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# Dichotomous Preferences and Power Set Extensions

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# Dichotomous preferences and power set extensions*

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**Abstract**

This paper is devoted to the study of how to extend a dichotomous partition of a universal set $X$ into good and bad objects to an ordering on the power set of $X$. We introduce a family of rules that naturally take into account the number of good objects and the number of bad objects, and provide axiomatic characterizations of two rules for ranking sets in such a context.

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## 1 Introduction

In this paper we study the question of how a decision maker ranks sets of objects (individuals, goods, etc.) in contexts where the a priori information

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he has about them is *dichotomous* in the sense that: (i) the decision maker partitions the set of all objects into "good" and "bad" items, and (ii) the said partition is the only information he uses in order to evaluate sets. In other words, degrees of *goodness or badness* are ignored and thus, only two indifference classes are considered. Situations where such a ranking can be of use include matching, the choice of assemblies, the election of new committee members, group identification and coalition formation, among others.

Barberà et al. (2001), Dimitrov et al. (2007), Kasher and Rubinstein (1997), or Samet and Schmeidler (2003), for instance, study society formation problems in which the distinction is between candidates who qualify for membership on the basis of the opinion of some founder or member of the society, and those who do not merit such a qualification. Dichotomy in this context would be especially meaningful if qualification for membership were based on a certain religious principle or political ideology. It would also be natural if societal decisions were settled by vote and the voters had only to decide for or against, as in Barberà et al. (2001). In other cases, the members of the society have to decide who is entitled to perform a certain activity within the group, such as driving a car or teaching at the university. Analogously, we might consider college admission problems, where the good (bad) objects might be all those students that do (do not) fulfil a certain academic requirement.

The objects over which the dichotomous partition is made need not necessarily be people, however. A certain religious doctrine might also serve as the criterion by which to partition a set of norms into good and bad. Similarly, one might consider the mix of day/night shifts allocated to a worker over a certain period of time, or whether the answers of a participant in a test or TV quiz are right or wrong. Finally, Bogomolnaia et al. (2005)

propose different examples to analyze allocation mechanisms for problems in which agents partition potential lottery outcomes dichotomously into good outcomes and bad ones.

At this point it is important, in any case, to notice that the specification of good and bad objects in most of these contexts implies homogeneity and full substitutability of the objects within a particular group.

Given such a dichotomous setting (in which each object is either good or bad), the specific question we ask is the following: how can one meaningfully extend this rudimentary information about preferences over single objects to an ordering on their power set? We answer this question by introducing three core axioms that naturally define a family of rules for ranking sets in this context and by presenting axiomatic characterizations of two different rules that belong to the defined family. Each of these rules takes into account the number of good objects and the number of bad objects in the corresponding sets under comparison; they differ in the way in which these two numbers are combined.

Moreover, each rule induces a unique *separable* preference relation over the set of all groups of objects. That is, given an arbitrary set of objects, the addition of a good object to this set always results in a higher ranked set, while the addition of a bad object results in a lower ranked set. Clearly then, the set of all good objects and the set of all bad objects, respectively, constitute the top and the bottom of the induced preference relations. Bearing this in mind, our results can be interpreted as an axiomatic characterization of two subclasses of the class of separable preferences, the latter being commonly used as a primitive in the analysis of voting situations (cf. Barberà et al. (1991), Berga et al. (2004), Ju (2003, 2005)) and coalition formation games (cf. Burani and Zwicker (2003), Dimitrov et al. (2006)).

On the other hand, the results can be also seen as a contribution to the problem of ranking sets of objects in the context of *choice under complete uncertainty*[1] for the special case in which outcomes are compared dichotomously as in Bogomolnaia et al. (2005).

## 2 Basic setup

We denote by $X$ the nonempty finite *set of objects*. These objects may be candidates considered for membership in a club, for example, or possible coalition partners, bills under legislative consideration, etc. We assume that each object is either good or bad, and that there is at least one good object and at least one bad object (cf. Fishburn (1992)). We denote by $G$ the set of all good objects in $X$; the set of all bad objects is $X \setminus G$.

