DOCUMENTOS DE TRABAJO

BILTOKI

D.T. 2011.01

A parallelizable algorithmic framework
for solving large scale multi-stage stochastic mixed 0-1 problems
under uncertainty.

Laureano F. Escudero, María Araceli Garín, María Merino y Gloria Pérez

Documento de Trabajo BILTOKI DT2011.01
Editado por el Departamento de Economía Aplicada III (Econometría y Estadística) de la Universidad del País Vasco.

Depósito Legal No.: BI-515/2011
ISSN: 1134-8984
URN: RePEc:ehu:biltok:201101

A parallelizable algorithmic framework for solving large scale multi-stage stochastic mixed 0-1 problems under uncertainty

Laureano F. Escudero[a], María Araceli Garín[b], María Merino[c], Gloria Pérez[c]

[a]*Dpto. de Estadística e Investigación Operativa, Universidad Rey Juan Carlos, Calle Tulipán, s/n, 28933 Móstoles (Madrid), Spain*
[b]*Dpto. de Economía Aplicada III, Universidad del País Vasco, Avenida Lehendakari Aguirre, 83, 48015 Bilbao (Vizcaya), Spain*
[c]*Dpto. de Matemática Aplicada, Estadística e Investigación Operativa, Universidad del País Vasco, Barrio Sarriena s/n, 48940 Leioa (Vizcaya), Spain*

Abstract

In this paper we present a parallelizable scheme of the Branch-and-Fix Coordination algorithm for solving medium and large scale multi-stage mixed 0-1 optimization problems under uncertainty. The uncertainty is represented via a nonsymmetric scenario tree. An information structuring for scenario cluster partitioning of nonsymmetric scenario trees is also presented, given the general model formulation of a multi-stage stochastic mixed 0-1 problem. The basic idea consists of explicitly rewriting the nonanticipativity constraints (NAC) of the 0-1 and continuous variables in the stages with common information. As a result an assignment of the constraint matrix blocks into independent scenario cluster submodels is performed by a so-called cluster splitting-compact representation. This partitioning allows to generate a new information structure to express the NAC which link the related clusters, such that the explicit NAC linking the submodels together is performed by a splitting variable representation. The new algorithm has been implemented in a C++ experimental code that uses the open source optimization engine *COIN-OR*, for solving the auxiliary linear and mixed 0-1 submodels. Some computational experience is reported to validate the new proposed approach. We give computational evidence of the model tightening effect that have preprocessing techniques in stochastic integer optimization as well, by using the probing and Gomory and clique cuts identification and appending schemes of the optimization engine.

Keywords: Multi-stage stochastic mixed 0-1 optimization, nonsymmetric scenario trees, implicit and explicit nonanticipativity constraints, splitting variable and compact representations, scenario cluster partitioning.

1. Introduction

Stochastic Optimization is actually one of the most robust tools for decision making. It is broadly used in real-world applications in a wide range of problems from different areas such as finance, scheduling, production planning, industrial engineering, capacity allocation, energy, air traffic, logistics, etc. The integer problems under uncertainty have been studied in [1, 14, 15, 18, 19, 21], for just citing a few references. An extended bibliography of Stochastic Integer Programming (SIP) has been collected in [20].

It is well known that a mixed 0-1 problem under uncertainty with a finite number of possible future scenarios has a mixed 0-1 Deterministic Equivalent Model (DEM), where the risk of providing a wrong solution is included in the model via a set of representative scenarios. However, as any graph representation of this type of multi-stage models can suggest, the scenario information structuring for this type of problems is more complex than for the approximation made by considering two-stage stochastic mixed 0-1 models. We should point out that the scenario tree in real-life problems is very frequently nonsymmetric and then, the traditional splitting variable representation for the nonanticipativity constraints (for short, NAC), see [1, 16], on the 0-1 and continuous variables does not appear readily accessible for manipulations that are required by the decomposition strategies. A new type of strategies is necessary for solving medium and large scale instances of the problem. The decomposition approaches that appear most promising are based on some forms of branching selection, and scenario cluster partitioning and bounding that definitively use the information about the separability of the problem, see our work in [6, 7].

In this work we present a stochastic mixed 0-1 optimization modeling approach and a parallelizable Branch-and-Fix Coordination (BFC) algorithm for solving general mixed 0-1 optimization problems under uncertainty, where it is represented by nonsymmetric scenario trees. One of its special features is the information structuring for generating, saving and manipulating the scenario cluster submodels in a mixture of splitting variable and compact representations. Given the structuring of the scenario clusters, the approach generates independent cluster submodels, then, allowing parallel computation for obtaining lower bounds to the optimal solution value as well as feasible solutions for the problem until getting the optimal one. (Tighter lower bounds can be obtained by following the lines presented in [9] by using Lagrangean decomposition approaches in a risk aversion environment). As a result, an assignment of the constraint matrix blocks into independent scenario cluster submodels is performed. We present a splitting variable representation with explicit NAC for linking the submodels together, and a compact representation for each submodel to treat the implicit NAC related to each of the scenario clusters. Then, the algorithm that we propose uses the Twin Node Family (TNF) concept, see [6, 7, 8], and it is specially designed for coordinating and reinforcing the branching nodes and the branching 0-1 variable selection strategies at each Branch-and-Fix (BF) tree. The nonsymmetric scenario tree which will be partitioned into smaller scenario cluster subtrees. The new proposal is denoted *Nonsymmetric BFC-MS* algorithm. We report some computational experience to validate the new approach by using a testbed of medium and large scale instances. We give computational evidence of the model tightening effect that have preprocessing techniques in stochastic integer optimization as well, by using the probing and Gomory and clique cuts identification and appending schemes of the open source optimization engine *COIN-OR*.

The remainder of the paper is organized as follows. Section 2 presents the multi-stage mixed 0-1 problem under uncertainty in a splitting variable representation as well as the required information about the variables by scenario cluster and stage. An illustrative example will be used through the paper to show the main ideas that are proposed in the decomposition framework. Section 3 shows how to generate the required information in order to know until what stage the cluster submodels have common information. In Section 4 a scheme for formulating the cluster submodels is presented, such that Section 5 gives the full model in splitting variable representation linking the submodels with the explicit non-anticipativity constraints. Section 6 presents the main steps of the so-called *Nonsymmetric BFC-MS* algorithm. Section 7 reports the computational experience using COIN-OR [13] to verify the effectiveness of the proposal. Section 8 concludes. Three appendices present the constraint matrices in detail for the illustrative example, and two more give the details of the order of storage of the variables in the model.

2. Splitting variable representation in stochastic optimization

Let us consider the following multi-stage deterministic mixed 0-1 model

$$
\begin{aligned}
&\min \sum_{t\in\mathcal{T}} a_t x_t + c_t y_t \\
&s.t.\ A_1 x_1 + B_1 y_1 = b_1 \\
&\quad A'_t x_{t-1} + A_t x_t + B'_t y_{t-1} + B_t y_t = b_t \quad \forall t \in \mathcal{T} - \{1\} \\
&\quad x_t \in \{0,1\}^{nx_t},\ \ y_t \in \boldsymbol{R}^{+ny_t},
\end{aligned} \tag{1}
$$

where $\mathcal{T}$ is the set of stages (without loss of generality, let us consider that a stage is only included by one time period), such that $T = |\mathcal{T}|$, x_t and y_t are the nx_t and ny_t dimensional vectors of the 0-1 and continuous variables, respectively, a_t and c_t are the vectors of the objective function coefficients, and A_t and B_t are the constraint matrices for stage t.

This model can be extended to consider uncertainty in some of the main parameters, in our case, the objective function, the *rhs* and the constraint matrix coefficients. To introduce the uncertainty in the parameters, we will use a scenario analysis approach. A scenario consists of a realization of all random variables in all stages, that is, a path through the scenario tree. In this sense, Ω will denote the set of scenarios, $\omega \in \Omega$ will represent a specific scenario, see Figure 1, and w^ω will denote the likelihood or probability assigned by the modeler to scenario ω, such that $\sum_{\omega\in\Omega} w^\omega = 1$. We say that two scenarios belong to the same group in a given stage provided that they have the same realizations of the uncertain parameters up to the stage. Following the *nonanticipativity principle*, see [1, 16], among others, both scenarios should have the same value for the related variables with the time index up to the given stage.

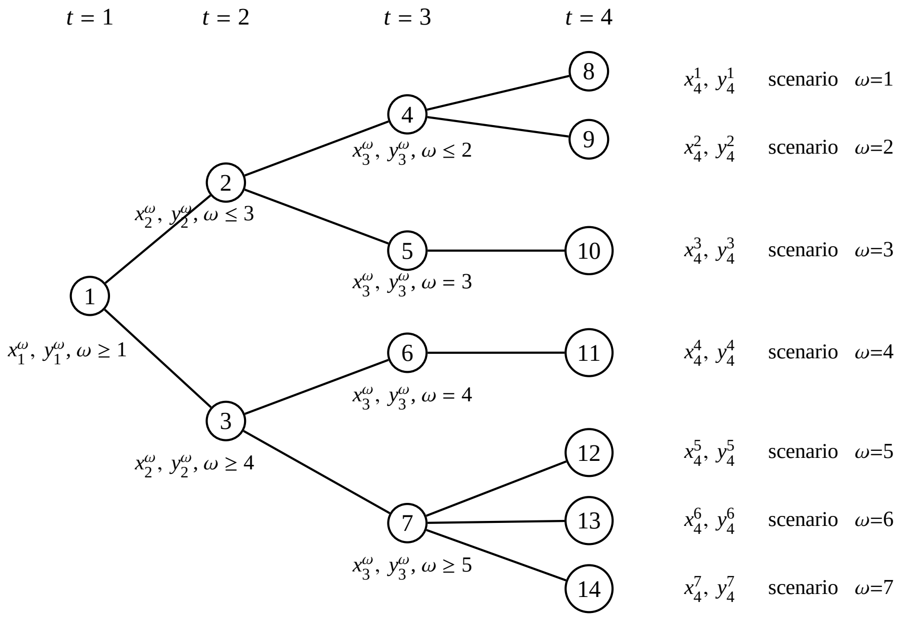


Figure 1: Scenario tree. Illustrative example.

Let also $\mathcal{G}$ denote the set of scenario groups (i.e., nodes in the underlying scenario tree), and $\mathcal{G}_t$ denote the subset of scenario groups that belong to stage $t \in \mathcal{T}$, such that $\mathcal{G} = \cup_{t\in\mathcal{T}} \mathcal{G}_t$. Ω_g denotes the set of scenarios in group g, for $g \in \mathcal{G}$. Note that the scenario group concept corresponds to the node concept in the underlying scenario tree.

The *splitting variable* representation of the DEM of the full recourse stochastic version related to the multi-stage deterministic problem (1) can be expressed as follows,

$$
\begin{aligned}
z_{MIP} = \min \sum_{\omega\in\Omega}\sum_{t\in\mathcal{T}} w^{\omega}\Big(a_t^{\omega}x_t^{\omega} + c_t^{\omega}y_t^{\omega}\Big) \\
s.t.\ A_1x_1^{\omega} + B_1y_1^{\omega} = b_1 \ \ \forall\omega\in\Omega \\
A_t'^{\omega}x_{t-1}^{\omega} + A_t^{\omega}x_t^{\omega} + B_t'^{\omega}y_{t-1}^{\omega} + B_t^{\omega}y_t^{\omega} = b_t^{\omega},\ \forall\omega\in\Omega,\ t\geq 2 \\
x_t^{\omega} - x_t^{\omega'} = 0,\ \forall\omega,\omega'\in\Omega_g : \omega\neq\omega',\ g\in\mathcal{G}_t,\ t\leq T-1 \\
y_t^{\omega} - y_t^{\omega'} = 0,\ \forall\omega,\omega'\in\Omega_g : \omega\neq\omega',\ g\in\mathcal{G}_t,\ t\leq T-1 \\
x_t^{\omega}\in\{0,1\}^{nx_t^{\omega}},\ \ y_t^{\omega}\in\boldsymbol{R}^{+ny_t^{\omega}},\ \forall\omega\in\Omega,\ t\in\mathcal{T}.
\end{aligned}
\tag{2}
$$

Following the nonanticipativity principle cited above for each stage t, the corresponding equalities must be satisfied,

$$
\begin{aligned}
A_t^{'\omega} = A_t^{'\omega'},\ A_t^{\omega} = A_t^{\omega'},\ B_t^{'\omega} = B_t^{'\omega'},\ B_t^{\omega} = B_t^{\omega'},\ b_t^{\omega} = b_t^{\omega'}, a_t^{\omega} = a_t^{\omega'}, c_t^{\omega} = c_t^{\omega'}, \\
\forall\omega,\omega'\in\Omega_g : \omega\neq\omega',\ g\in\mathcal{G}_t,\ 2\leq t\leq T-1.
\end{aligned}
\tag{3}
$$

Observe that for a given stage t, $A_t^{'\omega}$ and A_t^{ω} are the technology and recourse matrices for the x_t variables and $B_t^{'\omega}$ and B_t^{ω} are the corresponding ones for the y_t variables. Notice that $x_t^{\omega} - x_t^{\omega'} = 0$ and $y_t^{\omega} - y_t^{\omega'} = 0$ are the NAC. Finally, nx_t^{ω} and ny_t^{ω} denote the dimensions of the vectors of the variables x and y variables, respectively, related to stage t under scenario ω.

Definition 1. *A* Branch-and-Fix *(BF) tree associated with any scenario is the classical Branch-and-Bound tree in the integer optimization model for that scenario.*

As an additional notation, let $\mathcal{R}^{\omega}$ denote the BF tree associated with scenario ω, $\mathcal{A}^{\omega}$ be the set of active nodes in $\mathcal{R}^{\omega}$ for $\omega\in\Omega$, $\mathcal{I}$ the set of indices of the variables in any vector x_t^{ω}, and $(x_t^{\omega})_i$ the i-th variable in x_t^{ω}, for $t\in\mathcal{T}$, $\omega\in\Omega$, $i\in\mathcal{I}$.

Definition 2. *Two variables, say, $(x_t^{\omega})_i$ and $(x_t^{\omega'})_i$ are said to be* common *variables for the scenarios ω and ω', if $\omega,\omega'\in\Omega_g$, $g\in\mathcal{G}_t$, for $\omega\neq\omega'$, $t\in\mathcal{T}^-$, $i\in\mathcal{I}$. Notice that two* common *variables have nonzero elements in the NAC related to a given scenario group.*

Definition 3. *Any two nodes, say, $a\in\mathcal{A}^{\omega}$ and $a'\in\mathcal{A}^{\omega'}$ are said to be* twin *nodes with respect to a given scenario group if the paths from their root nodes to each of them in their own BF trees $\mathcal{R}^{\omega}$ and $\mathcal{R}^{\omega'}$, respectively, either having not yet branched on / fixed at their* common *variables, if any, or having the same 0-1 value for their branched on / fixed at their* common *variables $(x_t^{\omega})_i$ and $(x_t^{\omega'})_i$, for $\omega,\omega'\in\Omega_g, g\in\mathcal{G}_t, t\in\mathcal{T}^-, i\in\mathcal{I}$.*

Definition 4. *A* Twin Node Family *(TNF), say, $\mathcal{J}_f$ is a set of nodes such that any node is a* twin *node to all the other node members in the family, for $f\in\mathcal{F}$, where $\mathcal{F}$ is the set of the families.*

Definition 5. *A* candidate TNF *is a TNF whose members have not yet branched on / fixed at all their* common *variables.*

Definition 6. *A* TNF integer set *is a set of TNFs where all x variables take integer values, there is one node per each BF tree and the NAC $(x_t^{\omega})_i - (x_t^{\omega'})_i = 0$ are satisfied, $\forall\omega,\omega'\in\Omega_g, g\in\mathcal{G}_t, t\in\mathcal{T}^-, i\in\mathcal{I}$.*

The scenario tree information given in Figure 1 can also be represented and managed by using the vector $\boldsymbol{R}$ given in the following definition.

