Concave consumption function and precautionary wealth accumulation

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Abstract

This paper examines the theoretical foundations of precautionary wealth accumulation in a multi-period model where consumers face uninsurable earnings risk and borrowing constraints. We begin by characterizing the consumption function of individual consumers. We show that consumption function is concave when the utility function has strictly positive third derivative and the inverse of absolute prudence is a concave function. These conditions encompass all HARA utility functions with strictly positive third derivative as special cases. We then show that when consumption function is concave, a mean-preserving spread in earnings risk would encourage wealth accumulation at both the individual and aggregate levels.

Keywords: Consumption function, borrowing constraints, precautionary saving

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1 Introduction

This paper analyzes the optimal life-cycle saving behavior of risk-averse consumers who face uninsurable idiosyncratic earnings risk and borrowing constraints. The purpose of this study is to better understand the theoretical foundations of precautionary wealth accumulation. In particular, this study seeks to provide conditions under which an increase in the riskiness of earnings would lead to an increase in both individual and aggregate savings. Unlike most of the existing studies, we do not restrict our attention to HARA utility functions.[1] Instead, we explore general conditions on preferences under which precautionary saving would emerge.

Precautionary saving behavior has long been the subject of both empirical and theoretical studies. Recent empirical work show that precautionary saving plays a crucial role in explaining wealth accumulation over the life cycle.[2] In the theoretical literature, a large number of studies have analyzed the optimal response of saving to uninsurable income risk.[3] One intriguing finding from these studies is that precautionary saving behavior is closely related to the concavity of the consumption function. Intuitively, concavity means that there is an inverse relationship between the marginal propensity to consume and the total resources available for consumption. Thus, when facing the same fluctuation in earnings, poor consumers would have a larger response in consumption than rich consumers. This in turn implies a higher consumption volatility for the poor than the rich. Since risk-averse consumers dislike volatility in consumption, this creates an incentive for them to accumulate more wealth.[4] Huggett (2004) provides the first formal proof on the connection between concave consumption function and precautionary wealth accumulation. Using a canonical life-cycle model with purely transitory earnings risk, Huggett shows that when the consumption function is concave, an increase in earnings risk would induce each consumer to accumulate more assets. This in turn raises the average level of wealth at each stage of the life cycle.

Huggett's result suggests that one way to improve our understanding of precautionary saving behavior is to identify the conditions under which consumption function is concave.[5] Due to a lack of

[1] HARA is the acronym for hyperbolic absolute risk aversion. The HARA class of utility functions include some of the most commonly used utility functions in the existing studies, such as the constant-absolute-risk-aversion (CARA) utility function, the constant-relative-risk-aversion (CRRA) utility function, and the quadratic utility function. A formal definition of HARA utility function is stated in Section 3.3.

[2] See, for instance, Carroll and Samwick (1997, 1998), Gourinchas and Parker (2002), and Parker and Preston (2005). For a comprehensive survey of some earlier studies, see Browning and Lusardi (1996).

[3] Huggett (2004) provides an excellent review on some of the early studies. Heathcote *et al.* (2009) provide an extensive survey of the models with idiosyncratic risks and incomplete markets.

[4] See Carroll (1997) for more discussion on the implications of concave consumption function.

[5] This also raises the natural question of whether concave consumption function is consistent with empirical evidence.

closed-form solutions, existing studies typically rely on computational methods to derive the consumption function. Zeldes (1989), Deaton (1991) and Carroll (1992, 1997) are among the first studies that use these methods to show that consumption function is concave under constant-relative-risk-aversion utility. Carroll and Kimball (1996) is the first study that provides a rigorous analysis on this subject. These authors show that, in the absence of borrowing constraints, consumption function exhibits concavity when the utility function is drawn from the HARA class and has strictly positive third derivative. Huggett (2004) and Carroll and Kimball (2005) extend this result to an environment with borrowing constraints, while maintaining the HARA assumption.

Although existing studies focus exclusively on HARA utility functions, there is no compelling theoretical or empirical reason to confine ourselves to this class of utility functions. Adopting the HARA assumption is equivalent to assuming that consumers have linear absolute risk tolerance. There is no empirical evidence concerning the linearity or the curvature of absolute risk tolerance. In the theory of consumer preferences, the usual axioms do not imply a linear absolute risk tolerance. On the contrary, a growing number of studies show that the *nonlinearity* of absolute risk tolerance has important implications in both deterministic and stochastic models.[6] In the precautionary savings literature, it remains an open question whether the existing results on concave consumption function are still valid once we remove the HARA assumption. This paper provides the first general answer to this question.

