Finite Mixture Models

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Introduction

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- It concerns modeling a statistical distribution by a mixture (or weighted sum) of other distributions

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- It concerns modeling a statistical distribution by a mixture (or weighted sum) of other distributions
- Finite mixture models are also known as
  - latent class models
  - unsupervised learning models
- Finite mixture models are closely related to
  - intrinsic classification models
  - clustering
  - numerical taxonomy

- Heterogeneity of effects for different “classes” of observations
  - wine from different vineyards
  - healthy and sick individuals
  - normal and complicated pregnancies

- Estimating parameters of the distribution of lengths of halibut

- It is known that female halibut is longer, on average, than male fish and that the distribution of lengths is normal
- Gender cannot be determined at measurement
- Then distribution is a 2-component finite mixture of normals

- Estimating parameters of the distribution of lengths of halibut

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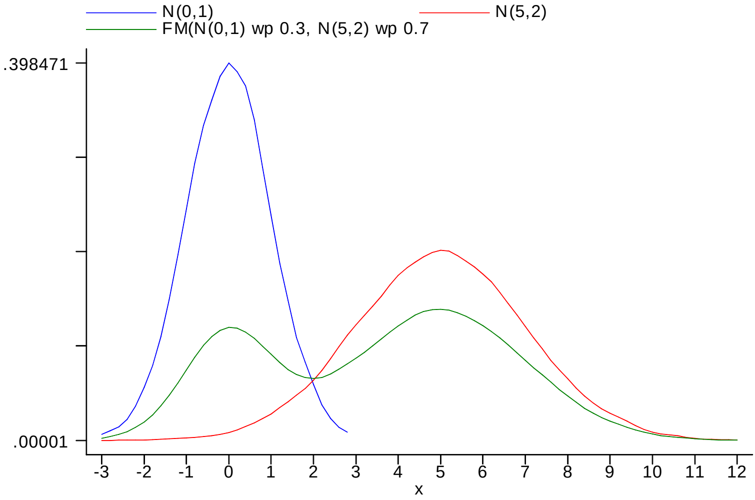
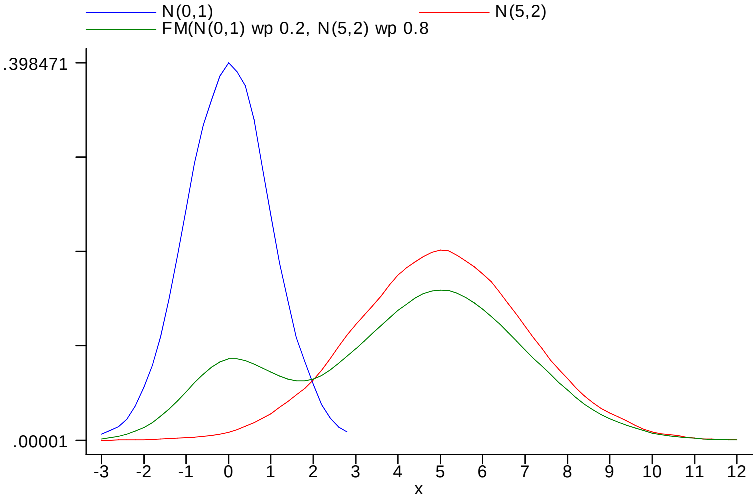
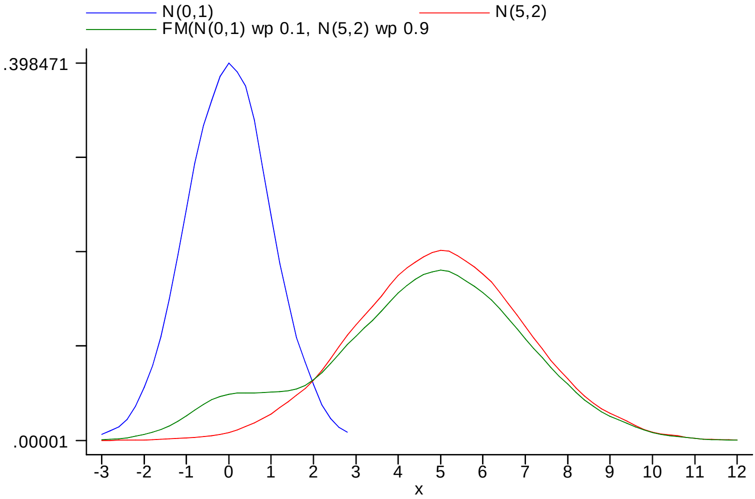
- A finite mixture model allows one to estimate:
  - mean lengths of male and female halibut
  - mixing probability


N(0,1)
N(5,2)
FM(N(0,1) wp 0.3, N(5,2) wp 0.7
.398471
.00001
-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12
x