Definition 7. *A general* **scenario tree** *in compact notation can be uniquely defined by $\mathcal{R} = (r(g)\ :\ g\in\cup_{t=1}^{T-1}\mathcal{G}_t)$, where $r(g)\in\boldsymbol{N}$ is the number of branches arising from the stage of group g, $g\leq|\mathcal{G}_{T-1}|$, to the next stage. That is,*

$$
\mathcal{R} = (\overbrace{r_{1|\mathcal{G}_1|}}^{t=1}\ |\ \overbrace{r_{21},\ r_{22},\ \ldots,\ r_{2|\mathcal{G}_2|}}^{t=2}\ |\ \overbrace{r_{31},\ r_{32},\ \ldots,\ r_{3|\mathcal{G}_3|}}^{t=3}\ |...|\ \overbrace{r_{T-1,1},\ r_{T-1,2},\ \ldots,\ r_{T-1,|\mathcal{G}_{T-1}|}}^{t=T-1}),
$$

where the number of groups for stage t, $|\mathcal{G}_t|$, corresponds to the sum of branches of the previous stage:

$$
|\mathcal{G}_1| = 1,\ |\mathcal{G}_{t+1}| = \sum_{i=1}^{|\mathcal{G}_t|} r_{ti},\ t\leq T-1
$$

A ***symmetric tree*** assumes that the number of branches is the same for all conditional distributions in the same stage, that is, for stage $t \leq T-1$, $r_{ti} = r_{tj}$, $\forall i \neq j$, $1 \leq i, j \leq |\mathcal{G}_t|$. A ***nonsymmetric tree*** is a non symmetric one. Moreover, without lost of generality in this work, we consider nonsymmetric trees, see below.

For the example given in [1], page 130, the scenario tree shown in Figure 1 can be defined by $\mathcal{R} = (2 \mid 2\ 2 \mid 2\ 1\ 1\ 3)$. So, $|\mathcal{G}_1| = 1$, $|\mathcal{G}_2| = 2$, $|\mathcal{G}_3| = 4$, $|\mathcal{G}_4| = 7 = |\Omega|$ and $T = 4$. The set of scenarios is $\Omega = \{1, 2, \ldots, 7\}$, and the subsets of scenario groups are $\mathcal{G}_1 = \{1\}$, $\mathcal{G}_2 = \{2,\ 3\}$, $\mathcal{G}_3 = \{4, 5, 6, 7\}$, $\mathcal{G}_4 = \{8, 9, \ldots, 14\}$ and $\mathcal{G} = \cup_{t=1}^{4} \mathcal{G}_t$. Finally, the scenarios in each group g, are: $\Omega_1 = \{1, \ldots, 7\}$, $\Omega_2 = \{1, 2, 3\}$, $\Omega_3 = \{4, \ldots, 7\}$, $\Omega_4 = \{1, 2\}$, $\Omega_5 = \{3\}$, $\Omega_6 = \{4\}$, $\Omega_7 = \{5, 6, 7\}$, $\Omega_8 = \{1\}$, $\Omega_9 = \{2\}$, $\Omega_{10} = \{3\}$, $\Omega_{11} = \{4\}$, $\Omega_{12} = \{5\}$, $\Omega_{13} = \{6\}$ and $\Omega_{14} = \{7\}$.

In general, for any multi-stage stochastic problem with T stages and $|\Omega|$ scenarios, the information about until what stage the scenario submodels have common information, and when the NAC must be explicit, is saved in the subsets $\mathcal{G}_t$ and Ω_g, $g \in G_t$, $t \in T$, i.e., in the scenario tree $\mathcal{R}$ or, alternatively, in the *scenario tree matrix*, defined below.

Definition 8. *The **scenario tree matrix**, $\mathcal{ST} \in M_{|\Omega| \times |\mathcal{G}|}$, is a matrix where the corresponding value for the pair (ω, g) gives the related stage t, such that*

$$\mathcal{ST}(\omega, g) = \begin{cases} t, & \text{if } \omega \in \Omega_g \text{ and } g \in \mathcal{G}_t \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Notice that the scenario tree matrix reproduces the structure given by the scenario tree $\mathcal{R}$. This matrix has been built by using the sets Ω_g and $\mathcal{G}_t$, i.e., the scenario tree $\mathcal{R}$, but these sets can be also generated from the matrix. For each stage $t \in \mathcal{T}$, we can obtain the set of scenario groups in such stage, $\mathcal{G}_t$, as the column of the position (ω, g), for which the corresponding element in the scenario tree matrix is equal to t; then $\mathcal{G}_t = \{g \in \mathcal{G} \mid \exists \omega \in \Omega : \ \mathcal{ST}(\omega, g) = t\}$. See also that the set of scenarios related to group g is $\Omega_g = \{\omega \in \Omega \mid \mathcal{ST}(\omega, g) \neq 0\}$. For our example, the scenario tree matrix, $\mathcal{ST}(\omega, g)$, is given in (5).

$$\mathcal{ST}(\omega, g) = \left(\begin{array}{c|cc|cccc|ccccccc} 1 & 2 & 0 & 3 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{array} \right). \tag{5}$$

3. Scenario clustering in nonsymmetric scenario trees

It is clear that the explicit representation of the NAC is not required for all pairs of scenarios in order to reduce the dimensions of model. In fact, we can represent implicitly the NAC for some pairs of scenarios in order to gain computational efficiency.

We will decompose the scenario tree into a subset of scenario clusters, where $\mathcal{P} = \{1, ..., q\}$ denotes the set of clusters and $q = |\mathcal{P}|$. Let Ω^p denote the set of scenarios that belongs to a generic cluster p, where $p \in \mathcal{P}$ and $\sum_{p=1}^{q} |\Omega^p| = |\Omega|$. It is clear that the criterion for scenario clustering in the sets, say, $\Omega^1, \ldots, \Omega^q$ is instance dependent. Moreover, we favor the approach that shows higher scenario clustering for greater number of scenario groups in common. In any case, notice that $\Omega^p \bigcap \Omega^{p'} = \emptyset$, $p, p' = 1, \ldots, q : p \neq p'$ and $\Omega = \cup_{p=1}^{q} \Omega^p$. Let also $\mathcal{G}^p \subset \mathcal{G}$ denote the set of scenario groups for cluster p, such that $\Omega_g \cap \Omega^p \neq \emptyset$ means that $g \in \mathcal{G}^p$, $\mathcal{G}_t^p = \mathcal{G}_t \cap \mathcal{G}^p$ denote the set of scenario groups for cluster $p \in \mathcal{P}$ in stage $t \in \mathcal{T}$.

We propose to choose the number of scenario clusters q as any value from the subset $\mathcal{Q} = \{|\mathcal{G}_1|, |\mathcal{G}_2|, \ldots, |\mathcal{G}_T|\}$. As we will see below, the value q will be associated with the number of stages with explicit NAC between cluster submodels.

Definition 9. *The **break stage** t^* is the stage t such that the number of scenario clusters is $q = |\mathcal{G}_{t^*+1}|$, where $t^*+1 \in \mathcal{T}$. Observe that cluster $p \in \mathcal{P}$ includes the scenarios that belong to group $g \in \mathcal{G}_{t^*+1}$, i.e., $\Omega^p = \Omega_g$.*

Definition 10. *The* scenario cluster *models are those that result from the relaxation of NAC until the break stage t^* in model (2).*

Notice that the choice of $t^* = 0$ corresponds to the full model and $t^* = T - 1$ corresponds to the scenario partitioning.

Definition 11. *The* ***cluster tree matrix*** *associated with the t^*–decomposition, $\mathcal{CT}^{t*} \in M_{q \times |\mathcal{G}|}$, is a matrix where the corresponding value for the pair (p, g) gives the related stage t, such that*

$$\mathcal{CT}^{t^*}(p,g) = \begin{cases} t, & \text{if } g \in \mathcal{G}_t^p \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Notice that $\mathcal{G}_t^p = \mathcal{G}_t \cap \mathcal{G}^p$, is the set of scenario groups for cluster $p \in \mathcal{P}$ in stage $t \in \mathcal{T}$.

Once decided the break stage, t^*, the corresponding cluster partition is given, and its structure is defined by the related cluster tree matrix.

Property 1. *For any stage $2 \le t \le t^* + 1$ and any cluster $p \in \mathcal{P}$, the cardinality of the subset of groups that belong to cluster p at stage t, $|\mathcal{G}_t^p|$ is always equal to 1.*

Property 2. *For any stage $t^* + 1 < t \le T$ and any cluster $p \in \mathcal{P}$, the cardinality of the subset of groups that belong to cluster p at stage t, $|\mathcal{G}_t^p|$ is greater than 1, unless if one realization of the uncertain parameters exactly occurs from stage $t - 1$ to stage t in cluster p, in which case it is also equal to 1.*

Notice that the subsets $\mathcal{G}^p$ and $\mathcal{G}_t$ and, consequently, $\mathcal{G}_t^p$ can be obtained from the cluster tree matrix given above. For each cluster $p \in \mathcal{P}$ (i.e., p–row in matrix $\mathcal{CT}^{t^*}$), the set of scenario groups $\mathcal{G}^p$ can be obtained as the set of columns in the t^*–cluster tree matrix with a nonzero element, i.e., $\mathcal{G}^p = \{g \in \mathcal{G} \mid \mathcal{CT}^{t^*}(p,g) \neq 0\}$. Similarly, the set $\mathcal{G}_t$ of scenario groups in each stage $t \in \mathcal{T}$ can be obtained as $\mathcal{G}_t = \{g \in \mathcal{G} \mid \exists p \in \mathcal{P} : \ \mathcal{CT}^{t^*}(p,g) = t\}$.

In the illustrative example depicted in Figure 1, three cases can be considered for generating the q cluster submodels where q can be chosen from the set of values $\{|\mathcal{G}_2|, |\mathcal{G}_3|, |\mathcal{G}_4|\} = \{2, 4, 7\}$, namely:

- **Case 1**. Let the break stage $t^* = 1$, then there are $q = |\mathcal{G}_2| = 2$ clusters, see Figure 2 and, then, two subsets of scenario groups, say $\mathcal{G}^1 = \{1, 2, 4, 5, 8, 9, 10\}$ and $\mathcal{G}^2 = \{1, 3, 6, 7, 11, 12, 13, 14\}$, where the scenarios in each set are $\Omega^1 = \{1, 2, 3\}$ and $\Omega^2 = \{4, 5, 6, 7\}$.

  The 1-cluster tree matrix is given in (7).

$$\mathcal{CT}^1(p,g) = \left(\begin{array}{c|cc|cccc|ccccccc} 1 & 2 & 0 & 3 & 3 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \end{array}\right). \tag{7}$$

- **Case 2**. Let the break stage $t^* = 2$, then there are $q = |\mathcal{G}_3| = 4$ clusters, see Figure 3 and, then, four subsets of scenario groups, say $\mathcal{G}^1 = \{1, 2, 4, 8, 9\}$, $\mathcal{G}^2 = \{1, 2, 5, 10\}$, $\mathcal{G}^3 = \{1, 3, 6, 11\}$, and $\mathcal{G}^4 = \{1, 3, 7, 12, 13, 14\}$, where the scenarios in each set are $\Omega^1 = \{1, 2\}$, $\Omega^2 = \{3\}$, $\Omega^3 = \{4\}$ and $\Omega^4 = \{5, 6, 7\}$.

  The 2-cluster tree matrix is given in (8).

$$\mathcal{CT}^2(p,g) = \left(\begin{array}{c|cc|cccc|ccccccc} 1 & 2 & 0 & 3 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 4 & 4 & 4 \end{array}\right). \tag{8}$$

- **Case 3**. Let the break stage $t^* = 3$, then there are $q = |\mathcal{G}_4| = 7$ clusters, see Figure 4 and, then, seven sets of scenario groups, say $\mathcal{G}^1 = \{1, 2, 4, 8\}$, $\mathcal{G}^2 = \{1, 2, 4, 9\}$, $\mathcal{G}^3 = \{1, 2, 5, 10\}$, $\mathcal{G}^4 = \{1, 3, 6, 11\}$, $\mathcal{G}^5 = \{1, 3, 7, 12\}$, $\mathcal{G}^6 = \{1, 3, 7, 13\}$ and $\mathcal{G}^7 = \{1, 3, 7, 14\}$, and seven sets of scenarios: $\Omega^1 = \{1\}$, $\Omega^2 = \{2\}$, $\ldots$, and $\Omega^7 = \{7\}$.

  Notice that the 3-cluster tree matrix $\mathcal{CT}^3$ is $\mathcal{ST}$, see (5).

Notice that in the above scenario cluster partitioning we favor the approach that shows higher scenario clustering for greater number of scenario groups in common.

4. Scenario cluster submodels

Let us assume that we have broken down the scenario tree into q clusters. Now, let us formulate the cluster submodels, and next the full mixed 0-1 DEM via splitting variable representation, so that the q cluster submodels are linked by the explicit NAC until stage t^*. For doing so, let $\mathbf{x}_t^p$ and $\mathbf{y}_t^p$ denote the vectors of the 0-1 and continuous variables, respectively, for scenario cluster $p \in \mathcal{P}$ and stage $t \in \mathcal{T}$. Let also nx_t^p and ny_t^p denote the number of 0-1 variables and number of continuous variables for the pair (p, t), respectively. For implementation purposes, the storage order of the variables is very important. We show in the Appendices D and E the order that we propose.

Definition 12. *The* ***representative scenario*** *for scenario group g in cluster p at stage t is the first ordered scenario in the scenario group, $\omega_g^p = min\{\omega \in \Omega_g\},\ g \in \mathcal{G}_t^p,\ p \in \mathcal{P},\ t \in \mathcal{T}$.*

The set of constraints is split such that the first block is related to the first stage, the second block represents the constraints related to the vectors of variables until stage $t^* + 1$ (i.e., stages with explicit NAC) that must be linked with their own replicas in all the other clusters $p' \in \mathcal{P}$, and the third block represents the constraints related to the vectors of variables from stage $t^* + 2$ (i.e., stages with implicit NAC).

$$
\begin{aligned}
& (MIP^p) \quad z^p = \sum_{t=1}^{T} w_t^p (a_t^p \mathbf{x}_t^p + c_t^p \mathbf{y}_t^p) \\
s.t. \quad & \mathbf{A}_1 \mathbf{x}_1^p + \mathbf{B}_1 \mathbf{x}_1^p = \mathbf{b}_1 \\
& \mathbf{A}_t^{'p} \mathbf{x}_{t-1}^p + \mathbf{A}_t^p \mathbf{x}_t^p + \mathbf{B}_t^{'p} \mathbf{y}_{t-1}^p + \mathbf{B}_t^p \mathbf{y}_t^p = \mathbf{b}_t^p, 2 \le t \le t^* + 1 \\
& [\mathbf{A}_t']^p \mathbf{x}_{t-1}^p + [\mathbf{A}_t]^p \mathbf{x}_t^p + [\mathbf{B}_t']^p \mathbf{y}_{t-1}^p + [\mathbf{B}_t]^p \mathbf{y}_t^p = \mathbf{b}_t^p,\ t^* + 1 < t \le T \\
& \mathbf{x}_t^p \in \{0,1\}^{nx_t^p},\ \ \mathbf{y}_t^p \in \boldsymbol{R}^{+ny_t^p},\ t \in \mathcal{T},
\end{aligned}
\tag{9}
$$

where w_t^p is the weight of cluster p in stage t to be expressed in (11).

