol{\beta}) = a_{22} + (a_{21} - a_{22})\Delta(\boldsymbol{\beta}).$$

We have all the ingredients of a game, hereafter referred to as a *heterogeneous game* and denoted by $\Gamma_x(\mathcal{M})$, where $0 < x < 1$ is the proportion of individuals of type I. We contrast the results obtained in the homogeneous game $\Gamma(\mathcal{M})$ with the ones in the heterogeneous one.

3 Evolutionarily stable strategies

An evolutionarily stable strategy (ESS) is a strategy that cannot be invaded by a mutant strategy. An evolutionarily stable strategy has to be a best response[2] to itself, and provides a strictly larger payoff than any strategy which is a best response to the evolutionarily stable strategy. If the inequality is not strict the strategy is neutrally stable.

In the homogeneous game $\Gamma(\mathcal{M})$, let $\mathcal{B}(\beta)$ denote the set of an individual's best responses to an opponent playing strategy β. The two conditions for a strategy α^* to be evolutionarily stable in $\Gamma(\mathcal{M})$ are: *(1)* $\alpha^* \in \mathcal{B}(\alpha^*)$ and *(2)* for any $\beta \in \mathcal{B}(\alpha^*)$ such that $\beta \neq \alpha^*$ we have $u(\alpha^*, \beta) > u(\beta, \beta)$. A strategy α^* that satisfies Condition *(1)* and weakly satisfies Condition *(2)*, i.e., such that $u(\alpha^*, \beta) \geq u(\beta, \beta)$, is called a neutrally stable strategy (NSS).

---

[2]Given a strategy played by the opponent, a best response is a strategy that yields the highest payoff.

Proposition 1 *Let $\Gamma(\mathcal{M})$ be a homogeneous game. (i) If $\mathcal{M}$ is a dom matrix, there is a unique ESS, which is a pure strategy (H if $a_1 > 0$ and D if $a_1 < 0$). (ii) If $\mathcal{M}$ is a coord matrix, H and D are the two unique ESS. (iii) If $\mathcal{M}$ is an anti matrix, strategy a is the unique ESS.*

Proof. *Let us rewrite (1) as*

$$u(\alpha, \beta) = \beta a_{21} + (1 - \beta)a_{22} + [a_1\beta - (1 - \beta)a_2]\,\alpha.$$

(i) If $\mathcal{M}$ is a dom matrix, $[a_1\beta - (1-\beta)a_2]$ is always strictly positive (if $a_1 > 0$) or strictly negative (if $a_1 < 0$). It is clear then that there is a dominant pure strategy: H if $a_1 > 0$ and D if $a_1 < 0$. This strategy is an ESS as it satisfies Condition (1) while Condition (2) becomes empty. (ii) If $\mathcal{M}$ is a coord matrix, from (1) we obtain

$$\mathcal{B}(\beta) = \begin{cases} [0,1] & \text{if } \beta = a \\ \{1\} & \text{if } \beta > a \\ \{0\} & \text{if } \beta < a. \end{cases}$$

The three strategies that satisfy Condition (1) are $\alpha = 1$, $\alpha = 0$, and $\alpha = a$. Condition (2) becomes empty for the first two strategies, while $\alpha = a$ does not satisfy Condition (2): for all $\beta \neq a$ that are best response to strategy a the difference $u(a,\beta) - u(\beta,\beta)$ should be strictly positive. But there exists such β where

$$u(a, \beta) - u(\beta, \beta) = -(a_1 + a_2)(\beta - a)^2 < 0.$$

Thus H and D are the only two ESS. (iii) If $\mathcal{M}$ is an anti matrix, from (1) we obtain

$$\mathcal{B}(\beta) = \begin{cases} [0,1] & \text{if } \beta = a \\ \{1\} & \text{if } \beta < a \\ \{0\} & \text{if } \beta > a. \end{cases}$$

Only strategy $\alpha = a$ satisfies Condition (1). It also satisfies Condition (2): the difference $u(a,\beta) - u(\beta,\beta)$ is strictly positive for any $\beta \neq a$. Indeed

$$u(a, \beta) - u(\beta, \beta) = -(a_1 + a_2)(\beta - a)^2 > 0.$$

Hence $\alpha^* = a$ is the only ESS. ∎

In the heterogeneous game $\Gamma_x(\mathcal{M})$, let $\mathcal{B}_x(\boldsymbol{\beta})$ denote the set of best responses to an opponent playing strategy $\boldsymbol{\beta}$. We define an ESS in the heterogeneous games as in Iñarra and Laruelle (2011) by extending the definition of ESS in homogeneous games. That is,

Definition 2 *Strategy $\boldsymbol{\alpha}^*$ is evolutionarily stable in $\Gamma_x(\mathcal{M})$ if and only if (1) $\boldsymbol{\alpha}^* \in \mathcal{B}_x(\boldsymbol{\alpha}^*)$, and (2) for any $\boldsymbol{\beta} \in \mathcal{B}_x(\boldsymbol{\alpha}^*)$ such that $\boldsymbol{\beta} \neq \boldsymbol{\alpha}^*$ we have $U(\boldsymbol{\alpha}^*, \boldsymbol{\beta}) > U(\boldsymbol{\beta}, \boldsymbol{\beta})$.*

From the equation of the expected payoff (4) it is clear that the best response will depend on $\Delta(\boldsymbol{\beta})$, and on the coefficients a_1 and a_2. Note that the function $g(\boldsymbol{\beta})$ plays no role. If $(a_1 + a_2)\Delta(\boldsymbol{\beta}) > a_2$, the best response is $\Delta(\boldsymbol{\alpha}) = 1$ (i.e., strategy HH), if $(a_1 + a_2)\Delta(\boldsymbol{\beta}) < a_2$, the best response is $\Delta(\boldsymbol{\alpha}) = 0$ (i.e., strategy DD), if $(a_1 + a_2)\Delta(\boldsymbol{\beta}) = a_2$, any strategy $\boldsymbol{\alpha}$ is a best response to $\boldsymbol{\beta}$. We have thus different ESS depending on the matrix as the following proposition shows.

