Absorbing sets in coalitional systems

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Abstract

The purpose of this paper is twofold: *First*, to present an approach and a solution for analyzing the stability of coalition structures: We define a coalitional system (a set and a binary relation on that set) that explains the transitions between coalition structures and we propose to solve these systems using the *absorbing sets solution* for abstract systems. *Second*, to perform an analysis of this approach to evidence its utility in determining the stable coalition structures for some socioeconomic problems. We find that the absorbing sets solution efficiently solves this class of coalitional systems.

Key words: Coalition structures, coalitional systems and absorbing sets solution.

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1 Introduction

This paper deals with the question of stability of coalition structures.

Let us start with a brief explanation of this problem. Players usually form coalitions because they find it profitable to do so and, in many socioeconomic situations, the profits that these coalitions derive are not independent of how the remaining players are organized. The emerging coalitions of players give rise to configurations (or coalition structures) with which some players may be satisfied, but others may want to change. The decision to move is motivated by the payoffs that players can obtain in each of the possible configurations. Thus players, trying to improve their situation, force the transition from one coalition structure to another. In this paper we consider how players (some or all) may move successively in discrete steps until they converge upon some stable coalition structures.

In particular, the purpose of this paper is twofold: *First*, to present an approach and a solution for analyzing this problem. The approach is simple: We endow an abstract system (a set of alternatives and a binary relation on that set) with a structure capable of explaining the transitions between coalition structures, that we call *coalitional systems*. We propose to solve these systems by means of the *absorbing sets solution* for abstract systems. *Second*, to perform an analysis of the proposed approach to evidence its utility in determining stable coalition structures. That is, we define a class of coalitional systems that we call *symmetric cooperative* which contains systems derived from coalition formation problems in socioeconomic situations. We find that the absorbing sets solution efficiently solves this class of coalitional systems.

Let us now be more explicit about the content of this paper. To analyze the stability of coalition structures we start with a set of players N and a partition function φ which associates each possible coalition structure (or partition of the set N) with a payoff vector. Then, by looking at these vectors, we define a binary relation on the set of coalition structures that formalizes the transitional process between them. However, we find that this binary relation allows the survival of non plausible transitions. Therefore, our following step is to remove these non plausible transitions, trying to leave only the credible ones. After this filtering process, we are left with a set of coalition structures and a binary relation on that set that is called a coalitional system. We believe that this coalitional system approach provides considerable insight into the dynamics of coalition formation.

We come now to the selected definition of stability. In this paper we consider that players are myopic in the sense that when confronted with a possible transition, they do not wonder about further deviations from the transition under consideration. Furthermore, we do not consider time horizon or explicit stopping criteria to end the transitional process that leads from one coalition structure to another. This means that players can change coalition structures indefinitely unless some natural stable coalition structure is reached. In accordance with these ideas we have selected the *absorbing sets solution* for solving our coalitional systems. Each absorbing set coincides with the *elementary dynamic solution*[1] for an abstract system introduced by Shenoy [1979] and the absorbing sets solution is the collection of all the absorbing sets. The stability notion of this solution implies that with any two alternatives in an absorbing set one dominates the other, if not directly then through a path. Moreover no alternative outside an absorbing set dominates an alternative in the set, even through a path. Hence this solution contains sets that either consist of only one alternative or whose alternatives are in a cycle. Of course, alternatives not in an absorbing set are ruled out as unstable. A nice property of this solution is that it always exists although, in general, it may not be unique. A second solution for abstract games is also considered: The *generalized stable sets solution* of van Deemen [1991]. The stability notion of this solution is that in each generalized stable set there is no dominance relation, not even through a path, between any two distinct alternatives. Moreover, every alternative outside a generalized stable set is dominated, either directly or through a path, by some alternative in the set. We find an interesting relation between the two solutions considered: Any set formed by picking up one element of each of the absorbing sets is generalized stable.

In the second part of this paper, we formulate some assumptions on the parti-

[1]The union of all distinct elementary dynamic solutions for an abstract game is the *dynamic solution* introduced by Shenoy [1979]. This solution, under the name of the *admissible set*, was previously defined by Kalai, Schmeidler and Pazner [1976]. Schwartz [1974] also introduces an equivalent definition. See also Kalai and Schmeidler [1977] for an analysis of the admissible set in social bargaining processes.

tion function which give rise to a class of coalitional systems that we call *symmetric cooperative*. For these systems, whose alternatives are coalition structures, there is *only one* absorbing set. Hence, the problem of multiplicity of absorbing sets is overcome. As we shall see, our results allow us to identify the structure of this set, that is, the type of coalition structures in the unique absorbing set, as well as the transitions between them. Moreover, the established relation between the absorbing sets and the generalized stable sets solutions shows that every coalition structure in the unique absorbing set is in turn, a generalized stable set according to van Deemen's definition. Some socioeconomic examples illustrate the interest of symmetric cooperative systems. The first, is a simple social organization model, which is introduced mainly for illustrative purposes. The second is a numerical example derived from the well-known standard Cournot oligopoly model. The approach followed in this paper allows us to determine the unique absorbing set for both examples.

The paper is organized as follows: The coalitional system approach is introduced in Section 2. In Section 3 we give the definition of the absorbing sets solution, of the generalized stable sets solution, as well as the relation between the two. Section 4 contains the definition of symmetric cooperative systems and the determination of the unique absorbing set for this class of systems. The two socioeconomic examples are presented in Section 5, while the proofs concerning the results of the social organization system are in the appendix.

2 Coalitional systems

In this section we describe an approach, called the coalitional system, for analyzing the stability of coalition structures.

As stated in the introduction, players usually form coalitions because they find it profitable to do so. These groupings of players give rise to coalition structures with which some players may be satisfied but others are not, and the later may want to move to other coalition structures. We analyze how some (or all) players may move successively in discrete steps until they converge upon some coalition structures which are stable.

We assume that the decision to move from one coalition structure to other is motivated by the payoffs that players can obtain in them. These payoffs are provided by a partition function[2]. Formally,

Let N be a set of players. Denote by $\mathcal{P}$ a coalition structure (or a partition of the set N) and by $\mathfrak{P}(N)$ the set of all coalition structures formed with the set N.

Definition 1 *A partition function is a function* $\varphi : \mathfrak{P}(N) \longrightarrow I\!R^N$. $\varphi(\mathcal{P}) = (\varphi_1(\mathcal{P}), ..., \varphi_n(\mathcal{P}))$ *denotes the vector of payoffs when the players form coalition structure* $\mathcal{P}$.

Taking into account these payoff vectors we define a binary relation over the set of coalition structures which defines a coalitional system. Thus,

Definition 2 *A coalitional system is a pair* $(\mathfrak{P}(N), sdom)$*, where* $sdom$ *is a binary relation defined on* $\mathfrak{P}(N)$.

Notice that a coalitional system is merely a specified abstract system. (An abstract system is a set of alternatives X and a binary relation R on that set[3]).

Let us start with an example in order to explain our approach.

Example 1 *Let* $N = \{1, 2, 3\}$ *and let* φ *be a partition function that gives the following payoff vectors:* $\varphi(\{1\}\{2\}\{3\})$=$(5, 4, 3)$, $\varphi(\{12\}\{3\})$=$(4, 4, 8)$, $\varphi(\{13\}\{2\})$=$(9, 4, 9)$, $\varphi(\{23\}\{1\})$=$(4, 7, 7)$, $\varphi(\{123\})$=$(6, 6, 6)$.

By looking at the payoff vectors, let us consider some of the transitions among coalition structures. For example, we can infer that coalitions of players 3, 13 and 23 can force transition from $\{123\}$ *to* $\{\{12\}\{3\}\}$, $\{\{13\}\{2\}\}$ *and* $\{\{23\}\{1\}\}$ *respectively; while it is not in the interest of any player to go from* $\{123\}$ *to* $\{\{1\}\{2\}\{3\}\}$. *Note also that although player 2 would like to go from* $\{\{13\}\{2\}\}$ *to* $\{\{23\}\{1\}\}$ *this will not happen, since with this move player 3 would be damaged. (Digraph 1 represents the transitions which are formalized in Definitions 3 and 4.)*

However, all these transitions do not seem equally credible. For example, transition from $\{\{1\}\{2\}\{3\}\}$ *to* $\{\{123\}\}$ *via coalition 123 seems implausible since players*

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[2]A definition of the partition function can be found in Lucas and Macceli [1978]. See also Lucas and Thrall [1963].

[3]These systems can be represented by means of a directed graph (digraph) the vertices of which are the elements of the set X and the arcs represent the binary relation between them. We use this representation in most of our examples.