The first block of constraint matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ is related to the first stage vectors of variables $\mathbf{x}_1^p$ and $\mathbf{y}_1^p$, respectively, whose *rhs* is $\mathbf{b}_1$.

The second block of constraint matrices $(\mathbf{A}_t^{'p}, \mathbf{A}_t^p, \mathbf{B}_t^{'p}, \mathbf{B}_t^p)$ is related to the stages $2 \le t$ until stage $t^* + 1$. For all the stages $t \le t^* + 1$, the weight of cluster p at stage t is $w_t^p = \sum_{\omega \in \Omega_g\ :\ g \in \mathcal{G}_t^p} w^\omega$. In a similar way, we can define the objective function coefficients a_t^p and c_t^p for $t \le t^* + 1$.

Finally, the third block represents the constraints for stages from $t^* + 2$ until the last one, T. In all of these stages, the nonanticipativity principle is implicity taken into account, since the submodel for each cluster is formulated via a compact representation. The constraint matrices $[\mathbf{A}_t']^p$ and $[\mathbf{B}_t']^p$, and $[\mathbf{A}_t]^p$ and $[\mathbf{B}_t]^p$ can be split into the $|\mathcal{G}_{t-1}^p|$ and $|\mathcal{G}_t^p|$ submatrices related to the scenarios groups in a given cluster p, respectively. For scenario group $g_i \in \mathcal{G}_t^p$, $i \in \{1, \ldots, |\mathcal{G}_t^p|\}$ in cluster p at stage t, let the representative scenario $\omega_{g_i}^p = min\ \{\omega \in \Omega_{g_i}\}$ to define the related block of matrices. In a similar way, the matrices $[\mathbf{B}_t']^p$ and $[\mathbf{B}_t]^p$ can be obtained.

Notice that the matrices $[\mathbf{A}_t']^p$ and $[\mathbf{B}_t']^p$ have $|G_{t-1}|$ columns, while the matrices $[\mathbf{A}_t]^p$ and $[\mathbf{B}_t]^p$ have $|G_t|$ columns. It can be observed that if there are explicit NAC in stage $t-1$, then $[\mathbf{A}_t']^p$ and $[\mathbf{B}_t']^p$ would be block diagonal matrices with the same number of columns as $[\mathbf{A}_t]^p$ and $[\mathbf{B}_t]^p$, that is, $|\mathcal{G}_t^p|$, see (10). But, since the NAC are implicitly considered, then the matrices become grouped matrices by columns, such that they will have in the same column the matrices $A_t'^{\omega}$ and $A_t'^{\omega'}$ for $\mathbf{x}_{t-1}^p$, and $B_t'^{\omega}$ and $B_t'^{\omega'}$ for $\mathbf{y}_{t-1}^p$, respectively, where $x_{t-1}^{\omega} = x_{t-1}^{\omega'}$ and $y_{t-1}^{\omega} = y_{t-1}^{\omega'}$ $\forall \omega, \omega' \in \Omega_g : \omega \neq \omega',\ g \in \mathcal{G}_t^p,\ t^* + 1 < t \le T$. Notice that these matrices can easily loose the diagonal block structure, see below and the Appendices A, B and C.

$$[\mathbf{A}'_t]^p = \underbrace{\begin{pmatrix} A_t^{'\omega^p_{g_1}} & 0 & \dots & 0 \\ 0 & A_t^{'\omega^p_{g_2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_t^{'\omega^p_{g_{|\mathcal{G}^p_{t-1}|}}} \end{pmatrix}}_{|\mathcal{G}^p_{t-1}| \text{ columns}}, \quad [\mathbf{A}_t]^p = \underbrace{\begin{pmatrix} A_t^{\omega^p_{g_1}} & 0 & 0 & \dots & 0 \\ 0 & A_t^{\omega^p_{g_2}} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_t^{\omega^p_{g_{|\mathcal{G}^p_t|}}} \end{pmatrix}}_{|\mathcal{G}^p_t| \text{ columns}} \tag{10}$$

Since $\mathbf{x}^p_t$, $\mathbf{y}^p_t$ and $\mathbf{b}^p_t$ have the dimension $|\mathcal{G}^p_t|$, the weight w^p_t of cluster p in stage t is as follows,

$$\mathbf{x}^p_t = \begin{pmatrix} x_t^{\omega^p_{g_1}} \\ x_t^{\omega^p_{g_2}} \\ \vdots \\ x_t^{\omega^p_{g_{|\mathcal{G}^p_t|}}} \end{pmatrix}, \mathbf{b}^p_t = \begin{pmatrix} b_t^{\omega^p_{g_1}} \\ b_t^{\omega^p_{g_2}} \\ \vdots \\ b_t^{\omega^p_{g_{|\mathcal{G}^p_t|}}} \end{pmatrix}, w^p_t = \begin{pmatrix} \sum_{\omega=\omega^p_{g_1}}^{\omega^p_{g_2}-1} w^\omega, \sum_{\omega=\omega^p_{g_2}}^{\omega^p_{g_3}-1} w^\omega, \dots, \sum_{\omega=\omega^p_{g_{|\mathcal{G}^p_t|}}}^{\omega=\overline{\omega}^p_t} w^\omega \end{pmatrix}, \tag{11}$$

where $\overline{\omega}^p_t = \max\{\omega \in \Omega_g \, g \in \mathcal{G}^p_t\}$ denotes the last ordered scenario in cluster p at stage t. Similarly, the objective function coefficients a^p_t and c^p_t can be defined.

The q cluster submodels (9) are linked by the NAC, that now can be formulated as follows,

$$\mathbf{x}^p_t - \mathbf{x}^{p'}_t = 0, \quad p \neq p', \ t \leq t^*, g \in \mathcal{G}_t, \ t = C\mathcal{T}^{t^*}(p,g) = C\mathcal{T}^{t^*}(p',g) \tag{12}$$

$$\mathbf{y}^p_t - \mathbf{y}^{p'}_t = 0, \quad p \neq p', \ t \leq t^*, g \in \mathcal{G}_t, \ t = C\mathcal{T}^{t^*}(p,g) = C\mathcal{T}^{t^*}(p',g). \tag{13}$$

Let us consider the three previous cases for the example depicted in Figure 1, where $T = 4$, $|\Omega| = 7$ and $|\mathcal{G}| = 14$.

- **Case 1**. Consider explicit NAC until stage $t^* = 1$ and, then, $q = 2$ clusters, whose scenario groups are given in Table 1. Using the 1-cluster tree matrix (7), the subset of scenario groups for cluster p and stage t, $\mathcal{G}^p_t$ can be determined. In this case, all of these subsets have a singleton element (see Property 1) until $t^* + 1 = 2$. And from $t^* + 2 = 3$ these subsets have one or more elements (see Property 2).

Table 1: Scenario groups for $q = 2$. Illustrative example

$\mathcal{G}^p_t$	$p = 1$	$p = 2$
$t = 1$	{1}	{1}
$t = 2$	{2}	{3}
$t = 3$	{4,5}	{6,7}
$t = 4$	{8,9,10}	{11,12,13,14}

Let us define the blocks of the matrices by stages. Obviously, the matrices for the first block (stage $t = 1$ and $q = 2$ cluster models) are are follows: $\mathbf{A}_1 := A_1$, $\mathbf{B}_1 := B_1$ and the *rhs* $\mathbf{b}_1 := b_1$.

The matrices for the second block (stages $2 \leq t \leq t^* + 1 = 2$) are are follows:

1. **For** $p = 1$**:** $\mathbf{A}_t^{'1} := A_t^{'\omega^1_g}$, $\mathbf{B}_t^{'1} := B_t^{'\omega^1_g}$, $\mathbf{A}^1_t := A_t^{\omega^1_g}$, $\mathbf{B}^1_t := B_t^{\omega^1_g}$, and $\mathbf{b}^1_t := b_t^{\omega^1_g}$, $2 \leq t \leq 2$, where the representative scenario for $t = 2$, $g \in \mathcal{G}^1_t = \{2\}$ is $\omega^1_2 = min\{\omega \in \Omega_2\} = 1$.
2. **For** $p = 2$**:** $\mathbf{A}_t^{'2} := A_t^{'\omega^2_g}$, $\mathbf{B}_t^{'2} := B_t^{'\omega^2_g}$, $\mathbf{A}^2_t := A_t^{\omega^2_g}$, $\mathbf{B}^2_t := B_t^{\omega^2_g}$, and $\mathbf{b}^2_t := b_t^{\omega^2_g}$, $2 \leq t \leq 2$, where the representative scenario for $t = 2$, $g \in \mathcal{G}^2_t = \{3\}$ is $\omega^2_3 = min\{\omega \in \Omega_3\} = 4$.

The matrices for the third block $[\mathbf{A}'_t]^p$ and $[\mathbf{A}_t]^p$ are as follows for stage $t^* + 1 = 2 < t \leq 4$:

1. **For** $p = 1$**:** For stage $t = 3$ and scenario group $g_i \in \mathcal{G}_3^1 = \{4,5\}$, $i \in \{1,\ldots,|\mathcal{G}_3^1|\} = \{1,2\}$, the representative scenario $\omega_{g_i}^1 = min\ \{\omega \in \Omega_{g_i}\}$ for group g_i is $\omega_4^1 = min\ \{\omega \in \Omega_4\} = 1$ for group $g_1 = 4$ and $\omega_5^1 = min\ \{\omega \in \Omega_5\} = 3$ for group $g_2 = 5$.
   Due to $x_2^1 = x_2^3$ from NAC, $[\mathbf{A}_3']^1 = \begin{pmatrix} A_3'^1 \\ A_3'^3 \end{pmatrix}$, $[\mathbf{A}_3]^1 = \begin{pmatrix} A_3^1 & 0 \\ 0 & A_3^3 \end{pmatrix}$ and the corresponding vectors of the x variables are $\mathbf{x}_2^1 = x_2^1$ and $\mathbf{x}_3^1 = \begin{pmatrix} x_3^1 \\ x_3^3 \end{pmatrix}$.
   For stage $t = 4$ and scenario group $g_i \in \mathcal{G}_4^1 = \{8,9,10\}$, $i \in \{1,\ldots,|\mathcal{G}_4^1|\} = \{1,2,3\}$, the representative scenario $\omega_{g_i}^1 = min\ \{\omega \in \Omega_{g_i}\}$ for group g_i is $\omega_8^1 = min\ \{\omega \in \Omega_8\} = 1$ for group $g_1 = 8$, $\omega_9^1 = min\ \{\omega \in \Omega_9\} = 2$ for group $g_2 = 9$ and $\omega_{10}^1 = min\ \{\omega \in \Omega_{10}\} = 3$ for group $g_3 = 10$.
   Due to $x_3^1 = x_3^2$ from NAC, $[\mathbf{A}_4']^1 = \begin{pmatrix} A_4'^1 & 0 \\ A_4'^2 & 0 \\ 0 & A_4'^3 \end{pmatrix}$, $[\mathbf{A}_4]^1 = \begin{pmatrix} A_4^1 & 0 & 0 \\ 0 & A_4^2 & 0 \\ 0 & 0 & A_4^3 \end{pmatrix}$ and the vectors of the x variables are $\mathbf{x}_3^1 = \begin{pmatrix} x_3^1 \\ x_3^3 \end{pmatrix}$ and $\mathbf{x}_4^1 = \begin{pmatrix} x_4^1 \\ x_4^2 \\ x_4^3 \end{pmatrix}$.
2. **For** $p = 2$**:** For stage $t = 3$ and scenario group $g_i \in \mathcal{G}_3^2 = \{6,7\}$, $i \in \{1,\ldots,|\mathcal{G}_3^2|\} = \{1,2\}$, the representative scenario $\omega_{g_i}^2 = min\ \{\omega \in \Omega_{g_i}\}$ for group g_i is $\omega_6^2 = min\ \{\omega \in \Omega_6\} = 4$ for group $g_1 = 6$ and $\omega_7^2 = min\ \{\omega \in \Omega_7\} = 5$ for group $g_2 = 7$.
   Due to $x_2^4 = x_2^5$ from NAC, $[\mathbf{A}_3']^2 = \begin{pmatrix} A_3'^4 \\ A_3'^5 \end{pmatrix}$, $[\mathbf{A}_3]^2 = \begin{pmatrix} A_3^4 & 0 \\ 0 & A_3^5 \end{pmatrix}$ and the corresponding vectors of the x-variables are $\mathbf{x}_2^2 = x_2^4$ and $\mathbf{x}_3^2 = \begin{pmatrix} x_3^4 \\ x_3^5 \end{pmatrix}$.
   For stage $t = 4$ and scenario group $g_i \in \mathcal{G}_4^2 = \{11,12,13,14\}$, $i \in \{1,\ldots,|\mathcal{G}_4^2|\} = \{1,2,3,4\}$, the representative scenario $\omega_{g_i}^2 = min\ \{\omega \in \Omega_{g_i}\}$ for group g_i is $\omega_{11}^2 = min\ \{\omega \in \Omega_{11}\} = 4$, for group $g_1 = 11$, $\omega_{12}^2 = min\ \{\omega \in \Omega_{12}\} = 5$ for group $g_2 = 12$, $\omega_{13}^2 = min\ \{\omega \in \Omega_{13}\} = 6$ for group $g_3 = 13$ and $\omega_{14}^2 = min\ \{\omega \in \Omega_{14}\} = 7$ for group $g_4 = 14$.
   Due to $x_3^5 = x_3^6 = x_3^7$ from NAC, $[\mathbf{A}_4']^2 = \begin{pmatrix} A_4'^4 & 0 \\ 0 & A_4'^5 \\ 0 & A_4'^6 \\ 0 & A_4'^7 \end{pmatrix}$, $[\mathbf{A}_4]^2 = \begin{pmatrix} A_4^4 & 0 & 0 & 0 \\ 0 & A_4^5 & 0 & 0 \\ 0 & 0 & A_4^6 & 0 \\ 0 & 0 & 0 & A_4^7 \end{pmatrix}$ and the vectors of the x-variables are $\mathbf{x}_3^2 = \begin{pmatrix} x_3^4 \\ x_3^5 \end{pmatrix}$ and $\mathbf{x}_4^2 = \begin{pmatrix} x_4^4 \\ x_4^5 \\ x_4^6 \\ x_4^7 \end{pmatrix}$.