The set of all subsets of $X$, including the empty set, will be denoted by $\mathcal{X}$. The elements of $\mathcal{X}$ are the (alternative) groups of objects an agent may be confronted with. The question now arises of how this agent ranks sets consisting of good and bad elements based on his partition $(G, X \setminus G)$ of $X$. Consequently, the problem to be analyzed is how to establish a reflexive, transitive and complete binary relation $\succsim$ on $\mathcal{X}$. For all $C, D \in \mathcal{X}$, $C \succsim D$ is to be interpreted as "$C$ is at least as good as $D$". The asymmetric and symmetric factors of $\succsim$ will be denoted by $\succ$ ("is better than") and $\sim$ ("is as good as"), respectively. Finally, we denote by $\mathcal{P}$ the set of all reflexive, transitive and complete binary relations on $\mathcal{X}$.

[1] In the problems of choice under complete uncertainty, or ignorance, the probabilities of the outcomes generated by each action are not taken into account. Therefore, each individual decision is simply described by the set of outcomes it generates (see Barberà et al. (2004) for motivation and a survey of this approach).

## 3 A family of rules

We start our analysis by introducing the following axioms:

*Local monotonicity towards good elements* (LM): There exists $A \in \mathcal{X} \setminus \{X\}$ such that $A \cup \{x\} \succ A$ for some $x \in G \setminus A$.

*Local aversion towards bad elements* (LA): There exists $A \in \mathcal{X} \setminus \{X\}$ such that $A \cup \{x\} \prec A$ for some $x \in X \setminus (A \cup G)$.

*Independence* (IND): For all $A, B \in \mathcal{X}$, and all $x \in X \setminus A$, $y \in X \setminus B$ with $x \in G \Leftrightarrow y \in G$, $A \succsim B \Leftrightarrow A \cup \{x\} \succsim B \cup \{y\}$.

Axiom LM states that we can always find some set, $A$, that can be improved by adding some good new element to it. This is a rather weak requirement if one assumes that good objects are valuable to the decision maker. Especially, the axiom becomes very plausible if one thinks of $A$ as the empty set and $x$ as any good element.

The second axiom, LA, expresses the idea that there is some situation in which the decision maker fears the addition of more bad elements. In particular, AV says that there exist some set, $A$, and some bad element not belonging to $A$, such that the former is worsened by the addition of the latter.

Finally, IND illustrates the effect of adding (or dropping) two elements that are "of the same type" in the sense that they are either both good or both bad. The axiom, which says that the original ranking between any two sets of objects is preserved under such a modification, is an adaptation to our context of other similar axioms often found in the literature on ranking sets (cf. Kannai and Peleg (1984) or Pattanaik and Xu (1990) among many others).

In fact, IND captures a double assumption: on the one hand it clearly proposes some idea of separability. On the other hand, given that it applies to every pair of elements, $x$ and $y$, belonging to the same category, it illustrates, albeit less explicitly, our second main assumption about the preferences over objects; namely, that no degrees of *goodness* or *badness* are considered.[2]

As it turns out, these three axioms generate a family of rules that are based on two numbers only - the number of good elements and the number of bad elements, where the former are positively weighted and the latter are negatively weighted.

**Theorem 1** *Let* $\succsim \in \mathcal{P}$ *satisfy IND, LM, and LA. Then, for all* $A, B \in \mathcal{X}$*,*

*(1)* $(|A \cap G| > |B \cap G|$ *and* $|A \setminus G| < |B \setminus G|)$ *implies* $A \succ B$*,*

*(2)* $(|A \cap G| \geq |B \cap G|$ *and* $|A \setminus G| \leq |B \setminus G|)$ *implies* $A \succsim B$*.*

The proof is presented in the appendix.

## 4 Characterization results

We now present axiomatic characterizations of two rules that belong to the above described family of rules.

The first rule turns out to result from the interplay of the core axioms introduced in the previous section and a robustness axiom that we are about to introduce. The idea behind this axiom works as follows. Imagine a situation in which a set of elements, $A$, consists *only* of a *proper* subset of bad elements from another set, $B$, which, as well as the bad elements, might also contain good ones. Imagine also that the decision maker, nevertheless, declares a strict preference for $A$ over $B$. We interpret this premise as revealing that

[2] The double dimension of IND might be made more explicit by splitting it into two different conditions: (1) For all $x \in X \setminus A$, $y \in X \setminus B$ with $x \in G \Leftrightarrow y \in G$, $\{x\} \sim \{y\}$; (2) For all $A, B \in \mathcal{X}$, and all $x \in X \setminus A$, $y \in X \setminus B$ with $\{x\} \sim \{y\}$, $A \succsim B \Leftrightarrow A \cup \{x\} \succsim B \cup \{y\}$.

the decision maker is insensitive to the presence of good elements in $B$, and that what prevails is the fact that $A$ contains fewer bad elements. In such a situation, therefore, we require that the decision-maker remains insensitive when a new element is added to the set $B$ in the sense that the corresponding strict preference is preserved.