1 and 3 would rather go to $\{\{13\}\{2\}\}$ *and they can make it. The idea behind this observation implies that a transition that might be carried out by a coalition will never occur when there exists a subcoalition whose members could get higher payoffs by going to another coalition structure. Thus, additionally to the capability to force a transition, internal consistency is also required. (Digraph 2 represents the internal consistent transitions which are formalized in Definitions 6 and 7.)*

But, the remaining transitions may still not be equally likely. Notice that transition from $\{\{123\}\}$ *to* $\{\{12\}\{3\}\}$ *is also non credible, since player 3 would rather jointly deviate with player 1 and reach* $\{\{13\}\{2\}\}$*. That is, an internal consistent transition will not occur if the players that force the transition are able to associate with some (or all) players outside deviating to a more profitable coalition structure. (Digraph 3 represents the strong dominant transitions which are formally defined in Definitions 8 and 9.)*

We next introduce the binary relation on $\mathfrak{P}(N)$. But prior to defining it, let us consider a second example in order to illustrate the difficulties of this task.

Example 2 *Let* N*=*$\{1,2,3,4,5,6\}$ *and let* φ *be a partition function such that* $\varphi(\{12\}\{34\}\{56\})=(1,2,3,4,5,6)$, $\varphi(\{1\}\{23\}\{4\}\{5\}\{6\})=(0,3,4,0,6,7)$,
$\varphi(\{12\}\{34\}\{5\}\{6\})=(0,0,0,0,7,8)$, $\varphi(\{12\}\{3\}\{456\})=(0,0,0,7,8,9)$.

Consider transition from $\{\{12\}\{34\}\{56\}\}$ *to* $\{\{1\}\{23\}\{4\}\{5\}\{6\}\}$*. In this case three different groups of players may force this transition, namely:* $\{235\}$, $\{236\}$ *and* $\{2356\}$*. Observe that players 2 and 3 have to agree on their union and both are in all three sets. This is because the wills of all players coming from different coalitions in* $\{\{12\}\{34\}\{56\}\}$ *and forming a new coalition in* $\{\{1\}\{23\}\{4\}\{5\}\{6\}\}$ *are needed to carry out the grouping action. However, to break coalition 56 only one players, either 5 or 6, is strictly necessary.*

If we analyze the transit from $\{\{12\}\{34\}\{56\}\}$ *to* $\{\{12\}\{34\}\{5\}\{6\}\}$ *we have that following three groups of players may force this transition:* $\{5\}$, $\{6\}$ *and* $\{56\}$ *while transition from* $\{\{12\}\{34\}\{56\}\}$ *to* $\{\{12\}\{3\}\{456\}\}$ *can be carried out only by means of the set* $\{456\}$*. (See Digraph 4.)*

Following the ideas suggested by this example in what follows we formalize the transitional process between the coalition structures. In such a transitional process we suppose that an agreement among all the involved players is needed to form a coalition, while players, (either individually or in a group) may freely leave a coalition[4]. We also assume that players who force a transition from one coalition structure to another will benefit with it.

Definition 3 [5] *Let* $\mathcal{P}$,$\mathcal{Q} \in \mathfrak{P}(N)$. *We say that* $\mathcal{P}$ *weakly dominates* $\mathcal{Q}$ *via* M *if there exists a set* $M \subseteq N$ *such that*

i) M *is the union of some (or all) coalitions in* $\mathcal{P}$ *that satisfies*
$\{P \backslash M : P \in \mathcal{P}\} = \{Q \backslash M : Q \in \mathcal{Q}\}$,
ii) $\varphi_j(\mathcal{P}) > \varphi_j(\mathcal{Q})$ *for all* $i \in M$.

The ideas lying behind this definition are the following: Condition *i)* says that only players who form coalitions in $\mathcal{P}$ have the capacity to make the transition from $\mathcal{Q}$ to $\mathcal{P}$ and that for a transition to occur, what is left once players in M are deleted from $\mathcal{Q}$ and $\mathcal{P}$, has to be equal. This last means that no player in $N \backslash M$ has made the transition from $\mathcal{Q}$ to $\mathcal{P}$; therefore, the players who have forced this transition are contained in M. Condition *ii)* says that transition from $\mathcal{Q}$ to $\mathcal{P}$ should be profitable for all players in M.

Since transition from $\mathcal{Q}$ to $\mathcal{P}$ may not be uniquely determined (see Example 2) we introduce the following definition:

Definition 4 *We say that* $\mathcal{Q}$ *weakly dominates* $\mathcal{P}$ *(*$\mathcal{P}$*wdom*$\mathcal{Q}$*) if there exists a collection of sets* $\mathcal{M} = \{M_1, ..., M_k\} \neq \emptyset$ *such that* $\mathcal{Q}$*wdom*$\mathcal{P}$ *via* M_i *for all* $i = 1, ..., k$.

Example 2 shows that not all players in $\mathcal{M}$ have the same decision power to force a transition. In the first transition players 2 and 3 are both necessary to carry out this action. In the second, we cannot identify any player as necessary to make the transition since either player 5 or player 6 is able to do it. However, in the last transition players 4, 5 and 6 are essential to make the transition.

[4] In treating the breaks other alternatives could have been taken. In fact, there are social situations where the mutual agreement of the involved agents about a break is needed. (Think of a marriage dissolution by the Rota Court in the Catholic Church.)

[5] This definition was introduced in Espinosa and Inarra [2000].

In what follows, we study the structure of collection $\mathcal{M}$ and we classify the players that make transitions between coalition structures possible.

Consider a weak dominant transition from $\mathcal{Q}$ to $\mathcal{P}$ by means of collection $\mathcal{M} = \{M_1, ...M_k\}$

Definition 5 *A player j that belongs to every M_i in $\mathcal{M}$ is called essential. A player $j \in M_i$ for which there exists a coalition M_k such that $j \notin M_k$ is called inessential.*

i) For every pair $M_i, M_j \in \mathcal{M}$ we have that $M_i \cup M_j \in \mathcal{M}$. Consequently the maximal set under inclusion $\bigcup\limits_{i=1}^{k} M_i$ also belongs to $\mathcal{M}$. Notice that this set identifies all the players, essential and inessential, who can make transition from $\mathcal{Q}$ to $\mathcal{P}$ possible.

ii) Let $P \in \mathcal{P}$ such that $\varphi_i(\mathcal{P}) > \varphi_i(\mathcal{Q})$ for all $i \in P$. If there is no $Q \in \mathcal{Q}$ such that $P \subseteq Q$ then $P \subseteq M$ for all M in $\mathcal{M}$. Notice that in this case coalition P is a union of players coming from different coalitions in $\mathcal{Q}$, all of them obtaining profits with transition from $\mathcal{Q}$ to $\mathcal{P}$. Since coalition P is in every set of $\mathcal{M}$ then players in it are essential in the transition from $\mathcal{Q}$ to $\mathcal{P}$.

iii) Let $P \in \mathcal{P}$ such that $\varphi_i(\mathcal{P}) > \varphi_i(\mathcal{Q})$ for all $i \in P$. If $P \subseteq Q$ then P is a break of Q. Notice that in a two part break if only players in P benefit from the transition then these players are essential. However, if players in P and $Q \backslash P$ benefit with the transition then all players in Q are inessential. In general, if a coalition Q breaks into several parts and all its players profit with that break, all players in Q are inessential, while the existence of just one coalition P in this break with at least one player who does not profit from the transition implies that all players in $Q \backslash P$ are essential.

Once we have established the weak dominant transitions from $\mathcal{Q}$ to all the coalition structures, we try to identify which of them are more likely to occur. In fact, Definition 4 only requires players in every M_i of $\mathcal{M}$ to gain with transition from $\mathcal{Q}$ to $\mathcal{P}$. Considering all the weak dominant transitions from a given coalition structure there may be reasons why one transition is more plausible than other. Here, we will discuss one such reason: Internal consistency.

Let $M \in \mathcal{M}$ and $M' \in \mathcal{M}'$ where $\mathcal{M}$ and $\mathcal{M}'$ are collections of sets which allow transitions from $\mathcal{Q}$ to $\mathcal{P}$ and from $\mathcal{Q}$ to $\mathcal{P}'$ respectively. Suppose that $M' \subseteq M$ and that players in M' obtain higher payoffs than in M. Then it seems unlikely that a transition from $\mathcal{Q}$ to $\mathcal{P}$ via M will take place, since the players in M' will probably not inform the players in $M \backslash M'$ about an upcoming transition to $\mathcal{P}'$, which they can make without their cooperation. If this happens for every M in $\mathcal{M}$ then transition from $\mathcal{Q}$ to $\mathcal{P}$ will be removed. In what follows we define this idea.