In this paper, we consider a model economy in which a large number of *ex ante* identical consumers face uninsurable idiosyncratic earnings risk and borrowing constraints over the life cycle. The exogenous earnings process can be decomposed into a permanent component and a purely transitory component. The consumers are identical before the earnings shocks are realized. In the first part of the analysis, we provide a detailed characterization of the consumption function for individual consumers. The main objective here is to derive a set of conditions on preferences such that the consumption function exhibits concavity. In the second part of the analysis, we explore the connection between concave consumption function and precautionary saving. In particular, we examine how changes in the riskiness of permanent

Using data from the Consumption Expenditure Survey, Gourinchas and Parker (2001) estimate the consumption function for U.S. households in various age groups and find that it is a concave function in liquid assets. Souleles (1999) and Johnson *et al.* (2006) examine the response of household consumption to income tax refunds and income tax rebates in the United States. Both studies find that households with few liquid assets tend to have a larger propensity to consume than those with more liquid assets, which is consistent with a concave consumption function.

[6] Gollier (2001) shows that the curvature of absolute risk tolerance is important in understanding the relationship between wealth inequality and equity premium. Gollier and Zeckhauser (2002) show that this curvature is crucial in understanding how the length of investment horizon would affect portfolio choice. Ghiglino and Venditti (2007) show that, in a deterministic growth model, the curvature of absolute risk tolerance is a key factor in determining the effect of wealth inequality on macroeconomic volatility.

and transitory earnings shocks would affect wealth accumulation at both the individual and aggregate levels. Similar to Huggett (2004), we focus on the effects of earnings risk on the average life-cycle profile of wealth, which captures the cross-sectional average level of asset holdings at different stages of the life cycle.

This paper presents two main findings. Our first major result states that, in the presence of borrowing constraints, consumption function at every stage of the life cycle exhibits concavity if the utility function has strictly positive third derivative and the inverse of absolute prudence is a concave function.[7] For any HARA utility function, the inverse of absolute prudence is a linear function. The stated conditions thus include all HARA utility functions with strictly positive third derivative as special cases. This makes clear that hyperbolic absolute risk aversion is not a necessary condition for concave consumption function. When comparing to Huggett (2004) and Carroll and Kimball (2005), our concavity result is applicable to a broader class of utility functions. For HARA utility functions, we show that the consumption function is *strictly* concave in the permanent component of the earnings shock when the borrowing constraint is not binding. This implies that the marginal propensity to consume out of permanent shock is less than unity and is strictly decreasing in the level of the shock. This provides a formal proof for the numerical findings in Carroll (2009).[8] Our second major result states that, when the consumption function at every stage of the life cycle is concave, a mean-preserving spread in either the permanent or transitory earnings shock would induce each consumer to accumulate more assets. This in turn raises the average level of wealth at different stages of the life cycle. Since all other factors (including prices) are held constant in this analysis, the increase in asset holdings is entirely driven by the precautionary motive.

To show that these results are tight, we provide a set of numerical examples in which the utility function has strictly positive third derivative but the inverse of absolute prudence exhibits local convexity. We show that under certain parameter values the consumption function in certain period is not globally concave. Using this result, we are able to construct examples in which a mean-preserving spread in the transitory earnings shock would lead to a *reduction* in aggregate savings. These examples show that if the inverse of absolute prudence is not globally concave, then precautionary saving may

[7] Following Kimball (1990), the coefficient of absolute prudence for a thrice differentiable utility function $u(\cdot)$ is defined as $\Pi(c) \equiv -u'''(c)/u''(c)$. The concavity of the inverse of $\Pi(\cdot)$ remains an open empirical question.

[8] Carroll (2009) numerically solves and simulates the standard buffer-stock saving model with CRRA utility function. He finds that the marginal propensity to consume out of permanent earnings shock is less than unity over a wide range of parameter values.

not occur even when the utility function has strictly positive third derivative.[9]

The current study is close in spirit to Huggett (2004). There are, however, two important differences between the two work. First, Huggett only considers purely transitory risk, while the present study makes a distinction between permanent and transitory risks. We show that this distinction is important in deriving the concavity results. Second, Huggett only considers HARA utility functions and does not explore the possibility of having concave consumption function under more general utility functions. Our results show that concave consumption function can be obtained even if we move beyond the HARA class of utility functions.

This rest of this paper is organized as follows. Section 2 presents the model environment and establishes some intermediate results. Section 3 establishes the concavity of the consumption function and contrasts our result to those in the existing studies. Section 4 examines how changes in the riskiness of permanent and transitory earnings risks would affect individual and aggregate savings. Section 5 presents a set of numerical examples to illustrate the theoretical results. Section 6 concludes.

2 The Model

Consider an economy inhabited by a continuum of *ex ante* identical consumers. The size of population is constant and is normalized to one. Each consumer faces a $(T+1)$-period planning horizon, where T is finite. The consumers have preferences over random consumption paths $\{c_t\}_{t=0}^{T}$ which can be represented by

$$E_0\left[\sum_{t=0}^{T}\beta^t u(c_t)\right], \tag{1}$$

where $\beta \in (0,1)$ is the subjective discount factor and $u(\cdot)$ is the (per-period) utility function. The domain of the utility function is given by $\mathcal{D} = [\underline{c}, \infty)$, where $\underline{c} \geq 0$ is a minimum consumption requirement.[10] The function $u : \mathcal{D} \to [-\infty, \infty)$ is once continuously differentiable, strictly increasing and strictly concave.[11] There is no restriction on the value of $u(\underline{c})$, which means the utility function can

[9] These examples also highlight an important difference between two-period models and general multi-period models. It is well-known that in two-period models, precautionary savings emerge if and only if the utility function has strictly positive third derivative. This result, however, is not true in models with more than two periods. We will discuss this point in greater detail in Section 4.

