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Computing Tournament Solutions using Relation Algebra and RelView*

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Abstract. We describe a simple computing technique for the tournament choice problem. It rests upon a relational modeling and uses the BDD-based computer system RelView for the evaluation of the relation-algebraic expressions that specify the solutions and for the visualization of the computed results. The Copeland set can immediately be identified using RelView's labeling feature. Relation-algebraic specifications of the Condorcet non-losers, the Schwartz set, the top cycle, the uncovered set, the minimal covering set, the Banks set, and the tournament equilibrium set are delivered. We present an example of a tournament on a small set of alternatives, for which the above choice sets are computed and visualized via RelView. The technique described in this paper is very flexible and especially appropriate for prototyping and experimentation, and as such very instructive for educational purposes. It can easily be applied to other problems of social choice and game theory.

Keywords: tournament, relational algebra, RelView, Copeland set, Condorcet non-losers, Schwartz set, top cycle, uncovered set, minimal covering set, Banks set, tournament equilibrium set

1 Introduction

Science is a process for obtaining new insights and building new knowledge in the form of testable explanations and predictions about the universe. Systematic experiments are an accepted means for doing science and they become increasingly important as one proceeds in investigations. In natural sciences they are

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used since centuries. Also in social sciences, which apply scientific methods to study human behavior and social patterns, experimental and empirical methods are of importance. But in the meantime they have also become important in formal sciences, like mathematics and theoretical computer science, for identifying properties and patterns, and for testing and especially falsifying conjectures. In this context, tool support is indispensable. Computer programs are used in numerous scientific fields to calculate results as well as to elucidate the underlying mathematical principles by means of visualization and animation. Frequently use is made of general computer algebra systems, like Maple and Mathematica. But also systems that focus on specific domains of applications are applied.

RelView (cf. [3, 18]) is such a so-called *specific purpose computer algebra system* for (heterogeneous) relation algebra in the sense of [22, 23] More precisely, RelView is a tool for the visualization and manipulation of relations, for prototyping and relational programming, and as such it appears to be very useful and appropriate for applications to social choice and game theory. In this system, computational tasks on relations can be described by short and concise programs which frequently consist of only a few lines that present the relation-algebraic expressions of the notions in question. Such programs are easy to alter in case of slightly changed specifications. Combining this with RelView's possibilities for visualization and stepwise execution of programs is suitable for experimentation and exploration, while avoiding unnecessary expenditure of work. Another advantage of the system is its implementation of relations via binary decision diagrams (BDDs) that proved to be superior to many other well-known implementations, like Boolean matrices, lists of pairs and lists of successor or predecessor lists. This leads to an amazing computational power, in particular if the solution of a hard problem requires the enumeration of a huge set of 'interesting objects' and a search through it. Applications in this regard can be found, e.g., in [2, 3, 17, 18].

In [4–6] we have combined relation algebra and RelView to solve some problems of social choice theory, viz. the formation of stable governments (see [21]) and the determination of the strength and influence of players (see, e.g., [16]). The first problem is a specific instance of one of the most interesting problems of social choice theory, viz. the computation of the set of most desirable alternatives according to an *asymmetric dominance relation* on given alternatives. Since the dominance relation may contain cycles, the concept of a *best alternative* that dominates all other alternatives is not applicable in most cases. Even undominated alternatives need not to exist. To get around these problems, in the literature so-called *choice sets* are considered that take over the role of the best/undominated alternative(s).

In this paper we show how certain choice sets can be computed using relation algebra and the RelView tool. We restrict our analysis to complete dominance relations, where each pair of different elements is related. Such relations are known as *tournament relations* and the elements of the choice sets are called *tournament winners*. But most of our results also hold in case of non-complete dominance relations or can easily be extended to them. In this paper, we de-

liver relation-algebraic specifications of the following choice sets: Condorcet non-losers, the Schwartz set, the top cycle, the uncovered set, the minimal covering set, the Banks set, and the tournament equilibrium set. Moreover, the above choice sets are visualized via RelView for a tournament on a small set of alternatives to give an impression of the features of RelView in this regard. Computing the choice sets just mentioned by hand in even this small example would already be a major task with a high risk of making mistakes. RelView guarantees us correct solutions, because it directly uses the mathematical relation-algebraic equations, which have been proved to be correct by formal calculations.

The remainder of the paper is organized as follows. The relation-algebraic preliminaries are introduced in Section 2. In Section 3 we first describe some well-known concepts for tournament winners. To give an impression of RelView's visualization potential with respect to the computation of choice sets, we then show a series of pictures produced by the tool. The corresponding relation-algebraic expressions are presented in Section 4. We demonstrate how to calculate them from formal logical problem specifications and how to translate them into the programming language of the tool. Section 5 sketches some generalizations and contains some concluding remarks.

2 Relation-algebraic Preliminaries

In this section we provide the relation-algebraic preliminaries as used throughout this paper. In particular, we focus on the modeling of sets and Cartesian products which are not commonly used and hence require some detailed explanation. More details can be found, for example, in [22, 23].

2.1 Fundamentals of Binary Relations

We write $R : X \leftrightarrow Y$ if R is a (binary) relation with source X and target Y, i.e., a subset of the Cartesian product $X \times Y$, and $[X \leftrightarrow Y]$ for the *type* of all these relations, i.e., the powerset $2^{X \times Y}$. We may consider R also as a Boolean matrix if X and Y are finite. This interpretation is well suited for many purposes and Boolean matrices are also used as one of the graphical representations of relations within RelView. Therefore, in this paper we often use Boolean matrix terminology and notation. In particular, we speak of rows, columns and entries of relations and write $R_{x,y}$ instead of $\langle x, y \rangle \in R$ or $x\, R\, y$.

We will use the following basic operations on relations (cf. [22, 23]): $\overline{R}$ (*complement*), $R \cup S$ (*union*), $R \cap S$ (*intersection*), R^{T} (*transposition*) and $R; S$ (*composition*). Furthermore, we will use the special relations O (*empty relation*), L (*universal relation*), and I (*identity relation*). Here we overload the symbols, i.e., we avoid the binding of types to them. Finally, if $R : X \leftrightarrow Y$ is included in $S : X \leftrightarrow Y$ we write $R \subseteq S$ and equality of R and S is denoted as $R = S$.

In order to reduce the use of brackets, it is generally agreed that composition binds stronger than union and intersection. So, for instance, $R \cup S; T$ stands for

$R \cup (S;T)$ and not for $(R \cup S);T$. Similarly, $R;S \cap T$ should be read as $(R;S) \cap T$ and not as $R;(S \cap T)$.

A relation $R : X \leftrightarrow X$ is *asymmetric* if $R \subseteq \overline{R^{\mathsf{T}}}$, *irreflexive* if $R \subseteq \bar{\mathsf{I}}$, *transitive* if $R;R \subseteq R$ and *complete* if $\bar{\mathsf{I}} \subseteq R \cup R^{\mathsf{T}}$. These are the relation-algebraic (or point-free) specifications of well-known properties which usually are defined point-wisely. For instance, $\bar{\mathsf{I}} \subseteq R \cup R^{\mathsf{T}}$ specifies that for all $x, y \in X$, from $x \neq y$ it follows $R_{x,y}$ or $R_{y,x}$, i.e., different elements are related via R. Asymmetry of R implies its irreflexivity, and in this case completeness is equivalent to $\bar{\mathsf{I}} = R \cup R^{\mathsf{T}}$.