Similarly, the matrices for the third block $[\mathbf{B'}_t]^p$ and $[\mathbf{B}_t]^p$ can be defined. The constraint matrix structure of the q cluster submodels is shown in Appendix A.

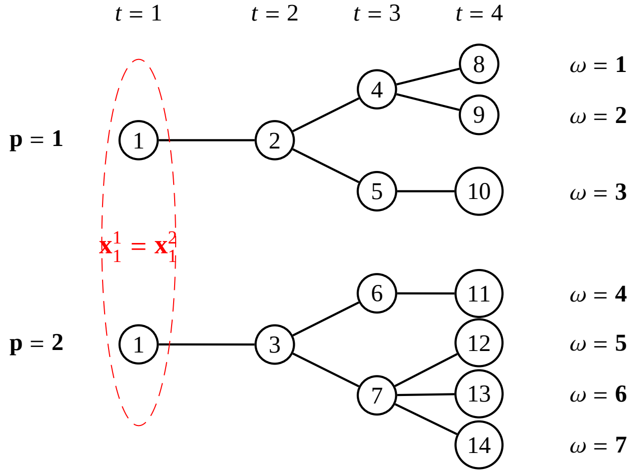


Figure 2: $\mathbf{x}_t^p$ variables with explicit NAC until $t^* = 1$. Illustrative example

- **Case 2**. Consider explicit NAC until stage $t^* = 2$ and, then, $q = 4$ clusters, whose scenario groups are given in Table 2. Using the 2-cluster tree matrix (8), the subset of scenario groups for cluster p and stage t, $\mathcal{G}_t^p$ can be determined. In this case, until $t^* + 1 = 3$ all of these subsets have a singleton element (see Property 1). And for $t^* + 2 = 4$ these subsets have one or more elements (see Property 2). Let us define the blocks of the matrices

Table 2: Scenario groups for $q = 4$. Illustrative example

$\mathcal{G}_t^p$	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$t = 1$	{1}	{1}	{1}	{1}
$t = 2$	{2}	{2}	{3}	{3}
$t = 3$	{4}	{5}	{6}	{7}
$t = 4$	{8,9}	{10}	{11}	{12,13,14}

by stages. Obviously, the matrices for the first block (stage $t = 1$ and $q = 4$ cluster models) are as follows: $\mathbf{A}_1 := A_1$, $\mathbf{B}_1 := B_1$ and the *rhs* $\mathbf{b}_1 := b_1$.

The matrices for the second block (stages $2 \leq t \leq t^* + 1 = 3$) are as follows:

1. **For** $p = 1$**:** $\mathbf{A}_t'^1 := A_t^{\prime\omega_g^1}$, $\mathbf{B}_t'^1 := B_t^{\prime\omega_g^1}$, $\mathbf{A}_t^1 := A_t^{\omega_g^1}$, $\mathbf{B}_t^1 := B_t^{\omega_g^1}$, and $\mathbf{b}_t^1 := b_t^{\omega_g^1}$, $2 \leq t \leq 3$, where the representative scenario for $t = 2$, $g \in \mathcal{G}_2^1 = \{2\}$ is $\omega_2^1 = min\{\omega \in \Omega_2\} = 1$ and for $t = 3$, $g \in \mathcal{G}_3^1 = \{4\}$ is $\omega_4^1 = min\{\omega \in \Omega_4\} = 1$.
2. **For** $p = 2$**:** $\mathbf{A}_t'^2 := A_t^{\prime\omega_g^2}$, $\mathbf{B}_t'^2 := B_t^{\prime\omega_g^2}$, $\mathbf{A}_t^2 := A_t^{\omega_g^2}$, $\mathbf{B}_t^2 := B_t^{\omega_g^2}$, and $\mathbf{b}_t^2 := b_t^{\omega_g^2}$, $2 \leq t \leq 3$, where the representative scenario for $t = 2$, $g \in \mathcal{G}_2^2 = \{2\}$ is $\omega_2^2 = min\{\omega \in \Omega_2\} = 1$ and for $t = 3$, $g \in \mathcal{G}_3^2 = \{5\}$ is $\omega_5^2 = min\{\omega \in \Omega_5\} = 3$.
3. **For** $p = 3$**:** $\mathbf{A}_t'^3 := A_t^{\prime\omega_g^3}$, $\mathbf{B}_t'^3 := B_t^{\prime\omega_g^3}$, $\mathbf{A}_t^3 := A_t^{\omega_g^3}$, $\mathbf{B}_t^3 := B_t^{\omega_g^3}$, and $\mathbf{b}_t^3 := b_t^{\omega_g^3}$, $2 \leq t \leq 3$, where the representative scenario for $t = 2$, $g \in \mathcal{G}_2^3 = \{3\}$ is $\omega_3^3 = min\{\omega \in \Omega_3\} = 4$ and for $t = 3$, $g \in \mathcal{G}_3^3 = \{6\}$ is $\omega_6^3 = min\{\omega \in \Omega_6\} = 4$.
4. **For** $p = 4$**:** $\mathbf{A}_t'^4 := A_t^{\prime\omega_g^4}$, $\mathbf{B}_t'^4 := B_t^{\prime\omega_g^4}$, $\mathbf{A}_t^4 := A_t^{\omega_g^4}$, $\mathbf{B}_t^4 := B_t^{\omega_g^4}$, and $\mathbf{b}_t^4 := b_t^{\omega_g^4}$, $2 \leq t \leq 3$, where the representative scenario for $t = 2$, $g \in \mathcal{G}_2^4 = \{3\}$ is $\omega_3^4 = min\{\omega \in \Omega_3\} = 4$ and for $t = 3$, $g \in \mathcal{G}_3^4 = \{7\}$ is $\omega_7^4 = min\{\omega \in \Omega_7\} = 5$.

The matrices for the third block $[\mathbf{A}_t']^p$ and $[\mathbf{A}_t]^p$ are as follows for stage $t^* + 1 = 3 < t \leq 4$:

1. **For** $p = 1$**:** For scenario group $g_i \in \mathcal{G}_4^1 = \{8,9\}$, $i \in \{1,\dots,|\mathcal{G}_4^1|\} = \{1,2\}$, the representative scenario $\omega^1_{g_i} = min\,\{\omega \in \Omega_{g_i}\}$ for group g_i is $\omega^1_8 = min\,\{\omega \in \Omega_8\} = 1$ for group $g_1 = 8$ and $\omega^1_9 = min\,\{\omega \in \Omega_9\} = 2$ for group $g_2 = 9$.
   Due to $x_3^1 = x_3^2$ from NAC, $[\mathbf{A}'_4]^1 = \begin{pmatrix} A'^1_4 \\ A'^2_4 \end{pmatrix}$, $[\mathbf{A}_4]^1 = \begin{pmatrix} A_4^1 & 0 \\ 0 & A_4^2 \end{pmatrix}$ and the corresponding vectors of the x variables are $\mathbf{x}_3^1 = x_3^1$ and $\mathbf{x}_4^1 = \begin{pmatrix} x_4^1 \\ x_4^2 \end{pmatrix}$.
2. **For** $p = 2$**:** For each scenario group $g_i \in \mathcal{G}_4^2 = \{10\}$, $i \in \{1,\dots,|\mathcal{G}_4^2|\} = \{1\}$, the representative scenario $\omega^2_{g_i} = min\,\{\omega \in \Omega_{g_i}\}$ for group g_i is $\omega^2_{10} = min\,\{\omega \in \Omega_{10}\} = 3$ for group $g_1 = 10$, $[\mathbf{A}'_4]^2 = \begin{pmatrix} A'^3_4 \end{pmatrix}$, $[\mathbf{A}_4]^2 = \begin{pmatrix} A_4^3 \end{pmatrix}$ and the corresponding vectors of the x variables are $\mathbf{x}_3^2 = x_3^3$, $\mathbf{x}_4^2 = x_4^3$.
3. **For** $p = 3$**:** For scenario group $g_i \in \mathcal{G}_4^3 = \{11\}$, $i \in \{1,\dots,|\mathcal{G}_4^3|\} = \{1\}$, the representative scenario $\omega^3_{g_i} = min\,\{\omega \in \Omega_{g_i}\}$ for group g_i is $\omega^3_{11} = min\,\{\omega \in \Omega_{11}\} = 4$ for group $g_1 = 11$, $[\mathbf{A}'_4]^3 = \begin{pmatrix} A'^4_4 \end{pmatrix}$, $[\mathbf{A}_4]^3 = \begin{pmatrix} A_4^4 \end{pmatrix}$ and the corresponding vectors of the x variables are $\mathbf{x}_3^3 = x_3^4$, $\mathbf{x}_4^3 = x_4^4$.
4. **For** $p = 4$**:** For scenario group $g_i \in \mathcal{G}_4^4 = \{12,13,14\}$, $i \in \{1,\dots,|\mathcal{G}_4^1|\} = \{1,2,3\}$, the representative scenario $\omega^4_{g_i} = min\,\{\omega \in \Omega_{g_i}\}$ for group g_i is $\omega^4_{12} = min\,\{\omega \in \Omega_{12}\} = 5$ for group $g_1 = 12$, $\omega^4_{13} = min\,\{\omega \in \Omega_{13}\} = 6$ for group $g_2 = 13$ and $\omega^4_{14} = min\,\{\omega \in \Omega_{14}\} = 7$ for group $g_3 = 14$.
   Due to $x_3^5 = x_3^6 = x_3^7$ from NAC, $[\mathbf{A}'_4]^4 = \begin{pmatrix} A'^5_4 \\ A'^6_4 \\ A'^7_4 \end{pmatrix}$, $[\mathbf{A}_4]^4 = \begin{pmatrix} A_4^5 & 0 & 0 \\ 0 & A_4^6 & 0 \\ 0 & 0 & A_4^7 \end{pmatrix}$ and the corresponding vectors of the x variables are $\mathbf{x}_3^4 = x_3^5$ and $\mathbf{x}_4^4 = \begin{pmatrix} x_4^5 \\ x_4^6 \\ x_4^7 \end{pmatrix}$.

Similarly, the matrices for the third block $[\mathbf{B'}_t]^p$ and $[\mathbf{B}_t]^p$ can be defined. The constraint matrix structure of the q cluster submodels is shown in Appendix B.

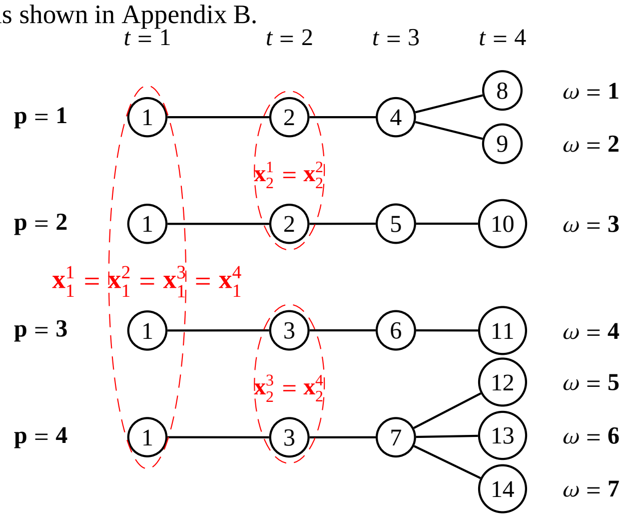


Figure 3: $\mathbf{x}_t^p$ variables with explicit NAC until $t^* = 2$. Illustrative example

- **Case 3**. Consider explicit NAC until stage $t^* = T - 1 = 3$ and, then, $q = 7$ clusters, whose scenario groups are given in Table 3. Using the scenario tree matrix (5), the subset of scenario groups for cluster p and stage t, $\mathcal{G}_t^p$ can be determined. In this case, all of these subsets have a singleton element (see Property 1).

  Let us define the blocks of the matrices by stages. Obviously, the matrices for the first block (stage $t = 1$ and $q = 7$ cluster models) are as follows: $\mathbf{A}_1 := A_1$, $\mathbf{B}_1 := B_1$ and the *rhs* $\mathbf{b}_1 := b_1$.

Table 3: Scenario groups for $q = 7$. Illustrative example

$\mathcal{G}_t^p$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 7$
$t = 1$	{1}	{1}	{1}	{1}	{1}	{1}	{1}
$t = 2$	{2}	{2}	{2}	{3}	{3}	{3}	{3}
$t = 3$	{4}	{4}	{5}	{6}	{7}	{7}	{7}
$t = 4$	{8}	{9}	{10}	{11}	{12}	{13}	{14}

The matrices for the second block (stages $2 \leq t \leq 4$) are as follows:

1. **For** $p = 1$**:** $\mathbf{A}_t^{\prime 1} := A_t^{\prime \omega_g^1}$, $\mathbf{B}_t^{\prime 1} := B_t^{\prime \omega_g^1}$, $\mathbf{A}_t^1 := A_t^{\omega_g^1}$, $\mathbf{B}_t^1 := B_t^{\omega_g^1}$, and $\mathbf{b}_t^1 := b_t^{\omega_g^1}$, where the representative scenario for $t = 2$, $g \in \mathcal{G}_2^1 = \{2\}$ is $\omega_2^1 = min\{\omega \in \Omega_2\} = 1$, for $t = 3$, $g \in \mathcal{G}_3^1 = \{4\}$ is $\omega_4^1 = min\{\omega \in \Omega_4\} = 1$, and for $t = 4$, $g \in \mathcal{G}_4^1 = \{8\}$ is $\omega_8^1 = min\{\omega \in \Omega_8\} = 1$.
2. **For** $p = 2$**:** $\mathbf{A}_t^{\prime 2} := A_t^{\prime \omega_g^2}$, $\mathbf{B}_t^{\prime 2} := B_t^{\prime \omega_g^2}$, $\mathbf{A}_t^2 := A_t^{\omega_g^2}$, $\mathbf{B}_t^2 := B_t^{\omega_g^2}$, and $\mathbf{b}_t^2 := b_t^{\omega_g^2}$, where the representative scenario for $t = 2$, $g \in \mathcal{G}_2^2 = \{2\}$ is $\omega_2^2 = min\{\omega \in \Omega_2\} = 1$, for $t = 3$, $g \in \mathcal{G}_3^2 = \{4\}$ is $\omega_4^2 = min\{\omega \in \Omega_4\} = 1$, and for $t = 4$, $g \in \mathcal{G}_4^2 = \{9\}$ is $\omega_9^2 = min\{\omega \in \Omega_9\} = 2$.
3. **For** $p = 3$**:** $\mathbf{A}_t^{\prime 3} := A_t^{\prime \omega_g^3}$, $\mathbf{B}_t^{\prime 3} := B_t^{\prime \omega_g^3}$, $\mathbf{A}_t^3 := A_t^{\omega_g^3}$, $\mathbf{B}_t^3 := B_t^{\omega_g^3}$, and $\mathbf{b}_t^3 := b_t^{\omega_g^3}$, where the representative scenario for $t = 2$, $g \in \mathcal{G}_2^3 = \{2\}$ is $\omega_2^3 = min\{\omega \in \Omega_2\} = 1$, for $t = 3$, $g \in \mathcal{G}_3^3 = \{5\}$ is $\omega_5^3 = min\{\omega \in \Omega_4\} = 3$, and for $t = 4$, $g \in \mathcal{G}_4^3 = \{10\}$ is $\omega_{10}^3 = min\{\omega \in \Omega_9\} = 3$.