[10] This specification encompasses those utility functions that are not defined at $c = 0$. One example is the Stone-Geary utility function which belongs to the HARA class and features a minimum consumption requirement. All the results in this paper remain valid if we set $\underline{c} = 0$.

[11] None of the results in this section require higher order differentiability of the utility function. These properties are required only in later sections.

be either bounded or unbounded below.

In each period $t \in \{0, 1, ..., T\}$, each consumer receives a random amount of labor endowment e_t, which they supply inelastically to work. Labor income at time t is then given by we_t, where $w > 0$ is a constant wage rate. The stochastic labor endowment is determined by $e_t = \widetilde{e}_t \varepsilon_t$, where $\widetilde{e}_t$ is a permanent component and ε_t is a purely transitory component.[12] The initial value of the permanent component $\widetilde{e}_0 > 0$ is given and is identical across all consumers. This component then evolves according to $\widetilde{e}_t = \widetilde{e}_{t-1} \nu_t$, where ν_t is drawn from a compact interval $\Lambda \equiv [\underline{\nu}, \overline{\nu}]$, with $\overline{\nu} > \underline{\nu} > 0$, according to the distribution $L_t(\cdot)$. Similarly, the transitory shock ε_t is drawn from a compact interval $\Xi \equiv [\underline{\varepsilon}, \overline{\varepsilon}]$, with $\overline{\varepsilon} > \underline{\varepsilon} > 0$, according to the distribution $G_t(\cdot)$.[13] Both $L_t(\cdot)$ and $G_t(\cdot)$ are well-defined distribution functions, which means they are both nondecreasing, right-continuous and satisfy the following conditions:

$$\lim_{\nu \to \underline{\nu}} L_t(\nu) = \lim_{\varepsilon \to \underline{\varepsilon}} G_t(\varepsilon) = 0 \quad \text{and} \quad \lim_{\nu \to \overline{\nu}} L_t(\nu) = \lim_{\varepsilon \to \overline{\varepsilon}} G_t(\varepsilon) = 1.$$

The random variables ε_t and ν_t are independent of each other, across time and across agents.

Under this specification, knowledge on both $\widetilde{e}_t$ and ε_t is required to determine the distribution of e_{t+1}. Thus, an individual's state at time t includes $z_t \equiv (\widetilde{e}_t, \varepsilon_t)$. Under the stated assumptions, the stochastic event $z_t \equiv (\widetilde{e}_t, \varepsilon_t)$ follows a Markov process with compact state space $Z_t \equiv \Delta_t \times \Xi$, where $\Delta_t \equiv \left[\underline{\xi}_t, \overline{\xi}_t\right]$ contains all possible realizations of $\widetilde{e}_t$.[14] Let $(Z_t, \mathcal{Z}_t)$ be a measurable space and $Q_t : (Z_t, \mathcal{Z}_t) \to [0, 1]$ be the transition function of the Markov process at time t. For any $z = (\widetilde{e}, \varepsilon) \in Z_t$ and for any $B \subseteq Z_{t+1}$,

$$Q_t(z, B) \equiv \Pr\{(\nu_{t+1}, \varepsilon_{t+1}) \in \Lambda \times \Xi : (\widetilde{e}\nu_{t+1}, \varepsilon_{t+1}) \in B\}.$$

It is straightforward to show that the transition function in each period satisfies the Feller property.

In light of the uncertainty in labor income, the consumers can only self-insure by borrowing or lending a single risk-free asset. The gross return from the asset is $(1 + r) > 0$.[15] Let a_t denote asset

[12] This dichotomy between permanent and transitory income shocks is commonly used in quantitative studies. Examples include Zeldes (1989), Carroll (1992, 1997, 2011), Ludvigson (1999), Gourinchas and Parker (2002), Storesletten *et al.* (2004a), among many others. Empirical studies on household earnings dynamics show that this specification fits the data well. See, for instance, Abowd and Card (1989), Storesletten *et al.* (2004b), and Moffitt and Gottschalk (2011).

[13] The age-dependent nature of $L_t(\cdot)$ and $G_t(\cdot)$ can be used to capture any age-specific differences in earnings across consumers. Hence, we do not include a separate life-cycle component in e_t.

[14] For any $t \in \{0, 1, ..., T\}$, the upper and lower bounds of Δ_t are given by $\overline{\xi}_t \equiv \widetilde{e}_0 \overline{\nu}^t$ and $\underline{\xi}_t \equiv \widetilde{e}_0 \underline{\nu}^t$, respectively.