Finally, we need the *transitive closure* $R^+ : X \leftrightarrow X$ of a relation $R : X \leftrightarrow X$. This is the least (with respect to inclusion) transitive relation of type $[X \leftrightarrow X]$ that contains R. Via the powers of R, inductively defined by $R^0 := \mathsf{I}$ and $R^{i+1} := R;R^i$ for all $i \in \mathbb{N}$, we can specify R^+ by $R^+ = \bigcup_{i>0} R^i$.

2.2 Modeling Sets

Relation algebra offers different ways of modeling sets and subsets. Our first modeling uses *vectors*, which are relations v with $v = v;\mathsf{L}$. Since for a vector the range is irrelevant, we consider in the following mostly vectors $v : X \leftrightarrow \mathbf{1}$ with a specific singleton set $\mathbf{1} = \{\perp\}$ as target and omit in such cases the subscript $\perp$, i.e., we write v_x instead of $v_{x,\perp}$. Such a vector can be considered as a Boolean matrix with exactly one column, i.e., as a Boolean column vector, and *represents* the subset $\{x \in X \mid v_x\}$ of X. A non-empty vector v is a *point* if $v;v^{\mathsf{T}} \subseteq \mathsf{I}$, i.e., if it is *injective*. This means that it represents a singleton subset of its source or an element from it if we identify a singleton set $\{x\}$ with the element x. In the Boolean matrix model, hence, a point $v : X \leftrightarrow \mathbf{1}$ is a Boolean column vector in which exactly one entry is 1.

As a second way to model sets, we will apply the relation-level equivalents of the set-theoretic symbol $\in$, that is, *membership-relations* $\mathsf{M} : X \leftrightarrow 2^X$. These specific relations are defined by demanding for all $x \in X$ and $Y \in 2^X$ that $\mathsf{M}_{x,Y}$ iff $x \in Y$. A simple Boolean matrix implementation of membership-relations requires an exponential number of bits. However, in [17] an implementation of $\mathsf{M} : X \leftrightarrow 2^X$ using BDDs is presented, where the number of BDD-vertices is linear in the size of the base set X. This implementation is part of RelView.

Finally, we will use embeddings for modeling sets. Given an injective function $\imath : Y \to X$ (in the usual mathematical sense), we may consider Y as a subset of X by identifying it with its image under $\imath$. If Y is actually a subset of X and $\imath$ is given as a relation of type $[Y \leftrightarrow X]$ such that $\imath_{y,x}$ iff $y = x$ for all $y \in Y$ and $x \in X$, then the vector $\imath^{\mathsf{T}};\mathsf{L} : X \leftrightarrow \mathbf{1}$ represents Y as a subset of X in the sense above. Clearly, the transition in the other direction is also possible, i.e., the generation of an *embedding-relation* $\mathit{inj}(v) : Y \leftrightarrow X$ from the vector representation $v : X \leftrightarrow \mathbf{1}$ of the subset Y of X such that for all $y \in Y$ and $x \in X$ we have $\mathit{inj}(v)_{y,x}$ iff $y = x$. We only have to remove from the identity relation $\mathsf{I} : X \leftrightarrow X$ all x-rows, where the element x ranges over the set $X \setminus Y$.

A combination of embedding-relations with membership-relations allows a *column-wise representation* of sets. More specifically, if the vector $v : 2^X \leftrightarrow \mathbf{1}$

represents a subset $\mathfrak{S}$ of 2^X in the sense above and we define the relation $S : X \leftrightarrow \mathfrak{S}$ by $S := \mathsf{M}; \mathit{inj}(v)^{\mathsf{T}}$, then for all $x \in X$ and $Y \in \mathfrak{S}$ we have $S_{x,Y}$ iff $x \in Y$. This means that the elements of $\mathfrak{S}$ are represented precisely by the columns of S. A further consequence is that $\overline{S^{\mathsf{T}}; \overline{S}} : \mathfrak{S} \leftrightarrow \mathfrak{S}$ is the relation-algebraic specification of set inclusion on $\mathfrak{S}$, that is, for all $Y, Z \in \mathfrak{S}$ we have the relationship $(\overline{S^{\mathsf{T}}; \overline{S}})_{Y,Z}$ iff $Y \subseteq Z$. From this, we obtain that the vector $\overline{(\overline{S^{\mathsf{T}}; \overline{S}} \cap \bar{\mathsf{I}}); \mathsf{L}} : \mathfrak{S} \leftrightarrow \mathbf{1}$ represents the subset $\mathfrak{S}_{\max}$ of the maximal sets of $\mathfrak{S}$, so that the relation

$$\mathit{MaxEnum}(S) := S; \mathit{inj}(\overline{(\overline{S^{\mathsf{T}}; \overline{S}} \cap \bar{\mathsf{I}}); \mathsf{L}})^{\mathsf{T}} : X \leftrightarrow \mathfrak{S}_{\max} \tag{1}$$

is a column-wise representation of the set $\mathfrak{S}_{\max}$ in the above sense. By transposing the set inclusion relation we get

$$\mathit{MinEnum}(S) := S; \mathit{inj}(\overline{(\overline{\overline{S}^{\mathsf{T}}; S} \cap \bar{\mathsf{I}}); \mathsf{L}})^{\mathsf{T}} : X \leftrightarrow \mathfrak{S}_{\min} \tag{2}$$

as column-wise representation of the set $\mathfrak{S}_{\min}$ that consists of the minimal sets of $\mathfrak{S}$.

2.3 Modeling Cartesian Products

Given a Cartesian product $X \times Y$ of two sets X and Y, there are two projection functions which decompose a pair $u = \langle u_1, u_2 \rangle$ into its first component u_1 and its second component u_2. For a relation-algebraic approach it is useful to consider instead of these functions the corresponding *projection relations* $\pi : X \times Y \leftrightarrow X$ and $\rho : X \times Y \leftrightarrow Y$ such that for all pairs $u \in X \times Y$ and elements $x \in X$ and $y \in Y$ we have $\pi_{u,x}$ iff $u_1 = x$ and $\rho_{u,y}$ iff $u_2 = y$. In the remainder of the paper we always assume pairs u to be of the form $\langle u_1, u_2 \rangle$.

The projection relations enable us to describe the well-known pairing operation of functional programming relation-algebraically in two versions. We only need one of them. The *left pairing* $[\![R, S]\!] : X \times Y \leftrightarrow Z$ of the relations $R : X \leftrightarrow Z$ and $S : Y \leftrightarrow Z$ is given by $[\![R, S]\!] := (\pi; R) \cap (\rho; S)$, where $\pi : X \times Y \leftrightarrow X$ and $\rho : X \times Y \leftrightarrow Y$ are as above. Point-wisely this specification says that $[\![R, S]\!]_{u,z}$ iff $R_{u_1,z}$ and $S_{u_2,z}$, for all $z \in Z$ and pairs $u \in X \times Y$.