Proposition 2 *Consider game $\Gamma_x(\mathcal{M})$. (i) If $\mathcal{M}$ is a dom matrix, there is only one ESS, which is HH if $a_1 > 0$, and DD if $a_1 < 0$. (ii) If $\mathcal{M}$ is a coord matrix, HH and DD are the only ESS. (iii) If $\mathcal{M}$ is an anti matrix, strategies $\hat{\boldsymbol{\alpha}}$ such that $\Delta(\hat{\boldsymbol{\alpha}}) = a$ are NSS.*

Proof. *(i)* If $\mathcal{M}$ is a dom matrix, rewriting the equation of the expected payoff (4) as

$$U(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left[a_1 \Delta(\boldsymbol{\beta}) - a_2(1 - \Delta(\boldsymbol{\beta}))\right]\Delta(\boldsymbol{\alpha}) + g(\boldsymbol{\beta})$$

we see that $a_1\Delta(\boldsymbol{\beta}) - a_2(1 - \Delta(\boldsymbol{\beta}))$ is always strictly positive (if $a_1 > 0$) or strictly negative (if $a_1 < 0$). Thus there is one dominant strategy: HH (recall that $\Delta(HH) = 1$) if $a_1 > 0$ or DD if $a_1 < 0$. Condition *(2)* becomes empty. *(ii)* If $\mathcal{M}$ is a coord matrix, from (4) we obtain

$$\mathcal{B}_x(\boldsymbol{\beta}) = \begin{cases} [0,1] \times [0,1] & \text{if } \Delta(\boldsymbol{\beta}) = a \\ \{(1,1)\} & \text{if } \Delta(\boldsymbol{\beta}) > a \\ \{(0,0)\} & \text{if } \Delta(\boldsymbol{\beta}) < a. \end{cases}$$

The strategies that satisfy Condition *(1)* are HH, DD, and all $\overline{\boldsymbol{\alpha}}$ such that $\Delta(\overline{\boldsymbol{\alpha}}) = a$. Condition *(2)* becomes empty for HH and DD, these two strategies are thus ESS. These are the only ESS as Condition *(2)* is not satisfied for strategies $\overline{\boldsymbol{\alpha}}$. Indeed as all $\boldsymbol{\beta}$ are best responses to $\overline{\boldsymbol{\alpha}}$, the difference $U(\overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}) - U(\boldsymbol{\beta}, \boldsymbol{\beta})$ should be strictly positive for any $\overline{\boldsymbol{\alpha}}$, and any $\boldsymbol{\beta} \neq \overline{\boldsymbol{\alpha}}$. But it is easy to see that for all $\overline{\boldsymbol{\alpha}}$ there exists $\boldsymbol{\beta} \neq \bar{\boldsymbol{\alpha}}$ such that:

$$U(\overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}) - U(\boldsymbol{\beta}, \boldsymbol{\beta}) = -(a_1 + a_2)\left(\Delta(\boldsymbol{\beta}) - a\right)^2 < 0.$$

(iii) If $\mathcal{M}$ is an anti matrix, we obtain

$$\mathcal{B}_x(\boldsymbol{\beta}) = \begin{cases} [0,1] \times [0,1] & \text{if } \Delta(\boldsymbol{\beta}) = a \\ \{(0,0)\} & \text{if } \Delta(\boldsymbol{\beta}) > a \\ \{(1,1)\} & \text{if } \Delta(\boldsymbol{\beta}) < a. \end{cases}$$

Strategies $\hat{\boldsymbol{\alpha}}$ such that $\Delta(\hat{\boldsymbol{\alpha}}) = a$ satisfy Condition *(1)*. For Condition *(2)* we compute

$$U(\hat{\boldsymbol{\alpha}}, \boldsymbol{\beta}) - U(\boldsymbol{\beta}, \boldsymbol{\beta}) = -(a_1 + a_2)\left(\Delta(\boldsymbol{\beta}) - a\right)^2.$$

This difference is strictly positive for any $\boldsymbol{\beta}$ such that $\Delta(\boldsymbol{\beta}) \neq a$. But for $\boldsymbol{\beta} \neq \widehat{\boldsymbol{\alpha}}$ such that $\Delta(\boldsymbol{\beta}) = a$ we have $U(\widehat{\boldsymbol{\alpha}}, \boldsymbol{\beta}) - U(\boldsymbol{\beta}, \boldsymbol{\beta}) = 0$. Therefore, strategies $\widehat{\boldsymbol{\alpha}}$ such that $\Delta(\widehat{\boldsymbol{\alpha}}) = a$ are NSS. ∎

Comparing the results for homogeneous and heterogeneous games we obtain that the probability of playing strategy H in $\Gamma(\mathcal{M})$ is substituted by the frequency of playing action H in $\Gamma_x(\mathcal{M})$. More precisely, $\boldsymbol{\alpha}^*$ is an ESS in the homogeneous game if and only if $\boldsymbol{\alpha}^*$ is either an NSS or an ESS in the heterogeneous game, where $\boldsymbol{\alpha}^*$ satisfies $\Delta(\boldsymbol{\alpha}^*) = \alpha^*$.

Theorem 1 *Consider games $\Gamma(\mathcal{M})$ and $\Gamma_x(\mathcal{M})$. (i) If $\boldsymbol{\alpha}^*$ is an ESS or an NSS in game $\Gamma_x(\mathcal{M})$, then $\Delta(\boldsymbol{\alpha}^*)$ is an ESS in game $\Gamma(\mathcal{M})$. (ii) If α^* is an ESS in game $\Gamma(\mathcal{M})$ and $\Delta(\boldsymbol{\alpha}) = \alpha^*$ has a unique solution $\boldsymbol{\alpha}^*$, then $\boldsymbol{\alpha}^*$ is an ESS in game $\Gamma_x(\mathcal{M})$. (iii) If α^* is an ESS in game $\Gamma(\mathcal{M})$ and $\Delta(\boldsymbol{\alpha}) = \alpha^*$ has multiple solutions, then all $\hat{\boldsymbol{\alpha}}$ such that $\Delta(\hat{\boldsymbol{\alpha}}) = \alpha^*$ are NSS in game $\Gamma_x(\mathcal{M})$*

Note that $\Delta(\boldsymbol{\alpha}) = \alpha^*$ has always at least one solution, the non-discriminating (with respect to types) strategy $\boldsymbol{\alpha} = (\alpha^*, \alpha^*)$. If it is the only solution, as for games with dom and coord matrices then (α^*, α^*) is an ESS. If there are other solutions, as for games with anti matrix, then all the solutions are NSS. The NSS such that $\widehat{\boldsymbol{\alpha}} \neq (\alpha^*, \alpha^*)$ correspond to discriminating strategies, when one type is discriminated with respect to the other type.

4 Replicator dynamics

The replicator dynamics study the stable state of the population. That is, it assumes that the population is decomposed in subgroups, each of which playing pure strategies. Payoffs represent the incremental effect from playing the game in question on an individual's fitness, measured as the number of offspring per time unit. It is also assumed that each offspring inherits its parent's strategy - strategies breed true. The population state is the repartition of the population into the different subgroups. The replicator dynamics looks for the repartitions that are stable over time.

In the homogeneous game, $\Gamma(\mathcal{M})$, there are two pure strategies, H and D. The total population at time t, that we denote $n(t)$, is decomposed into $n(t) = n_H(t) + n_D(t)$ where $n_H(t)$ is the number of individuals who are currently programmed to play strategy H and $n_D(t)$ is the number of individuals who are currently programmed to play strategy D. A state can simply be represented by the population share programmed to play strategy H that we denote $\gamma(t)$. In the following we will drop t and write $n_H = n\gamma$ and $n_D = n(1 - \gamma)$.