*Robustness* (ROB): For all $A, B \in \mathcal{X}$ with $A \subset (B \setminus G)$ and for all $x \in X$, $A \succ B \Rightarrow A \succ B \cup \{x\}$.

As shown in our next theorem, the addition of ROB to IND, LM, and LA results in the characterization of the *bad-elements-priority rule* $\succsim^{bp} \in \mathcal{P}$, which was first defined in Dimitrov at al. (2003)[3]:

For all $A, B \in \mathcal{X}$,

$$A \succsim^{bp} B \text{ iff } \begin{cases} |A \setminus G| < |B \setminus G|, \\ \text{or} \\ |A \setminus G| = |B \setminus G| \text{ and } |A \cap G| \geq |B \cap G|. \end{cases}$$

**Theorem 2** *Let $\succsim \in \mathcal{P}$. Then $\succsim$ satisfies IND, LM, LA, and ROB if and only if $\succsim = \succsim^{bp}$.*

The proof is presented in the appendix.

In order to demonstrate the *independence* of the axioms used for the characterization of $\succsim^{bp}$, consider the following examples:

$\neg$(IND): Let $|X| \geq 3$. For all $A, B \in \mathcal{X}$, let $\succsim$ be defined as follows: (1) if $|A| \geq 3$ and $|B| \geq 3$, then $A \sim B$, (2) if $|A| < 3$ and $|B| \geq 3$, then $A \succ B$, (3) if $|A| < 3$ and $|B| < 3$, then $\succsim = \succsim^{bp}$.

$\neg$(LM): For all $A, B \in \mathcal{X}$, $A \succsim B$ iff $|A \setminus G| \leq |B \setminus G|$.

[3] The cited paper does not contain a suitable axiomatic characterization of the proposed rule.

$\neg$(LA): For all $A, B \in \mathcal{X}$, $A \succsim B$ iff (1) $|A \setminus G| > |B \setminus G|$, or (2) $|A \setminus G| = |B \setminus G|$ and $|A \cap G| \geq |B \cap G|$.

$\neg$(ROB): For all $A, B \in \mathcal{X}$, $A \succsim B$ iff (1) $|A \cap G| > |B \cap G|$, or (2) $|A \cap G| = |B \cap G|$ and $|A \setminus G| \leq |B \setminus G|$.

The second rule from the family of rules described in the previous section is of an additive nature and can be introduced by means of the following axiom.

*Local dichotomy* (LD): There exists $A \in \mathcal{X}_{\emptyset} \setminus \{X\}$ such that $A \setminus \{x\} \sim A \cup \{y\}$ for some $x \in A \cap G$ and some $y \in X \setminus (A \cup G)$.

Condition LD is a much weaker version of a dichotomy axiom used in Dimitrov et al. (2004). LD states that there exist some set, $A$, a good element $x$, and a bad element $y$, such that the decision maker considers the non-inclusion of the good element $x$ in $A$ and the inclusion of the bad element $y$ in $A$ to be indifferent. In other words, the axiom displays a local perfect substitution between "the presence of a good element" and "the absence of a bad element" for some $A \in \mathcal{X}_{\emptyset} \setminus \{X\}$, $x \in A$ and $y \in X \setminus A$ .

We are now ready to present the characterization of the difference rule, $\succsim^d \in \mathcal{P}$, defined as follows[4]:

For all $A, B \in \mathcal{X}$,

$$A \succsim^d B \text{ iff } |A \cap G| - |A \setminus G| \geq |B \cap G| - |B \setminus G| .$$

**Theorem 3** *Let* $\succsim \in \mathcal{P}$. *Then* $\succsim$ *satisfies IND, LM, and LD if and only if* $\succsim = \succsim^d$.