[15] In the existing studies, it is typical to assume that w and r are deterministic and time-invariant. The time-invariant

holdings at time t. A consumer is said to be in debt if a_t is negative. In each period t, the consumers are subject to the budget constraint

$$c_t + a_{t+1} = we_t + (1+r)\, a_t, \tag{2}$$

and the borrowing constraint: $a_{t+1} \geq -\underline{a}_{t+1}$. The parameter $\underline{a}_{t+1} \geq 0$ represents the maximum amount of debt that a consumer can owe at time t. The borrowing limits are period-specific (or, more precisely, age-specific) for the following reason. In life-cycle models, consumers are typically forbidden to die in debt. Thus, the borrowing limit in the terminal period must be $\underline{a}_{T+1} = 0$. But the borrowing limit in all other periods can be different from zero. Throughout this paper, we maintain the following assumptions on the borrowing limits: $\underline{a}_t \geq 0$, for all t, $\underline{a}_{T+1} = 0$ and

$$w\underline{e}_t - (1+r)\, \underline{a}_t + \underline{a}_{t+1} > \underline{c}, \qquad \text{for all } t, \tag{3}$$

where $\underline{e}_t \equiv \underline{\xi}_t \times \underline{\varepsilon}$ is the lowest possible value of e_t. The intuitions of (3) are as follows. Consider a consumer who faces the worst possible state at time t, i.e., $a_t = -\underline{a}_t$ and $e_t = \underline{e}_t$. The highest attainable consumption in this particular state is $c_t = w\underline{e}_t - (1+r)\, \underline{a}_t + \underline{a}_{t+1}$. Condition (3) then ensures that a consumer can meet the minimum consumption requirement even in the worst possible state. The same condition also ensures that any debt at time t can be repaid in the future even if a consumer receives the lowest labor income in all future periods, i.e.,

$$\sum_{j=0}^{T-t} \frac{w\underline{e}_{t+j} - \underline{c}}{(1+r)^j} - (1+r)\, \underline{a}_t > 0, \qquad \text{for all } t.$$

2.1 Consumers' Problem

Given the prices w and r, the consumers' problem is to choose sequences of consumption and asset holdings, $\{c_t, a_{t+1}\}_{t=0}^{T}$, so as to maximize the expected lifetime utility in (1), subject to the budget constraint in (2), the minimum consumption requirement $c_t \geq \underline{c}$, the borrowing constraint $a_{t+1} \geq -\underline{a}_{t+1}$ for all t, and the initial conditions: $a_0 \geq -\underline{a}_0$ and $\widetilde{e}_0 > 0$.

assumption is not essential for our results. Specifically, all of our proofs can be easily modified to handle any price sequences $\{w_t, r_t\}_{t=0}^{T}$ that are deterministic, bounded above and strictly positive.

Define a sequence of assets $\{\overline{a}_t\}_{t=0}^{T}$ according to

$$\overline{a}_{t+1} = w\overline{e}_t + (1+r)\,\overline{a}_t - \underline{c}, \qquad \text{for all } t,$$

where $\overline{e}_t \equiv \overline{\xi}_t \times \overline{\varepsilon}$ is the highest possible value of e_t, and $\overline{a}_0 = a_0$. This sequence specifies the amount of asset available in every period for a consumer who receives the highest possible labor income $w\overline{e}_t$ and consumes only the minimum requirement $\underline{c}$ in every period. Since $(1+r) > 0$, this sequence is monotonically increasing and bounded above by

$$\overline{a}_T \equiv (1+r)^T a_0 + \sum_{j=0}^{T-1} (1+r)^{T-1-j} \left(w\overline{e}_j - \underline{c}\right),$$

which is finite as T is finite. It is straightforward to show that any feasible sequence of assets $\{a_t\}_{t=0}^{T}$ must be bounded above by $\{\overline{a}_t\}_{t=0}^{T}$. Hence, the state space of asset in every period t can be restricted to the compact interval $A_t = \left[-\underline{a}_t, \overline{a}_t\right]$.

In any given period, the state of a consumer is summarized by $s = (a, z)$, where a denotes his asset holdings at the beginning of the period, and $z \equiv (\widetilde{e}, \varepsilon)$ is the current realization of the earnings shocks.[16] The set of all possible states at time t is given by $S_t = A_t \times Z_t$.

Define a set of value functions $\{V_t\}_{t=0}^{T}$, $V_t : S_t \to [-\infty, \infty]$ for each t, recursively according to

$$V_t(a, z) = \max_{c \in \left[\underline{c}, x(a,z) + \underline{a}_{t+1}\right]} \left\{ u(c) + \beta \int_{Z_{t+1}} V_{t+1}\left[x(a, z) - c, z'\right] Q_t\left(z, dz'\right) \right\}, \tag{P1}$$

where $x(a, z) \equiv we(z) + (1+r)\,a$ and $e(z)$ is the level of labor endowment under $z \equiv (\widetilde{e}, \varepsilon)$. The variable $x(a, z)$ is often referred to as *cash-in-hand* in the existing studies. In the terminal period, the value function is given by

$$V_T(a, z) = u\left[we(z) + (1+r)\,a\right], \qquad \text{for all } (a, z) \in S_T.$$