   $\vdots$

7. **For** $p = 7$**:** $\mathbf{A}_t^{\prime 7} := A_t^{\prime \omega_g^7}$, $\mathbf{B}_t^{\prime 7} := B_t^{\prime \omega_g^7}$, $\mathbf{A}_t^7 := A_t^{\omega_g^7}$, $\mathbf{B}_t^7 := B_t^{\omega_g^7}$, and $\mathbf{b}_t^7 := b_t^{\omega_g^7}$, where the representative scenario for $t = 2$, $g \in \mathcal{G}_2^7 = \{3\}$ is $\omega_3^7 = min\{\omega \in \Omega_3\} = 4$, for $t = 3$, $g \in \mathcal{G}_3^7 = \{7\}$ is $\omega_7^7 = min\{\omega \in \Omega_7\} = 5$, and for $t = 4$, $g \in \mathcal{G}_4^7 = \{14\}$ is $\omega_{14}^7 = min\{\omega \in \Omega_{14}\} = 7$.

Moreover, since $t^* + 1 = T = 4$, there is not a third block of constraints.

Similarly, the matrices for the third block $[\mathbf{B'}_t]^p$ and $[\mathbf{B}_t]^p$ can be defined. The constraint matrix structure of the $q = 7$ cluster submodels is shown in Appendix C.

5. SIP mixed 0-1 model with nonsymmetric scenario trees

The decomposition in scenario clusters of the DEM (2) can be given by the mixture of a splitting variable representation (between the cluster submodels) and a compact representation (for each of them), such that the objective function value Z_{MIP} of the full model can be obtained as the sum of the related objective function values for each scenario cluster, z^p (9). So, $Z_{MIP} = \sum_{p=1}^{q} z^p$ subject to the NAC (12)-(13) between the clusters.

An external structure of information must be defined, via the so-called *representative cluster set* and the *predecessor cluster matrix*. Both elements are required by the asymmetry of the scenario cluster partitioning; see below.

First, remind that $\mathcal{P}$ is the set of the q scenario clusters, and let us consider a *representative cluster set*, $\mathcal{P}^t$ for stage $t \in \mathcal{T}$. The main aim is to determine the vectors of variables without replicas for $t = 1, ..., t^*$. Each element in set $\mathcal{P}^t$ is the representative scenario cluster of the clusters that belong to group g at stage t. Wlog, the first ordered cluster associated with scenario group $g \in \mathcal{G}_t$ can be considered as the representative cluster, such that $\mathcal{P}^t = \{p_1^t,\ p_2^t, \ldots p_{|\mathcal{G}_t|}^t\}$, where $p_g^t = \{\min\ p \mid g \in \mathcal{G}_t^p,\ p \in \mathcal{P}\}$. Notice that the required information for the definition of $\mathcal{P}^t$ is given in the corresponding t^*-cluster tree matrix. See also that $\mathcal{P}^t = \mathcal{P}, \forall t > t^*$. For $t \in \{1, \ldots, t^*\}$, the number of elements in such set coincides with the number of scenario groups, i.e., $|\mathcal{P}^t| = |\mathcal{G}_t|$, in particular, $\mathcal{P}^1 = \{1\}$. Moreover, each set is included in the corresponding set for the next stage, $\mathcal{P}^1 \subset \mathcal{P}^2 \subset \ldots \subset \mathcal{P}^{t^*} \subset \mathcal{P}^{t^*+1} \subseteq \ldots \subseteq \mathcal{P}^T = \mathcal{P}$.

In the illustrative example, the set of representative clusters for Case 1 ($t^* = 1$, $q = 2$), they are: $\mathcal{P}^1 = \{1\}$ and $\mathcal{P}^2 = \mathcal{P}^3 = \mathcal{P}^4 = \mathcal{P} = \{1, 2\}$. For Case 2 ($t^* = 2$, $q = 4$) they are: $\mathcal{P}^1 = \{1\}$, $\mathcal{P}^2 = \{1, 3\}$ and $\mathcal{P}^3 = \mathcal{P}^4 = \mathcal{P} = \{1, 2, 3, 4\}$. Finally, for Case 3 ($t^* = 3$, $q = 7$) these sets are: $\mathcal{P}^1 = \{1\}$, $\mathcal{P}^2 = \{1, 4\}$, $\mathcal{P}^3 = \{1, 3, 4, 5\}$ and $\mathcal{P}^4 = \mathcal{P} = \{1, 2, \ldots, 7\}$.

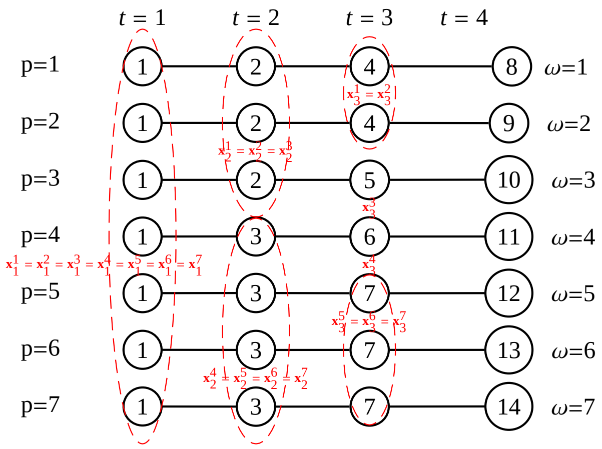


Figure 4: $\mathbf{x}_t^p$ variables with explicit NAC until $t^* = 3$. Illustrative example

Second, for the hybrid formulation between the full model and the scenario cluster submodels, the predecessor cluster of the representative cluster p in stage t, $pred(t-1, p_g^t)$, can be defined via the *predecessor cluster matrix.*

Definition 13. *The* ***predecessor cluster matrix*** *associated to the* t^**−decomposition,* $pred \in M_{t^* \times |\mathcal{P}|}$ *is a matrix where the corresponding value for the pair* (t, p) *gives the predecessor cluster of the representative cluster* p *in stage* t*.*

At each row $t = 1, ..., t^*$*, the matrix is computed from the sets of representative clusters at the related stage,* $\mathcal{P}^t$*.*

$$pred(t, \cdot) = (\ \overbrace{1 \ \cdots 1}^{p_1^t \,\le\, p \,<\, p_2^t} \quad \overbrace{p_2^t \cdots p_2^t}^{p_2^t \,\le\, p \,<\, p_3^t} \ \ \ldots \ \overbrace{p^t_{|\mathcal{P}^t|} \cdots p^t_{|\mathcal{P}^t|}}^{p^t_{|\mathcal{P}^t|} \,\le\, p \,\le\, q}\ )$$

So, the function ϕ for calculating the predecessor cluster of a given scenario cluster p can be defined, taking into account that the given cluster must be a representative cluster p_g^t at stage t. So, $\phi(p_g^t) = pred(t-1, p_g^t)$.

In the illustrative example, the predecessor cluster matrix, $pred \in M_{1\times 2}$ for Case 1 ($t^* = 1$, $q = 2$) is $pred(t,p) = \begin{pmatrix} 1 & 1 \end{pmatrix}$. For Case 2 ($t^* = 2$, $q = 4$), $pred \in M_{2\times 4}$ is $pred(t,p) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix}$. And, finally, for Case 3 ($t^* = 3$, $q = 7$), $pred \in M_{3\times 7}$ is $pred(t,p) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 4 & 4 & 4 \\ 1 & 1 & 3 & 4 & 5 & 5 & 5 \end{pmatrix}$.

Definition 14. *The* cluster splitting-compact *representation is the splitting variable formulation that is extended between all the scenario cluster submodels and the compact representation into each scenario cluster.*

By using the previous elements, the full DEM can be formulated in a cluster splitting-compact representation as follows,

$$(MIP) \quad Z_{MIP} = \min \ a_1\mathbf{x}_1^1 + c_1\mathbf{x}_1^1 + \sum_{t=2}^{t^*} \sum_{p=p_1^t}^{p^t_{|\mathcal{P}^t|}} w_t^p(a_t^p\mathbf{x}_t^p + c_t^p\mathbf{y}_t^p) + \sum_{t=t^*+1}^{T} \sum_{p=1}^{q} w_t^p(a_t^p\mathbf{x}_t^p + c_t^p\mathbf{y}_t^p)$$

$$
\begin{aligned}
s.t.\quad & \mathbf{A}_1\mathbf{x}_1^1 + \mathbf{B}_1\mathbf{y}_1^1 = b_1^1 \\
& \mathbf{A}_t^{'p}\mathbf{x}_{t-1}^{\phi(p)} + \mathbf{A}_t^p\mathbf{x}_t^p + \mathbf{B}_t^{'p}\mathbf{y}_{t-1}^{\phi(p)} + \mathbf{B}_t^p\mathbf{y}_t^p = \mathbf{b}_t^p, \qquad p \in \mathcal{P}^t,\ 2 \le t \le t^* + 1 \\
& [\mathbf{A}_t']^p\mathbf{x}_{t-1}^p + [\mathbf{A}_t]^p\mathbf{x}_t^p + [\mathbf{B}_t']^p\mathbf{y}_{t-1}^p + [\mathbf{B}_t]^p\mathbf{y}_t^p = \mathbf{b}_t^p, \quad p \in \mathcal{P},\ t^* + 1 < t \le T \\
& \mathbf{x}_t^p = \mathbf{x}_t^{p'},\ \ p \neq p',\ p, p' \in \mathcal{P}^t, t \le t^* \\
& \mathbf{y}_t^p = \mathbf{y}_t^{p'},\ \ p \neq p',\ p, p' \in \mathcal{P}^t, t \le t^* \\
& \mathbf{x}_t^p \in \{0, 1\}^{nx_t^p},\ \ \mathbf{y}_t^p \in \boldsymbol{R}^{+ny_t^p},\ p \in \mathcal{P},\ t \in \mathcal{T},
\end{aligned}
\tag{14}
$$

where the vectors $\mathbf{x}_t^p$ and $\mathbf{y}_t^p$ for stage t such that $2 \le t \le t^* + 1$ have as many replicas as groups in the scenario tree for each scenario cluster. The matrices $\mathbf{A}_t^{'p}$, $\mathbf{B}_t^{'p}$, $\mathbf{A}_t^p$ and $\mathbf{B}_t^p$ have been defined in the scenario cluster model (9). The order of the storage of the variables is shown in Appendices D and E.

6. Nonsymmetric BFC-MS Algorithm

Before executing the proposed algorithm for solving the original multi-stage stochastic mixed 0-1 problem, it is required to fix the data structuring, see Figure 5. A decision has to be made on fixing the break stage t^* for considering the splitting variable representation (i.e., the stages with explicit NAC) and, consequently, the number of clusters $q \in \mathcal{Q}$, where $\mathcal{Q} = \{|\mathcal{G}_1|, |\mathcal{G}_2|, ..., |\mathcal{G}_T| = |\Omega|\}$. Notice that those clusters will be explicitly linked by NAC until the stage t^*. Remind that this selection fixes the way to build the cluster submodels. Observe also that $q = |\mathcal{G}_T|$ means that the scenario cluster strategy is not to be used.

Step 0: Inputs: scenario tree $\mathcal{R}$ and number of variables $n_{xt},\ n_{yt}$.

Step 1: Define $\mathcal{T},\ \Omega,\ \mathcal{G},\ \mathcal{G}_t\ \forall t \in \mathcal{T},\ \Omega_g\ \forall g \in \mathcal{G},\ \mathcal{ST}$ and weights $w^\omega\ \forall \omega \in \Omega$.

Step 2: Decide break stage t^* and uniquely $q = |\mathcal{G}_{t^*+1}|$.

Step 3: Define $\mathcal{G}^p,\ \mathcal{G}_t^p,\ \mathcal{CT}^{t^*},\ \Omega^p,\ \omega_g^p,\ \overline{w}_t^p,\ \mathcal{P}^t,\ pred(p,t),\ n_{xt}^p,\ n_{yt}^p\ \forall p \in \mathcal{P}, t \in \mathcal{T}$.

Step 4: Generate/read the full model (14).

Step 5: Generate the cluster submodels (9).

Figure 5: Data structuring

The *Nonsymmetric BFC-MS* algorithm allows that the number of 0-1 variables at each stage, nx_t (and, then, the number of continuous variables ny_t) may be different from one cluster to another, except for the stages $t = 1, 2, ..., t^*$ where the number of variables has to be same for all clusters, because in those stages the cluster variables are scenario variables and, then, replicas. So, let us split the time horizon in two parts, the first one includes the stages $t = 1, 2, ..., t^*$, and the second part includes the other stages in set $\mathcal{T}$. Then, the algorithm satisfies the implicit NAC on the x and y variables for the set of stages $t = t * +1, ..., T$ at each iteration by solving the cluster submodels (9) with any state-of-the-art MIP optimization package. Notice that the NAC are relaxed in those models for the stages $t = 1, 2, ..., t^*$, such that its satisfaction is performed by using a Branch-and-Fix Coordination (BFC) type of algorithm [6] and, so, it is guaranteed that the algorithm obtains the optimal solution for the DEM (2) of the stochastic problem.

At each TNF integer set two new models can be defined as in our previous works [6, 7], but with a substantial difference since in the new approach the break stage t^* defines the x variables to fix at their 0-1 variables in the first model and, additionally, it defines in the second model the x variables whose integrality is to be relaxed. That stage also defines the y variables whose NAC are to be explicitly satisfied in both models.