To see how the population share changes over time, we need to compute the payoff obtained with each pure strategy in state γ. It is immaterial for an individual whether she

interacts with an individual drawn at random from a polymorphic population or with an individual playing the mixed strategy γ. A random match when the population is in state γ is therefore equivalent to meeting an opponent playing γ. An individual playing H gets as expected payoff $u(H, \gamma)$ while an individual playing D gets $u(D, \gamma)$.

If reproduction takes place continuously over time, then the birthrate at any time t, of individuals programmed to pure strategies, with dots for time derivation and suppressing time arguments, results in the following population dynamics:

$$\dot{n}_H = n_H u(H, \gamma) = n\gamma u(H, \gamma).$$

Similarly

$$\dot{n}_D = n_D u(D, \gamma) = n(1 - \gamma)u(D, \gamma)$$

and

$$\dot{n} = \dot{n}_H + \dot{n}_D = n\left[\gamma u(H, \gamma) + (1 - \gamma)u(D, \gamma)\right] = nu(\gamma, \gamma).$$

The equation relative to $\dot{n}_D$ is redundant to study the evolution of the repartition into the two subgroups of population. We keep the equation $n_H = n\gamma$ that we derive with respect to time

$$n\dot{\gamma} = \dot{n}_H - \gamma\dot{n} = n\gamma u(H, \gamma) - n\gamma u(\gamma, \gamma) = n\gamma\left[u(H, \gamma) - u(\gamma, \gamma)\right].$$

The evolution of the state γ is thus given by the following equation

$$\dot{\gamma} = \gamma\left[u(H, \gamma) - u(\gamma, \gamma)\right]. \tag{5}$$

We obtain a single equation.[3] The conditions for a state γ^* to be asymptotically stable are (1) γ^* is a critical point, that is, $\dot{\gamma}(\gamma^*) = 0$ and (2) $\frac{\partial\dot{\gamma}}{\partial\gamma}(\gamma^*) < 0$.

The following proposition gives the asymptotically stable states for a homogeneous game depending on the matrix of the game.

Proposition 3 *Let a homogeneous game $\Gamma(\mathcal{M})$ with replicator dynamics (5). (i) If $\mathcal{M}$ is a dom matrix, there is a unique asymptotically stable state, $\gamma^* = 1$ if $a_1 > 0$, and $\gamma^* = 0$ if $a_1 < 0$. (ii) If $\mathcal{M}$ is a coord matrix, there are two asymptotically stable states, $\gamma_1^* = 0$ and $\gamma_2^* = 1$. (iii) If $\mathcal{M}$ is an anti matrix, there is a unique asymptotically stable state, $\gamma^* = a$.*

---

[3]By contrast in the heterogeneous case we will obtain a system of equations.

Proof. (5) can be rewritten using (1) as

$$\dot{\gamma} = \gamma(1 - \gamma)\left[a_1\gamma - a_2(1 - \gamma)\right].$$

If $a_1a_2 < 0$ the set of critical points is $\{0, 1\}$, while it is $\{0, a, 1\}$ if $a_1a_2 > 0$. The second step is to check the sign of the derivative at the critical points. The derivative is given by:

$$\frac{\partial\dot{\gamma}}{\partial\gamma}(\gamma) = \begin{cases} (a_1 + a_2)\left[-3\gamma^2 + 2\gamma(1 + a) - a\right] & \text{if } a_1 + a_2 \neq 0 \\ (2\gamma - 1)a_2 & \text{if } a_1 + a_2 = 0 \end{cases}$$

and

$$\frac{\partial\dot{\gamma}}{\partial\gamma}(1) = -a_1, \qquad \frac{\partial\dot{\gamma}}{\partial\gamma}(0) = -a_2, \qquad \frac{\partial\dot{\gamma}}{\partial\gamma}(a) = \frac{a_1a_2}{a_1+a_2}.$$

The sign of the derivatives depends on the matrix. (i) If $\mathcal{M}$ is a dom matrix, $\gamma^* = 1$ is asymptotically stable if $a_1 > 0$ and $\gamma^* = 0$ is asymptotically stable if $a_1 < 0$ (recall that a is not a critical point). (ii) If $\mathcal{M}$ is a coord matrix, both $\gamma_1^* = 0$ and $\gamma_2^* = 1$ are asymptotically stable while $\gamma = a$ is not. (iii) If $\mathcal{M}$ is an anti matrix, only $\gamma^* = a$ is asymptotically stable. ∎

We proceed now to the heterogeneous game, $\Gamma_x(\mathcal{M})$. As the number of replicator dynamic equations is based on the number of pure strategies, here we will have a system instead of a single equation. First let us state the conditions for the stability of a non-linear system of k equations:

$$\begin{cases} \dot{\mu}_1 &= f_1(\mu_1, .., \mu_k) \\ ... \\ \dot{\mu}_k &= f_k(\mu_1, .., \mu_k) \end{cases} \tag{6}$$

One condition is based on the Jacobian matrix of the system, which we denote $\Omega(\boldsymbol{\mu})$ at state $\boldsymbol{\mu} = (\mu_1, .., \mu_k)$. Recall that

$$\Omega(\boldsymbol{\mu}) = \begin{pmatrix} \frac{\partial f_1}{\partial \mu_1}(\boldsymbol{\mu}) & ... & \frac{\partial f_1}{\partial \mu_k}(\boldsymbol{\mu}) \\ \\ \frac{\partial f_k}{\partial \mu_1}(\boldsymbol{\mu}) & ... & \frac{\partial f_k}{\partial \mu_k}(\boldsymbol{\mu}) \end{pmatrix}$$

We will denote by $\lambda_1\left[\Omega(\boldsymbol{\mu})\right], ..., \lambda_k\left[\Omega(\boldsymbol{\mu})\right]$ the eigenvalues of the Jacobian matrix $\Omega(\boldsymbol{\mu})$. The eigenvalues can have an imaginary part, so $Re[y]$ stands for the real part of y. The conditions for a non linear system to be asymptotically stable in the neighborhood of a stationary state are twofold.

Definition 3 *The state $\boldsymbol{\mu}^* = (\mu_1^*, .., \mu_k^*)$ is locally asymptotically stable in the non linear system (6) if the following two conditions are satisfied:*

1. $\dot{\mu}_i(\boldsymbol{\mu}^*) = 0$ *for* $i = 1, ..., k$ *and*

2. $Re[\lambda_i [\Omega(\boldsymbol{\mu}^*)]] < 0$ *for* $i = 1, ..., k$.

The first condition states that state $\boldsymbol{\mu}^*$ is stationary, while the second condition guarantees that any trajectory of the system starting close enough to $\boldsymbol{\mu}^*$ will converge to $\boldsymbol{\mu}^*$ as $t \to \infty$.