Define a set of optimal policy correspondences for consumption $\{g_t\}_{t=0}^{T}$ according to

$$g_t(a, z) \equiv \operatorname*{arg\,max}_{c \in \left[\underline{c}, x(a,z) + \underline{a}_{t+1}\right]} \left\{ u(c) + \beta \int_{Z_{t+1}} V_{t+1}\left[x(a, z) - c, z'\right] Q_t\left(z, dz'\right) \right\}, \tag{4}$$

[16] For the results in this section, the distinction between $\widetilde{e}$ and ε is immaterial. Thus, we express individual state as $s = (a, z)$, instead of $s = (a, \widetilde{e}, \varepsilon)$, throughout this section.

for all $(a, z) \in S_t$ and for all t. Given $g_t(a, z)$, the optimal choices of a_{t+1} are given by

$$h_t(a, z) \equiv \left\{a' : a' = x(a, z) - c, \text{ for some } c \in g_t(a, z)\right\}. \tag{5}$$

Our first theorem summarizes the main properties of the value functions. The first part of the theorem states that the value function in every period t is bounded and continuous on S_t. This is true even when the utility function $u(\cdot)$ is unbounded below. Boundedness of the value functions ensures that the conditional expectation in (P1) is well-defined. Continuity of the objective function in (P1) ensures that the optimal policy correspondence $g_t(\cdot)$ is non-empty and upper hemicontinuous. The second part of the theorem establishes the strict monotonicity and strict concavity of $V_t(\cdot, z)$. Strict concavity of $V_t(\cdot, z)$ then implies that $g_t(\cdot, z)$ and $h_t(\cdot, z)$ are single-valued functions for all $z \in Z_t$.

The last part of the theorem establishes the differentiability of $V_t(\cdot, z)$. Specifically, this result states that $V_t(\cdot, z)$ is not only differentiable in the interior of A_t, but also (right-hand) differentiable at the endpoint $-\underline{a}_t$. This property is important because, as is well-known in this literature, a consumer may choose to exhaust the borrowing limit in certain states.[17] In other words, the consumers' problem may have corner solutions in which $h_{t-1}(a, z) = -\underline{a}_t$ for some $(a, z) \in S_{t-1}$. Thus, the first-order condition of (P1) has to be valid even when $h_{t-1}(a, z) = -\underline{a}_t$. This requires the value function $V_t(\cdot, z)$ to be right-hand differentiable at $a = -\underline{a}_t$. Note that the standard result in Stokey, Lucas and Prescott (1989) Theorem 9.10 only establishes the differentiability of the value function in the interior of the state space. Thus, additional effort is needed to establish this result.[18]

Additional conditions are imposed in part (iii) of Theorem 1 to ensure that $g_t(a, z) > \underline{c}$ for all $(a, z) \in S_t$. Specifically, the proof of part (iii) uses an intermediate result which states that if the utility function satisfies the Inada condition $u'(\underline{c}+) \equiv \lim_{c \to \underline{c}+} u'(c) = +\infty$, or the consumers are impatient so that $\beta(1 + r) \leq 1$, then it is never optimal to consume only the minimum consumption requirement $\underline{c}$.[19] The intuition of the Inada condition is well understood. When $\beta(1 + r) \leq 1$, the loss in utility incurred by reducing current consumption to $\underline{c}$ is always greater than the gain from increased future

[17] This result is formally proved in a number of studies, including Mendelson and Amihud (1982), Aiyagari (1994), Huggett and Ospina (2001) and Rabault (2002).

[18] Another related point is that $V_t(\cdot, z)$ is differentiable on $[-\underline{a}_t, \overline{a}_t)$ even if the policy function $g_t(\cdot, z)$ is not differentiable at the point where the borrowing constraint becomes binding. For a more detailed discussion on this, see Schechtman (1976) p.221-222.

[19] The same assumption is also used in Huggett (2004). In the buffer-stock savings model pioneered by Deaton (1991) and Carroll (1992, 1997), it is typical to assume that consumers are impatient so that $\beta(1 + r) \leq 1$. It is also possible to avoid this type of corner solution by imposing $u(\underline{c}) = -\infty$.

consumption. Hence, it is never optimal to choose $\underline{c}$. It follows that $h_t(a,z)$ can never reach the upper bound $\overline{a}_{t+1}$ in any period t. Hence, there is no need to consider corner solutions in which $h_{t-1}(a,z) = \overline{a}_t$, and the (left-hand) differentiability of $V_t(\cdot, z)$ at $a = \overline{a}_t$. Unless otherwise stated, all proofs can be found in Appendix A.

Theorem 1 *The following results hold for all* $t \in \{0, 1, ..., T\}$.

(i) The value function $V_t(a,z)$ *is bounded and continuous on* S_t.

(ii) For all $z \in Z_t$, $V_t(\cdot, z)$ *is strictly increasing and strictly concave on* A_t.