So, first, let the MIP (15) that results after fixing in model (14) the x **variables for the stages up to the break stage** t^* at the 0-1 related values for a given TNF integer set. In the new model, $\overline{x}$ will denote the 0-1 values of the respective vector x, it can be expressed in cluster splitting-compact representation as follows,

$$
\begin{aligned}
(MIP^{TNF}) \quad z^{TNF} = \min \ & a_1\mathbf{x}_1^1 + c_1\mathbf{x}_1^1 + \sum_{t=2}^{t^*}\sum_{p=p_1^t}^{p^t_{|\mathcal{P}^t|}} w_t^p(a_t^p\mathbf{x}_t^p + c_t^p\mathbf{y}_t^p) + \sum_{t=t^*+1}^{T}\sum_{p=1}^{q} w_t^p(a_t^p\mathbf{x}_t^p + c_t^p\mathbf{y}_t^p) \\
s.t. \quad & \mathbf{A}_1\mathbf{x}_1^1 + \mathbf{B}_1\mathbf{y}_1^1 = b_1^1 \\
& \mathbf{A}_t^{'p}\mathbf{x}_{t-1}^{\phi(p)} + \mathbf{A}_t^p\mathbf{x}_t^p + \mathbf{B}_t^{'p}\mathbf{y}_{t-1}^{\phi(p)} + \mathbf{B}_t^p\mathbf{y}_t^p = \mathbf{b}_t^p, \qquad p \in \mathcal{P}^t,\ 2 \le t \le t^* + 1 \\
& [\mathbf{A}'_t]^p\mathbf{x}_{t-1}^p + [\mathbf{A}_t]^p\mathbf{x}_t^p + [\mathbf{B}'_t]^p\mathbf{y}_{t-1}^p + [\mathbf{B}_t]^p\mathbf{y}_t^p = \mathbf{b}_t^p, \quad p \in \mathcal{P},\ t^* + 1 < t \le T \\
& \mathbf{x}_t^p = \overline{\mathbf{x}}_t^p,\ p \in \mathcal{P},\ t \in \mathcal{T},\ t \le t^* \\
& \mathbf{y}_t^p = \mathbf{y}_t^{p'},\ \ p \neq p',\ p, p' \in \mathcal{P}^t, t \le t^* \\
& \mathbf{x}_t^p \in \{0,1\}^{nx_t^p},\ p \in \mathcal{P},\ t \in \mathcal{T},\ t > t^* \\
& \mathbf{y}_t^p \in \boldsymbol{R}^{+ny_t^p},\ p \in \mathcal{P},\ t \in \mathcal{T}.
\end{aligned} \tag{15}
$$

The second MIP model to solve at each TNF integer set corresponds to the case in which, not all the x **variables for the stages up to the break stage** t^* have been branched on / fixed at in the current TNF. In this case, the new MIP model (16) will allow the x variables to take fractional values between 0 and 1, if they are not yet branched on / fixed at the current TNF. In the new model, $\tilde{x}$ will denote the 0-1 values of the subset, say, $\tilde{\mathbf{X}}$ of the x variables which have been already branched on / fixed at. The new model, also in cluster splitting-compact representation, can be expressed as follows,

$$
\begin{aligned}
(MIP^{f}) \quad z^{f} = \min \ & a_1\mathbf{x}_1^1 + c_1\mathbf{x}_1^1 + \sum_{t=2}^{t^*}\sum_{p=p_1^t}^{p^t_{|\mathcal{P}^t|}} w_t^p(a_t^p\mathbf{x}_t^p + c_t^p\mathbf{y}_t^p) + \sum_{t=t^*+1}^{T}\sum_{p=1}^{q} w_t^p(a_t^p\mathbf{x}_t^p + c_t^p\mathbf{y}_t^p) \\
s.t. \quad & \mathbf{A}_1\mathbf{x}_1^1 + \mathbf{B}_1\mathbf{y}_1^1 = b_1^1 \\
& \mathbf{A}_t^{'p}\mathbf{x}_{t-1}^{\phi(p)} + \mathbf{A}_t^p\mathbf{x}_t^p + \mathbf{B}_t^{'p}\mathbf{y}_{t-1}^{\phi(p)} + \mathbf{B}_t^p\mathbf{y}_t^p = \mathbf{b}_t^p, \qquad p \in \mathcal{P}^t,\ 2 \le t \le t^* + 1 \\
& [\mathbf{A}'_t]^p\mathbf{x}_{t-1}^p + [\mathbf{A}_t]^p\mathbf{x}_t^p + [\mathbf{B}'_t]^p\mathbf{y}_{t-1}^p + [\mathbf{B}_t]^p\mathbf{y}_t^p = \mathbf{b}_t^p, \quad p \in \mathcal{P},\ t^* + 1 < t \le T \\
& \mathbf{x}_t^p = \tilde{\mathbf{x}}_t^p,\ \mathbf{x} \in \tilde{\mathbf{X}}, p \in \mathcal{P},\ t \le t^* \\
& \mathbf{x}_t^p = \mathbf{x}_t^{p'},\ \mathbf{x} \notin \tilde{\mathbf{X}},\ p \neq p',\ p, p' \in \mathcal{P}^t, t \le t^* \\
& \mathbf{y}_t^p = \mathbf{y}_t^{p'},\ \ p \neq p',\ p, p' \in \mathcal{P}^t, t \le t^* \\
& 0 \le \mathbf{x}_t^p \le 1,\ \mathbf{x} \notin \tilde{\mathbf{X}},\ p \in \mathcal{P},\ t \le t^* \\
& \mathbf{x}_t^p \in \{0,1\}^{nx_t^p},\ p \in \mathcal{P},\ t \in \mathcal{T},\ t > t^* \\
& \mathbf{y}_t^p \in \boldsymbol{R}^{+ny_t^p},\ p \in \mathcal{P},\ t \in \mathcal{T}.
\end{aligned} \tag{16}
$$

The specific BFC scheme that we propose is based on branching on the 0-1 x variables for the stages $t = 1, 2, ..., t^*$ along the scenario cluster related trees and simultaneously coordinating the satisfaction of the related NAC for all the TNFs. Lower bounds of the optimal solution value for the original problems are obtained by solving the MIP problems (9) and (16) in a certain order, see below. Feasible solutions to the original DEM (2) are obtained by solving the MIP problem (15) until getting the optimal solution. We should point out that, for computational efficiency reasons, we consider the implicit NAC in the problems (15) and (16).

Step 0: **(Initializations)** $\overline{Z}_{MIP} := +\infty,\ t := 0,\ i = 0.$

Step 1: **(Root node)** Append cuts and solve the independent preprocessed MIP cluster submodels (9).
$\underline{Z}_0 = \sum_{p=1}^{q} z^p$.
If x^p variables do not satisfy (12), then go to Step 2.
If y^p variables do not satisfy (13), then go to Step 7.
Otherwise, $\overline{Z}_{MIP} := \underline{Z}_0$, **STOP**.

Step 2: **(First iteration)** Initialize $t := 1$, $i := 1$ and go to Step 5.

Step 3: **(Next stage)** Reset $t := t + 1$. If $t > t^*$, then go to Step 9.

Step 4: **(Next node)** Reset $i := i + 1$. If $i > n_{xt}$, then go to Step 3.

Step 5: **(Branch to 0)** Branch $x_{ti}^p := 0, \forall p \in \mathcal{P} \mid \mathcal{CT}^{t^*}(p, g) = t.$

Step 6: **(Cluster submodels)** Append cuts and solve the independent preprocessed MIP cluster submodels (9).
$\underline{Z}_i = \sum_{p=1}^{q} z^p$.
If $\underline{Z}_i \geq \overline{Z}_{MIP}$, then go to Step 8.
If x^p variables do not satisfy (12), then go to Step 4.
If y^p variables satisfy (13), then update $\overline{Z}_{MIP} := \underline{Z}_i$ and go to Step 8.

Step 7: **(TNF models)**
Append cuts and solve the MIP model(15) to satisfy the NAC for the y variables.
Update $\overline{Z}_{MIP} := \min\{z^{TNF}, \overline{Z}_{MIP}\}$.
If $t = t^*$ and $i = n_{xt}$, then go to Step 8.
Append cuts and solve the MIP model (16) to satisfy the NAC for the y variables,
where the integrality is relaxed on the non-yet branched on / fixed at x variables.
If $z^f < \overline{Z}_{MIP}$ and all the relaxed x variables are 0-1, $\overline{Z}_{MIP} := z^f$, go to Step 8.
If $z^{TNF} = z^f$ or $z^f \geq \overline{Z}_{MIP}$, then go to Step 8, otherwise go to Step 4.

Step 8: **(Prune)** Prune the branch. If $x_{ti}^p = 0,\ p \in \mathcal{P} \mid \mathcal{CT}^{t^*}(p, g) = t$, then go to Step 11.

Step 9: **(Previous node)** Reset $i := i - 1$.
If $i = 0$ and $t = 0$ then the optimal solution value $\overline{Z}_{MIP}$ has been found, **STOP**.
If $i = 0$, then $t := t - 1$ and go to Step 4.

Step 10: **(Check)** If $x_{ti}^p = 1, \forall i \leq n_{xt},\ t \leq t^*,\ p \in \mathcal{P} \mid \mathcal{CT}^{t^*}(p, g) = t$, then go to Step 9.

Step 11: **(Branch to 1)** Branch $x_{ti}^p = 1,\ p \in \mathcal{P} \mid \mathcal{CT}^{t^*}(p, g) = t$. Then go to Step 6.

Figure 6: Nonsymmetric BFC-MS Algorithm

It is well known that one of the most important contributions to the advancement of the theory and applications of deterministic integer optimization has been the development of the preprocessing techniques for solving large scale instances in affordable computing effort, due to the tightening of the models and, so, reducing the LP feasible space without eliminating any feasible integer solution that potentially could become the optimal one. Some of the key ingredients in preprocessing are the probing techniques [11, 12, 17] and schemes for identifying and appending Gomory cuts [2, 3, 10] and clique cuts [4], among other important schemes. So, our algorithm for solving large instances of the mixed integer DEM takes benefit from the processing techniques of the optimization engine of choice. They are used for solving the auxiliary mixed integer submodels related to the scenario clusters. The difference in computing time by using preprocessing compared with the alternative that does not use it is crucial in solving large scale instances.

Figure 6 shows the main steps of the *Nonsymmetric BFC-MS* algorithm. The strategies for selecting the branching Twin Node Family (TNF) and fixing the 0-1 variables across the BF trees have been taken from our previous work [7], given the good results that have been obtained for symmetric scenario trees.

7. Computational experience

The proposed approach has been implemented in a C++ experimental code. It uses the open source optimization engine *COIN-OR* for solving the LP relaxation and mixed 0-1 submodels, in particular, we have used the functions: Clp (LP solver), Cbc (MIP solver), Cgl (Cut generator), Osi, OsiClp, OsiCbc and CoinUtils. As a result the total computing time for obtaining the optimal solution of the original DEM has been improved strongly, see below.

The computational experiments were conducted in a Workstation Debian Linux (kernel v2.6.26 with 64 bits), 2 processors Xeon 5355 (Quad Core with 2x4 cores), 2.664 Ghz and 16 Gb of RAM.

Table 4 gives the dimensions of the DEM of the full stochastic model in compact representation for difficult medium and large scale problems. Table 5 gives $[\mu]$, integer part of the mean μ and σ, the standard deviation for the dimensions of the cluster submodels; so, we can observe the variability of the nonsymmetric clusters. The headings are as follows: *m*, number of constraints; *nx*, number of 0-1 variables; *ny*, number of continuous variables; *nel*, number of nonzero coefficients in the constraint matrix; and *dens*, constraint matrix density (in %).

Table 4: Testbed problem dimensions

Inst.	*m*	*nx*	*ny*	*nel*	*dens*
P1	696	160	376	1550	0.42
P2	1202	530	241	3053	0.33
P3	7282	1878	4152	20818	0.05
P4	16172	4270	9340	53257	0.02
P5	23907	5560	11675	68937	0.02
P6	32914	6672	14010	105854	0.02
P7	2085	450	1155	9105	0.27
P8	4696	1090	2516	9935	0.06
P9	11298	2668	5962	25262	0.03
P10	16870	4600	10430	42015	0.02
P11	31648	7984	17676	83252	0.01
P12	40020	8847	19377	100680	0.01
P13	5256	1176	2904	12861	0.06
P14	11121	2538	6045	27315	0.03
P15	14570	3370	7830	32508	0.02
P16	28176	6584	15008	62934	0.01
P17	45844	10794	24256	102480	0.01
P18	76424	18108	40208	170954	0.00

Table 5: Testbed cluster-subproblem dimensions

Inst.	$[\mu_m]$	σ_m	$[\mu_{nx}]$	σ_{nx}	$[\mu_{ny}]$	σ_{ny}	$[\mu_{nel}]$	σ_{nel}	μ_{dens}	σ_{dens}
P1	133	29.94	283	6.57	68	14.62	275	62.12	2.31	0.68
P2	496	67.77	230	20.41	101	16.33	1227	171.46	0.76	0.08
P3	869	305.42	193	70.06	431	149.96	2145	766.93	0.46	0.20
P4	1788	578.65	397	130.53	876	280.11	4961	1617.42	0.24	0.09
P5	2815	20.77	561	4.15	1181	8.31	6953	50.67	0.14	0.00
P6	3823	28.24	673	4.98	1417	9.97	10675	78.08	0.13	0.00
P7	750	187.13	160	43.01	415	104.16	3236	859.05	0.80	0.20
P8	643	190.64	138	41.02	320	94.11	1259	372.83	0.49	0.21
P9	1241	537.03	269	116.78	602	256.60	2544	1097.09	0.30	0.18
P10	2007	454.02	516	145.94	1172	330.98	4711	1333.12	0.15	0.03
P11	3322	1207.93	729	265.57	1618	582.71	7608	2758.76	0.11	0.05
P12	3748	1454.9	740	287.61	1623	625.81	8423	3264.84	0.12	0.06
P13	950	259.56	199	55.92	492	137.05	2171	609.62	0.37	0.14
P14	1751	543.56	365	113.59	871	267.64	3930	1216.77	0.20	0.06
P15	1973	617.25	423	132.62	984	306.15	4081	1275.4	0.17	0.09
P16	3403	983.54	733	212.10	1673	482.39	7010	2025.25	0.09	0.03
P17	5000	2216.05	1081	479.50	2431	1075.64	10266	4548.74	0.08	0.05
P18	5126	1966.84	824	316.76	1830	699.81	8604	3300.17	0.07	0.03

Table 6: Computational results. Stochastic solution

Instance	q	$\|\Omega\|$	$\|\mathcal{G}\|$	Z_{LP}	Z_0	z_{MIP}	GAP_{LP}	GAP_0	tt_{LP}	tt_0
P1	6	52	80	4395695	4654305	4654305	5.9	0	0.0	0.4
P2	3	6	12	75103.6	58589.1	58585.1	22.0	0.0	0.0	14.9
P3	10	247	313	5691.3	442336	573848	9982.9	29.7	0.1	4.6
P4	11	347	427	11601.4	725490	903367	7686.7	24.5	0.4	24.4
P5	10	1001	1112	4977.8	385471	468277	9307.4	21.5	0.7	32.6
P6	10	1001	1112	6116.5	540241	653638	10586	21.0	0.9	54.8
P7	3	13	30	20210.9	964395	973038	4714.4	0.9	0.0	8.9
P8	8	377	545	3156.8	156064	156064	4843.7	0.0	0.1	2.7
P9	10	1021	1334	3829.5	239683	239683	6158.9	0.0	0.5	8.7
P10	9	674	920	5757.0	394469	505729	8684.6	28.2	0.5	40.2
P11	11	1569	1996	5474.1	401435	401435	7233.4	0.0	1.5	78.2
P12	12	2388	2949	3209.4	318391	370024	11429.5	16.8	2.7	24.8
P13	6	208	392	8071.8	371498	372296	4512.3	0.2	0.1	4.3
P14	7	523	846	6157.3	339381	339381	5411.8	0.0	0.3	3.3
P15	8	1140	1685	3941.7	212593	212593	5293.5	0.0	0.7	19.0
P16	9	2372	3292	3521.9	258977	258977	7253.3	0.0	2.4	78.8
P17	10	4063	5397	2629.0	303900	303900	11459.5	0.0	6.0	2.7
P18	11	7058	9054	3824.7	318958	318958	8239.5	0.0	17.9	6.2

Table 6 shows some results of our computational experimentation. The headings are as follows: q, number of clusters; $|\Omega|$, number of scenarios; $|\mathcal{G}|$, number of scenario groups; Z_{LP}, solution value of the LP relaxation of the original DEM problem in compact representation; $Z_0 = \sum_p z^p$, optimal expected solution value at the root node obtained by solving independently the cluster submodels; z_{MIP}, optimal solution value of the original DEM problem; GAP_{LP}, optimality gap defined as $\frac{z_{MIP}-Z_{LP}}{Z_{LP}}$ (in %); GAP_0, optimality gap defined as $\frac{z_{MIP}-Z_0}{Z_0}$ (in %); and tt_{LP} and tt_0, elapsed time (in seconds) to obtain the Z_{LP} and Z_0 solutions, respectively. We can observe the very big value for GAP_{LP} and the very small value for GAP_0. The latter lower bound is often optimal or very closed to the optimal solution value (see also Figure 8, where the distance GAP_0 in the first iteration is very small). Intuitively, small values of GAP_0 improve the convergence speed of the algorithm. Observe that the computing time tt_0 is very small, except for the instances P11 and P16.