The proof is presented in the appendix.

[4] See Dimitrov et al. (2004) for a different characterization of the same rule.

In order to check the *independence* of the axioms used for the characterization of $\succsim^d$, the reader may consider the following examples:

$\neg$(IND): Let $X = \{x, y\}$, $G = \{x\}$, and consider the following ranking on $\mathcal{X} : \emptyset \sim \{x, y\} \sim \{x\} \succ \{y\}$.

$\neg$(LM): For all $A, B \in \mathcal{X}$, $A \sim B$.

$\neg$(LD): For all $A, B \in \mathcal{X}$, $A \succsim B$ iff $|A| \geq |B|$.

As we show in Lemma 4 in the appendix, $\succsim \in \mathcal{P}$ satisfies axioms IND, LM, and LD if and only if $\succsim$ satisfies IND, LA, and LD. Thus, we have the following alternative characterization of the difference rule.

**Theorem 4** *Let* $\succsim \in \mathcal{P}$. *Then* $\succsim$ *satisfies IND, LA, and LD if and only if* $\succsim = \succsim^d$.

To check the *independence* of the above three axioms characterizing $\succsim^d$, we may take the following examples:

$\neg$(IND): Let $X = \{x, y\}$, $G = \{x\}$, and consider the following ranking on $\mathcal{X} : \emptyset \sim \{x, y\} \sim \{y\} \succ \{x\}$.

$\neg$(LA): For all $A, B \in \mathcal{X}$, $A \sim B$.

$\neg$(LD): For all $A, B \in \mathcal{X}$, $A \succsim B$ iff $|A| \leq |B|$.

# 5 Concluding remarks

Among the different ways in which the decision maker may evaluate sets of objects containing both good and bad elements, we have presented two plausible solutions derived from a common axiomatic basis. These core axioms are *Independence* (IND), *Local monotonicity towards good elements* (LM), and *Local aversion towards bad elements* (LA), and they determine a whole family of rules. Then, imposing the *Robustness* axiom (ROB) we obtain a characterization of the bad-elements-priority rule. Finally, if, instead of (ROB), we use *Local dichotomy* (LD) and either (LM) and (IND), or (LA) and (IND), then an additive rule that maximizes the difference between the number of good and bad elements is obtained.

Our model performs an axiomatic analysis based on a very elementary partition of the set of objects into good and bad items. A natural step forward in this research is to advance in defining the structure of the decision maker's information about the alternatives. For instance, the simple information structure described in this paper could be enriched by embedding a similarity relation (cf. Pattanaik and Xu (2000)) on the sets of good and bad objects. This would allow discrimination among different subgroups of good (bad) objects and would also enable consideration of extensions to the rules characterized here.

# 6 Appendix

This section collects the proofs of all theorems that appear in the text. In what follows, for all $S \subseteq X$ and all $k \in \{1, \ldots, |S|\}$, we denote by $(S)_k$ any subset of $S$ with $k$ elements.

We will first prove the following two lemmas.

**Lemma 1** *Let $\succsim \in \mathcal{P}$ satisfy IND and LM. Then $B \cup E \succ B$ for all $B \in \mathcal{X}$ and all $E \subseteq (G \setminus B) \setminus \{\emptyset\}$.*

**Proof of Lemma 1.** Take $\succsim \in \mathcal{P}$ as above and let $E = \{e_1, \ldots, e_n\}$. By LM, there exists $A \in \mathcal{X} \setminus \{X\}$ such that $A \cup \{x\} \succ A$ for some $x \in G \setminus A$.

Repeated application of IND implies that $\{x\} \succ \emptyset$ for some $x \in G \setminus A$. By reflexivity, $\emptyset \sim \emptyset$ and by IND, $\{x\} \sim \{e_1\}$. Thus, by transitivity, $\{e_1\} \succ \emptyset$. Now, by applying IND, $B \cup \{e_1\} \succ B$. Repeating the same argument with $e_2$ we obtain $B \cup \{e_1, e_2\} \succ B \cup \{e_1\}$, and by the same argument repeated $(n-2)$-times and transitivity, we get $B \cup E \succ B$. ■