We end this section with a relation-algebraic construction that establishes a Boolean lattice isomorphism between the two types $[X \leftrightarrow Y]$ and $[X \times Y \leftrightarrow \mathbf{1}]$ via the projection relations $\pi : X \times Y \leftrightarrow X$ and $\rho : X \times Y \leftrightarrow Y$. It is given by the function $\mathit{vec} : [X \leftrightarrow Y] \to [X \times Y \leftrightarrow \mathbf{1}]$, where $\mathit{vec}(R) := ((\pi; R) \cap \rho); \mathsf{L}$, and defines the vector $\mathit{vec}(R) : X \times Y \leftrightarrow \mathbf{1}$ corresponding to the relation $R : X \leftrightarrow Y$. Using a point-wise notation, this definition says that for all pairs $u \in X \times Y$ we have R_{u_1,u_2} iff $\mathit{vec}(R)_u$.

3 Winners of a Tournament

In our setting a *tournament* is an asymmetric and complete relation $D : A \leftrightarrow A$, where A is a *finite* set of alternatives. If $D_{x,y}$ we say that x *dominates* (or beats)

y. In the social choice literature this is frequently denoted as $x \succ y$. Due to this notation, for a relation $R : A \leftrightarrow A$ and a set $X \in 2^A$, an alternative x is said to be *maximal* in X with respect to R if for all $y \in X$ from $R_{y,x}$ it follows that $R_{x,y}$. Notice, that this is not the straightforward generalization of the notion 'maximal' of order theory to arbitrary relations, although the two definitions coincide if R is in fact a 'greater-than' strict-order relation. However, also social choice theory uses the common definition of 'maximal' for sets (and the dual notion of 'minimal'). Given a set $\mathfrak{S}$ of sets, $X \in \mathfrak{S}$ is maximal in $\mathfrak{S}$ if $X \subseteq Y$ implies $X = Y$, for all $Y \in \mathfrak{S}$.

3.1 Choice Sets

If the tournament $D : A \leftrightarrow A$ is a transitive relation, it is a complete strict-order relation and this ensures, since the set A is finite, the existence of a so-called *Condorcet winner* that dominates all other elements[1]. When there is no Condorcet winner, deciding which alternatives are most desirable may be a difficult task. Different methods have been proposed to compute a set of such alternatives, i.e., a choice set. In what follows we list some prominent examples, well known from the literature.

- The *Copeland set* $CO(D)$ (introduced in [10]) consists of the alternatives x for which the cardinality of the set of elements that are dominated by x is maximal.
- A *Condorcet non-loser* is an alternative that dominates at least one other alternative. These elements form the largest non-trivial choice set, denoted as $CNL(D)$ in [9].
- The *Schwartz set* $SC(D)$ (introduced in [24]) is the union of all minimal subsets X of A such that $X \neq \emptyset$ and there is no $x \in A \setminus X$ and no $y \in X$ such that x dominates y.

To specify the next important choice set, we first generalize the notion of dominance from alternatives to sets of alternatives and define $X \in 2^A$ as *dominating set* if $X \neq \emptyset$ and all $x \in X$ dominate all $y \in A \setminus X$. It is known that the dominating sets form a chain in the order $(2^A, \subseteq)$.

- The *top cycle* $TC(D)$ (because of the results of [15, 27] also called *Good set* or *Smith set*) is the unique minimal dominating set.

Also for the specification of the next two choice sets we need new notions. Given $X \in 2^A$ and $x, y \in X$, we say that x *covers* y in X if $D_{x,y}$ and for all $z \in X$ from $D_{y,z}$ it follows $D_{x,z}$. Furthermore, we say that $X \in 2^A$ is *internally stable* if there is no pair $x, y \in X$ such that x covers y in X, and *externally stable* if for all $x \in A \setminus X$ there exists $y \in X$ such that y covers x in $X \cup \{x\}$.

[1] In terms of order theory, a transitive tournament $D : A \leftrightarrow A$ is a linear strict-order relation on A and a Condorcet winner is the greatest element of A, if $D_{x,y}$ means that x is greater than y.

- The *uncovered set* $UC(D)$ (proposed in [14, 19]) consists of the uncovered alternatives, i.e., all $x \in A$ such that no alternative from A covers x in A.
- A covering set is a set that is both internally stable and externally stable. In [13] it is shown that in tournaments there exists a unique *minimal covering set*, denoted as $MC(D)$.

To specify the next choice set, we define $X \in 2^A$ to be a *transitive set* if the restriction of D to X is transitive, i.e., $D_{x,y}$ and $D_{y,z}$ implies $D_{x,z}$, for all $x, y, z \in X$.

- The choice set $BA(D)$, proposed in [1] and nowadays called *Banks set*, consists of the maximal elements with respect to D of the maximal transitive subsets of A. The latter sets are called Banks trajectories.

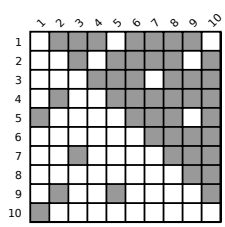
Also for the specification of our last choice set we need auxiliary notions. Given $X \in 2^A$ and $y \in A$, we denote by $D^{X,y}$ the set of all alternatives from X which dominate y and by $D_{|D^{X,y}} : A \leftrightarrow A$ the restriction of D to this set.

- For each $X \in 2^A$ the *tournament equilibrium set* $TEQ_X(D)$ of [25] is recursively defined by $TEQ_X(D) = MTC_X(T)$, where $MTC_X(T)$ consists of the maximal elements of X with respect to the *asymmetric part* $T^+ \cap \overline{(T^+)^\mathsf{T}}$ of the transitive closure T^+ and the underlying relation $T : A \leftrightarrow A$ is defined by $T_{x,y}$ iff $x \in TEQ_{D^{X,y}}(D_{|D^{X,y}})$ and $y \in X$, for all $x, y \in A$.

The subset $TEQ_A(D)$ of A is called the tournament equilibrium set of D and T is the corresponding TEQ-relation. Notice, that the recursive specification of T terminates due to the finiteness of A. The termination case is $X = \emptyset$ and here T equals the empty relation.

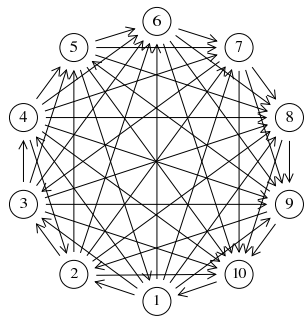
3.2 Visualization via RelView

To demonstrate the visualization potential of RelView, we consider a small example, viz. the tournament $D : A \leftrightarrow A$ on the set $A := \{1, \ldots, 10\}$ that in the so-called relation window of RelView is depicted as follows:

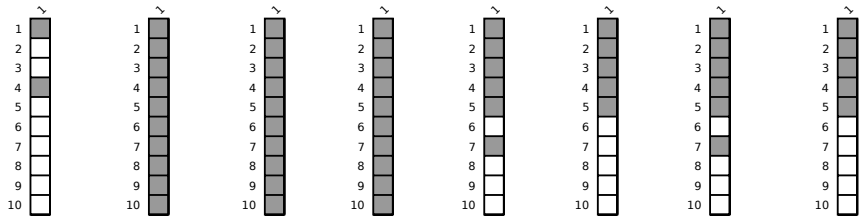


In this RelView matrix a black square means a 1-entry and a white square means a 0-entry. So, for instance, the first row shows that alternative 1 dominates all alternatives except 1, 5 and 10 and the last column shows that alternative 10 is dominated by all alternatives except 1 and 10. From the arrangement of the black squares in the matrix it immediately becomes clear that D is complete, asymmetric and irreflexive. The next picture shows the representation of

the tournament D as directed graph, again produced by RelView using an implemented algorithm that draws the vertices of the graph on a circle.