Table 7: *Nonsymmetric BFC-MS* performance vs *B&B*

	Clustering by break stage		*Nonsymmetric BFC-MS*			*B&B*	
Instance	T	Q	$nTNF$	tt	tt_C	tt	tt_C
P1	4	$\{1, 6, 21, 52\}$	1	0.4	0.3	4000.2	0.8
P2	4	$\{1, 2, 3, 6\}$	114	198.1	138.8	1304.2	1304.2
P3	4	$\{1, 10, 55, 247\}$	8	21.8	1.7	41.4	1.7
P4	4	$\{1, 11, 68, 347\}$	16	171.6	11.7	1530.8	19.4
P5	4	$\{1, 10, 100, 1001\}$	8	162.1	8.8	448.5	13.4
P6	4	$\{1, 10, 100, 1001\}$	10	229.5	8.5	889.7	48.4
P7	5	$\{1, 3, 5, 8, 13\}$	81	142.5	41.8	188.3	35.9
P8	5	$\{1, 8, 35, 124, 377\}$	1	2.7	0.9	272.3	6.1
P9	5	$\{1, 10, 55, 247, 1021\}$	1	9.1	1.4	100.0	4.5
P10	5	$\{1, 9, 46, 190, 674\}$	10	206.6	45.8	7992.7	296.4
P11	5	$\{1, 11, 68, 3347, 1569\}$	1	80.2	14.9	12113.1	126.8
P12	5	$\{1, 12, 81, 467, 2388\}$	7	513.8	66.8	3566.2(*)	867.5
P13	6	$\{1, 6, 21, 52, 104, 208\}$	3	13.2	2.6	1304.2	10.2
P14	6	$\{1, 7, 28, 81, 206, 523\}$	1	14.2	3.5	—	22.9
P15	6	$\{1, 8, 35, 124, 377, 1140\}$	1	19.7	4.9	7226.3(*)	19.2
P16	6	$\{1, 9, 46, 190, 674, 2372\}$	1	81.2	26.4	628.5(*)	48.5(*)
P17	6	$\{1, 10, 55, 247, 1021, 4063\}$	1	152.8	8.7	1897.3	67.3
P18	6	$\{1, 11, 68, 347, 1569, 7058\}$	1	377.0	24.1	—	202.9

—: Time limit exceeded (6 hours)

(*): Optimum not reached after 6 hours of computing, time for obtaining a 0.05 quasi-optimal soln

Table 7 shows the efficiency and stability of the *Nonsymmetric BFC-MS* algorithm. The headings are as follows: T, number of stages; Q, set of possible number of clusters; $nTNF$, number of TNFs; *B&B*, plain use of the Branch-and-Bound procedure for the full model by using the Cbc function of *COIN-OR*; and tt and tt_C, total elapsed time (in seconds) without and with preprocessing (in our case, it consists of using probing techniques and schemes for identifying and appending Gomory cuts and clique cuts implemented in the functions of *COIN-OR*). Although other break stages have been considered, we have obtained the best results for the break stage $t^* = 1$ and, then, $q = |\mathcal{G}_2|$ for both without and with preprocessing options. Although for lack of space we do not report all the detailed results (but they are available upon request to the authors), it is worthy to remark that for the scenario partition $t^* + 1 = T$ ($q = |\Omega|$) i.e., no cluster partition is considered, the execution of all the instances exceeded the time limit of six hours of computing. We can observe (1) the efficiency of using the preprocessing techniques and (2) the astonishing small computing time required by the *Nonsymmetric BFC-MS* algorithm, such that it clearly outperforms the plain use of the optimization engine of choice.

Figure 7 depicts the elapsed time (seconds) in increasing order of the number of 0-1 variables related to the testbed whose results are reported in Table 7. The times for solving the problems using B&B plus cuts, using the algorithm BFC and using BFC plus cuts have been represented in red, blue and green.

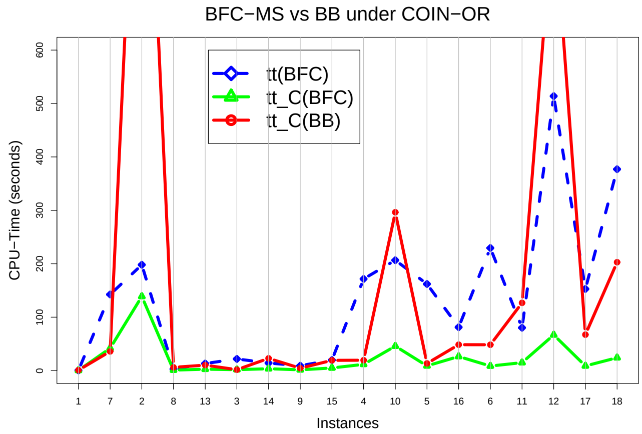


Figure 7: CPU-time for BFC-MS vs BB (increasing order of nx)

Figure 8 depicts an illustrative example of the convergence of the algorithm. It corresponds to the instance P7. The procedure exploits the TNF branching selection and fixing of the 0-1 variables along the groups $g \in \cup_{t=1}^{t^*} \mathcal{G}_t$ at the stages. 81 $nTNF$ are required to be branched until obtaining the optimal solution. It obtains lower bounds (given by the cluster submodels, in vertical) and upper bounds (in horizontal stair steps) of the optimal solution value (red dashed line) for the mixed 0-1 problems at the TNFs. The vertical empty lines correspond to branches that are pruned due to infeasibilities.

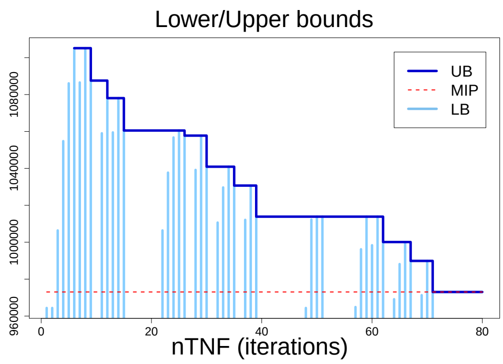


Figure 8: BFC-MS convergence

8. Conclusions and future work

In this work a modeling approach and an exact Branch-and-Fix Coordination algorithmic framework, so-called *Nonsymmetric BFC-MS*, have been proposed for solving multi-stage mixed 0-1 problems under uncertainty in the parameters, being the uncertainty represented by scenarios trees. It can appear in any coefficient of the objective function, constraint matrix and right-hand–side at any stage. The 0-1 and continuous variables can also appear at any stage. The approach treats the uncertainty by scenario cluster analysis, allowing the scenario tree to be nonsymmetric. This last feature has not been considered in the literature that we are aware of. However, in our opinion, it is crucial for solving medium and large scale problems, since the real-life mixed integer optimization problems under uncertainty that, at least, we have encountered have very frequently nonsymmetric scenarios to represent the uncertainty. As expected the efficiency of the preprocessing techniques (i.e., probing and Gomory and clique cuts identification and appending schemes) is remarkable for the cluster submodels to be solved at the candidate Twin Node Families (TNF) and the TNF integer sets. The computational time that we report for solving large scale multi-stage stochastic mixed 0-1 problems is very small and it seems to validate the new approach. On the other hand, it clearly outperforms the plain use of the optimization engine of choice.

As a future work we are considering Lagrangean Decomposition (LD) as a powerful tool for iteratively obtaining strong lower bounds to the optimal solution value of the submodels to be solved at the candidate TNFs and TNF integer sets. The key point in LD is the dualization of the nonanticipativity constraints (NAC) in the scenario cluster submodels, see in [9] the good results that have been obtained for highly combinatorial stochastic problems. Another point of future research derives from the observation of the independent character of the cluster submodels, such that it paves the way to use parallel computing for solving them so that the result could be a parallelized *Nonsymmetric BFC-MS* algorithm.

9. Acknowledgements

This research has been partially supported by the projects ECO2008-00777 ECON from the Ministry of Education and Science, Grupo de Investigación IT-347-10 from the Basque Government, URJC-CM-2008-CET-3703 and RIESGOS CM from Comunidad de Madrid, and PLANIN MTM2009-14087-C04-01 from Ministry of Science and Innovation, Spain. We would like to express our gratefulness to Prof. Dr. F. Tusell for making easier the access to the Quantitative Economics Laboratory (Economic Sciences School, UPV/EHU) to perform and check the computing experience.

Appendix A. Constraint matrix structure for the illustrative example. Case 1: $t^* = 1$ and $q = 2$ clusters

$$
\begin{pmatrix}
A_1 & & & & & & \\
A_2^{\prime 1} & A_2^1 & & & & & \\
 & A_3^{\prime 1} & A_3^1 & & & & \\
 & A_3^{\prime 3} & & A_3^3 & & & \\
 & & A_4^{\prime 1} & & A_4^1 & & \\
 & & A_4^{\prime 2} & & & A_4^2 & \\
 & & & A_4^{\prime 3} & & & A_4^3
\end{pmatrix}
\cdot
\begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_3^3 \\ x_4^1 \\ x_4^2 \\ x_4^3 \end{pmatrix}
+
\begin{pmatrix}
B_1 & & & & & & \\
B_2^{\prime 1} & B_2^1 & & & & & \\
 & B_3^{\prime 1} & B_3^1 & & & & \\
 & B_3^{\prime 3} & & B_3^3 & & & \\
 & & B_4^{\prime 1} & & B_4^1 & & \\
 & & B_4^{\prime 2} & & & B_4^2 & \\
 & & & B_4^{\prime 3} & & & B_4^3
\end{pmatrix}
\cdot
\begin{pmatrix} y_1^1 \\ y_2^1 \\ y_3^1 \\ y_3^3 \\ y_4^1 \\ y_4^2 \\ y_4^3 \end{pmatrix}
=
\begin{pmatrix} b_1 \\ b_2^1 \\ b_3^1 \\ b_3^3 \\ b_4^1 \\ b_4^2 \\ b_4^3 \end{pmatrix}
$$

$$
\begin{pmatrix}
A_1 & & & & & & & \\
A_2^{\prime 4} & A_2^4 & & & & & & \\
 & A_3^{\prime 4} & A_3^4 & & & & & \\
 & A_3^{\prime 5} & & A_3^5 & & & & \\
 & & A_4^{\prime 4} & & A_4^4 & & & \\
 & & & A_4^{\prime 5} & & A_4^5 & & \\
 & & & A_4^{\prime 6} & & & A_4^6 & \\
 & & & A_4^{\prime 7} & & & & A_4^7
\end{pmatrix}
\cdot
\begin{pmatrix} x_1^1 \\ x_2^4 \\ x_3^4 \\ x_3^5 \\ x_4^4 \\ x_4^5 \\ x_4^6 \\ x_4^7 \end{pmatrix}
+
\begin{pmatrix}
B_1 & & & & & & & \\
B_2^{\prime 4} & B_2^4 & & & & & & \\
 & B_3^{\prime 4} & B_3^4 & & & & & \\
 & B_3^{\prime 5} & & B_3^5 & & & & \\
 & & B_4^{\prime 4} & & B_4^4 & & & \\
 & & & B_4^{\prime 5} & & B_4^5 & & \\
 & & & B_4^{\prime 6} & & & B_4^6 & \\
 & & & B_4^{\prime 7} & & & & B_4^7
\end{pmatrix}
\cdot
\begin{pmatrix} y_1^1 \\ y_2^4 \\ y_3^4 \\ y_3^5 \\ y_4^4 \\ y_4^5 \\ y_4^6 \\ y_4^7 \end{pmatrix}
=
\begin{pmatrix} b_1 \\ b_2^4 \\ b_3^4 \\ b_3^5 \\ b_4^4 \\ b_4^5 \\ b_4^6 \\ b_4^7 \end{pmatrix}
$$

Notation. $A_t^{\prime\omega}$, A_t^ω, $B_t^{\prime\omega}$, B_t^ω: scenario matrices, b_t^ω: scenario *rhs* for the corresponding stage and x_t^ω, y_t^ω: scenario 0-1 and continuous variables, respectively.

Appendix B. Constraint matrix structure for the illustrative example. Case 2: $t^* = 2$ and $q = 4$ clusters

$$
\begin{aligned}
&\begin{pmatrix} A_1 & & & & \\ A_2^{\prime 1} & A_2^1 & & & \\ & A_3^{\prime 1} & A_3^1 & & \\ & & A_4^{\prime 1} & A_4^1 & \\ & & A_4^{\prime 2} & & A_4^2 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \\ x_4^2 \end{pmatrix} + \begin{pmatrix} B_1 & & & & \\ B_2^{\prime 1} & B_2^1 & & & \\ & B_3^{\prime 1} & B_3^1 & & \\ & & B_4^{\prime 1} & B_4^1 & \\ & & B_4^{\prime 2} & & B_4^2 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^1 \\ y_3^1 \\ y_4^1 \\ y_4^2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^1 \\ b_3^1 \\ b_4^1 \\ b_4^2 \end{pmatrix} \\
&\begin{pmatrix} A_1 & & & \\ A_2^{\prime 1} & A_2^1 & & \\ & A_3^{\prime 3} & A_3^3 & \\ & & A_4^{\prime 3} & A_4^3 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^3 \\ x_4^3 \end{pmatrix} + \begin{pmatrix} B_1 & & & \\ B_2^{\prime 1} & B_2^1 & & \\ & B_3^{\prime 3} & B_3^3 & \\ & & B_4^{\prime 3} & B_4^3 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^1 \\ y_3^3 \\ y_4^3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^1 \\ b_3^3 \\ b_4^3 \end{pmatrix} \\
&\begin{pmatrix} A_1 & & & \\ A_2^{\prime 4} & A_2^4 & & \\ & A_3^{\prime 4} & A_3^4 & \\ & & A_4^{\prime 4} & A_4^4 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^4 \\ x_3^4 \\ x_4^4 \end{pmatrix} + \begin{pmatrix} B_1 & & & \\ B_2^{\prime 4} & B_2^4 & & \\ & B_3^{\prime 4} & B_3^4 & \\ & & B_4^{\prime 4} & B_4^4 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^4 \\ y_3^4 \\ y_4^4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^4 \\ b_3^4 \\ b_4^4 \end{pmatrix} \\
&\begin{pmatrix} A_1 & & & & & \\ A_2^{\prime 4} & A_2^4 & & & & \\ & A_3^{\prime 5} & A_3^5 & & & \\ & & A_4^{\prime 5} & A_4^5 & & \\ & & A_4^{\prime 6} & & A_4^6 & \\ & & A_4^{\prime 7} & & & A_4^7 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^4 \\ x_3^5 \\ x_4^5 \\ x_4^6 \\ x_4^7 \end{pmatrix} + \begin{pmatrix} B_1 & & & & & \\ B_2^{\prime 4} & B_2^4 & & & & \\ & B_3^{\prime 5} & B_3^5 & & & \\ & & B_4^{\prime 5} & B_4^5 & & \\ & & B_4^{\prime 6} & & B_4^6 & \\ & & B_4^{\prime 7} & & & B_4^7 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^4 \\ y_3^5 \\ y_4^5 \\ y_4^6 \\ y_4^7 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^4 \\ b_3^5 \\ b_4^5 \\ b_4^6 \\ b_4^7 \end{pmatrix}
\end{aligned}
$$

Notation. $A_t^{\prime\omega}$, A_t^{ω}, $B_t^{\prime\omega}$, B_t^{ω}: scenario matrices, b_t^{ω}: scenario *rhs* for the corresponding stage and x_t^{ω}, y_t^{ω}: scenario 0-1 and continuous variables, respectively.