Definition 6 *We say that $\mathcal{P}$wdom$\mathcal{Q}$ via M is internally consistent if there is no other $\mathcal{P}'$ such that $\mathcal{P}'$wdom$\mathcal{Q}$ via M' with $M' \subseteq M$ and $\varphi_i(\mathcal{P}') > \varphi_i(\mathcal{P})$ for all $i \in M'$.*

Definition 7 *We say that $\mathcal{P}$wdom$\mathcal{Q}$ is internally consistent if there is at least one $M \in \mathcal{M}$ such that $\mathcal{P}$wdom$\mathcal{Q}$ via M is internally consistent.*

In Example 2 we observe that starting from $\{\{12\}\{34\}\{56\}\}$, inessential players 5 and 6 may force transition to $\{\{12\}\{34\}\{5\}\{6\}\}$, blocking the weakly dominant transition to $\{\{1\}\{23\}\{4\}\{5\}\{6\}\}$ via $\{235\}$, $\{236\}$ and $\{2356\}$, hence the transition to coalition structure $\{\{1\}\{23\}\{4\}\{5\}\{6\}\}$ will not occur and should be removed.

Once more, we may still observe the survival of non credible transitions. That is, even if transition from $\mathcal{Q}$ to $\mathcal{P}$ is internally consistent there may still be reasons why it may not be plausible. For example, it may happen that the players in M seeking cooperation with some (or all) players outside forming for example M' can obtain greater payoffs, say in $\mathcal{P}'$. In this case players in M will prefer to go to $\mathcal{P}'$, provided that this transition is also internally consistent, rather than to $\mathcal{P}$. This idea is formalized below.

Definition 8 *Let transition from $\mathcal{Q}$ to $\mathcal{P}$ via M be internally consistent. We say that $\mathcal{P}$ strongly dominates $\mathcal{Q}$ via M if there is no other internally consistent transition from $\mathcal{Q}$ to $\mathcal{P}'$ via M' with $M \subseteq M'$ and $\varphi_i(\mathcal{P}') > \varphi_i(\mathcal{P})$ for all $i \in M'$.*

Definition 9 *We say that $\mathcal{P}$ strongly dominates $\mathcal{Q}$ ($\mathcal{P}$sdom$\mathcal{Q}$) if there is at least one $M \in \mathcal{M}$ such that $\mathcal{P}$sdom$\mathcal{Q}$ via M.*

Observe that transitions from $\{\{12\}\{34\}\{56\}\}$ to $\{\{12\}\{34\}\{5\}\{6\}\}$ and to $\{\{12\}\{3\}\{456\}\}$ are both internally consistent in Example 2. However, since players 4, 5 and 6 prefer to go to $\{\{12\}\{3\}\{456\}\}$ then this last coalition structure strongly dominates $\{\{12\}\{34\}\{56\}\}$ while the other not.

Thus, we have that Definitions 4, 7 and 9 give rise to the coalitional system $(\mathfrak{P}(N), sdom)$ introduced in Definition 2.

In what follows we show some properties of the binary relation sdom.

Theorem 1 *The binary relation sdom on $\mathfrak{P}(N)$ is neither reflexive, complete nor transitive, but asymmetric*[6].

Proof. It is immediate that sdom on $\mathfrak{P}(N)$ is not reflexive. Example 3 shows that it is neither complete nor transitive. To prove asymmetry requires a bit more work. Let $\mathcal{P}$wdom$\mathcal{Q}$. Then there exists at least one set M such that $\mathcal{P}$wdom$\mathcal{Q}$ via M. In this case we show there is no set R such that $\mathcal{Q}$wdom$\mathcal{P}$ via R.

Since all players in M have higher payoffs in $\mathcal{P}$ than in $\mathcal{Q}$ then any set that could force transition from $\mathcal{P}$ to $\mathcal{Q}$ via R should satisfy that $R \cap M = \emptyset$. On the other hand, Condition *i)* of Definition 3 requires M to be the union of some (or all) coalitions in $\mathcal{P}$ such that $\mathcal{P} \backslash M = \mathcal{Q} \backslash M$ and we know that players M are organized differently in $\mathcal{Q}$ than in $\mathcal{P}$ (otherwise they would not force transition from $\mathcal{Q}$ to $\mathcal{P}$.) But in this case we have that $\mathcal{P} \backslash R \neq \mathcal{Q} \backslash R$, and consequently there is no set R such that $\mathcal{Q}$wdom$\mathcal{P}$ via R, and $\mathcal{Q}$ does not weakly dominate $\mathcal{P}$.

Since the set of strong dominant transitions is a subset of the weak dominant transitions, we have that sdom is asymmetric. ■

Let us finish this section with a discussion on the strong dominance relation.

[6]A binary relation R on X is reflexive if for all $x \in X$, xRx. It is complete if for all $x, y \in X$, xRy or yRx. It is transitive if for all $x, y, z \in X$, xRy & $yRz \Rightarrow xRz$. It is assymetric if for all $x, y \in X$ $xRy \Rightarrow \neg yRx$.

In this paper, players are myopic in the sense that they only think of those transitions that can be made in just one stage; they do not consider in their decisions what is going to happen later on. Consequently, from a given coalition structure, and by looking at the payoff vectors, we are able to identify those transitions which are more likely to occur and rule out the non plausible ones. As we have described above, our selection of transitions is made in two consecutive steps.

> In the first step the validity of any weak dominant transition via M is checked against any other weak dominant transition via subsets of M. (See Examples 1 and 2).

That is, an internal consistency criterion is considered. This criterion differs from the one usually applied in game theory, which accounts for further deviations from the deviation under consideration[7]. In our setting, we analyze whether deviating from a to b is more likely than deviating from a to c. We do not consider whether after deviating from a to b another deviation to c is going to occur. The following example shows that the two approaches are different.

Example 3 *Let N=*$\{$*1,2,3,4,5,6*$\}$ *be a set of players and consider that function* φ *gives the following payoff vectors:* $\varphi(\{12\}\{34\}\{56\}) = (1,1,1,1,1,1)$, $\varphi(\{135\}\{2\}\{4\}\{6\}) = (2,0,2,0,2,0)$, $\varphi(\{13\}\{2\}\{4\}\{5\}\{6\}) = (3,0,3,0,0,0)$.

In accordance with Definitions 4, 7 and 9, transition from $\{\{12\}\{34\}\{56\}\}$ *to* $\{\{135\}\{2\}\{4\}\{5\}\}$ *can be forced by essential players 1, 3 and 5, while there will be not a transition from* $\{\{12\}\{34\}\{56\}\}$ *to* $\{\{13\}\{2\}\{4\}\{5\}\{6\}\}$*. However, transition from* $\{\{135\}\{2\}\{4\}\{5\}\}$ *to* $\{\{13\}\{2\}\{4\}\{5\}\{6\}\}$ *can be made by essential players 1 and 3.*

With farsighted players it could be argued that from $\{\{12\}\{34\}\{56\}\}$ *to* $\{\{135\}\{2\}\{4\}\{6\}\}$ *transition via* $\{135\}$ *would not occur since there is a second transition from* $\{\{135\}\{2\}\{4\}\{5\}\}$ *to* $\{\{13\}\{2\}\{4\}\{5\}\{6\}\}$ *that would damage player 5, an essential player in making the first transition under consideration. (See Digraph 5.)*

[7] See Chew [1994] for a study of farsighted stability.

In the second step the validity of any internally consistent transition via M is checked against any other internal consistent transition via supersets of M. (See Examples 1 and 2.)

That is, an external consistency criterion is considered. The idea behind this criterion is that after players in M have accounted for any possible internal defections, they take into account the possibility of going to other coalition structures with some (or all) of the remaining players.

Once we have established all the weak dominant transitions from a given coalition structure to all the others, it is interesting to learn whether the application of this two-step filtering process assures that some transitions are left. As we shall see the lemma presented below answers this question in the affirmative. However, the simultaneous application of both internal and external consistency criteria to the set of weak dominant transitions from a given coalition structure does not guarantee that at least one strong dominant transition remains. The following example shows this.