N(0,1)
N(5,2)
FM(N(0,1) wp 0.2, N(5,2) wp 0.8
.398471
.00001
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
11
12
x

Introduction

A graphical view



- Characteristics of wine by cultivar
- Infant Birthweight - two types of pregnancies “normal” and “complicated”
- Medical Services - two types of consumers “healthy” and “sick”
- Public goods experiments - selfish, reciprocal, and altruist
- Stock Returns in “typical” and “crisis” times
- Using somatic cell counts to classify records from healthy or infected goats
- Models of internet traffic

- FMM is a semiparametric / nonparametric estimator of the density (Lindsay)
- Experience suggests that usually only few latent classes are needed to approximate density well (Heckman)
- In practice FMM are flexible extensions to basic parametric models
    - can generate skewed distributions from symmetric components
    - can generate leptokurtic distributions from mesokurtic ones

Outline of talk

- Introduction
- Model
    - Formulation
    - Estimation
    - Popular densities
    - Properties
- Examples
    - Color of wine
    - Birthweight and prenatal care
    - Medical care

- The density function for a C-component finite mixture is

$$f(y|\mathbf{x};\theta_1,\theta_2,...,\theta_C;\pi_1,\pi_2,...,\pi_C) = \sum_{j=1}^{C} \pi_j f_j(y|\mathbf{x};\theta_j)$$

where $0 < \pi_j < 1$, and $\sum_{j=1}^{C} \pi_j = 1$

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- More generally

$$f(y|\mathbf{x};\mathbf{z};\theta_1,\theta_2,...,\theta_C;\pi_1,\pi_2,...,\pi_C) = \sum_{j=1}^{C} \pi_j(\mathbf{z}) f_j(y|\mathbf{x};\theta_j)$$

- Maximum likelihood

$$\max_{\pi,\theta} \ln L = \sum_{i=1}^{N} \left(\log\left(\sum_{j=1}^{C} \pi_j f_j(y|\theta_j)\right) \right)$$

Model
Estimation

- Maximum likelihood

$$\max_{\pi,\theta} \ln L = \sum_{i=1}^{N} \left(\log(\sum_{j=1}^{C} \pi_j f_j(y|\theta_j) \right)$$

- Trick to ensure $0 < \pi_j < 1$, and $\sum_{j=1}^{C} \pi_j = 1$

$$\pi_j = \frac{\exp(\gamma_j)}{\exp(\gamma_1) + \exp(\gamma_2) + ... + \exp(\gamma_{C-1}) + 1}$$

- Maximum likelihood

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- EM
- Bayesian MCMC

Popular mixture component densities

- Normal (Gaussian)
- Poisson
- Gamma
- Negative Binomial
- Student-t
- Weibull

- Conditional mean:

$$E(y_i|\mathbf{x}_i) = \sum_{j=1}^{C} \pi_j \lambda_j \text{ where } \lambda_j = E_j(y_i|\mathbf{x}_i)$$

- Conditional mean:

$$\mathsf{E}(y_i|\mathbf{x}_i) = \sum_{j=1}^{C} \pi_j \lambda_j \text{ where } \lambda_j = \mathsf{E}_j(y_i|\mathbf{x}_i)$$

- Marginal effects:

$$\frac{\partial \mathsf{E}_j(y_i|\mathbf{x}_i)}{\partial \mathbf{x}_i} = \frac{\partial \lambda_j}{\partial \mathbf{x}_i} \longrightarrow \text{within component}$$

$$\frac{\partial \mathsf{E}(y_i|\mathbf{x}_i)}{\partial \mathbf{x}_i} = \sum_{j=1}^{C} \pi_j \frac{\partial \lambda_j}{\partial \mathbf{x}_i} \longrightarrow \text{overall}$$

Model

Some basic properties

- Prior probability that observation y_i belongs to component c:

$$\Pr[y_i \in \text{population } c | \mathbf{x}_i, \boldsymbol{\theta}] = \pi_c$$

$c = 1, 2, ..C$

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$c = 1, 2, ..C$

- Posterior probability that observation y_i belongs to component c:

$$\Pr[y_i \in \text{population } c|\mathbf{x}_i, y_{i;}\boldsymbol{\theta}] = \frac{\pi_c f_c(y_i|\mathbf{x}_i, \boldsymbol{\theta}_c)}{\sum_{j=1}^{C} \pi_j f_j(y_i|\mathbf{x}_i, \boldsymbol{\theta}_j)}$$

$c = 1, 2, ..C$

- The number of components has to be specified - we usually have little theoretical guidance
- Even if prior theory suggests a particular number of components we may not be able to reliably distinguish between some of the components
- In some cases additional components may simply reflect the presence of outliers in the data
- Likelihood function may have multiple local maxima