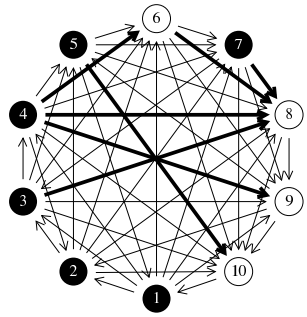


The next eight pictures show eight RelView vectors which represent eight subsets of the set A. These subsets are, from left to right, the Copeland set, the Condorcet non-losers, the Schwartz set, the top cycle, the uncovered set, the minimal covering set, the Banks set and the tournament equilibrium set. For instance, from the first four vectors we obtain that $CO(D) = \{1, 4\}$ and $CNL(D) = SC(D) = TC(D) = A$.

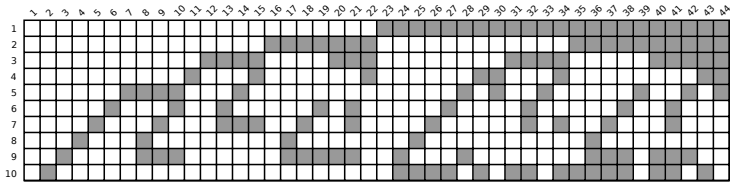


$CO(D)$ $CNL(D)$ $SC(D)$ $TC(D)$ $UC(D)$ $MC(D)$ $BA(D)$ $TEQ_A(D)$

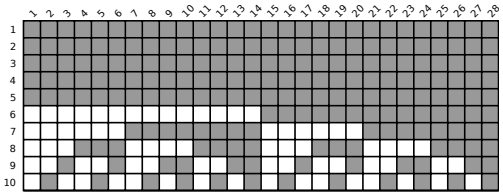
The subsequent picture shows again the tournament D as a directed graph. To visualize the uncovered set, in the graph an edge from x to y is drawn boldface iff x covers y in A. Then the initial vertices of this subgraph, drawn in black, precisely correspond to the elements of the uncovered set $UC(D)$.



With regard to covering sets, RelView computed the following column-wise representation of the 44 internally stable subsets of A.

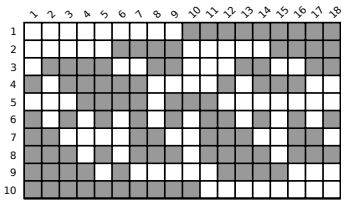


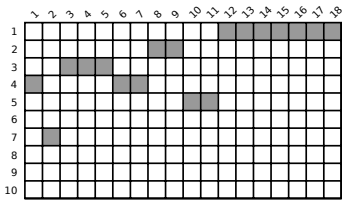
As column-wise representation of the externally stable subsets of A we obtained the following 10×28 RelView matrix.



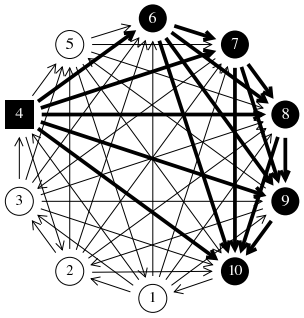
The only column that both matrices have in common (as last and as first column, respectively) is the above vector representing the minimal (and only) covering set $MC(D)$ of the tournament D.

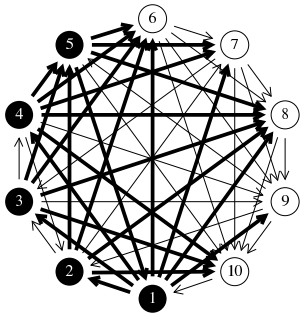
Concerning the Banks set $BA(D)$, the RelView tool computed 338 transitive subsets of A, where exactly 18 of them are maximal. The left one of the following two RelView matrices shows the column-wise representation of the 18 Banks trajectories, and the 18 columns of the RelView matrix on the right are the representations of the corresponding maximal elements with respect to D as points. The union of the 18 points yields the Banks set's vector representation.





We close the series of pictures with the following two directed graphs, again produced by the RelView tool.





Each of these directed graphs represents once more the tournament D, with certain vertices and edges again being marked as in the above graph that visualizes the cover relationships and the uncovered set. In the directed graph on the left the vertices of the first Banks trajectory of the above column-wise representation are drawn black, with the maximal element of the trajectory with respect to D highlighted furthermore as square. In order that the transitivity of the trajectory becomes easily identifiable, the restriction of D to it is additionally emphasized by boldface edges. In the graph on the right the black edges designate the tournament equilibrium set $TEQ_A(D)$ and the boldface edges describe the TEQ-relation T.

4 Relation-algebraic Specifications of Choice Sets

If RelView depicts a relation as a Boolean matrix, then it is able to mark its rows and columns for explanatory purposes. So far, we have only shown consecutive row and/or column numbers. But also the numbers of 1-entries can be attached as labels. This immediately allows to identify the Copeland set with the help of the system. It is also easy to compute the vector representation of the Copeland set by a small RelView program that uses the pre-defined operations for the comparison of relations with regard to the number of their 1-entries.

In this section we show how the remaining choice sets of Section 3 can be specified by relation-algebraic expressions. To illustrate how easy then the translation into RelView code is, we demonstrate this for the specification of the Schwartz set. For the remainder of this section, let $D : A \leftrightarrow A$ be a fixed tournament.

4.1 Condorcet Non-losers

Given a relation $R : X \leftrightarrow Y$, the vector $R; \mathsf{L} : X \leftrightarrow \mathbf{1}$ is called the domain of R, since it represents the set of those elements $x \in X$ that are in relationship $R_{x,y}$ to at least one element $y \in X$. Since D is irreflexive, $D_{x,y}$ implies $x \neq y$. In view of the definition of a Condorcet non-loser this immediately leads to

$$cnl(D) := D; \mathsf{L} : A \leftrightarrow \mathbf{1} \tag{3}$$

as relation-algebraic specification of the set $CNL(D)$, where L is the universal vector of type $[A \leftrightarrow \mathbf{1}]$.