Example 4 *Let N=$\{$1,2,3,4,$\}$ and let φ be a partition function such that φ($\{$123$\}\{$4$\}$)=(1,1,1,1), φ($\{$1234$\}$)=(4,4,4,4), φ($\{$1$\}\{$23$\}\{$4$\}$)=(5,0,0,0), φ($\{$13$\}\{$2$\}\{$4$\}$)=(6,0,2,0) and φ($\{$12$\}\{$3$\}\{$4$\}$)=(0,0,3,0).*

In this example we analyze the four weak dominant transitions that can be made from coalition structure $\{\{$123$\}\{$4$\}\}$. Thus, transition from $\{\{$123$\}\{$4$\}\}$ to $\{\{$1234$\}\}$ via $\{$1234$\}$ is removed by internal consistency using transition to $\{\{$1$\}\{$23$\}\{$4$\}\}$ via $\{$1$\}$. This transition in turn is removed by external consistency using transition to $\{\{$13$\}\{$2$\}\{$4$\}\}$ via $\{$13$\}$, which is also taken out by internal consistency by transition to $\{\{$12$\}\{$3$\}\{$4$\}\}$ via $\{$3$\}$. Moreover, this last transition is removed using external consistency by means of transition to $\{\{$1234$\}\}$. Therefore no weak dominant transitions are left using both criteria simultaneously. (See Digraph 6.)

Additionally, we believe that the idea of applying first the internal consistency criterion looking at what happens inside M and after looking outside M is consistent with some social situations[8].

[8]For example, prior to take any decision, any group of allied countries analyzes its internal consistency and only then are alliances with other countries taken into account.

Now let us present a lemma which guarantees that our filtering process when applied to the weak dominant transitions from a given coalition structure allows the survival of at least one transition.

Lemma 2 *If $\mathcal{Q}wdom\mathcal{P}$ then there exists a coalition structure $\mathcal{P}'$ such that $\mathcal{P}'sdom\mathcal{Q}$.*

Proof. Assume that $\mathcal{P}$wdom$\mathcal{Q}$. Then there exists a collection $\mathcal{M}$ that can force transition from $\mathcal{Q}$ to $\mathcal{P}$. If this transition is not internally consistent, then, by Definitions 4 and 7, we have that for every set in $\mathcal{M}$ there are subsets of players in other coalition structures which weakly dominate $\mathcal{Q}$, with higher payoffs. Thus, we may have a sequence of coalition structures each of which is removed by another until an internally consistent transition is reached. Since every set in $\mathcal{M}$ is removed by a subset of players, cycling in the sequence is impossible, and an internally consistent transition will be reached in a finite number of steps.

If the last coalition structure in the sequence above strongly dominates $\mathcal{Q}$, then we are done. If not, there exists a sequence of coalition structures each of which is removed by another until a strong dominant transition is reached. This will necessarily happen in a finite number of steps since every set in $\mathcal{M}$ is removed by a superset, which again makes cycling in the sequence impossible. ■

To conclude we want to stress that the removal of the non credible transitions may yet leave some 'incompatible' transitions. That is, considering the set of transitions from one coalition structure some players may be involved in two or more different transitions. (In Example 1, there are two strongly dominant transitions from $\{\{1,2,3\}\}$ to $\{\{13\}\{2\}\}$ and to $\{\{23\}\{1\}\}$ via $\{13\}$ and $\{23\}$ respectively and player 3 is an essential player in both of them.) All in all, we think that even though the filtering process we have defined does not allow us to know exactly what transitions are going to happen, an important number of non credible transitions have been removed, making the subsequent application of solution concepts easier.

3 The absorbing sets solution for abstract systems

In the first part of this section we define the absorbing sets solution for abstract games. Then, we define a second solution: The generalized stable sets solution and we establish the relation between the two. As we shall see this relationship will be of use in the following section.

Let (X, R) be an abstract system. For $a, b \in X$, aRb means that a *dominates* b.

A *path* from a to b in X is a sequence of alternatives $a = a_0, a_1, a_2, ...a_m = b \in X$ such that $a_{i-1}Ra_i$ for all $i \in \{1, \ldots, m\}$.

Let R^T be the *transitive closure* of R (*i.e.* aR^Tb means that there is a path from a to b).

Now let us consider the definition of an *absorbing set*.

Definition 10 *Let (X, R) be an abstract system. A nonempty $A \subseteq X$ is called an absorbing set if*

i) for all $a, b \in A$ $(a \neq b)$: aR^Tb,

ii) there is no $b \in X \backslash A$ and $a \in A$ such that bR^Ta.

The *absorbing sets solution* for an abstract system (X, R) is the collection of all its absorbing sets.

Each of the absorbing sets satisfies two conditions. Condition *i)* says that in any two alternatives in an absorbing set one dominates the other, if not directly then through a path. Condition *ii)* says that no alternative outside the absorbing set dominates an alternative in the set, even through a path.

The notion of stability lying on the absorbing sets solution may be understood as follows. Suppose that at some point in time an alternative in an absorbing set A is reached, then all alternatives in A will be visited an infinite number of times, while no alternative outside A will ever be visited again.

Now, we show that every abstract system has at least one absorbing set. Considering *Theorem* 1 in Kalai and Schmeidler [1977] [9] which states that if X is finite the admissible set (which is the union of all its absorbing sets) is always nonempty, we can state (without proof) the following theorem:

Theorem 3 *Let (X, R) be an abstract game. Then (X, R) has at least one absorbing set.*

[9] See also *Theorem* 2.5 in Shenoy [1979] .

Let us now define the generalized stable sets solution and analyze its relation with the absorbing sets solution.

As is known, the von Neumann and Morgenstern stable sets solution suffers from the drawback that not every abstract game has a stable set. One can show however, that this solution always exists if the binary relation R is transitive. This property was utilized by van Deemen $[1991]$ to introduce yet another solution concept for abstract systems, the *generalized stable sets solution*[10]. Formally,

Definition 11 *Let (X, R) be an abstract system. A nonempty $A \subseteq X$ is called a generalized stable set if it is a stable set for (X, R^T), or equivalently if*

i) for all $a, b \in A (a \neq b)$: $\neg a R^T b$,

ii) for all $b \in X \backslash A$ there exists $a \in A$ such that $a R^T b$.

The *generalized stable sets solution* for an abstract system (X, R) is the collection of all its generalized stable sets.

Condition *i)* says that in any two distinct alternatives in the generalized stable set neither dominates the other, not even through a path. Condition *ii)* says that every alternative outside this set is dominated by some alternative in the generalized stable set, either directly or through a path.

A nice property of this solution is that every abstract system has at least one generalized stable set.

In what follows, we establish the relationship between the generalized stable sets solution and the absorbing sets solution.

Notice first that Conditions *i)* of the two solutions are exactly opposite. Moreover we find that any set formed by picking up one element of each of the absorbing sets is generalized stable. Let us see this.

Theorem 4 *Let $\{A_1, ..., A_k\}$ be the absorbing sets solution of the abstract system (X, R). Then S is a generalized stable set if and only if $|S \cap A_i| = 1$ for $i = 1, ..., k$.*

[10]This solution does not give a proper generalization of the $vN\&M$ stable set, *i.e.* even if a stable set exists, it is not necessarily a generalized stable set.

Proof. Let $S \subseteq X$. We will show first that S cannot be a generalized stable set if there exists an absorbing set A_i for which $|S \cap A_i| \neq 1$. So, let A_i be such an absorbing set and assume that $|S \cap A_i| = 0$, *i.e.* $S \subseteq X \backslash A_i$. Now, choose $a \in A_i$. Then, by Condition *ii)* of the absorbing set definition we know that there is no $b \in S$ such that $bR^T a$. Then S cannot be a generalized stable set, since its Condition *ii)* is violated. Now assume that $|S \cap A_i| > 1$. Then choose $a, b \in S \cap A_i$. By Condition *i)* of the absorbing set definition, we have $aR^T b$. Then S cannot be a generalized stable set, since its Condition *i)* is violated.

Now assume that $|S \cap A_i| = 1$ for $i = 1, ..., k$. To prove that Condition *i)* of a generalized stable set holds, let $a, b \in S$. Then a and b must be in different absorbing sets, say $a \in A_i$ and $b \in A_j$. Applying Condition *ii)* of the absorbing set definition to A_i we obtain $\neg bR^T a$, and applying it to A_j we obtain $\neg aR^T b$. To prove that Condition *ii)* of a generalized stable set holds, notice the following remark: An element in X not belonging to any absorbing set is dominated (directly or through a path) by the elements of at least one absorbing set. Now, let $S \cap A_i = a_i$ for $i = 1, ..., k$ and consider the set $X \backslash S$. If an element of $X \backslash S$, call it b, is in any A_i, then by Condition *i)* of the absorbing set definition we have $a_i R^T b$. However, if b is not in any A_i then by the previous remark we have $a_i R^T b$. Hence, any element in $X \backslash S$ is dominated (directly or through a path) by an a_i, and Condition *ii)* of the generalized stable set definition follows. ■

The following corollary will be of interest in Section 4.