- Parameterize $\gamma_j = Z\alpha_j$ in

$$\pi_j = \frac{\exp(\gamma_j)}{\exp(\gamma_1) + \exp(\gamma_2) + \ldots + \exp(\gamma_{C-1}) + 1}$$

- Parameterizing mixing probabilities
  - may lead to finite sample identification issues
  - may lead to computational difficulties

- Estimate models with 2 and then more components
- At each step calculate

$$AIC = -2\log(L) + 2K$$
$$BIC = -2\log(L) + K\log(N)$$

- Pick the model with the smallest AIC, BIC

- Stata package `fmm`

```
fmm depvar [indepvars] [if] [in] [weight],
components(#) mixtureof(density)
```

- where density is one of

  `gamma`
  `negbin1`
  `negbin2`
  `normal`
  `poisson`
  `studentt`

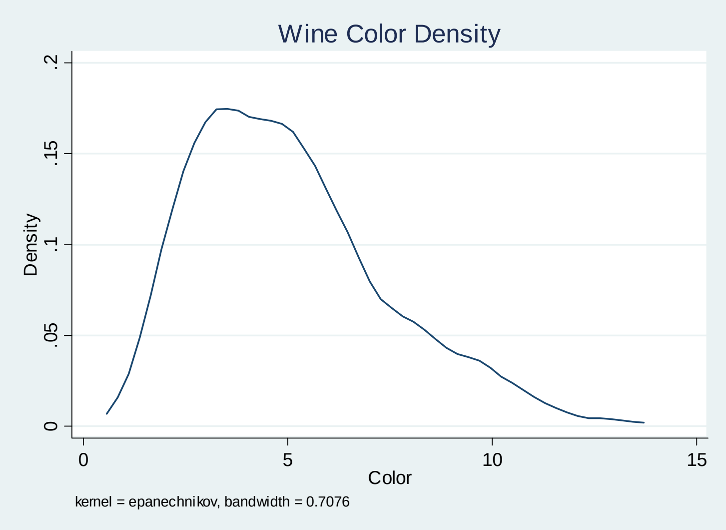
- `predict` and `mfx` give predictions and marginal effects of means, component means, prior and posterior probabilities

Example 1
Color of Wine

Results of a chemical analysis of wines grown in the same region in Italy but derived from three different cultivars

Cultivar	Freq.	% of total	Color intensity (mean)
1	59	33.15	5.528
2	71	39.89	3.086
3	48	26.97	7.396
Total	178	100	5.058

Example 1
Color of Wine



- Finite mixture of Normals with 3 components

$$
\begin{aligned}
& f(y_i | \theta_1, \theta_2, ..., \theta_C; \pi_1, \pi_2, ..., \pi_C) \\
= \quad & \sum_{j=1}^{C} \pi_j \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{1}{2\sigma_j^2} (y_i - x_i \beta_j)^2 \right)
\end{aligned}
$$

Example 1

Color of Wine

Parameter	component 1	component 2	component 3
Constant	4.929	7.548	2.803
	(0.334)	(0.936)	(0.244)
π	0.365	0.312	0.323
	(0.176)	(0.117)	(0.107)

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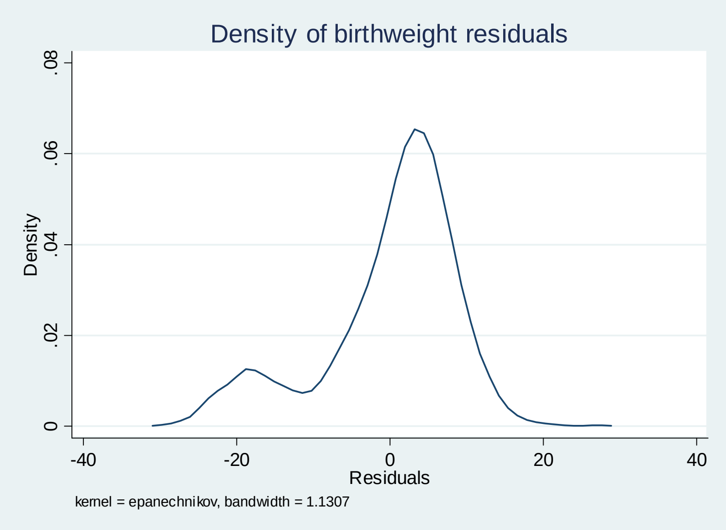
	Posterior probability (median)		
Cultivar	component 1	component 2	component 3
1	0.737	0.195	9.00e-5
2	0.048	0.023	0.923
3	0.030	0.970	7.54e-14

Example 2

Infant Birthweight and Prenatal Care

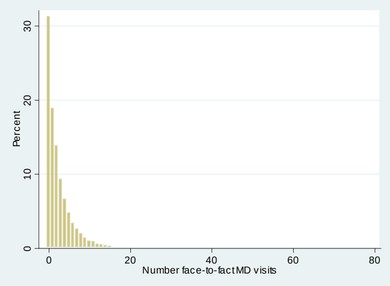
- Data from the National Maternal and Infant Health Survey
- Number of observations: 5219
- Number of covariates: 12