4.1 First RD approach

In game $\Gamma_x(\mathcal{M})$ there are four pure strategies, HH, HD, DH and DD. The population can be decomposed into $n = n_{HH} + n_{HD} + n_{DH} + n_{DD}$, where n_{HH} is the number of individuals programmed to play HH, etc, with

$$n_{HH} = n\theta_{HH}, \qquad n_{HD} = n\theta_{HD}, \qquad n_{DH} = n\theta_{DH}, \text{ and} \qquad n_{DD} = n\theta_{DD}. \qquad (7)$$

The population state is the repartition of the population into the different strategies that can be represented by $\boldsymbol{\theta} = (\theta_{HH}, \theta_{HD}, \theta_{DH})$ as $\theta_{DD} = 1 - \theta_{HH} - \theta_{HD} - \theta_{DH}$. Graphically the set of possible $\boldsymbol{\theta}$ is the tetrahedron showed in Figure 1.
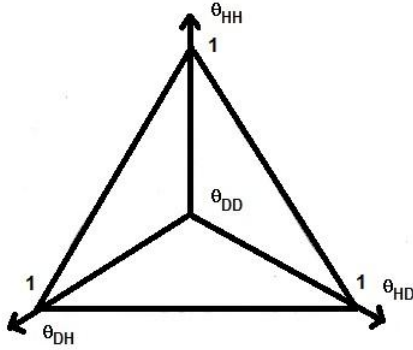

Figure 1: Tetrahedron representing the state space.

The vertices of the tetrahedron correspond to a monomorphic population, i.e., only one strategy (or genotype) survives in the population. Outside the vertices, the population is in a polymorphic state, i.e., at least two strategies co-exist.

To construct the replicator dynamics we need the payoff received when playing the pure strategies. Preliminary to the question of the payoff is the question of the strategy used by an opponent drawn at random in state $\boldsymbol{\theta}$. If the population is in state $\boldsymbol{\theta}$, those who play HH and those who play HD will play H when facing an individual of type I. Thus H against an

individual of type I is played with probability $\theta_{HH} + \theta_{HD}$. Those who play HH and those who play DH will play H against an opponent of type II. Thus H against an individual of type II is played with probability $\theta_{HH} + \theta_{DH}$. In state $\boldsymbol{\theta}$ it is as if an individual was playing against an opponent playing $\boldsymbol{\beta}(\boldsymbol{\theta}) = (\theta_{HH} + \theta_{HD}, \theta_{HH} + \theta_{DH})$. Therefore if the individual plays $\boldsymbol{\alpha}$ the expected payoff that she gets is $U(\boldsymbol{\alpha}, \boldsymbol{\beta}(\boldsymbol{\theta}))$ in state $\boldsymbol{\theta}$. The evolution of the repartition of the population is thus given by

$$
\begin{aligned}
\dot{n}_{HH} &= n_{HH}\, U(HH, \boldsymbol{\beta}(\boldsymbol{\theta})) \\
\dot{n}_{HD} &= n_{HD}\, U(HD, \boldsymbol{\beta}(\boldsymbol{\theta})) \\
\dot{n}_{DH} &= n_{DH}\, U(DH, \boldsymbol{\beta}(\boldsymbol{\theta})) \\
\dot{n}_{DD} &= n_{DD}\, U(DD, \boldsymbol{\beta}(\boldsymbol{\theta}))
\end{aligned}
$$

while the total population varies as

$$
\dot{n} = \dot{n}_{HH} + \dot{n}_{HD} + \dot{n}_{DH} + \dot{n}_{DD} = n\, U(\boldsymbol{\beta}(\boldsymbol{\theta}), \boldsymbol{\beta}(\boldsymbol{\theta})).
$$

The last equality is obtained by using (4) and (7).

Now we proceed to solve the system, dropping the equation relative to $\dot{n}_{DD}$ as it is redundant. We derive the equations in (7) to obtain

$$
\begin{aligned}
n\dot{\theta}_{HH} &= \dot{n}_{HH} - \theta_{HH}\dot{n} = n\theta_{HH}\left[U(HH, \boldsymbol{\beta}(\boldsymbol{\theta})) - U(\boldsymbol{\beta}(\boldsymbol{\theta}), \boldsymbol{\beta}(\boldsymbol{\theta}))\right] \\
n\dot{\theta}_{HD} &= \dot{n}_{HD} - \theta_{HD}\dot{n} = n\theta_{HD}\left[U(HD, \boldsymbol{\beta}(\boldsymbol{\theta})) - U(\boldsymbol{\beta}(\boldsymbol{\theta}), \boldsymbol{\beta}(\boldsymbol{\theta}))\right] \\
n\dot{\theta}_{DH} &= \dot{n}_{DH} - \theta_{DH}\dot{n} = n\theta_{DH}\left[U(DH, \boldsymbol{\beta}(\boldsymbol{\theta})) - U(\boldsymbol{\beta}(\boldsymbol{\theta}), \boldsymbol{\beta}(\boldsymbol{\theta}))\right].
\end{aligned}
$$

Again substituting (4) into the previous equations we get

$$
\begin{aligned}
\dot{\theta}_{HH} &= \theta_{HH}\left[(a_1 + a_2)\Delta(\boldsymbol{\beta}(\boldsymbol{\theta})) - a_2\right]\left[1 - \Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))\right] \\
\dot{\theta}_{HD} &= \theta_{HD}\left[(a_1 + a_2)\Delta(\boldsymbol{\beta}(\boldsymbol{\theta})) - a_2\right]\left[x - \Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))\right] \\
\dot{\theta}_{DH} &= \theta_{DH}\left[(a_1 + a_2)\Delta(\boldsymbol{\beta}(\boldsymbol{\theta})) - a_2\right]\left[1 - x - \Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))\right]
\end{aligned}
\qquad (8)
$$

where by (3)

$$
\Delta(\boldsymbol{\beta}(\boldsymbol{\theta})) = \Delta(\theta_{HH} + \theta_{HD}, \theta_{HH} + \theta_{DH}) = \theta_{HH} + x\theta_{HD} + (1-x)\theta_{DH}.
$$

We now study the dynamics of the system of equations given by (8). One should notice that we can have zero eigenvalues at the vertices making these points non-hyperbolic. But these are particular cases for specific values of the parameters x, a_1 and a_2. We ignore these cases and focus the analysis on the cases where the system of ordinary differential equations is hyperbolic at all vertices of the tetrahedron. We carry out the analysis using Cartesian coordinates instead of barycentric coordinates.

Theorem 2 *Let a heterogeneous game $\Gamma_x(\mathcal{M})$ with replicator dynamics (8). (i) If $\mathcal{M}$ is a dom matrix, there is a unique asymptotically stable point, which is $\boldsymbol{\theta^*} = (1,0,0)$ if $a_1 > 0$ and $\boldsymbol{\theta^*} = (0,0,0)$ if $a_1 < 0$. (ii) If $\mathcal{M}$ is a coord matrix, there are two asymptotically stable points: $\boldsymbol{\theta_1^*}=(1,0,0)$ and $\boldsymbol{\theta_2^*}=(0,0,0)$. (iii) If $\mathcal{M}$ is an anti matrix, there are infinite non-isolated neutrally stable points, all $\tilde{\boldsymbol{\theta}}$ such that $\Delta(\boldsymbol{\beta}(\tilde{\boldsymbol{\theta}})) = a$.*

Proof.