Corollary 5 *Let $A = \{a_1, ..., a_k\}$ be the unique absorbing set of the abstract system (X, R). Then each set $\{a_i\}$ $i = 1, ..., k$ is generalized stable.*

4 Symmetric cooperative systems

In this section we introduce a class of coalitional systems that we call *symmetric cooperative*. As we shall see in Section 5 this class contains some systems derived from coalition formation problems in socioeconomic contexts.

Now let us summarize the results of this section.

We find that each symmetric cooperative system has exactly *one* absorbing set which contains the grand coalition. This last property is desirable, since by assumption in our class of systems the grand coalition is the efficient outcome. Furthermore, we are able to identify the type of coalition structures in the absorbing set as well as the transitions between them. Finally by Corollary 5 we have that each of the coalition structures in the unique absorbing set is a generalized stable set in the sense of van Deemen.

In what follows we establish four assumptions on the partition function φ preceded by an explanatory idea. The coalitional system derived from these restrictions on φ is called *symmetric cooperative.*

First, let us introduce some notation.

Let $\mathcal{P} = \{P_1, ..., P_l\}$ be a coalition structure. For the sake of simplicity, we consider that coalitions in $\mathcal{P}$ are arranged in non increasing order according to their size, that is, $|P_1| \geq |P_2| \geq ... \geq |P_l|$.

All players forming a coalition receive the same payoff.

A.1: Let P_k be a coalition in $\mathcal{P}$ and let i, j be two players such that $i, j \in P_k$. Then $\varphi_i(P_k,\mathcal{P}) = \varphi_j(P_k, \mathcal{P})$.

Taking into account this assumption, hereafter we denote by $\varphi_i(P_k, \mathcal{P})$ the payoff of any player i in P_k, $P_k \in \mathcal{P}$, by $\varphi(\{l\}, \mathcal{P})$ the payoff of player l in $\mathcal{P}$, and by $\varphi_i(\{N\})$ the payoff of any player in $\{N\}$ where $\mathcal{P} = \{N\}$.

All players in equal sized coalitions of a coalition structure $\mathcal{P}$ receive the same payoff. Players in coalitions of smaller size of a coalition structure $\mathcal{P}$ receive strictly greater payoffs than players in coalitions of larger sizes.

A.2: Let P_k, P_r be two distinct coalitions in $\mathcal{P}$ such that $|P_r| = |P_k|$. Then $\varphi_i(P_k, \mathcal{P}) = \varphi_i(P_r, \mathcal{P})$. If $|P_r| < |P_k|$. Then $\varphi_i(P_k, \mathcal{P}) < \varphi_i(P_r, \mathcal{P})$.

Now assume that total cooperation is the ”efficient coalition structure”. Of course, this assumption does not imply that every player in the grand coalition receives a payoff greater than the payoff he receives in any other coalition structure. In that case the problem of stability would be trivial. Our analysis is focused rather on situations where some players are interested in deviating from the efficient outcome.

The sum of the payoffs over all players is maximal when they form the grand coalition.

A.3: $|N|\,\varphi_i(\{N\}) > \sum_{i\in N}\varphi_i(\mathcal{P})$.

Notice that by assumption **A.2,** we know that the players in the smallest coalition of any coalition structure receive the greatest payoff. Hence, these players are the first candidates to object to the transition to the grand coalition (and also to every other coalition structure dominated by $\{N\}$). This suggests the definition of a set of non dominated coalition structures, denoted by $\mathfrak{nD}$, that plays an important role in the obtaining of our results. Formally

$$\mathfrak{nD} = \{\mathcal{P} \in \mathfrak{P}(N) : \varphi_i(P_l, \mathcal{P}) \geq \varphi_i(\{N\}).$$

In words, $\mathfrak{nD}$ contains those coalition structures not dominated by the grand coalition. However in this set there are two types of coalition structure: *i)* Coalition structures whose single deviators give rise to coalition structures again in $\mathfrak{nD}$. In this case, by Assumption **A.3**, the coalition structures generated have the following characteristic: ”The payoff of any single deviator is higher than the payoff of any player in the largest coalition of any other coalition structure in $\mathfrak{nD}$”. *ii)* Coalition structures whose single deviators give rise to coalition structures not in $\mathfrak{nD}$. In this second case the coalition structures generated may not have the characteristic mentioned and what the following assumption does is simply to require them to do so.

Single deviation is always preferred to be in the worst position of any coalition structure in $\mathfrak{nD}$.

A.4: Let $\mathcal{Q} \in \mathfrak{n}\mathfrak{D}$ and let $\mathcal{Q}^d$ be a coalition structure that arises when a single player deviates from $\mathcal{Q}$. Then

$$\varphi(\{l\}, \mathcal{Q}^d) \geq \varphi_i(P_1, \mathcal{P}) \text{ for all } \mathcal{P} \in \mathfrak{n}\mathfrak{D}.$$

From this we have the following remark.

Remark 1 *Assumption **A.4** implies that single deviation from any coalition structure in $\mathfrak{n}\mathfrak{D}$ is profitable. (To see this replace $\mathcal{P}$ by $\mathcal{Q}$).*[11]

In what follows we introduce Lemmas 6 and 7 to be used in Theorem 8, the main result of this section.

Lemma 6 *Suppose that $\{N\}$wdom$\mathcal{Q}$. Then either $\{N\}$sdom$\mathcal{Q}$ or there exists a coalition structure $\mathcal{P}$ such that $\mathcal{P}$sdom$\mathcal{Q}$, where the largest coalitions in $\mathcal{P}$ are at most the size of the largest coalitions in $\mathcal{Q}$.*

Proof. If $\{N\}sdom\mathcal{Q}$ then Lemma 6 follows. If this is not the case, then by Lemma 2 there exists a coalition structure $\mathcal{P}$ and a set of players $M \subsetneq N$ such that $\mathcal{P}sdom\mathcal{Q}$ via M. Notice that $\mathcal{P} \in \mathfrak{n}\mathfrak{D}$. On the other hand Assumptions **A.2** and **A.3** imply that $\varphi_i(P_1, \mathcal{P}) \leq \varphi_i(\{N\})$. Hence $P_1 \cap M = \emptyset$, that is players in P_1 cannot force transition from $\mathcal{Q}$ to $\mathcal{P}$. Since players in P_1 are not in M (see Definition 5 *i)* and *ii)*) the cardinality of P_1 cannot be greater than the cardinality of Q_1. So we have proved that the largest coalition in $\mathcal{P}$ is at most the size of the largest coalition in $\mathcal{Q}$. It also easily follows that if there are several equal sized largest coalitions in $\mathcal{P}$ and in $\mathcal{Q}$, then the number of these coalitions in $\mathcal{P}$ is at most the number in $\mathcal{Q}$. ■

Lemma 7 *If $\{N\}$ does not weakly dominate $\mathcal{Q}$ then $\mathcal{Q}^d$sdom$\mathcal{Q}$.*

Proof. First notice that if $\{N\}$ does not weakly dominate $\mathcal{Q}$ then Remark 1 guarantees that $\mathcal{Q}^d$ weak dominates $\mathcal{Q}$. Now assume that $\mathcal{Q}^d$ does not strongly dominate $\mathcal{Q}$. Then by Lemma 2 there exists a coalition structure $\mathcal{P}$ and a set of players M such

[11]Notice that this assumption does not imply that single deviation from any coalition structure in $\mathfrak{n}\mathfrak{D}$ will be the only possible transition. There may be others to coalition structures also in $\mathfrak{n}\mathfrak{D}$, as stated in the proof of Lemma 7 and emphasized in the final conclusion of this theorem.

that $\mathcal{P} sdom \mathcal{Q}$ via M. Notice that $\mathcal{P} \in \mathfrak{n}\mathfrak{D}$. Additionally, players in M are either single deviators from any coalition of $\mathcal{Q}$ receiving $\varphi(\{l\}, \mathcal{Q}^d)$ or/and singletons in $\mathcal{Q}$ receiving more than $\varphi_i(\{N\})$. By Assumption **A.4** we know that $\varphi(\{l\}, \mathcal{Q}^d) \geq \varphi_i(P_1, \mathcal{P})$ and since $\mathcal{P} \in \mathfrak{n}\mathfrak{D}$ we also know that $\varphi_i(\{N\}) \geq \varphi_i(P_1, \mathcal{P})$. Consequently $P_1 \cap M = \emptyset$ and $P_1 \subset Q_i$, $Q_i \in \mathcal{Q}$. Since $|Q_i| > 1$ then a player $k \in (P_1 \cap Q_i)$ will single deviate from $\mathcal{Q}$ giving rise to $\mathcal{Q}^d$. This player will not be member of M and therefore $\mathcal{Q}^d$ will strongly dominate $\mathcal{Q}$. So we have arrived at a contradiction and Lemma 7 follows. ■

Theorem 8 *Every symmetric cooperative system has exactly one absorbing set and that set contains* $\{N\}$.