Appendix C. Constraint matrix structure for the illustrative example. Case 3: $t^* = 3$ and $q = 7$ clusters

Case 3 The $q = 7$ clusters are as follows:

$$\begin{pmatrix} A_1 & & & \\ A_2^{\prime 1} & A_2^1 & & \\ & A_3^{\prime 1} & A_3^1 & \\ & & A_4^{\prime 1} & A_4^1 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \end{pmatrix} + \begin{pmatrix} B_1 & & & \\ B_2^{\prime 1} & B_2^1 & & \\ & B_3^{\prime 1} & B_3^1 & \\ & & B_4^{\prime 1} & B_4^1 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^1 \\ y_3^1 \\ y_4^1 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^1 \\ b_3^1 \\ b_4^1 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & & & \\ A_2^{\prime 1} & A_2^1 & & \\ & A_3^{\prime 1} & A_3^1 & \\ & & A_4^{\prime 2} & A_4^2 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^2 \end{pmatrix} + \begin{pmatrix} B_1 & & & \\ B_2^{\prime 1} & B_2^1 & & \\ & B_3^{\prime 1} & B_3^1 & \\ & & B_4^{\prime 2} & B_4^2 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^1 \\ y_3^1 \\ y_4^2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^1 \\ b_3^1 \\ b_4^2 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & & & \\ A_2^{\prime 1} & A_2^1 & & \\ & A_3^{\prime 3} & A_3^3 & \\ & & A_4^{\prime 3} & A_4^3 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^3 \\ x_4^3 \end{pmatrix} + \begin{pmatrix} B_1 & & & \\ B_2^{\prime 1} & B_2^1 & & \\ & B_3^{\prime 3} & B_3^3 & \\ & & B_4^{\prime 3} & B_4^3 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^1 \\ y_3^3 \\ y_4^3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^1 \\ b_3^3 \\ b_4^3 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & & & \\ A_2^{\prime 4} & A_2^4 & & \\ & A_3^{\prime 4} & A_3^4 & \\ & & A_4^{\prime 4} & A_4^4 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^4 \\ x_3^4 \\ x_4^4 \end{pmatrix} + \begin{pmatrix} B_1 & & & \\ B_2^{\prime 4} & B_2^4 & & \\ & B_3^{\prime 4} & B_3^4 & \\ & & B_4^{\prime 4} & B_4^4 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^4 \\ y_3^4 \\ y_4^4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^4 \\ b_3^4 \\ b_4^4 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & & & \\ A_2^{\prime 4} & A_2^4 & & \\ & A_3^{\prime 5} & A_3^5 & \\ & & A_4^{\prime 5} & A_4^5 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^4 \\ x_3^5 \\ x_4^5 \end{pmatrix} + \begin{pmatrix} B_1 & & & \\ B_2^{\prime 4} & B_2^4 & & \\ & B_3^{\prime 5} & B_3^5 & \\ & & B_4^{\prime 5} & B_4^5 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^4 \\ y_3^5 \\ y_4^5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^4 \\ b_3^5 \\ b_4^5 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & & & \\ A_2^{\prime 4} & A_2^4 & & \\ & A_3^{\prime 5} & A_3^5 & \\ & & A_4^{\prime 6} & A_4^6 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^4 \\ x_3^5 \\ x_4^6 \end{pmatrix} + \begin{pmatrix} B_1 & & & \\ B_2^{\prime 4} & B_2^4 & & \\ & B_3^{\prime 5} & B_3^5 & \\ & & B_4^{\prime 6} & B_4^6 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^4 \\ y_3^5 \\ y_4^6 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^4 \\ b_3^5 \\ b_4^6 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & & & \\ A_2^{\prime 4} & A_2^4 & & \\ & A_3^{\prime 5} & A_3^5 & \\ & & A_4^{\prime 7} & A_4^7 \end{pmatrix} \cdot \begin{pmatrix} x_1^1 \\ x_2^4 \\ x_3^5 \\ x_4^7 \end{pmatrix} + \begin{pmatrix} B_1 & & & \\ B_2^{\prime 4} & B_2^4 & & \\ & B_3^{\prime 5} & B_3^5 & \\ & & B_4^{\prime 7} & B_4^7 \end{pmatrix} \cdot \begin{pmatrix} y_1^1 \\ y_2^4 \\ y_3^5 \\ y_4^7 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^4 \\ b_3^5 \\ b_4^7 \end{pmatrix}$$

Notation. $A_t^{\prime\omega}$, A_t^{ω}, $B_t^{\prime\omega}$, B_t^{ω}: scenario matrices, b_t^{ω}: scenario *rhs* for the corresponding stage and x_t^{ω}, y_t^{ω}: scenario 0-1 and continuous variables, respectively.

Appendix D. Storage of the variables $\mathbf{x}_{ti}^{p}$ and $\mathbf{y}_{ti}^{p}$ in the cluster submodel p

Table D.8: Storage order of the variables in cluster p

Explicit NAC

Stage 1

Variable	Position
x_{11}	1
$\ldots$	$\ldots$
$\ldots$	$\ldots$
$x_{1n_{x_1}^p}$	$n_{x_1}^p$

Stage t

$\ldots$
$\ldots$
$\ldots$
$\ldots$
$\ldots$

Stage t^*

Variable	Position
x_{t^*1}	$\sum_{t=1}^{t^*-1} n_{x_t}^p + 1$
$\ldots$	$\ldots$
$\ldots$	$\ldots$
$x_{t^*n_{x_{t^*}}^p}$	$\sum_{t=1}^{t^*} n_{x_t}^p$

Implicit NAC

Stage t^*+1

Variable	Position
$x_{t^*+1,1}$	$\sum_{t=1}^{t^*} n_{x_t}^p + 1$
$\ldots$	$\ldots$
$x_{t^*+1,n_{x_{t^*+1}}^p}$	$\sum_{t=1}^{t^*+1} n_{x_t}^p$

Stage t

$\ldots$
$\ldots$
$\ldots$
$\ldots$
$\ldots$

Stage T

Variable	Position
x_{T1}	$\sum_{t=1}^{T-1} n_{x_t}^p + 1$
$\ldots$	$\ldots$
$x_{Tn_{x_T}^p}$	$\sum_{t=1}^{T} n_{x_t}^p = n_x^p$

Explicit NAC

Stage 1

Variable	Position
y_{11}	$n_x^p + 1$
$\ldots$	$\ldots$
$\ldots$	$\ldots$
$y_{1n_{y_1}^p}$	$n_x^p + n_{y_1}^p$

Stage t

$\ldots$
$\ldots$
$\ldots$
$\ldots$
$\ldots$

Stage t^*

Variable	Position
y_{t^*1}	$n_x^p + \sum_{t=1}^{t^*-1} n_{y_t}^p + 1$
$\ldots$	$\ldots$
$\ldots$	$\ldots$
$y_{t^*n_{y_{t^*}}^p}$	$n_x^p + \sum_{t=1}^{t^*} n_{y_t}^p$

Implicit NAC

Stage t^*+1

Variable	Position
$y_{t^*+1,1}$	$n_x^p + \sum_{t=1}^{t^*} n_{y_t}^p + 1$
$\ldots$	$\ldots$
$y_{t^*+1,n_{y_{t^*+1}}^p}$	$n_x^p + \sum_{t=1}^{t^*+1} n_{y_t}^p$

Stage t

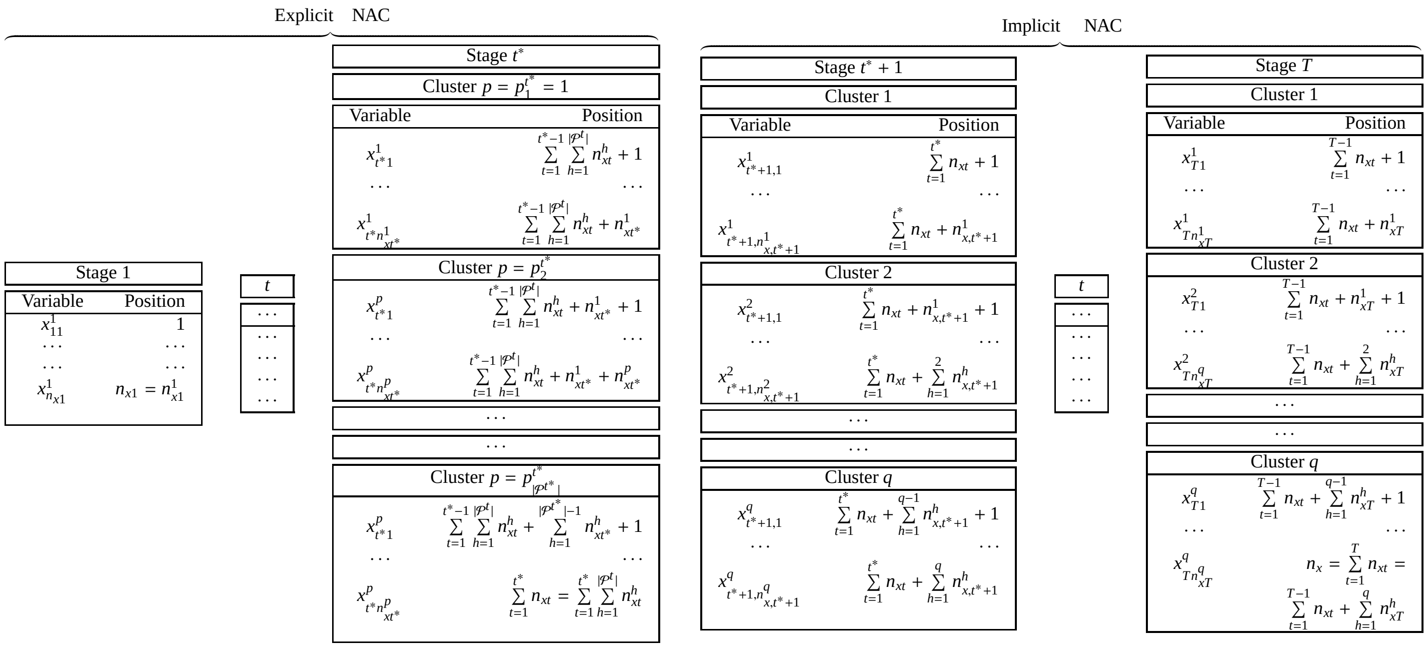
$\ldots$
$\ldots$
$\ldots$
$\ldots$
$\ldots$

Stage T

Variable	Position
y_{T1}	$n_x^p + \sum_{t=1}^{T-1} n_{y_t}^p + 1$
$\ldots$	$\ldots$
$y_{Tn_{y_T}^p}$	$n_x^p + n_y^p = n_x^p + \sum_{t=1}^{T} n_{y_t}^p$

Note: For simplicity we have eliminated the upperindex p in the name of the variables corresponding to the related cluster.

Appendix E. Storage of the 0-1 variables $\mathbf{x}_{ti}^{p}$ in the full model



Note: After the n_x 0-1 variables, the continuous variables y are stored in similar way.

References

[1] Birge JR, Louveaux FV. *Introduction to Stochastic Programming*. Springer; 1997.

[2] Cornuejols G. Revival of the Gomory cuts in the 1990s. *Operations Research* 2007; 149:63-66.

[3] Cornuejols G. Valid inequalities for mixed integer linear programs. *Mathematical Programming Ser. B* 2008; 112(1):3-44.

[4] Crowder H, Johnson EL, Padberg M. Solving large-scale zero-one linear programming problems. *Annals of Operations Research* 1983; 31:803-834.

[5] Escudero LF, Garín A, Merino M, Pérez G. A general algorithm for solving two-stage stochastic mixed 0-1 first stage problems. *Computers and Operations Research* 2009; 36:2590-2600.

[6] Escudero LF, Garín A, Merino M, Pérez G. BFC-MSMIP: an exact Branch-and-Fix Coordination approach for solving multistage stochastic mixed 0-1 problems. *TOP* 2009; 17(1):96-122.

[7] Escudero LF, Garín A, Merino M, Pérez G. On BFC-MSMIP strategies for scenario cluster partitioning, and twin node family branching selection and bounding for multistage stochastic mixed integer programming. *Computers and Operations Research* 2010; 37:738-753.

[8] Escudero LF, Garín A, Merino M, Pérez G. A note on the implementation of the BFC-MSMIP algorithm in C++ by using COIN-OR as an optimization engine. *Working paper serie Biltoki DT.2010.02, UPV/EHU, Spain* 2010; http://econpapers.repec.org/paper/ehubiltok/201002.htm.

[9] Escudero LF, Landete M, Rodriguez-Chia A. Stochastic Set Packing Problem. *European Journal of Operational Research* 2010; doi:10.1016/ejor.2010.11.022.

[10] Gomory RE. An algorithm for integer solutions to linear programs. In R.L. Graves and P. Wolfe, editors. *Recent Advances in Mathematical Programming*. McGraw-Hill 1963; p. 269-302.

[11] Guignard M, Johnson EL, Spielberg K. Logical processing in integer programming. *Annals of Operations Research* 2005; 140:263-304.

[12] Guignard M, Spielberg K. Logical reduction methods in zero-one programming. Minimal preferred variables. *Operations Research* 1981; 29:49-74.

[13] INFORMS. COIN-OR: COmputational INfrastructure for Operations Research. http://www.coin-or.org; 2010.

[14] Klein Haneveld W, van der Vlerk Kang M. Stochastic integer programming: General models and algorithms. *Annals of Operations Research* 1999; 85:39-57.

[15] Ntaimo L, Sen S. The million variable 'march' for stochastic combinatorial optimization. *Journal of Global Optimization* 2005; 32:385-400.

[16] Rockafellar RT, Wets RJB. Scenario and policy aggregation in optimisation under uncertainty. *Mathematics of Operations Research* 1991; 16:119-147.

[17] Savelsbergh MWP. Preprocessing and probing techniques for mixed integer programming problems. *ORSA Jornal of Computing* 1994; 6:445-454.

[18] Schultz R. Stochastic programming with integer variables. *Mathematical Programming Ser. B* 2003; 97:285-309.

[19] Schultz R, Tiedemann S. Conditional value-at-risk in stochastic programs with mixed integer recourse. *Mathematical Programming Serie B* 2006; 105:365-386.

[20] van der Vlerk MH. *Stochastic Integer Programming Bibliography.* http://www.eco.rug.nl/mally/biblio/sip.html, 1996-2007.

[21] Wallace SW, Ziemba WT, editors. *Applications of Stochastic Programming.* MPS-SIAM Series on Optimization; 2005.