1. From (8) it is easy to see that the set of isolated critical points is: $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. In order to see their stability let the Jacobian matrix $\Omega(\boldsymbol{\theta})$

$$
\Omega(\boldsymbol{\theta}) = \begin{pmatrix} \frac{\partial \dot{\theta}_{HH}}{\partial \theta_{HH}}(\boldsymbol{\theta}) & \frac{\partial \dot{\theta}_{HH}}{\partial \theta_{HD}}(\boldsymbol{\theta}) & \frac{\partial \dot{\theta}_{HH}}{\partial \theta_{DH}}(\boldsymbol{\theta}) \\[2ex] \frac{\partial \dot{\theta}_{HD}}{\partial \theta_{HH}}(\boldsymbol{\theta}) & \frac{\partial \dot{\theta}_{HD}}{\partial \theta_{HD}}(\boldsymbol{\theta}) & \frac{\partial \dot{\theta}_{HD}}{\partial \theta_{DH}}(\boldsymbol{\theta}) \\[2ex] \frac{\partial \dot{\theta}_{DH}}{\partial \theta_{HH}}(\boldsymbol{\theta}) & \frac{\partial \dot{\theta}_{DH}}{\partial \theta_{HD}}(\boldsymbol{\theta}) & \frac{\partial \dot{\theta}_{DH}}{\partial \theta_{DH}}(\boldsymbol{\theta}) \end{pmatrix}
$$

with

$$
\begin{aligned}
\frac{\partial \dot{\theta}_{HH}}{\partial \theta_{HH}}(\boldsymbol{\theta}) &= (1 - \Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) \left[(a_1 + a_2)\Delta(\boldsymbol{\beta}(\boldsymbol{\theta})) - a_2 \right] \\
&\quad + \theta_{HH} \left[(a_1 + a_2)(1 - 2\Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) + a_2 \right] \\
\frac{\partial \dot{\theta}_{HH}}{\partial \theta_{HD}}(\boldsymbol{\theta}) &= x\theta_{HH} \left[(a_1 + a_2)(1 - 2\Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) + a_2 \right] \\
\frac{\partial \dot{\theta}_{HH}}{\partial \theta_{DH}}(\boldsymbol{\theta}) &= (1 - x)\theta_{HH} \left[(a_1 + a_2)(1 - 2\Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) + a_2 \right] \\
\frac{\partial \dot{\theta}_{HD}}{\partial \theta_{HH}}(\boldsymbol{\theta}) &= \theta_{HD} \left[(a_1 + a_2)(x - 2\Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) + a_2 \right] \\
\frac{\partial \dot{\theta}_{HD}}{\partial \theta_{HD}}(\boldsymbol{\theta}) &= (x - \Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) \left[(a_1 + a_2)\Delta(\boldsymbol{\beta}(\boldsymbol{\theta})) - a_2 \right] \\
&\quad + x\theta_{HD} \left[(a_1 + a_2)(x - 2\Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) + a_2 \right] \\
\frac{\partial \dot{\theta}_{HD}}{\partial \theta_{DH}}(\boldsymbol{\theta}) &= (1 - x)\theta_{HD} \left[(a_1 + a_2)(x - 2\Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) + a_2 \right] \\
\frac{\partial \dot{\theta}_{DH}}{\partial \alpha_{HH}}(\boldsymbol{\theta}) &= \theta_{DH} \left[(a_1 + a_2)(1 - x - 2\Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) + a_2 \right] \\
\frac{\partial \dot{\theta}_{DH}}{\partial \theta_{HD}}(\boldsymbol{\theta}) &= x\theta_{DH} \left[(a_1 + a_2)(1 - x - 2\Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) + a_2 \right] \\
\frac{\partial \dot{\theta}_{DH}}{\partial \theta_{DH}}(\boldsymbol{\theta}) &= (1 - x - \Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) \left[(a_1 + a_2)\Delta(\boldsymbol{\beta}(\boldsymbol{\theta})) - a_2 \right] + \\
&\quad + (1 - x)\theta_{DH} \left[(a_1 + a_2)(1 - x - 2\Delta(\boldsymbol{\beta}(\boldsymbol{\theta}))) + a_2 \right]
\end{aligned}
$$

14

Now we evaluate the Jacobian matrix at the different critical points:

$$\Omega(1,0,0) = \begin{pmatrix} -a_1 & -xa_1 & -(1-x)a_1 \\ 0 & -(1-x)a_1 & 0 \\ 0 & 0 & -xa_1 \end{pmatrix}$$

The eigenvalues are $\lambda_1\left[\Omega(1,0,0)\right] = -a_1$, $\lambda_2\left[\Omega(1,0,0)\right] = -(1-x)a_1$, and $\lambda_3\left[\Omega(1,0,0)\right] = -xa_1$. They have no imaginary part, and have all the same sign, negative if $a_1 > 0$. If $\mathcal{M}$ is a dom matrix with $a_1 > 0$ or if $\mathcal{M}$ is a coord matrix, then $(1,0,0)$ is asymptotically stable. Similarly the eigenvalues of $\Omega(0,1,0)$ are $\lambda_1\left[\Omega(0,1,0)\right] = (1-x)(a_1 x + a_2 x - a_2)$, $\lambda_2\left[\Omega(0,1,0)\right] = -x(a_1 x + a_2 x - a_2)$, and $\lambda_3\left[\Omega(0,1,0)\right] = -(a_1 x + a_2 x - a_2)$. The eigenvalues do not have the same sign. Thus $(0,1,0)$ cannot be asymptotically stable. The eigenvalues of $\Omega(0,0,1)$ are $\lambda_1\left[\Omega(0,0,1)\right] = x(a_1 - a_1 x - a_2 x)$, $\lambda_2\left[\Omega(0,0,1)\right] = -(1-2x)(a_1 - a_1 x - a_2 x)$, and $\lambda_3\left[\Omega(0,0,1)\right] = -(1-x)(a_1 - a_1 x - a_2 x)$. The eigenvalues do not have the same sign. Thus $(0,0,1)$ cannot be asymptotically stable. Finally the eigenvalues of $\Omega(0,0,0)$ are $\lambda_1\left[\Omega(0,0,0)\right] = -a_2$, $\lambda_2\left[\Omega(0,0,0)\right] = -xa_2$, and $\lambda_3\left[\Omega(0,0,0)\right] = -(1-x)a_2$. They have no imaginary part, and have all the same sign, negative if $a_2 > 0$ (i.e., $a_1 < 0$). If $\mathcal{M}$ is a dom matrix with $a_1 < 0$ or if $\mathcal{M}$ is a coord matrix, then $(0,0,0)$ is asymptotically stable.