(iii) Suppose either $u'(\underline{c}+) = +\infty$ *or* $\beta(1+r) \leq 1$. *Then the function* $V_t(\cdot, z)$ *is continuously differentiable on* $[-\underline{a}_t, \overline{a}_t)$ *for all* $z \in Z_t$. *Let* $p_t(a,z)$ *denote the derivative of* $V_t(a,z)$ *with respect to* a. *Then* $p_t(a,z) = (1+r)u'[g_t(a,z)]$, *for all* $(a,z) \in [-\underline{a}_t, \overline{a}_t) \times Z_t$.

Our next theorem establishes some basic properties of the policy functions. The first part of Theorem 2 states that $g_t(a,z)$ is strictly greater than the minimum consumption requirement $\underline{c}$ for all $(a,z) \in S_t$. As mentioned above, this follows from the assumption that either $u'(\underline{c}+) = +\infty$ or $\beta(1+r) \leq 1$. The second part of the theorem establishes the Euler equation for consumption. This equation plays a central role in establishing the concavity of the consumption function. The third part of the theorem states that the consumption function at every stage of the life cycle is strictly increasing in the current state.

Theorem 2 *Suppose either* $u'(\underline{c}+) = +\infty$ *or* $\beta(1+r) \leq 1$. *Then the following results hold for all* $t \in \{0, 1, ..., T\}$.

(i) It is never optimal to consume the minimum requirement, i.e., $g_t(a,z) > \underline{c}$ *for all* $(a,z) \in S_t$.

(ii) For all $(a,z) \in S_t$, *the policy functions* $g_t(a,z)$ *and* $h_t(a,z)$ *satisfy the Euler equation*

$$u'[g_t(a,z)] \geq \beta(1+r) \int_{Z_{t+1}} u'\left[g_{t+1}\left(h_t(a,z), z'\right)\right] Q_t\left(z, dz'\right), \tag{6}$$

with equality holds if $h_t(a,z) > -\underline{a}_{t+1}$.

(iii) The consumption function $g_t(\cdot)$ *is a strictly increasing function.*

2.2 Changing the Curvature of Marginal Utility

We now examine how changes in the curvature of the marginal utility function would affect consumers' optimal consumption behavior. The same issue has been previously studied by Kimball (1990). Using a two-period model, Kimball shows that an increase in the convexity of the marginal utility function, as measured by the coefficient of absolute prudence $\Pi(c) \equiv -u'''(c)/u''(c)$, would lead to a reduction in current consumption and an increase in current savings. Theorem 3 below generalizes Kimball's result to a multi-period setting in the presence of borrowing constraints, without invoking the second and third derivatives of the utility function.

Let $u(\cdot)$ and $v(\cdot)$ be two utility functions defined on $\mathcal{D}$ that are once continuously differentiable, strictly increasing and strictly concave. Define the marginal utility functions $\phi(c) \equiv u'(c)$ and $\varphi(c) \equiv v'(c)$ on $\mathcal{D}$. Under the stated assumptions, both $\phi(\cdot)$ and $\varphi(\cdot)$ are continuous, strictly positive and strictly decreasing. In addition, the inverse functions $\phi^{-1}(\cdot)$ and $\varphi^{-1}(\cdot)$ are also continuous and strictly decreasing. A consumer with utility function $v(\cdot)$ is said to be (strictly) more prudent than one with $u(\cdot)$ if there exists an increasing and (strictly) convex function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$, with $\Upsilon(0) = 0$, such that

$$\varphi(c) \equiv \Upsilon[\phi(c)], \qquad \text{for all } c \in \mathcal{D}. \tag{7}$$

If both $u(\cdot)$ and $v(\cdot)$ are thrice differentiable, then this definition implies that $v(\cdot)$ has a greater coefficient of absolute prudence than $u(\cdot)$.

Suppose the conditions in Theorems 1 and 2 are satisfied. Then there exists a unique set of policy functions $\{g_t(a, z; u)\}_{t=0}^{T}$ for the consumer with $u(\cdot)$. Similarly, there exists a unique set of policy functions $\{g_t(a, z; v)\}_{t=0}^{T}$ for the consumer with $v(\cdot)$. Our next theorem states that, holding other things constant, a more prudent consumer would consume less (and hence save more) in every period than a less prudent one.

Theorem 3 *Let $u(\cdot)$ and $v(\cdot)$ be two utility functions defined on $\mathcal{D}$ that are once continuously differentiable, strictly increasing and strictly concave. Suppose $\beta(1 + r) \leq 1$. If $v(\cdot)$ is more prudent than $u(\cdot)$, then*

$$g_t(a, z; v) \leq g_t(a, z; u), \tag{8}$$

for all $(a, z) \in S_t$ and for all $t \in \{0, 1, ..., T\}$. If $v(\cdot)$ is strictly more prudent than $u(\cdot)$, then strict inequality holds in (8) for all $(a, z) \in S_t$ and for $t \in \{0, 1, ..., T-1\}$.