4.2 The Schwartz Set

In [11] it is shown that an alternative $x \in A$ is in the Schwartz set $SC(D)$ iff it is a maximal element of A with respect to the asymmetric part $D^+ \cap \overline{(D^+)^\mathsf{T}}$ of D^+. Having a relation-algebraic specification $max(R, v) : A \leftrightarrow \mathbf{1}$ at hand, that yields for $R : A \leftrightarrow A$ and $v : A \leftrightarrow \mathbf{1}$ the vector representation of the maximal

elements with respect to R of the set v represents, from this characterization we immediately get

$$schwartzSet(D) := max(D^+ \cap \overline{(D^+)^\mathsf{T}}, \mathsf{L}) : A \leftrightarrow \mathbf{1} \tag{4}$$

as vector representation of the Schwartz set $SC(D)$ of D. In (4) the universal vector $\mathsf{L} : A \leftrightarrow \mathbf{1}$ represents the entire set A. It remains to realize $max(R, v)$ for given $R : A \leftrightarrow A$ and $v : A \leftrightarrow \mathbf{1}$. To solve this task, we assume that v represents the set $X \in 2^A$. We start with a formal logical definition of 'maximal' and then use well-known correspondences between relation-algebraic and logical constructions.

$$\begin{aligned} x \text{ maximal in } X \text{ w.r.t. } R &\iff x \in X \wedge \forall\, y : y \in X \wedge R_{y,x} \rightarrow R_{x,y} \\ &\iff v_x \wedge \forall\, y : v_y \wedge R_{y,x} \rightarrow R_{x,y} \\ &\iff v_x \wedge \neg\exists\, y : R^\mathsf{T}{}_{x,y} \wedge \overline{R}_{x,y} \wedge v_y \\ &\iff (v \cap \overline{(R^\mathsf{T} \cap \overline{R}); v})_x \end{aligned}$$

This yields the relation-algebraic specification

$$max(R, v) := v \cap \overline{(R^\mathsf{T} \cap \overline{R}); v} : A \leftrightarrow \mathbf{1} \tag{5}$$

and we are done. If we translate the above two relation-algebraic specifications into the programming language of RelView, we obtain the following code, where the meaning of the RelView operations '`&`', '`-`', '`^`', '`*`' and '`Ln1`' directly follows from a comparison of the specifications and the code.

```
max(R,v) = v & -((R^ & -R)*v).

schwartzSet(D)
  DECL R
  BEG  R = trans(D)
       RETURN max(R & -R^,Ln1(D))
  END.
```

Here `max` is a relational function and `schwartzSet` is a relational program. The difference is that in RelView programs local declarations (of variables, functions, products) and statements (like assignments and loops) may be used, whereas RelView functions are defined as common in mathematics. In the program `schwartzSet` the only purpose of the local variable `R` is to avoid a two-fold computation of the transitive closure.

Using the technique we will apply later on to compute the choice sets $MC(D)$ and $BA(D)$, it is also possible to get a column-wise representation of those sets that are used in Section 3.1 to define the Schwartz set as their union. Of course, this leads to an algorithm that is much less efficient than the computation by means of (4). Its advantage is that in combination with RelView it may be used to visualize the original idea behind the Schwartz set for educational purposes. This remark also holds for the choice set we consider next.

4.3 The Top Cycle

In [8] the authors prove that the top cycle $TC(D)$ equals the set of maximal elements of A with respect to the transitive closure of the so-called *weak dominance relation* of D. The latter relation is defined as $\overline{D^{\mathsf{T}}}$. If we combine the description of [8] with the function *max* specified in (5), we get at once

$$topCycle(D) := max(\overline{D^{\mathsf{T}}}, \mathsf{L}) : A \leftrightarrow \mathbf{1} \tag{6}$$

as vector representation of the top cycle $TC(D)$, where in (6) again $\mathsf{L} : A \leftrightarrow \mathbf{1}$ is the universal vector that represents the entire set A.

4.4 The Uncovered Set

To obtain a relation-algebraic specification of the uncovered set $UC(D)$, we first develop a relation $C : A \leftrightarrow A$ that describes the cover-relationships on the entire set A. Starting with the logical specification of the notion in question, for all $x, y \in A$ we get for the *covering relation* C that

$$\begin{aligned} C_{x,y} &\iff D_{x,y} \wedge \forall\, z : D_{y,z} \rightarrow D_{x,z} \\ &\iff D_{x,y} \wedge \neg\exists z : D_{y,z} \wedge \overline{D}_{x,z} \\ &\iff (D \cap \overline{\overline{D}\,; D^{\mathsf{T}}})_{x,y} \end{aligned}$$

where the quantified variable z ranges over A. This shows $C = D \cap \overline{\overline{D}\,; D^{\mathsf{T}}}$. A similar calculation proves that, for $x \in A$ given, there is no $y \in A$ with $C_{y,x}$ iff $(\overline{C^{\mathsf{T}}; \mathsf{L}})_x$, where $\mathsf{L} : A \leftrightarrow \mathbf{1}$. Hence, $\overline{C^{\mathsf{T}}; \mathsf{L}} : A \leftrightarrow \mathbf{1}$ represents the set $UC(D)$ and unfolding C in this expression followed by simple transformations concerning transposition leads to

$$uncovSet(D) := \overline{D^{\mathsf{T}} \cap \overline{D; \overline{\overline{D^{\mathsf{T}}}}}\,; \mathsf{L}} : A \leftrightarrow \mathbf{1} \tag{7}$$

as relation-algebraic specification of a vector that represents the uncovered set $UC(D)$ of D.

4.5 The Minimal Covering Set

We start the development of a relation-algebraic specification of the set $MC(D)$ with the calculation of a vector that represents the set of internally stable sets. A formalization of $X \in 2^A$ to be internally stable in first-order logic requires more than three variables. Since pure relation algebra in the sense of Section 2.1 can express exactly those first-order logic formulae which contain at most two free variables and in which at most three variables occur (see [28]), projection relations or equivalent notions (like pairing operations) have to be used. To reach

our goal, we calculate as given below, where $\mathsf{M} : A \leftrightarrow 2^A$ is the membership-relation, x, y and z range over A and u ranges over $A \times A$.

$$
\begin{aligned}
& X \text{ internally stable} \\
\iff\ & \neg\exists\, x, y : x \in X \wedge y \in X \wedge D_{x,y} \wedge \forall\, z : z \in X \wedge D_{y,z} \rightarrow D_{x,z} \\
\iff\ & \neg\exists\, u : u_1 \in X \wedge u_2 \in X \wedge D_{u_1,u_2} \wedge \forall\, z : z \in X \wedge D_{u_2,z} \rightarrow D_{u_1,z} \\
\iff\ & \neg\exists\, u : \mathsf{M}_{u_1,X} \wedge \mathsf{M}_{u_2,X} \wedge vec(D)_u \wedge \neg\exists\, z : \mathsf{M}_{z,X} \wedge D_{u_2,z} \wedge \overline{D}_{u_1,z} \\
\iff\ & \neg\exists\, u : vec(D)_u \wedge [\![\mathsf{M},\mathsf{M}]\!]_{u,X} \wedge \neg\exists\, z : [\![\overline{D}, D]\!]_{u,z} \wedge \mathsf{M}_{z,X} \\
\iff\ & \neg\exists\, u : vec(D)_u \wedge [\![\mathsf{M},\mathsf{M}]\!]_{u,X} \wedge (\,\overline{[\![\overline{D}, D]\!];\mathsf{M}}\,)_{u,X} \\
\iff\ & \neg\exists\, u : vec(D)_u \wedge ([\![\mathsf{M},\mathsf{M}]\!] \cap \overline{[\![\overline{D}, D]\!];\mathsf{M}}\,)_{u,X} \\
\iff\ & (\,\overline{vec(D)^{\mathsf{T}}; ([\![\mathsf{M},\mathsf{M}]\!] \cap \overline{[\![\overline{D}, D]\!];\mathsf{M}}\,)}^{\mathsf{T}}\,)_X
\end{aligned}
$$

From this equivalence we obtain the vector we are looking for as follows.

$$
instabSets(D) := \overline{vec(D)^{\mathsf{T}}; ([\![\mathsf{M},\mathsf{M}]\!] \cap \overline{[\![\overline{D}, D]\!];\mathsf{M}}\,)}^{\mathsf{T}} : 2^A \leftrightarrow \mathbf{1}
$$