**Lemma 2** *Let $\succsim \in \mathcal{P}$ satisfy IND and LA. Then $B \cup E \prec B$ for all $B \in \mathcal{X}$ and all $E \subseteq (X \setminus (B \cup G)) \setminus \{\emptyset\}$.*

**Proof of Lemma 2.** The proof is similar to the proof of Lemma 1 except that LA is applied instead of LM. ■

**Theorem 1** *Let $\succsim \in \mathcal{P}$ satisfy IND, LM, and LA. Then, for all $A, B \in \mathcal{X}$,*

*(1) $(|A \cap G| > |B \cap G|$ and $|A \setminus G| < |B \setminus G|)$ implies $A \succ B$,*

*(2) $(|A \cap G| \geq |B \cap G|$ and $|A \setminus G| \leq |B \setminus G|)$ implies $A \succsim B$.*

**Proof of Theorem 1.** (1) Let $|A \cap G| > |B \cap G|$ and $|A \setminus G| < |B \setminus G|$. By reflexivity, $\emptyset \sim \emptyset$. If $|B \cap G| = 0$ (i.e., $B \cap G = \emptyset$), $A \cap G \succ \emptyset$ follows from Lemma 1 with $A \cap G$ in the role of $E$, i.e., we have $A \cap G \succ B \cap G$ in this case. If $|B \cap G| = s > 0$, the application of IND $s$-times results in $(A \cap G)_s \sim B \cap G$. By Lemma 1, with $(A \cap G) \setminus (A \cap G)_s$ in the role of $E$, we have $A \cap G \succ (A \cap G)_s$. This, by transitivity, results in $A \cap G \succ B \cap G$. Therefore, whether $B \cap G \neq \emptyset$ or $B \cap G = \emptyset$, we have that $A \cap G \succ B \cap G$.

By Lemma 2 and $|B \setminus G| > 0$ (i.e., $B \setminus G \neq \emptyset$), we have $B \cap G \succ B$. If $A \setminus G = \emptyset$, we have by transitivity that $A \succ B$. Suppose now that $|A \setminus G| = v$. Starting from $A \cap G \succ B \cap G$ and applying $v$-times IND, we obtain $A \succ (B \cap G) \cup (B \setminus G)_v$. By Lemma 2, with $(B \setminus G) \setminus (B \setminus G)_v$ in the role of $E$, $(B \cap G) \cup (B \setminus G)_v \succ B$. By transitivity, $A \succ B$.

(2) The case in which $|A \cap G| > |B \cap G|$ and $|A \setminus G| < |B \setminus G|$ was proved in the previous paragraph. Thus, we will distinguish the three re-

maining possible cases:

(2.1) $|A \cap G| > |B \cap G|$ and $|A \setminus G| = |B \setminus G|$,

(2.2) $|A \cap G| = |B \cap G|$ and $|A \setminus G| < |B \setminus G|$, and

(2.3) $|A \cap G| = |B \cap G|$ and $|A \setminus G| = |B \setminus G|$.

(2.1) As in the first part of the proof, it can be proved, by using reflexivity, IND and LM, that $A \cap G \succ B \cap G$. If $|A \setminus G| = |B \setminus G| = 0$, then it follows directly that $A \succ B$. If $|A \setminus G| = |B \setminus G| = u > 0$, by IND repeated $u$-times, $A \succ B$.

(2.2) Let $|A \setminus G| = u$. By reflexivity, $\emptyset \sim \emptyset$, and applying $u$-times IND, $A \setminus G \sim (B \setminus G)_u$. By Lemma 2, with $(B \setminus G) \setminus (B \setminus G)_u$ in the role of $E$, $(B \setminus G)_u \succ B \setminus G$. By transitivity, $A \setminus G \succ B \setminus G$. Applying IND $|A \cap G| = |B \cap G|$-times, $A \succ B$.

(2.3) From reflexivity, $\emptyset \sim \emptyset$, and applying IND $|A \cap G| = |B \cap G|$-times, $A \cap G \sim B \cap G$. Again by IND applied $|A \setminus G| = |B \setminus G|$-times, $A \sim B$. ■

**Theorem 2** *Let* $\succsim \in \mathcal{P}$. *Then* $\succsim$ *satisfies IND, LM, LA, and ROB if and only if* $\succsim = \succsim^{bp}$.