Proof. It is sufficient to show that starting from an arbitrary coalition structure $\mathcal{Q}$ we can always arrive at coalition structure $\{N\}$, *i.e.* there is a sequence of coalition structures $\{N\} = \mathcal{Q}^k, ..., \mathcal{Q}^0 = \mathcal{Q}$ such that $\mathcal{Q}^j sdom \mathcal{Q}^{j-1}$ for all $j = 1, ..., k$.

Two cases can be considered, $\mathcal{Q} \in \mathfrak{n}\mathfrak{D}$ and $\mathcal{Q} \notin \mathfrak{n}\mathfrak{D}$.

Case i) $\mathcal{Q} \in \mathfrak{n}\mathfrak{D}$. In this case $\{N\}$ does not weakly dominate $\mathcal{Q}$. Then by Lemma 7, we know that $\mathcal{Q}^d sdom \mathcal{Q}$. This type of deviation occurs as long as the resulting coalition structure from single deviation belongs to $\mathfrak{n}\mathfrak{D}$. Hence, there will be a sequence of coalition structures each of which deviates from the coalition of largest size until we arrive at a coalition structure which is dominated by $\{N\}$, a possibility that we analyze in *Case ii)*.

Case ii) $\mathcal{Q} \notin \mathfrak{n}\mathfrak{D}$. Now $\mathcal{Q}$ is weakly dominated by $\{N\}$. In this case, by Lemma 6, either $\{N\} sdom \mathcal{Q}$ (in which case we are in $\{N\}$ and Theorem 8 is done) or there is a coalition structure, say $\mathcal{Q}^t$, whose largest coalitions are at most the size of the largest coalitions in $\mathcal{Q}$. Notice that Q^t belongs to $\mathfrak{n}\mathfrak{D}$. Hence, if transition to $\{N\}$ is not directly reached then we arrive at a coalition structure whose largest coalitions do not increase. We are back in *Case i)* and single deviation process from the largest coalition starts.

But the process described necessarily has an end. Therefore, in a finite number of steps we arrive at a coalition structure strongly dominated by $\{N\}$. ■

Corollary 9 *In the absorbing set of a symmetric cooperative system, either* $\{N\}$ *is* *the unique coalition structure of the set or there is at least one coalition structure dominated by* $\{N\}$.

Proof. By Theorem 8 we know that the unique absorbing set of a symmetric cooperative system contains $\{N\}$. Assume that $\{N\}$ is not the unique element of the set. Then the by definition of absorbing sets, in any two elements of the set one strongly dominates the other, if not directly then through a path. Then at least one coalition structure has to be strongly dominated by $\{N\}$. ■

Notice that this result can be interpreted as analogous to the prisoner dilemma result. It exhibits an undesirable consequence of players' myopic behavior, which may lead players to transit to dominated outcomes.

Additionally, we want to emphasize that the result that the grand coalition is in the unique absorbing set is not trivial. In the following example, we can see that the absorbing set may not contain $\{N\}$ even though it is the efficient outcome.

Example 5 *Assume a partition function* φ *that satisfies assumptions* ***A.1***, ***A.2*** *and* ***A.3.*** *These assumptions allow us to write the partition function considering only the sizes of coalitions in the coalition structures. That is, for any coalition structure* $\mathcal{P} = \{P_1, ..., P_l\}$*, let* $(p_1, ..., p_l)$ *be the vector whose entries represent the sizes of coalitions in* $\mathcal{P}$*, call it coalition-size vector. Denote by* $\varphi(p_i)$ *the payoff of any player in* P_i*. Now consider the following numerical example. Let* $\{1, 2, 3, 4, 5, 6, 7\}$ *be the set of players and assume that function* φ *gives the following payoff vectors:* $\varphi(7) = (50)$, $\varphi(6, 1) = (30, 70)$, $\varphi(5, 2) = (35, 60)$, $\varphi(4, 3) = (40, 55)$, $\varphi(5, 1, 1) = (25, 65, 65)$, $\varphi(4, 2, 1) = (30, 20, 65)$, $\varphi(3, 3, 1) = (35, 35, 65)$, $\varphi(3, 2, 2) = (35, 40, 40)$, $\varphi(4, 1, 1, 1) = (20, 53, 53, 53)$, $\varphi(3, 2, 1, 1) = (25, 30, 53, 53)$, $\varphi(2, 2, 2, 1) = (30, 30, 30, 53)$, $\varphi(3, 1, 1, 1, 1) = (15, 40, 40, 40, 40)$, $\varphi(2, 2, 1, 1, 1) = (20, 20, 35, 35, 35)$, $\varphi(2, 1, 1, 1, 1, 1) = (10, 25, 25, 25, 25, 25)$, $\varphi(1, 1, 1, 1, 1, 1, 1) = (15, 15, 15, 15, 15, 15, 15)$.

Digraph 7 shows the absorbing set for the symmetric cooperative system of this example. This set is formed by coalition structures represented by the following coalition-size vectors: $(5, 2)$, $(4, 3)$, $(5, 1, 1)$, $(4, 2, 1)$, $(3, 3, 1)$, $(4, 1, 1, 1)$, $(3, 2, 1, 1)$. *Notice that the efficient outcome* (7) *is not in the absorbing set.*

Now, applying the generalized stable sets solution to the symmetric cooperative systems we have the following result.

Proposition 10 *Each coalition structure in the unique absorbing set is a generalized stable set.*

Proof. Recall that the definition of the generalized stable sets solution has to satisfy two conditions. The first requires that neither of any two distinct alternatives in each generalized stable set dominates the other, even through a path. By Corollary 5, we know that each generalized stable set contains only one coalition structure, hence this condition is satisfied by vacuity. The second condition requires that any alternative outside a generalized stable set is dominated by some alternative in that set, either directly or through a path, which in the present case is obviously true. ■

Let us conclude this section emphasizing the type of coalition structures and the transitions in the absorbing set of a symmetric cooperative system.

i) $\{N\}$ is contained in the unique absorbing set. *ii)* Coalition structures in the set $\mathfrak{n}\mathfrak{D}$ are certainly related through single deviations. They may also be related through transitions which lead to more ”internally balanced” coalition structures, that is coalition structures whose largest coalitions do not grow. *iii)* There is also at least one transition from a coalition structure not in $\mathfrak{n}\mathfrak{D}$ to the grand coalition. Moreover, if from a coalition structure not in $\mathfrak{n}\mathfrak{D}$, the transition to the grand coalition does not exist, then there will be another transition which will lead to a more balanced coalition structure in $\mathfrak{n}\mathfrak{D}$.

5 Examples

In this section we introduce a model and a numerical example, both giving rise to symmetric cooperative systems. The first, mainly illustrative, shows clearly the results obtained in the previous section. The second is a numerical example for 5 players that corresponds to the well-known standard Cournot oligopoly model.

5.1 A social organization system

Let N be a group of countries which consider the possibility of forming social organizations (coalition structures) with the aim of obtaining common benefits. Think for example of the groupings of countries for opening markets, for example the European Union, the American Common Market and coalitions of countries for defense such as NATO.

Consider that the benefits of the entire group depend on the degree of cooperation of its members, measured by the number of social institutions they organize. In particular assume for the sake of simplicity that the degree of cooperation is represented by the following linear function: $u(\mathcal{P}) = |N| - (|\mathcal{P}| - 1)$ for all $\mathcal{P} \in \mathfrak{P}(N)$ where $|N| = n$ is the number of countries participating in the social organization and $|\mathcal{P}|$ the number of social institutions formed by them. Hence, $u(\{N\}) = n$ denotes the benefits of the unique social organization derived when total cooperation is reached, while $u(\mathcal{P}) = 1$ denotes the benefits when the degenerated social organization formed by singletons takes place.

Furthermore, assume that social norms and custom impose an equal division of the profits from cooperation. In accordance with this idea we assume that coalitional profits are $\frac{1}{|\mathcal{P}|}$ of total benefits and that each country receives $\frac{1}{|P_i|}$ of coalitional benefits. Hence in this model the payoff of each country is given by

$$u(P_i, \mathcal{P}) = \frac{|N| - (|\mathcal{P}| - 1)}{|P_i|\,|\mathcal{P}|}.$$

Proposition 11 *The payoff function u satisfies Assumptions **A.1-A.4**.*

Proof. See the appendix ■

Let us illustrate this model with a numerical example.