Next, we treat the externally stable sets in the same manner using M, x, y, z and u as above. Because of the $\forall\exists\forall$-structure of the initial logical specification of 'externally stable', the derivation and the resulting relation-algebraic expression are a bit more complex than in the above case. We start with an auxiliary calculation. Its first step is a consequence of the irreflexivity of the tournament D and the type of the universal relation L introduced in the fifth step is $[\mathbf{1} \leftrightarrow A]$.

$$
\begin{aligned}
& \exists\, y : y \in X \wedge D_{y,x} \wedge \forall\, z : z \in X \cup \{x\} \wedge D_{x,z} \rightarrow D_{y,z} \\
\iff\ & \exists\, y : y \in X \wedge D_{y,x} \wedge \forall\, z : z \in X \wedge D_{x,z} \rightarrow D_{y,z} \\
\iff\ & \exists\, u : u_1 \in X \wedge u_2 = x \wedge D_{u_1,u_2} \wedge \forall\, z : z \in X \wedge D_{u_2,z} \rightarrow D_{u_1,z} \\
\iff\ & \exists\, u : u_2 = x \wedge D_{u_1,u_2} \wedge u_1 \in X \wedge \neg\exists\, z : \overline{D}_{u_1,z} \wedge D_{u_2,z} \wedge z \in X \\
\iff\ & \exists\, u : \rho_{u,x} \wedge vec(D)_u \wedge (\pi;\mathsf{M})_{u,X} \wedge \neg\exists\, z : [\![\overline{D}, D]\!]_{u,z} \wedge \mathsf{M}_{z,X} \\
\iff\ & \exists\, u : (\rho \cap vec(D);\mathsf{L})_{u,x} \wedge (\pi;\mathsf{M})_{u,X} \wedge (\,\overline{[\![\overline{D}, D]\!];\mathsf{M}}\,)_{u,X} \\
\iff\ & \exists\, u : (\rho \cap vec(D);\mathsf{L})^{\mathsf{T}}{}_{x,u} \wedge (\pi;\mathsf{M} \cap \overline{[\![\overline{D}, D]\!];\mathsf{M}}\,)_{u,X} \\
\iff\ & ((\rho \cap vec(D);\mathsf{L})^{\mathsf{T}}; (\pi;\mathsf{M} \cap \overline{[\![\overline{D}, D]\!];\mathsf{M}}\,))_{x,X}
\end{aligned}
$$

This yields, again with L being of type $[\mathbf{1} \leftrightarrow A]$:

$$
\begin{aligned}
& X \text{ externally stable} \\
\iff\ & \forall\, x : x \notin X \rightarrow \exists\, y : y \in X \wedge D_{y,x} \wedge \forall\, z : z \in X \cup \{x\} \wedge D_{x,z} \rightarrow D_{y,z} \\
\iff\ & \forall\, x : \overline{\mathsf{M}}_{x,X} \rightarrow ((\rho \cap vec(D);\mathsf{L})^{\mathsf{T}}; (\pi;\mathsf{M} \cap \overline{[\![\overline{D}, D]\!];\mathsf{M}}\,))_{x,X} \\
\iff\ & \neg\exists\, x : \overline{\mathsf{M}}_{x,X} \wedge (\,\overline{(\rho \cap vec(D);\mathsf{L})^{\mathsf{T}}; (\pi;\mathsf{M} \cap \overline{[\![\overline{D}, D]\!];\mathsf{M}}\,)}\,)_{x,X} \\
\iff\ & \neg\exists\, x : \mathsf{L}_{\perp,x} \wedge (\,\overline{\overline{\mathsf{M}} \cap \overline{(\rho \cap vec(D);\mathsf{L})^{\mathsf{T}}; (\pi;\mathsf{M} \cap \overline{[\![\overline{D}, D]\!];\mathsf{M}}\,)}}\,)_{x,X} \\
\iff\ & (\,\overline{\mathsf{L};\, \overline{\mathsf{M} \cup (\rho \cap vec(D);\mathsf{L})^{\mathsf{T}}; (\pi;\mathsf{M} \cap \overline{[\![\overline{D}, D]\!];\mathsf{M}}\,)}}^{\mathsf{T}}\,)_X
\end{aligned}
$$

As a consequence of this equivalence we get the following relation-algebraic specification of a vector that represents the set of externally stable sets.

$$exstabSets(D) := \overline{\mathsf{L};\ \overline{\mathsf{M} \cup (\rho \cap vec(D);\mathsf{L})^{\mathsf{T}};(\pi;\mathsf{M} \cap \overline{[\![\,\overline{D}\,,D];\mathsf{M}}\,)}}^{\mathsf{T}} : 2^A \leftrightarrow \mathbf{1}$$

The intersection of the two relation-algebraic specifications we just have developed represents the set $\mathfrak{C}(D)$ of all covering sets of D. In the next step we apply the technique of Section 2.2 to obtain from this intersection a relation that column-wisely represents $\mathfrak{C}(D)$. Here is the result.

$$CovSets(D) := \mathsf{M}; inj(instabSets(D) \cap exstabSets(D))^{\mathsf{T}} : A \leftrightarrow \mathfrak{C}(D)$$

From Section 2.2 we also know that an application of the function *MinEnum* of (2) selects from the relation $CovSets(D)$ those columns which represent minimal covering sets. In tournaments there is precisely one minimal covering set and this is the reason for the target $\mathbf{1}$ of the relation-algebraic specification

$$mincovSet(D) := MinEnum(CovSets(D)) : A \leftrightarrow \mathbf{1} \tag{8}$$

of a vector that represents the minimal covering set $MC(D)$ of D.

4.6 The Banks Set

The first step towards a relation-algebraic specification of $BA(D)$ is the development of a vector that represents the set $\mathfrak{T}(D)$ of transitive sets of 2^A with respect to D. We assume $X \in 2^A$ and calculate as given below, where $\mathsf{M} : A \leftrightarrow 2^A$ is the membership-relation, x, y and z range over A and u ranges over $A \times A$.

$$\begin{aligned}
& X \text{ transitive} \\
\iff\ & \forall\, x,y,z : x \in X \wedge y \in X \wedge z \in X \wedge D_{x,y} \wedge D_{y,z} \rightarrow D_{x,z} \\
\iff\ & \neg\exists\, x,y,z : x \in X \wedge y \in X \wedge z \in X \wedge D_{x,y} \wedge D_{y,z} \wedge \overline{D}_{x,z} \\
\iff\ & \neg\exists\, u,z : u_1 \in X \wedge u_2 \in X \wedge z \in X \wedge D_{u_1,u_2} \wedge D_{u_2,z} \wedge \overline{D}_{u_1,z} \\
\iff\ & \neg\exists\, u : \mathsf{M}_{u_1,X} \wedge \mathsf{M}_{u_2,X} \wedge vec(D)_u \wedge \exists\, z : \mathsf{M}_{z,X} \wedge \overline{D}_{u_1,z} \wedge D_{u_2,z} \\
\iff\ & \neg\exists\, u : [\![\mathsf{M},\mathsf{M}]\!]_{u,X} \wedge vec(D)_u \wedge \exists\, z : [\![\,\overline{D}\,,D]\!]_{u,z} \wedge \mathsf{M}_{z,X} \\
\iff\ & \neg\exists\, u : vec(D)_u \wedge [\![\mathsf{M},\mathsf{M}]\!]_{u,X} \wedge ([\![\,\overline{D}\,,D];\mathsf{M})_{u,X} \\
\iff\ & (\,\overline{vec(D)^{\mathsf{T}};([\![\mathsf{M},\mathsf{M}] \cap [\![\,\overline{D}\,,D];\mathsf{M})}^{\mathsf{T}}\,)_X
\end{aligned}$$