**Proof of Theorem 2.** It is not difficult to check that $\succsim^{bp}$ satisfies the four axioms. Suppose now that $\succsim \in \mathcal{P}$ satisfies IND, LM, LA, and ROB. We have to prove that, for all $A, B \in \mathcal{X}$,

(1) $|A \setminus G| < |B \setminus G|$ implies $A \succ B$,

(2) ($|A \setminus G| = |B \setminus G|$ and $|A \cap G| > |B \cap G|$) implies $A \succ B$, and

(3) ($|A \setminus G| = |B \setminus G|$ and $|A \cap G| = |B \cap G|$) implies $A \sim B$.

(1) Let $|A \setminus G| = u$ and $|B \setminus G| = v$, with $v > u$. By reflexivity and IND applied $u$-times, $A \setminus G \sim (B \setminus G)_u$. By Lemma 2, with $(B \setminus G) \setminus (B \setminus G)_u$ in the role of $E$, $(B \setminus G)_u \succ B \setminus G$. By transitivity, $A \setminus G \succ B \setminus G$.

Now, let us consider the following partitions of $A \setminus G$ and $B \setminus G$ :

$$\begin{aligned} A \setminus G &= (A \setminus G)^1 \cup (A \setminus G)^2 , \\ B \setminus G &= (B \setminus G)^1 \cup (B \setminus G)^2 , \end{aligned}$$

where

$$\begin{aligned} (A \setminus G)^1 &= \{x \in A \setminus G \mid x \in B \setminus G\} , \\ (A \setminus G)^2 &= \{x \in A \setminus G \mid x \in X \setminus (B \setminus G)\} , \\ (B \setminus G)^1 &= \{x \in B \setminus G \mid x \in A \setminus G\} = (A \setminus G)^1 , \\ (B \setminus G)^2 &= \{x \in B \setminus G \mid x \in X \setminus (A \cup G)\} . \end{aligned}$$

Let $(A \setminus G)^1 = \{a_1^-, \ldots, a_{u_1}^-\} = (B \setminus G)^1$, $(A \setminus G)^2 = \{a_{u_1+1}^-, \ldots, a_u^-\}$, $(B \setminus G)^2 = \{b_{u_1+1}^-, \ldots, b_v^-\}$. Note that, by hypothesis, $|(B \setminus G)^2| > |(A \setminus G)^2|$. Consider $\{b_{u_1+1}^-, \ldots, b_u^-\} \subseteq (B \setminus G)^2$. Then $((A \setminus G)^1 \cup \{b_{u_1+1}^-, \ldots, b_u^-\}) \setminus (A \setminus G)^2 \sim A \setminus G$ by Theorem 1. By transitivity, $((A \setminus G)^1 \cup \{b_{u_1+1}^-, \ldots, b_u^-\}) \setminus (A \setminus G)^2 \succ B \setminus G$. By ROB, $((A \setminus G)^1 \cup \{b_{u_1+1}^-, \ldots, b_u^-\}) \setminus (A \setminus G)^2 \succ B$. Now, if $A \cap G = \emptyset$, then $A \sim A \setminus G$ by reflexivity, and by transitivity, $A \succ B$. If $A \cap G \neq \emptyset$, then, by Lemma 1, $A \succ A \setminus G$, and by transitivity, $A \succ B$.

(2) Let $A \setminus G = \{a_1^-, \ldots, a_u^-\}$, $B \setminus G = \{b_1^-, \ldots, b_u^-\}$, $A \cap G = \{a_1^+, \ldots, a_r^+\}$, and $B \cap G = \{b_1^+, \ldots, b_s^+\}$, $r > s$. By Theorem 1, $(A \setminus G) \cup \{a_1^+, \ldots, a_s^+\} \sim B$. By Lemma 1, $A \succ (A \setminus G) \cup \{a_1^+, \ldots, a_s^+\}$. Again by transitivity, $A \succ B$.

(3) By Theorem 1, $A \sim B$. ■

**Theorem 3** *Let $\succsim \in \mathcal{P}$. Then $\succsim$ satisfies IND, LM, and LD if and only if $\succsim = \succsim^d$.*

We will first prove the following two lemmas.