Example 6 *Let $N = \{1, 2, 3, 4\}$ be the set of countries. Let (4) (3,1) (2,2) (2,1,1) and (1,1,1,1) be the coalition-size vectors representing the coalition structures that can be formed with the set N. (For example (3,1) indicates a coalition structure formed by a coalition of 3 countries and a coalition of 1 country respectively). According to our social organization model the profit vectors corresponding to the coalition-size vectors above are:*

$(1,1,1,1)$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$, $(\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4})$, $(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3})$, $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$.

Digraph 8 represents the symmetric cooperative system associated with this example, where the absorbing set has been determined.

5.2 An example on Cournot oligopoly system[12]

Consider a set of firms with identical cost structure competing à la Cournot. Market demand is assumed to be $Q = a - p$, $a > 0$, where Q is the aggregate output and p the price. We assume that any group of firms may form a coalition and in that case share its coalitional profit equally. The cost function for any coalition of firms is assumed to be $C(Q_T) = cQ_T$, where $c \in [0, a)$ and Q_T is the output of coalition T. Given a coalition structure $\mathcal{P}$ (a market structure in this setting) we also assume that any coalition of firms $T \subseteq N$ maximizes profits. By solving this problem for each $T \in \mathcal{P}$ we obtain the following equilibrium profits:

$$\pi(T, \mathcal{P}) = \frac{(a-c)^2}{|T|\,(1 + |\mathcal{P}|)^2}.$$

$\pi(T, \mathcal{P})$ denotes the profits of a firm in a coalition of T firms in the market structure $\mathcal{P}$.[13]

Notice that for this model only the market structure formed by singletons can be sustained as a Nash equilibrium, since any single deviation is profitable. However, the result obtained with our approach is different, since the market structure formed by singletons will not necessarily be in the absorbing set.

Example 7 *Let* $N = \{1, 2, 3, 4, 5\}$ *be the set of firms. Suppose, for the sake of simplicity that* $a = 1$ *and* $c = 0$*. Coalition-size vectors* (5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$, $(1,1,1,1,1)$ *represent the market structures that can be formed with* 5 *identical firms. For example,* $(3,2)$ *indicates a duopoly formed by* 3 *and* 2

[12] See Bloch [1996], Ray and Vohra [1997], Espinosa and Inarra [2000] and Espinosa, Grafe and Inarra [2001] for a study of the stability of market structures for a Cournot oligopoly model with fixed costs. See also Yi [1997].

[13] The Cournot oligopoly model derives to a symmetric cooperative system. In order to prove this it can be shown that the Cournot profit function satisfies the four assumptions established on the partition function of Section 4 but this is studied in another paper. See Inarra, Kuipers and Olaizola [2001].

firms respectively. The payoff vectors derived from the profit function are:
$(\frac{1}{20},\frac{1}{20},\frac{1}{20},\frac{1}{20},\frac{1}{20})$, $(\frac{1}{36},\frac{1}{36},\frac{1}{36},\frac{1}{36},\frac{1}{9})$, $(\frac{1}{27},\frac{1}{27},\frac{1}{27},\frac{1}{18},\frac{1}{18})$, $(\frac{1}{48},\frac{1}{48},\frac{1}{48},\frac{1}{16},\frac{1}{16})$,
$(\frac{1}{32},\frac{1}{32},\frac{1}{32},\frac{1}{32},\frac{1}{16})$, $(\frac{1}{50},\frac{1}{50},\frac{1}{25},\frac{1}{25},\frac{1}{25})$, $(\frac{1}{36},\frac{1}{36},\frac{1}{36},\frac{1}{36},\frac{1}{36})$.

Digraph 9 represents the symmetric cooperative system associated with this example, where the absorbing set has been identified.

It can be easily seen that the market structures of the absorbing set for this symmetric cooperative system are given by: (5), $(4,1)$, $(3,1,1)$ *and* $(2,1,1,1)$*, while* $(3,2)$, $(2,2,1)$ *and* $(1,1,1,1,1)$ *represent non stable market structures.*

6 Appendix

6.1 Proof of proposition 9

We show that function u satisfies the assumptions established in Section 4.

Assumptions **A.1** and **A.2** are satisfied since u is strictly decreasing in the number of coalitions and on the number of players in each coalition. It can easily be seen that Assumption **A.3** is also satisfied. To prove Assumption **A.4** needs a bit more work. This assumption requires:

$$\begin{aligned} \varphi(\{l\}, \mathcal{Q}^d) &\geq \varphi_i(P_1, \mathcal{P}) \text{ for all } \mathcal{P} \in \mathfrak{u}\mathfrak{D}, \\ \text{where } \mathfrak{n}\mathfrak{D} &= \{\mathcal{P} \in \mathfrak{P}(N) : \varphi_i(P_l, \mathcal{P}) > \varphi_i\{(N\}\}. \end{aligned}$$

We first study the set $\mathfrak{n}\mathfrak{D}$.

Let $|N| = n$ and $|\mathcal{P}| = r$, and let the smallest coalition in $\mathcal{P}$ be a singleton, *i.e.* $P_l = \{l\}$. In our model the inequality

$$\varphi(\{l\}, \mathcal{P}) \geq \varphi_i(\{N\}) \text{ is equivalent to } r \leq \frac{n+1}{2}.$$

Given n we determine the maximal r, denoted by r^*, that satisfies the last inequality. Since the sizes of coalition structures $r = 1, ..., n$ are integers we have:

$$r^* = \frac{n-1}{2} \text{ if } n \text{ is odd and } r^* = \frac{n}{2} \text{ if } n \text{ is even.}$$

As function u is strictly decreasing in r, if Assumption **A.4** is satisfied for r^* then it is also satisfied for any coalition structure of size $r < r^*$ in $\mathfrak{n}\mathfrak{D}$. (See the explanation that follows Assumption **A.4**). Additionally, simple algebraic calculations show that every coalition structure of size r^* in $\mathfrak{n}\mathfrak{D}$ contains a singleton.

Now, let us check whether Assumption **A.4** is satisfied for r^*.

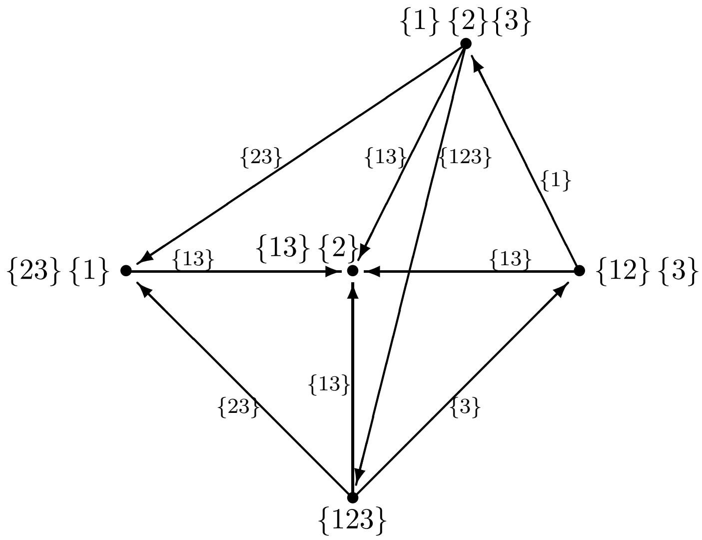
Case i) n is odd. In this case the payoff of a single deviator from a coalition structure of size r^* is exactly 1. Since the payoff of players in the largest coalitions of all the coalition structures in $\mathfrak{n}\mathfrak{D}$ are strictly smaller than 1 Assumption **A.4** is satisfied for this case.

Case ii) n is even. In this case the exact payoff of the single deviator from any coalition structure of size r^* is $\frac{n}{n+2}$. On the other hand we have

$$\max\{\varphi_i(P_1, \mathcal{P})\ for\ all\ \mathcal{P} \in \mathfrak{n}\mathfrak{D}\} = \frac{n-1}{n+2}.$$

Since $\frac{n}{n+2} > \frac{n-1}{n+2}$, Assumption **A.4** is satisfied and Proposition 9 follows.