2. The isolated critical points are the only critical points if $\mathcal{M}$ is a dom matrix, while points $\tilde{\boldsymbol{\theta}}$ such that $\Delta(\boldsymbol{\beta}(\tilde{\boldsymbol{\theta}})) = a$ are also critical points if $\mathcal{M}$ is a coord matrix or an anti matrix. The infinite non-isolated critical points $\tilde{\boldsymbol{\theta}}$ are unstable when $\mathcal{M}$ is a coord matrix and neutrally stable when $\mathcal{M}$ is an anti matrix. In the former case, when the population is at state $\boldsymbol{\xi}$ such that $\Delta(\boldsymbol{\beta}(\boldsymbol{\xi})) = a$, any small shock would move the population state away from $\boldsymbol{\xi}$ towards one of the sinks located either at $(1,0,0)$ or $(0,0,0)$. In the latter case, when the population is at state $\boldsymbol{\xi}$, a shock would move the population from state $\boldsymbol{\xi}$ to another state $\boldsymbol{\varphi}$ with $\Delta(\boldsymbol{\beta}(\boldsymbol{\varphi})) = a$ close enough to ensure stability (but thus violating asymptotic stability).

∎

Graphically the points $\tilde{\boldsymbol{\theta}}$ form a plane π that does not intersect the interior of the tetrahedron if $\mathcal{M}$ is a dom matrix but does otherwise. Based on the analysis of the eigenvalues above, when plane π does not cross the tetrahedron, there is necessarily one single asymptotically stable vertice (which is an evolutionary equilibrium) and one single unstable vertice, while the remaining two vertices act like saddles but in $\mathbb{R}^3$. The latter are $(0,1,0)$ and $(0,0,1)$

for any $\mathcal{M}$. When $\mathcal{M}$ is a dom matrix with $a_1 > 0$, the asymptotically stable vertice is $(1, 0, 0)$ while $(0, 0, 0)$ is unstable and vice-versa when $a_1 < 0$. Both $(1, 0, 0)$ and $(0, 0, 0)$ are asymptotically stable for a coord matrix and unstable for an anti matrix.

When plane π does intercept the tetrahedron, the latter becomes divided in two subspaces, S_1 and S_2, in such a way that $(1, 0, 0) \in S_1$ and $(0, 0, 0) \in S_2$. To view this, at the edge of the tetrahedron $\overline{\theta_{HH}\theta_{DD}}$ we have both $\theta_{HD} = \theta_{DH} = 0$. π crosses this edge at the point $\theta_{HH} = a$. Given $a \in (0, 1)$ for both cases when $\mathcal{M}$ is either a coord or an anti matrix, π separates $(1, 0, 0)$ and $(0, 0, 0)$ into two different subspaces.

4.2   Second RD approach

An alternative way to construct a replicator dynamics in an heterogeneous game $\Gamma_x(\mathcal{M})$ is to split the n individuals into two subgroups for each type of opponent: the individuals who play H versus those who play D when contesting against a type I. That is, $n = n_I^H + n_I^D$, where n_I^H is the number of individuals who play H when facing an opponent of type I and n_I^D refers to the subgroup of individuals who play D when facing an opponent of type I. But we can also divide the population into two subgroups with respect to their behaviour when contesting against a type II opponent. Following the same reasoning and notation, $n = n_{II}^H + n_{II}^D$. And we have

$$n_I^H = n\gamma_I \qquad\qquad n_{II}^H = n\gamma_{II}$$

$$n_I^D = n(1 - \gamma_I) \qquad n_{II}^D = n(1 - \gamma_{II}).$$

The population state is the repartition of the population into the pure action H against each type of opponent. It can be represented by $\boldsymbol{\gamma} = (\gamma_I, \gamma_{II})$.

In this approach the strategy played by an opponent drawn at random in state $\boldsymbol{\gamma}$ is simply $\boldsymbol{\gamma}$, it is as if the opponent played H against an individual of type I with probability γ_I and an individual of type II with probability γ_{II}.

As the pure actions are only specified for one type of opponent, we have to specify what is done against the other type. Here we assume that the individual, when facing the other type behaves as an average individual.

If reproduction takes place continuously over time, then the birthrate at any time t of individuals programmed to play H or D against types I and II results in the following

population dynamics

$$\dot{n}_I^H = n_I^H \ U((H,\gamma_{II}),\boldsymbol{\gamma}) \qquad \dot{n}_{II}^H = n_{II}^H \ U((\gamma_I,H),\boldsymbol{\gamma})$$

$$\dot{n}_I^D = n_I^D \ U((D,\gamma_{II}),\boldsymbol{\gamma}) \qquad \dot{n}_{II}^D = n_{II}^D \ U(\gamma_I,D),\boldsymbol{\gamma})$$

$$\dot{n} = \dot{n}_I^H + \dot{n}_I^D \qquad\qquad \dot{n} = \dot{n}_{II}^H + \dot{n}_{II}^D$$

Substituting (4) into $\dot{n} = \dot{n}_I^H + \dot{n}_I^D$ we obtain

$$\dot{n} = n \left[\gamma_I \ U((H,\gamma_{II}),\boldsymbol{\gamma}) + (1-\gamma_I) \ U((D,\gamma_{II}),\boldsymbol{\gamma})\right] = n \ U(\boldsymbol{\gamma},\boldsymbol{\gamma})$$

In order to obtain $\dot{\gamma}_I$ we derive $n_I^H = n\gamma_I$ to obtain $n\dot{\gamma}_I = \dot{n}_I^H - \gamma_I \ \dot{n}$ and, similarly, $n\dot{\gamma}_{II} = \dot{n}_{II}^H - \gamma_{II} \ \dot{n}$. Using (4),

$$
\begin{aligned}
n\dot{\gamma}_I &= \dot{n}_I^H - \gamma_I \dot{n} = n\gamma_I \ U((H,\gamma_{II}),\boldsymbol{\gamma}) - n\gamma_I \ U(\boldsymbol{\gamma},\boldsymbol{\gamma}) \\
&= n\gamma_I \left[U((H,\gamma_{II}),\boldsymbol{\gamma}) - U(\boldsymbol{\gamma},\boldsymbol{\gamma})\right] = n\gamma_I \left[(a_1+a_2)\Delta(\boldsymbol{\gamma}) - a_2\right]\left[\Delta(H,\gamma_{II}) - \Delta(\boldsymbol{\gamma})\right]
\end{aligned}
$$

We proceed similarly for $\dot{\gamma}_{II}$ and obtain respectively

$$
\begin{aligned}
\dot{\gamma}_I &= \left[(a_1+a_2)\Delta(\boldsymbol{\gamma}) - a_2\right]\gamma_I(1-\gamma_I)x \\
\dot{\gamma}_{II} &= \left[(a_1+a_2)\Delta(\boldsymbol{\gamma}) - a_2\right]\gamma_{II}(1-\gamma_{II})(1-x).
\end{aligned}
\tag{9}
$$

Note that at the limit case, when $x = 0$, the system reduces to $\dot{\gamma}_I = 0$ and $\dot{\gamma}_{II} = \gamma_{II}(1 - \gamma_{II})\left[(a_1+a_2)\gamma_{II} - a_2\right]$, the last equation being the replicator dynamics of the homogeneous population game. The other limit case ($x = 1$) is similar with $\dot{\gamma}_{II} = 0$.