**Lemma 3** *Let $\succsim \in \mathcal{P}$ satisfy IND and LD, and let $A, B \in \mathcal{X}$ be such that*

$B = A \cup E$ *with* $|E \cap G| = |E \setminus G|$. *Then* $A \sim B$.

**Proof of Lemma 3.** Take $\succsim \in \mathcal{P}$ as above and let $E \cap G = \{e_1^+, \ldots, e_n^+\}$, $E \setminus G = \{e_1^-, \ldots, e_n^-\}$.

If $|E \cap G| = |E \setminus G| = 0$, the lemma follows by reflexivity. If $|E \cap G| = |E \setminus G| > 0$, we have by LD that there exists $F \in \mathcal{X}_\emptyset \setminus \{X\}$ such that $F \setminus \{x\} \sim F \cup \{y\}$ for some $x \in F \cap G$ and some $y \in X \setminus (F \cup G)$. Applying IND repeatedly, we have $\emptyset \sim \{x, y\}$ for some $x \in F \cap G$ and some $y \in X \setminus (F \cup G)$. By reflexivity, $\emptyset \sim \emptyset$ and by IND applied twice, $\{x, y\} \sim \{e_1^+, e_1^-\}$. Thus, by transitivity, $\emptyset \sim \{e_1^+, e_1^-\}$. By IND, $\{e_2^+, e_2^-\} \sim \{e_1^+, e_2^+, e_1^-, e_2^-\}$ and by transitivity, $\emptyset \sim \{e_1^+, e_2^+, e_1^-, e_2^-\}$. Repeating the same argument $(n-2)$-times and by transitivity, we have $\emptyset \sim E$. Thus, by IND, $A \sim A \cup E$, i.e., $A \sim B$. ■

**Lemma 4** $\succsim \in \mathcal{P}$ *satisfies IND, LM, and LD if and only if it satisfies IND, LA, and LD.*

**Proof of Lemma 4.** Let $\succsim \in \mathcal{P}$ satisfy IND, LM, and LD. We have to show that $\succsim$ also satisfies LA. Notice first that Lemma 1 and Lemma 3 hold. In particular, by Lemma 1 we have that $\{x, y\} \succ \{x\}$ for all $x \in X$ and all $y \in G \in \mathcal{X}_\emptyset$ with $y \neq x$. Furthermore, by Lemma 3, we have $\emptyset \sim \{x, y\}$ for all $G \in X_\emptyset$, all $x \in G$, and all $y \in X \setminus G$.

In order to prove that $\succsim$ satisfies LA, we first prove the following Claim: $\{x\} \succ \{x, w\}$ for all $x \in X$, all $G \in \mathcal{X}_\emptyset \setminus \{X\}$ and all $w \in X \setminus G$ with $w \neq x$. This would demonstrate that, for each $G \in \mathcal{X}_\emptyset \setminus \{X\}$, there exists $A \in \mathcal{X} \setminus \{X\}$ such that $A \cup \{w\} \prec A$ for some $w \in X \setminus (A \cup G)$, as required by LA. In particular, $A$ could be any singleton included in $G$.

Consider first the case in which $|X| = 2$ and let $X = \{x, w\}$ with $w \in X \setminus G$. Since $G$ is nonempty, this means that $G = \{x\}$. Then, by Lemma

1, $\{x,w\} \succ \{w\}$. By IND, $\{x\} \succ \emptyset$. On the other hand, by Lemma 3, $\emptyset \sim \{x,w\}$. Thus, by transitivity, $\{x\} \succ \{x,w\}$.

Suppose next that $|X| \geq 3$, and take $x, w$ as in the Claim. If $G \neq \{x\}$, then there exists $z \in G$, $z \neq x$. By Lemma 3, $\emptyset \sim \{z,w\}$, and by Lemma 1 we have $\{z,w\} \succ \{w\}$. By transitivity, $\emptyset \succ \{w\}$, and by IND, $\{x\} \succ \{x,w\}$. If $G = \{x\}$, then, by Lemma 1, $\{x,w\} \succ \{w\}$. By IND, $\{x\} \succ \emptyset$. On the other hand, by Lemma 3, $\emptyset \sim \{x,w\}$. Thus, by transitivity, $\{x\} \succ \{x,w\}$.