As a consequence we get as desired vector representation of the set $\mathfrak{T}(D)$ the following one.

$$transSets(D) := \overline{vec(D)^{\mathsf{T}};([\![\mathsf{M},\mathsf{M}] \cap [\![\,\overline{D}\,,D];\mathsf{M})}^{\mathsf{T}} : 2^A \leftrightarrow \mathbf{1}$$

Next, we apply again the results of Section 2.2 to obtain from this vector, first, the column-wise representation

$$TransSets(D) := \mathsf{M}; inj(transSets(D))^{\mathsf{T}} : A \leftrightarrow \mathfrak{T}(D).$$

of the set $\mathfrak{T}(D)$ and, afterwards, by means of the function *MaxEnum* of (1) the column-wise representation

$$\mathit{BanksTraj}(D) := \mathit{MaxEnum}(\mathit{TransSets}(D)) : A \leftrightarrow \mathfrak{T}(D)_{\max}$$

of the set $\mathfrak{T}(D)_{\max}$ of the maximal sets of $\mathfrak{T}(D)$. Each column of the relation $\mathit{BanksTraj}(D)$ represents precisely one Banks trajectory. Let B be an abbreviation for $\mathit{BanksTraj}(D)$. Since the restriction of D to each Banks trajectory X is a strict-order relation, an alternative x is maximal in X with respect to D iff $x \in X$ and there is no $y \in X$ with $D_{y,x}$. Notice, that $x \in X$ and $B_{x,X}$ are equivalent and the same holds for $y \in X$ and $B_{y,X}$. With the help of these properties we get therefore for all $x \in A$ that

$$\begin{aligned} x \in BA(D) &\iff \exists\, X : B_{x,X} \wedge \neg\exists\, y : B_{y,X} \wedge D_{y,x} \\ &\iff \exists\, X : B_{x,X} \wedge (\,\overline{D^{\mathsf{T}}; B}\,)_{x,X} \wedge \mathsf{L}_X \\ &\iff ((B \cap \overline{D^{\mathsf{T}}; B}\,); \mathsf{L})_x. \end{aligned}$$

In this calculation $\mathsf{L} : \mathfrak{T}(D)_{\max} \leftrightarrow \mathbf{1}$ is the universal vector that represents the set of Banks trajectories, X ranges over $\mathfrak{T}(D)_{\max}$ and y ranges over A. The relation-algebraic specification

$$\mathit{banksSet}(D) := (B \cap \overline{D^{\mathsf{T}}; B}\,); \mathsf{L}, \text{ where } B := \mathit{BanksTraj}(D) \tag{9}$$

of the vector representation of the Banks set $BA(D)$ of D is an immediate consequence of the latter calculation.

4.7 The Tournament Equilibrium Set

Finally, we demonstrate how to specify the tournament equilibrium set with relation-algebraic means in such a way that this yields a recursive algorithm for computing this choice set. Decisive for its definition is the relation $T : A \leftrightarrow A$ such that, given $X \in 2^A$, it holds that

$$T_{x,y} \iff x \in \mathit{TEQ}_{D^{X,y}}(D_{|D^{X,y}}) \wedge y \in X$$

for all $x, y \in A$; see Section 3.1. A little reflection shows that this point-wise specification is equivalent to the point-free equation

$$T = \bigcup_{y \in X} \mathit{TEQ}_{D^{X,y}}(D_{|D^{X,y}}) \times \{y\}.$$

The relation T depends on the relation D and the set X. To make this fact explicit, we use the functional notation $T(D, X)$ instead of the simple T. In a similar manner we write $\mathit{TEQ}(D_{|D^{X,y}}, D^{X,y})$ instead of $\mathit{TEQ}_{D^{X,y}}(D_{|D^{X,y}})$. As result of these functional notations we get T and TEQ as functions

$$T : [A \leftrightarrow A] \times 2^A \to [A \leftrightarrow A] \qquad \mathit{TEQ} : [A \leftrightarrow A] \times 2^A \to 2^A$$

which fulfill for all $D : A \leftrightarrow A$ and $X \in 2^A$ the recursive equation

$$T(D, X) = \bigcup_{y \in X} TEQ(D_{|D^{X,y}}, D^{X,y}) \times \{y\}.$$

The computation of the tournament equilibrium set of D starts with the call $T(D, A)$. As a consequence, during the entire evaluation process the first argument of T is the restriction of the tournament D to the second argument of T. Since A is assumed to be finite, eventually both arguments become empty and we get $T(\mathsf{O}, \emptyset) = \mathsf{O}$ in this termination case.

To obtain versions of the functions T and TEQ that completely work on relations, we model their second arguments and the results of TEQ via vectors, i.e., we consider the variant

$$TeqRel : [A \leftrightarrow A] \times [A \leftrightarrow \mathbf{1}] \to [A \leftrightarrow A]$$

of the function T and the variant

$$teqSet : [A \leftrightarrow A] \times [A \leftrightarrow \mathbf{1}] \to [A \leftrightarrow \mathbf{1}]$$

of the function TEQ. Then the set $TEQ(D, A) \in 2^A$ corresponds to the vector $teqSet(D, \mathsf{L}) : A \leftrightarrow \mathbf{1}$, where $\mathsf{L} : A \leftrightarrow \mathbf{1}$, and hence, the latter represents the tournament equilibrium set of D as desired.

If we assume $v : A \leftrightarrow \mathbf{1}$ to be the vector representation of $X \in 2^A$, then the relation-algebraic version of the equation $TEQ_X(D) = MTC_X(T)$ of Section 3.1 looks as follows, where the function max is as defined in (5).

$$teqSet(D, v) = max(R \cap \overline{R^{\mathsf{T}}}, v), \text{ where } R := TeqRel(D, v)^+. \tag{10}$$

The empty set is represented by the empty vector. As a consequence, the termination case $T(\mathsf{O}, \emptyset) = \mathsf{O}$ of the function T translates into

$$TeqRel(\mathsf{O}, \mathsf{O}) = \mathsf{O}. \tag{11}$$

To solve the entire task, what remains is the translation of the above recursive equation into the language of relation algebra. To this end, assume again $X \in 2^A$ to be represented by $v : A \leftrightarrow \mathbf{1}$. Then every element y of X corresponds to a point p such that $p \subseteq v$. Using this fact, $TeqRel(D, v) = \bigcup_p teqSet(D', w); p^{\mathsf{T}}$ is the relation-algebraic version of the recursive equation, provided p ranges over all points contained in v, the vector $w : A \leftrightarrow \mathbf{1}$ represents the set $\{x \in X \mid D_{x,y}\}$, where y is represented by p, and $D' : A \leftrightarrow A$ is the restriction of D to this set. It is easy to verify that $w = D; p \cap v$. In Boolean matrix terminology the restriction of D to the set w represents is obtained by "zeroing out" in D all 1-entries outside the square $w; w^{\mathsf{T}}$, i.e., by changing them to 0. This leads to $D' = D \cap w; w^{\mathsf{T}}$, so that, finally, the relation-algebraic version of the recursive equation is

$$TeqRel(D, v) = \bigcup_p teqSet(D \cap w; w^{\mathsf{T}}, w); p^{\mathsf{T}}, \text{ where } w := D; p \cap v. \tag{12}$$

In the programming language of RelView the big union of (12) can easily be implemented via a while-loop through all points contained in the vector v; this leads to a cascade-like recursion.