The interpretation of Theorem 3 depends crucially on the value of $\beta(1+r)$. When $\beta(1+r) < 1$, an increase in the convexity of the marginal utility function has two effects on consumption, namely intertemporal substitution effect and precautionary effect. The former is evident from the fact that Theorem 3 holds even in a deterministic environment.[20] When $\beta(1+r) = 1$, the curvature of the marginal utility function has no effect on consumption in a deterministic environment. In this case, the result in Theorem 3 is purely driven by the precautionary effect. The results in this section thus illustrate a close relationship between the convexity of the marginal utility function and the precautionary motive of saving. Specifically, a more convex marginal utility function implies a stronger precautionary motive.

3 Concavity of Consumption Function

3.1 Main Theorem

In this section, we provide a set of conditions on the utility function $u(\cdot)$ such that the consumption function at every stage of the life cycle exhibits concavity. From this point onwards, we will focus on utility functions that are thrice continuously differentiable, strictly increasing and strictly concave. The main results of this section are summarized in Theorem 5. These results cover two groups of utility functions: (i) quadratic utility functions, or those with $u'''(c) = 0$ throughout the domain $\mathcal{D}$, and (ii) utility functions with strictly positive third derivative throughout its domain. For the latter class of utility functions, an additional condition is needed in order to establish the desired result. It is shown that this additional condition is satisfied by a large class of utility functions, including (but is not limited to) all HARA utility functions with strictly positive third derivative.

Recall that $\phi(\cdot)$ is the marginal utility function, i.e., $\phi(c) \equiv u'(c)$ for $c \in \mathcal{D}$. If the utility function is thrice differentiable with nonzero third derivative, then we can define $\eta : \mathcal{D} \to \mathbb{R}$ according to

$$\eta(c) = \frac{\left[\phi'(c)\right]^2}{\phi''(c)} \equiv \frac{\left[u''(c)\right]^2}{u'''(c)}, \tag{9}$$

and $\Phi : \mathbb{R}_+ \to \mathbb{R}$ according to

$$\Phi(m) \equiv \eta\left[\phi^{-1}(m)\right]. \tag{10}$$

Both $\eta(\cdot)$ and $\Phi(\cdot)$ are strictly positive if $u'''(\cdot)$ is strictly positive. Within this group of utility functions, we confine our attention to those that satisfy the following assumption.

[20] See the proof of Theorem 3 for more discussion on this point.

Assumption A Let $N > 1$ be a positive integer. Let $\boldsymbol{\mu}$ be a discrete probability measure with masses $(\mu_1, ..., \mu_N)$ on a set of points $(\psi_1, ..., \psi_N) \in \mathbb{R}_+^N$. Then the function $\Phi(\cdot)$ defined by (9) and (10) satisfies

$$\Phi\left[\beta(1+r)\sum_{n=1}^{N}\mu_n\psi_n\right] \geq \beta(1+r)\sum_{n=1}^{N}\mu_n\Phi(\psi_n). \tag{11}$$

Before stating the main theorem, we first explain the implications of this assumption. We will proceed in two steps. First, we identify a specific class of functions $\Phi(\cdot)$ that are consistent with this assumption. Next, we identify utility functions that can generate these $\Phi(\cdot)$. We begin with two special cases. First, if $\Phi(m) \equiv bm$ for some strictly positive real number b, then the condition in (11) holds with equality for any $\beta(1+r) > 0$. This seemingly trivial example turns out to have great importance. In Section 3.3, it is shown that this simple form of $\Phi(\cdot)$ encompasses all HARA utility functions with strictly positive third derivative. Second, if $\beta(1+r) = 1$, then the inequality in (11) becomes Jensen's inequality. Thus, any concave function $\Phi(\cdot)$ is consistent with Assumption A. The following lemma extends this result to the general case where $\beta(1+r) \leq 1$, under the assumption that $u'''(\cdot)$ is strictly positive.[21] This result provides the basis for finding a broader class of utility functions that satisfy Assumption A. This will be explained more fully in Section 3.4.

Lemma 4 *Suppose $u'''(\cdot) > 0$ and $\beta(1+r) \leq 1$. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by (9) and (10). If $\Phi(\cdot)$ is concave, then Assumption A is satisfied.*

We are now ready to state the main results of this section. Building on our earlier findings, Theorem 5 states that the consumption functions $\{g_t(a, \widetilde{e}, \varepsilon)\}_{t=0}^T$ are concave in $(a, \widetilde{e})$ and in (a, ε) if the utility function $u(\cdot)$ belongs to either one of the following categories: (i) quadratic utility functions, or (ii) utility functions with strictly positive third derivative that satisfy Assumption A.

Theorem 5 *Suppose the conditions in Theorem 2 are satisfied. Suppose the utility function $u(\cdot)$ is thrice continuously differentiable, strictly increasing, strictly concave and satisfies one of the following conditions: (i) $u'''(\cdot) = 0$, or (ii) $u'''(\cdot) > 0$ and Assumption A. Then the following results hold for all $t \in \{0, 1, ..., T\}$.*

(i) For any $\varepsilon \in \Xi$, the consumption function $g_t(a, \widetilde{e}, \varepsilon)$ is concave in $(a, \widetilde{e})$.