Suppose next that $\succsim \in \mathcal{P}$ satisfies IND, LA, and LD. The proof that $\succsim$ also satisfies LM is analagous to the above one by virtue of the fact that in this case Lemma 2 and Lemma 3 hold. ■

**Corollary 1** *Let $\succsim \in \mathcal{P}$ satisfy IND, LM, and LD. Then the statement in Lemma 2 holds, that is, $B \cup E \prec B$ for all $B \in \mathcal{X}$ and all $E \subseteq (X \setminus (B \cup G)) \setminus \{\emptyset\}$.*

**Proof Theorem 3.** It can be easily checked that $\succsim^d$ satisfies the three axioms. Suppose now that $\succsim \in \mathcal{P}$ satisfies IND, LM, and LD. We have to prove that, for all $A, B \in \mathcal{X}$,

(1) $|A \cap G| - |A \setminus G| > |B \cap G| - |B \setminus G|$ implies $A \succ B$, and

(2) $|A \cap G| - |A \setminus G| = |B \cap G| - |B \setminus G|$ implies $A \sim B$.

Let $|A \cap G| = r$, $|B \cap G| = s$, $|A \setminus G| = u$, $|B \setminus G| = v$.

(1) In this case $r - u > s - v$. We consider the following three possible cases:

(1.1) $r > u$ and $s > v$,

(1.2) $r > u$ and $s \leq v$,

(1.3) $r \leq u$ and $s < v$.

(1.1) Let $r > u$ and $s > v$. By reflexivity, $\emptyset \sim \emptyset$. By Lemma 3, $(A \cap G)_u \cup (A \setminus G) \sim \emptyset$. Also by Lemma 3, $(B \cap G)_v \cup (B \setminus G) \sim \emptyset$. Thus, by transitivity, $(A \cap G)_u \cup (A \setminus G) \sim (B \cap G)_v \cup (B \setminus G)$. Given that

$r-u>s-v$, by IND applied $(s-v)$-times $(A\cap G)_{u+s-v}\cup(A\setminus G)\sim(B\cap G)_{v+s-v}\cup(B\setminus G)$, i.e., $(A\cap G)_{u+s-v}\cup(A\setminus G)\sim B$. By Lemma 1, $A\succ(A\cap G)_{u+s-v}\cup(A\setminus G)$, and by transitivity, $A\succ B$.

(1.2) Let $r>u$ and $s\leq v$. As in case (1.1), by reflexivity, Lemma 3 and transitivity we get $(A\cap G)_u\cup(A\setminus G)\sim(B\cap G)\cup(B\setminus G)_s$. By Lemma 1, $A\succ(A\cap G)_u\cup(A\setminus G)$, and, if $s<v$, by Lemma 2,

$$(B\cap G)\cup(B\setminus G)_s\succ B.$$

If $s=v$, $(B\cap G)\cup(B\setminus G)_s=B$. In any case, by transitivity, $A\succ B$.

(1.3) Let $r\leq u$ and $s<v$. As before, by reflexivity, Lemma 3 and transitivity we get $(A\cap G)\cup(A\setminus G)_r\sim(B\cap G)\cup(B\setminus G)_s$. Since $r-u>s-v$, then $u-r<v-s$. Then we can apply IND $(u-r)$-times obtaining $(A\cap G)\cup(A\setminus G)_{r+u-r}\sim(B\cap G)\cup(B\setminus G)_{s+u-r}$. That is, $A\sim(B\cap G)\cup(B\setminus G)_{s+u-r}$. By Lemma 3,

$$(B\cap G)\cup(B\setminus G)_{s+u-r}\succ B.$$

Then, by transitivity, $A\succ B$.

(2) In this case $r-u=s-v$. If $r\geq u$ $(s\geq v)$, then, as in case (1), by reflexivity, Lemma 3 and transitivity we get

$$(A\cap G)_u\cup(A\setminus G)\sim(B\cap G)_v\cup(B\setminus G),$$

and by IND applied $(r-u)(=s-v)$-times, $A\sim B$.

If $r<u$ $(s<v)$, then, by reflexivity, Lemma 3 and transitivity we get

$$(A\cap G)\cup(A\setminus G)_r\sim(B\cap G)\cup(B\setminus G)_s,$$

and by IND applied $(u-r)(=v-s)$-times, $A\sim B$. ■

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