The following theorem gives the stable states for a heterogeneous game depending on the matrix of the game, based on the second model of replicator dynamics.

Theorem 3 *Let a heterogeneous game $\Gamma_x(\mathcal{M})$ with replicator dynamics (9). (i) If $\mathcal{M}$ is a dom matrix, there is a unique asymptotically stable point, which is $\boldsymbol{\gamma}^* = (1,1)$ if $a_1 > 0$ and $\boldsymbol{\gamma}^* = (0,0)$ if $a_1 < 0$. (ii) If $\mathcal{M}$ is a coord matrix, there are two asymptotically stable points: $\boldsymbol{\gamma}_1^* = (1,1)$ and $\boldsymbol{\gamma}_2^* = (0,0)$. (iii) If $\mathcal{M}$ is an anti matrix, there are infinite non-isolated neutrally stable points, all $\tilde{\boldsymbol{\gamma}}$ such that $\Delta(\tilde{\boldsymbol{\gamma}}) = a$.*

Proof. Analyzing system (9), the system has always four isolated critical points: $(1,0)$, $(0,1)$, $(1,1)$ and $(0,0)$, which are the corners of the state space. The Jacobian matrix Ω

$$\Omega(\boldsymbol{\gamma}) = \begin{pmatrix} \frac{\partial \dot{\gamma}_I}{\partial \gamma_I}(\boldsymbol{\gamma}) & \frac{\partial \dot{\gamma}_I}{\partial \gamma_{II}}(\boldsymbol{\gamma}) \\[2mm] \frac{\partial \dot{\gamma}_{II}}{\partial \gamma_I}(\boldsymbol{\gamma}) & \frac{\partial \dot{\gamma}_{II}}{\partial \gamma_{II}}(\boldsymbol{\gamma}) \end{pmatrix}$$

with

$$\frac{\partial \dot{\gamma}_I}{\partial \gamma_I}(\boldsymbol{\gamma}) = x(1 - 2\gamma_I)\left[(a_1 + a_2)\Delta(\boldsymbol{\gamma}) - a_2\right] + \gamma_I(1 - \gamma_I)x^2(a_1 + a_2)$$

$$\frac{\partial \dot{\gamma}_I}{\partial \gamma_{II}}(\boldsymbol{\gamma}) = x(1 - x)\gamma_I(1 - \gamma_I)(a_1 + a_2)$$

$$\frac{\partial \dot{\gamma}_{II}}{\partial \gamma_I}(\boldsymbol{\gamma}) = x(1 - x)\gamma_{II}(1 - \gamma_{II})(a_1 + a_2)$$

$$\frac{\partial \dot{\gamma}_{II}}{\partial \gamma_{II}}(\boldsymbol{\gamma}) = (1 - x)(1 - 2\gamma_{II})\left[(a_1 + a_2)\Delta(\boldsymbol{\gamma}) - a_2\right] + \gamma_{II}(1 - \gamma_{II})(1 - x)^2(a_1 + a_2)$$

The Jacobian matrix Ω evaluated at the critical point $(1, 1)$, $\Omega(1, 1)$, gives the eigenvalues $\lambda_i^{(1,1)} = \{-xa_1, -(1 - x)a_1\}$, having both the same signs, and are negative if $a_1 > 0$. Thus $(1, 1)$ is an asymptotically stable point if $a_1 > 0$. $\Omega(0, 0)$ leads to $\lambda_i^{(0,0)} = \{-xa_2, -(1 - x)a_2\}$. The eigenvalues have the same signs, and are negative if $a_2 > 0$ (i.e., $a_1 < 0$). Thus $(0, 0)$ is an asymptotically stable point if $a_1 < 0$. The eigenvalues corresponding to $\Omega(1, 0)$ are $\lambda_i^{(1,0)} = \{-x(xa_1 + xa_2 - a_2), (1 - x)(xa_1 + xa_2 - a_2)\}$, hence the eigenvalues have opposite signs. Thus $(1, 0)$ is a saddle and the same is true for $(0, 1)$ given that $\Omega(0, 1)$ has eigenvalues $\lambda_i^{(0,1)} = \{x(a_1 - xa_1 - xa_2), -(1 - x)(a_1 - xa_1 - xa_2)\}$. If $\mathcal{M}$ is a dom matrix the isolated critical points are the only critical points, while points $\tilde{\boldsymbol{\gamma}}$ such that $\Delta(\tilde{\boldsymbol{\gamma}}) = a$ are also critical points if $\mathcal{M}$ is a coord matrix or an anti matrix. Graphically the points $\tilde{\boldsymbol{\gamma}}$ form a straight line π with infinite non-isolated critical points that does not intersect the interior of the $[0, 1] \times [0, 1]$ square if $\mathcal{M}$ is a dom matrix but does otherwise. Moreover, when $\mathcal{M}$ is a coord matrix, $\dot{\gamma}_I > 0$ and $\dot{\gamma}_{II} > 0$ hold in the region of the $[0, 1] \times [0, 1]$ square where $\Delta(\boldsymbol{\gamma}) > a$ and the vector field necessarily converges to the corner $(1, 1)$ as $t \to \infty$. On the other hand, when $\Delta(\boldsymbol{\gamma}) < a$, we have $\dot{\gamma}_I < 0$ and $\dot{\gamma}_{II} < 0$ and the vector field converges to the corner $(0, 0)$ as $t \to \infty$. When $\mathcal{M}$ is an anti matrix, the vector field flows in the opposite direction as described above and converges to the critical points along the straight line π when $t \to \infty$. The critical points along $\Delta(\tilde{\boldsymbol{\gamma}}) = a$ are thus neutrally stable. ∎

These stability patterns can be seen in figure 2. Linking the results of this section with those of section 3, we obtain the following result for heterogeneous games whose proof is omitted.