5 Conclusion

We have presented the relation-algebraic approach to the tournament choice problem. The relation-algebraic specifications of several choice sets, well-known in the literature on social choice theory, have been delivered. All that happened in a formal and goal-directed way. This drastically reduces the danger of making errors. Another clear advantage of this technique is the support by the RelView system, which evaluates the relation-algebraic expressions and visualizes the computed results in a very efficient and elegant way. Notice, that even for our simple example of Section 3.2 it would be very hard to compute the considered choice sets by hand, with a high chance of making mistakes. Another interesting property of RelView is that it allows randomly to generate tournaments, even with specific properties. For experiments it is also very important that RelView allows to test the validity of arbitrary Boolean combinations of relation-algebraic inclusion. Since, for $R, S : X \leftrightarrow Y$ given, we have

$$R \subseteq S \iff \overline{\mathsf{L}; (R \cap \overline{S}); \mathsf{L}} = \mathsf{L} \qquad \neg(R \subseteq S) \iff \overline{\mathsf{L}; (R \cap \overline{S}); \mathsf{L}} = \mathsf{O}$$

(cf. the *Tarski rule* in [22]) and since the Boolean operations $\cup$, $\cap$ and $\overline{}$ on $[\mathbf{1} \leftrightarrow \mathbf{1}]$ correspond to the logical connectives $\vee$, $\wedge$ and $\neg$, each such formula can immediately be translated into an expression that only has the relations $\mathsf{O} : \mathbf{1} \leftrightarrow \mathbf{1}$ and $\mathsf{L} : \mathbf{1} \leftrightarrow \mathbf{1}$ as possible values and the formula is true (false) iff the expression has the universal relation (empty relation) as its value.

As already mentioned, the relation-algebraic approach can easily be applied to other social choice problems. In particular, we could use relation algebra and next RelView to compute and experiment with other choice sets mentioned in the literature, e.g., in [12]. Examples are von Neumann-Morgenstern stable sets, maximal sets, undominated sets, dominant sets and ultimate uncovered sets. All these solutions can easily be specified relation-algebraically.

There is no consistent definition of covering in the social choice literature; see e.g., [12, 20] where a lot of examples and corresponding references are given. The notion of covering we have used in Section 3.1 is *downward covering* in the sense of [7]. In [7] also *upward covering* and *bidirectional covering* are investigated[2]. On tournaments all three variants coincide, but on non-complete asymmetric dominance relations they may be different. Changing from downward covering to the other variants using relation-algebra requires only marginal modifications of the relation-algebraic specifications and, consequently, of the RelView programs obtained from them. For example, the uncovered set with respect to upwards covering is obtained by replacing in the relation-algebraic specification (7) the left residual $\overline{D; \overline{D^{\mathsf{T}}}}$ by the right residual $\overline{D^{\mathsf{T}}; \overline{D}}$. If in (7) instead of the left residual the intersection $\overline{D; \overline{D^{\mathsf{T}}}} \cap \overline{D^{\mathsf{T}}; \overline{D}}$ of both residuals is used, the uncovered set with respect to bidirectional covering is represented.

[2] Alternative x *upward covers* alternative y in X if $D_{x,y}$ and for all $z \in X$ from $D_{z,x}$ it follows $D_{z,y}$, and x *bidirectional covers* y in X if $D_{x,y}$ and for all $z \in X$ from $D_{y,z}$ it follows $D_{x,z}$ and from $D_{z,x}$ it follows $D_{z,y}$.

In the case of covering sets the changes are of a similar type. E.g., to compute minimal covering sets with respect to upward covering using (8), in the auxiliary specifications *instabSets*(D) and *exstabSets*(D) of Section 4.5 only the left pairing $[\![\overline{D}, D]$ has to be changed into $[\![D^{\mathsf{T}}, \overline{D}^{\mathsf{T}}]$. A replacement of $[\![\overline{D}, D]$ by $[\![\overline{D}, D] \cup [\![D^{\mathsf{T}}, \overline{D}^{\mathsf{T}}]$ in *instabSets*(D) and *exstabSets*(D) computes minimal covering sets with respect to bidirectional covering. Notice, however, that for non-complete asymmetric dominance relations uniqueness of a minimal covering set is not guaranteed for upward covering and downward covering; only in the bidirectional case there is always a unique minimal covering set (cf. [7]).

To give a last example of the flexibility of the approach, we consider once more the specification of the Banks set. For asymmetric dominance relations (so-called *weak tournaments*) in [12] four variants are discussed. The notion we have introduced in Section 3.1 corresponds to the third variant of [12]. It is not hard to modify the specification of Section 4.6 to obtain also the other variants. E.g., the second variant of [12] considers instead of transitive subsets those subsets $X \in 2^A$ for which the restriction of D to X is *negatively transitive*. Since the latter means that $\overline{D}$ is transitive, a relation-algebraic specification of the second variant is obtained if in (9) the definition $B := \mathit{BanksTraj}(D)$ is replaced by $B := \mathit{BanksTraj}(\overline{D})$. The first variant of [12] works with transitive and total subsets. This only demands for a relation-algebraic specification of total sets, which is even more easy to calculate than *BanksTraj*(D), since

$$\forall\, x : x \in X \rightarrow (\forall\, y : y \in X \rightarrow (D_{x,y} \vee D_{y,x}))$$

can be expressed in pure relation algebra because of the criterion mentioned in Section 4.5.

An important choice set we have not mentioned so far is the *Slater set*, introduced in [26]. It consists of all undominated elements of those linear strict-order relations (called *Slater orders*) that share as many pairs with the given tournament as possible. We have developed a relation-algebraic specification of the Slater set by, first, relation-algebraically specifying all linear strict-order relations as a vector of type $[A \leftrightarrow A] \leftrightarrow \mathbf{1}$ and, next, combining this with a relation-algebraic specification of set intersection and the size-comparison relation on 2^A that relates X and Y iff $|X| \leq |Y|$. But the result is rather inefficient and only works in the case of very small sets of alternatives. At best it can be used for educational purposes.

As already mentioned in the introduction, RelView is an ideal tool for experimentation and exploration. In spite of the fact that it implements relations very efficiently, it cannot compete with special algorithms tailored for hard problems of social choice theory. Presently, we therefore mainly use the system to refine the specifications we have developed and to investigate and explore further concepts of social choice theory and related notions. But we also develop RelView further to make it more applicable by expanding its interface in such a way that it is possible to outsource program logic into small problem-specific plug-ins. By specific plug-ins based on known algorithms from the social choice literature, we hope in the future to be able to deal also with large problems

which cannot be solved with the present version of the tool, thereby keeping up the advantages of the system.

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