6.2 Digraphs



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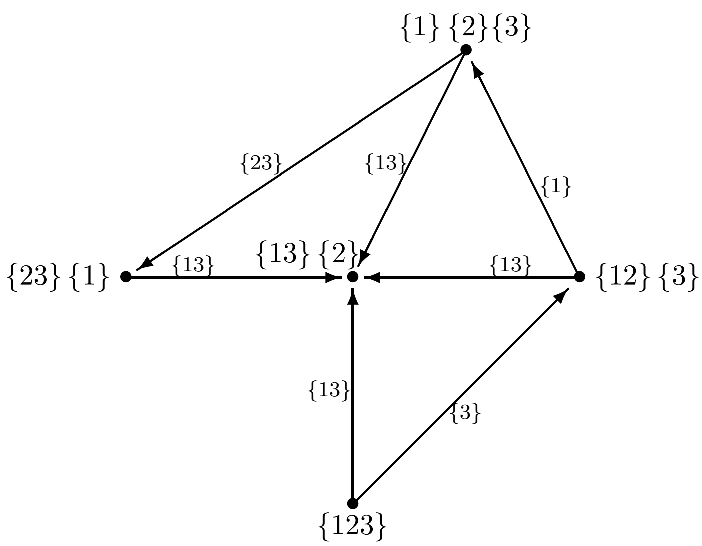

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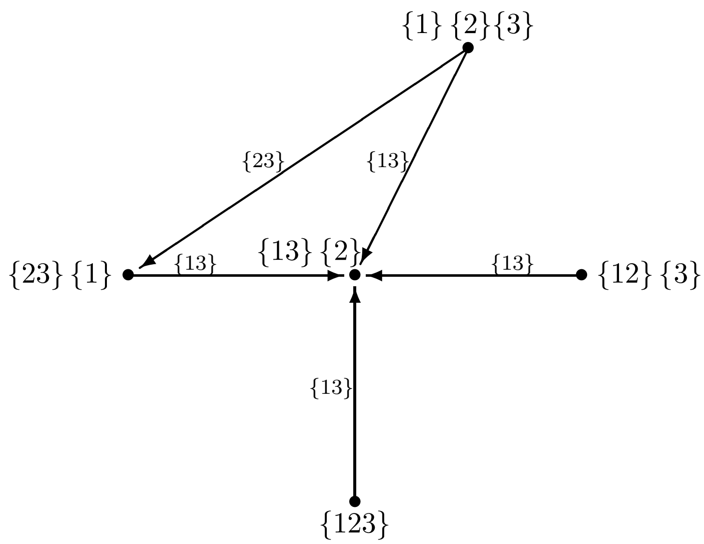

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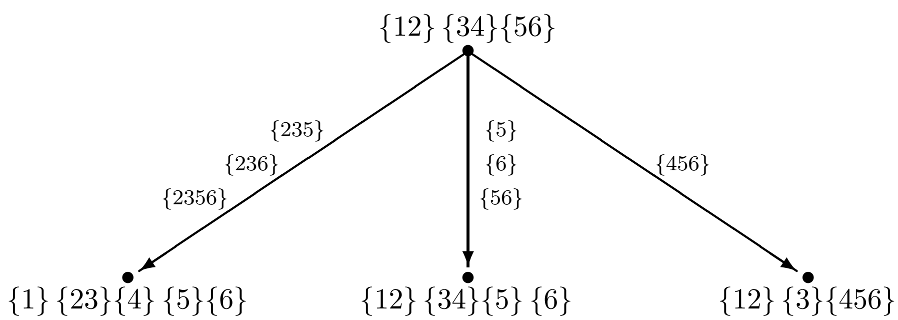

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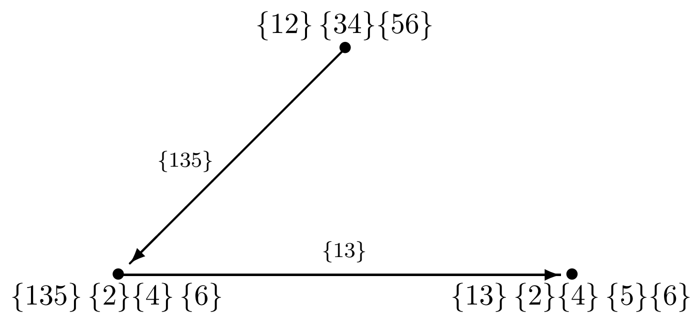

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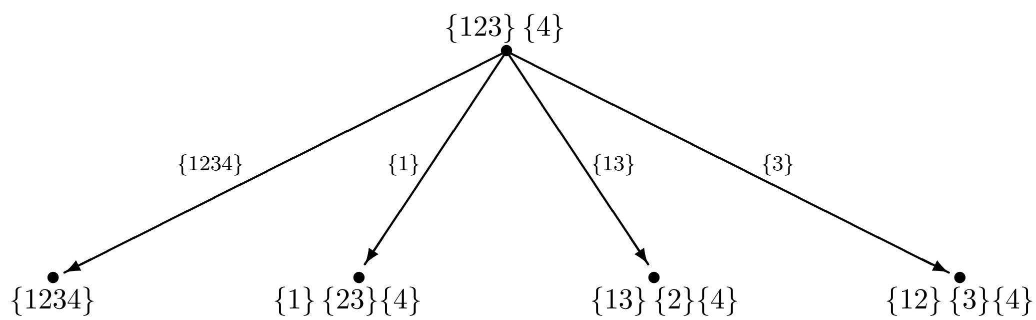
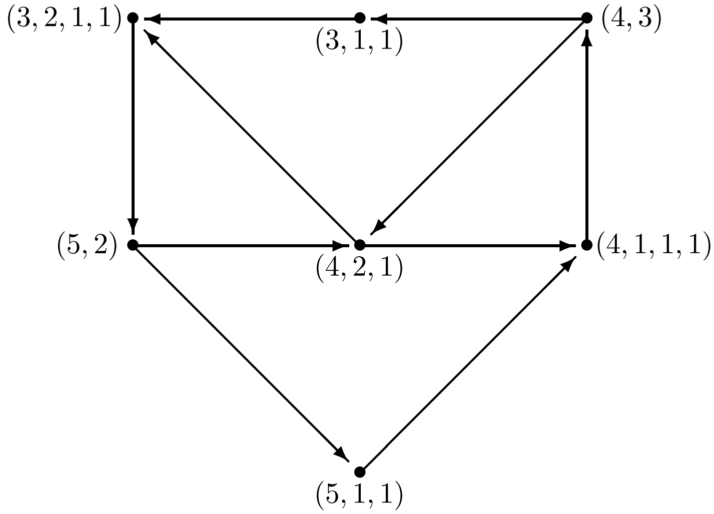
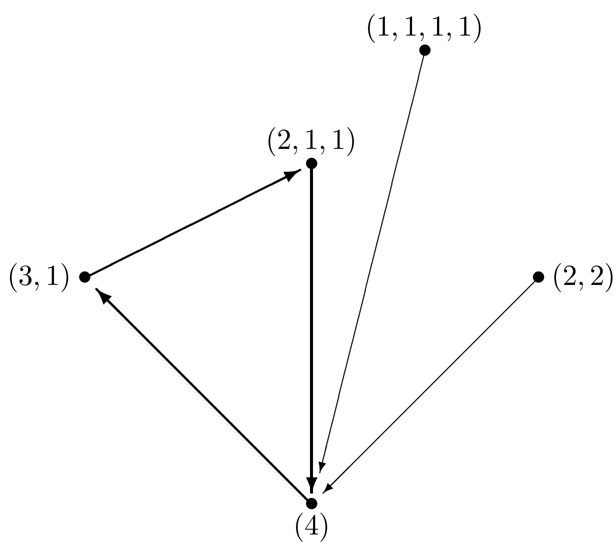


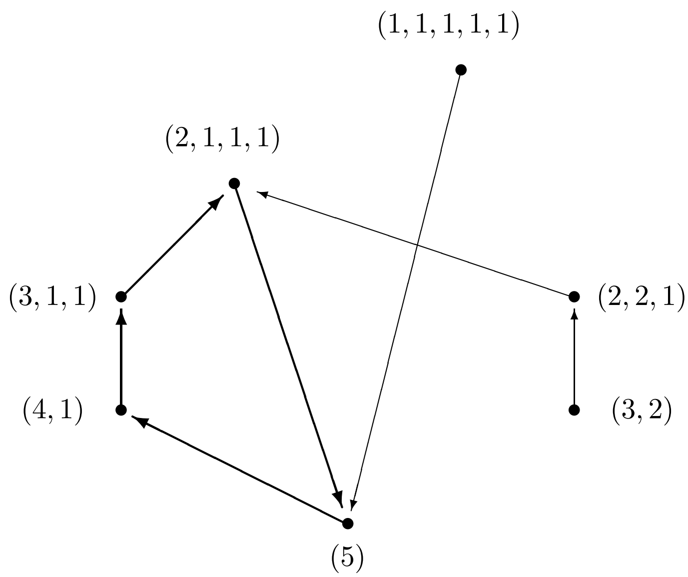
Digraph 6



Digraph 7



Digraph 8



Digraph 9

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