Proposition 4 *Let a game* $\Gamma_x(\mathcal{M})$. *A strategy* $\boldsymbol{\alpha}^*$ *is an ESS iff the state* $\boldsymbol{\alpha}^*$ *is an asymptotically stable state in replicator dynamics (9). A strategy* $\boldsymbol{\alpha}^*$ *is an NSS iff the state* $\boldsymbol{\alpha}^*$ *is a neutrally stable state in replicator dynamics (9).*

The above proposition means that a well established result in the literature can be extended for heterogeneous games with incomplete information regarding the players' own type
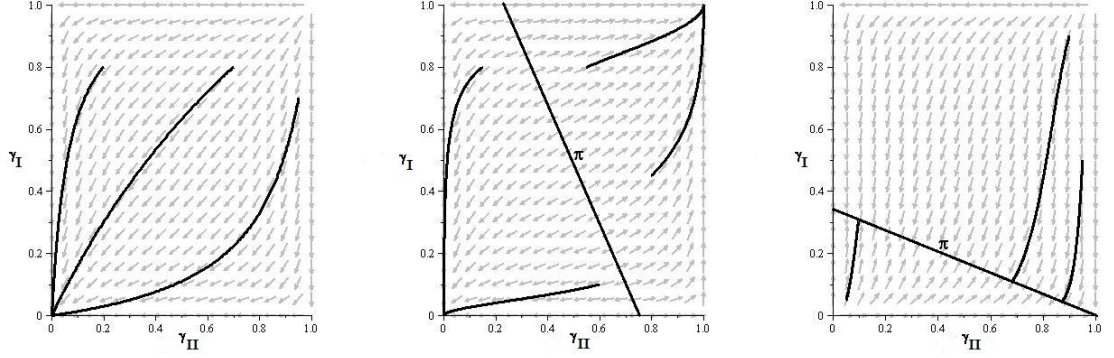
Figure 2: from left to right: dynamics for the cases when $\mathcal{M}$ is a dom matrix ($a_1 < 0$), coord matrix and anti matrix.

and symmetric two-strategy pairwise contests.[4]

4.3   Comparison between the two RD approaches

Proposition 5 *Let a heterogeneous game $\Gamma_x(\mathcal{M})$. We have that $\boldsymbol{\theta^*} = (\theta_{HH}^*, \theta_{HD}^*, \theta_{DH}^*)$ is a stable point of (8) iff $\boldsymbol{\gamma^*} = (\gamma_I^*, \gamma_{II}^*)$ is a stable point of (9), where $\gamma_I^* = \theta_{HH}^* + \theta_{HD}^*$ and $\gamma_{II}^* = \theta_{HH}^* + \theta_{DH}^*$.*

Proof. It is easy to check that given that all coefficients are non-negative and smaller or equal to 1, we have $\boldsymbol{\theta^*} = (1, 0, 0)$ iff $\boldsymbol{\gamma^*} = (1, 1)$, $\boldsymbol{\theta^*} = (0, 1, 0)$ iff $\boldsymbol{\gamma^*} = (1, 0)$, $\boldsymbol{\theta^*} = (0, 0, 1)$ iff $\boldsymbol{\gamma^*} = (0, 1)$, $\boldsymbol{\theta^*} = (0, 0, 0)$ iff $\boldsymbol{\gamma^*} = (0, 0)$. Moreover $\boldsymbol{\beta}(\boldsymbol{\theta}) = \boldsymbol{\gamma}$. In consequence $\Delta(\boldsymbol{\beta}(\boldsymbol{\theta})) = a$ iff $\Delta(\boldsymbol{\gamma}) = a$. ∎

The proposition above links both approaches of replicator dynamics, showing that the population evolves to the same asymptotically (neutrally) stable states, independently of which replicator dynamics setup is used. In the first model, the individual inherits a clearly defined behaviour toward both types of opponents (pure actions when contesting against both types I or II). On the other hand, in the second model, the natural selection was modeled using an inheritance mechanism in which individuals inherit a strategy according to which the behaviour against one type of opponent is well defined (a pure action H or D) but the behaviour against the other type is undefined (the individual randomizes her action). For example, an individual inherits the H behaviour against a type I opponent while the behaviour toward a type II opponent is random.

---

[4]See Taylor and Jonker (1978) and Bomze and Weibull (1995).

5 Conclusion

In this paper we studied evolutionary stability in all classes of two person symmetric games with two strategies, two possible types of individuals and incomplete information with respect to their own type. We called such games heterogeneous games in contrast with the well known homogeneous version with complete information in which all individuals in the population are of the same type. We started with static evolutionary games and then extended the study to the use of replicator dynamics.

We showed that the only class of such games in which the results differ from their classic homogeneous versions with complete information is the class of anti-coordination games. The latter is the only class of game in which the condition for evolutionary stable strategies vanishes. Instead, we found infinite NSS.

We also showed that in such games of incomplete information, the natural selection process can be modeled using two different replicator dynamics setups. The first setup relies on an inheritance mechanism solely based on pure strategies while the alternative replicator dynamics relies on both pure and mixed strategies. The strategies define the two actions that an individual will adopt when contesting against the two different types of opponents. Both dynamics lead to the same results.

References

[1] Arrowsmith, D. K., Place, C. M. (1998), *Dynamical systems, differential equations, maps and chaotic behaviour*, Chapman & Hall/CRC; Boca Raton, Florida, USA.

[2] Bomze, I. M., Weibull, J. W. (1995), Does Neutral Stability Imply Lyapunov Stability?, *Games and Economic Behavior* **11**, 173-192.

[3] Cressman, R., Morrison, W. G., Wen, J. F. (1998), On the Evolutionary Dynamics of Crime, *Canadian Journal of Economics* **31**, 1101-1117.

[4] Dennis, R. L., Newberry, R. C., Cheng, H. W., Estevez, I. (2008), Appearance Matters: Artificial Marking Alters Aggression and Stress, *Poultry Science* **87**, 1939.1946.

[5] Friedman, D. (1998), On Economic Applications of Evolutionary Game Theory, *Journal of Evolutionary Economics* **8**, pp. 15-43.

[6] Iñarra, E., Laruelle, A. (2011), Discriminating by Tagging: Artificial Distinction, Real Discrimination, *IKERLANAK, Working Paper Series: IL.50/11.*

[7] Ji, Q., Xian-Jia, W. (2011), Evolutionary Games in a Generalized Moran Process With Arbitrary Selection Strength and Mutation, *Chinese Physics B* **20**, 030203.

[8] Maynard Smith, J., Price, G. R. (1973), The Logic of Animal Conflicts, *Nature* **246**, 15-18.

[9] Nowak, M. A. (2006), *Evolutionary Dynamics: exploring the equations of life*, Harvard University Press; Cambridge, Massachusetts, USA.

[10] Platkowski, T., Zakrzewski, J. (2011), Asymptotically Stable Equilibrium and Limit Cycles in the Rock-Paper-Scissors Game in a Population of Players With Complex Personalities, *Physica A* **390**, 4219-4226.

[11] Taylor, P., Jonker, L. (1978), Evolutionarily Stable Strategies and Game Dynamics, *Mathematical Biosciences* **40**, 145-156.

[12] Weibull, J. W. (1995), *Evolutionary Game Theory,* MIT Press; Cambridge, Massachusetts, USA.

[13] Xu, C., Ji, M., Yap., Y. J., Zheng, D., Hui, P. M. (2011), Costly Punishment and Cooperation in the Evolutionary Snowdrift Game, *Physica A* **390**, 1607-1614.