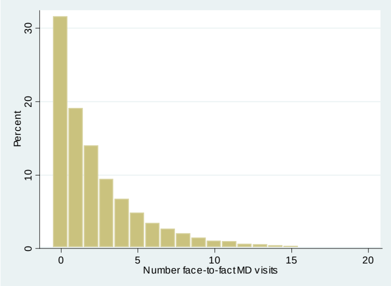

Density of birthweight residuals
Density
0
.02
.04
.06
.08
-40
-20
0
20
40
Residuals
kernel = epanechnikov, bandwidth = 1.1307

Variable	OLS	FMM	
		component 1	component 2
black	-1.213**	-1.231**	-0.775*
	(0.312)	(0.215)	(0.393)
edu	0.353**	0.292**	0.040
	(0.074)	(0.050)	(0.102)
numdead	-1.181**	-0.170	-0.585**
	(0.163)	(0.117)	(0.171)
onsethat	-0.501**	-0.294*	0.006
	(0.183)	(0.127)	(0.234)
π		0.864	0.136
se(π)		(0.005)	(0.005)

- Data from the RAND Health Insurance Experiment
- Number of observations: 20186
- Number of covariates: 17

Example 3
Medical Care Use





- The density of the C-component finite mixture is specified as

$$
\begin{aligned}
& f(y_i|\theta_1,\theta_2,...,\theta_C;\pi_1,\pi_2,...,\pi_C) \\
= & \sum_{j=1}^{C}\pi_j\frac{\Gamma(y_i+\psi_{j,i})}{\Gamma(\psi_{j,i})\Gamma(y_i+1)}\left(\frac{\psi_{j,i}}{\lambda_{j,i}+\psi_{j,i}}\right)^{\psi_{c,i}}\left(\frac{\lambda_{j,i}}{\lambda_{j,i}+\psi_{j,i}}\right)^{y_i}
\end{aligned}
$$

where $\lambda=\exp(x\beta)$ and $\psi=(1/\alpha)\lambda^k$

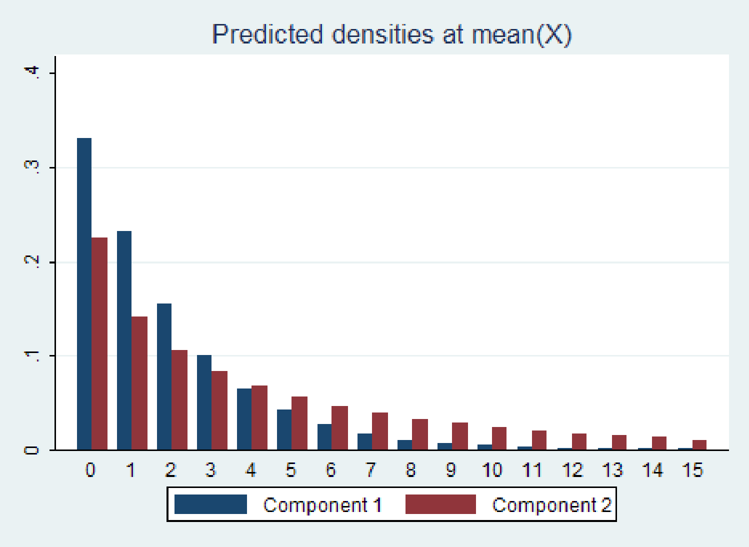
- $k=1$ NB-2
- $k=0$ NB-1
- $k=0$ fits best

Parameter Estimates

	nb1	fmm nb1	
		component 1	component 2
logc	-0.149*	-0.203*	-0.024
	(0.012)	(0.020)	(0.031)
educdec	0.023*	0.027*	0.015
	(0.003)	(0.005)	(0.010)
disea	0.021*	0.019*	0.033*
	(0.001)	(0.002)	(0.004)
π		0.802	0.198
		(0.037)	(0.037)
log L	–42405	–42037	
BIC	84999	84461	

Marginal Effects

	nb1	fmm nb1		
	overall	overall	component 1	component 2
$E(y\|\bar{\mathbf{x}})$	2.561	2.511	1.887	5.038
logc	-0.382*	-0.331*	-0.382*	-0.121
	(0.030)	(0.032)	(0.032)	(0.158)
educdec	0.058*	0.056*	0.052*	0.073
	(0.007)	(0.009)	(0.008)	(0.053)
disea	0.054*	0.062*	0.036*	0.167*
	(0.003)	(0.004)	(0.004)	(0.024)



Predicted densities at mean(X)
0
.1
.2
.3
.4
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
Component 1
Component 2

Example 3
Medical Care Use