(ii) For any $\widetilde{e} \in \Delta_t$, the consumption function $g_t(a, \widetilde{e}, \varepsilon)$ is concave in (a, ε).

[21] When $\beta(1+r) < 1$, the inequality in (11) is not Jensen's inequality. In particular, it is not satisfied by *all* concave functions $\Phi(\cdot)$, but only by those with $\Phi(0) \geq 0$. The additional requirement $\Phi(0) \geq 0$ is fulfilled if $u'''(\cdot) > 0$.

The proof of this theorem can be found in Section 3.2. Here we only provide a heuristic discussion on the main ideas of the proof. As stated in Theorem 2, the set of consumption functions is characterized by the Euler equation in (6). More specifically, using the consumption function in the terminal period as the starting point, one can derive the consumption function in all preceding periods recursively using the Euler equation. This essentially defines an operator which maps the consumption function at time $t+1$ to that at time t.[22] When the utility function is quadratic, the Euler equation is linear in both $g_t(\cdot)$ and $g_{t+1}(\cdot)$. This, together with the fact that consumption function in the terminal period is linear in $(a, \widetilde{e})$ and in (a, ε), implies that the consumption function in all preceding periods are (piecewise) linear in $(a, \widetilde{e})$ and in (a, ε).[23] For utility functions with strictly positive third derivative, Assumption A is sufficient to ensure that the Euler operator preserves concavity. In other words, if $g_{t+1}(\cdot)$ is concave in $(a, \widetilde{e})$ and in (a, ε), then $g_t(\cdot)$ is also concave in $(a, \widetilde{e})$ and in (a, ε).

Three additional remarks are in order. First, Theorem 5 states that the consumption function is jointly concave in current asset holdings and one of the income shocks, when the other is held constant. In particular, $g_t(a, \widetilde{e}, \varepsilon)$ is not jointly concave in $(a, \widetilde{e}, \varepsilon)$. This happens because the stochastic labor endowment $e \equiv \widetilde{e}\varepsilon$ is not jointly concave in $(\widetilde{e}, \varepsilon)$. Consequently, the consumption function in the terminal period is not jointly concave in $(a, \widetilde{e}, \varepsilon)$. It also means that the graph of the budget correspondence $\mathcal{B}_t(a, \widetilde{e}, \varepsilon) \equiv \left\{c : \underline{c} \leq c \leq w\widetilde{e}\varepsilon + (1+r)a + \underline{a}_{t+1}\right\}$ is not a convex set in any given period. Second, concavity in $(a, \widetilde{e})$ implies concavity in (a, ε), but the converse is not true in general. This point is explained more fully in Section 3.2. This result highlights the significance of distinguishing between permanent shock and purely transitory shock. Third, Theorem 5 only establishes the weak concavity of the consumption functions. In Section 3.5, we present a set of sufficient conditions under which the consumption functions are strictly concave in $(a, \widetilde{e})$ and in (a, ε).

3.2 Proof of Theorem 5

In this section, we focus on part (i) of the theorem. We will explain why this result implies the result in part (ii) but not vice versa. The main ideas of the proof are as follows. For any $\varepsilon \in \Xi$ and

[22] This is often referred to as the Euler operator. For a detailed characterization of this operator, see Deaton (1991) and Rabault (2002).

[23] If the borrowing constraint is never binding, then the consumption function in any period $t < T$ is linear in $(a, \widetilde{e})$ and in (a, ε) when the utility function is quadratic. If the borrowing constraint is binding in some states, then the consumption function is kinked and piecewise linear in these arguments.

$t \in \{0, 1, ..., T\}$, the function $g_t(a, \widetilde{e}, \varepsilon)$ is concave in $(a, \widetilde{e})$ if and only if its hypograph,

$$\mathcal{H}_t(\varepsilon) \equiv \{(c, a, \widetilde{e}) \in \mathcal{D} \times A_t \times \Delta_t : c \leq g_t(a, \widetilde{e}, \varepsilon)\},$$

is a convex set. The first step of the proof is to derive an alternate but equivalent expression for $\mathcal{H}_t(\varepsilon)$.[24] This alternate expression is favored because it is more tractable. For each $(a, \widetilde{e}, \varepsilon) \in S_t$, define the constraint set

$$\mathcal{B}_t(a, \widetilde{e}, \varepsilon) \equiv \left\{c : \underline{c} \leq c \leq x(a, \widetilde{e}, \varepsilon) + \underline{a}_{t+1}\right\}.$$

For each $\varepsilon \in \Xi$, define a set $\mathcal{G}_t(\varepsilon)$ such that (i) $\mathcal{G}_t(\varepsilon)$ is a subset of $\mathcal{D} \times A_t \times \Delta_t$, and (ii) any $(c, a, \widetilde{e})$ in $\mathcal{G}_t(\varepsilon)$ satisfies $c \in \mathcal{B}_t(a, \widetilde{e}, \varepsilon)$ and

$$\phi(c) \geq \beta(1+r) \int_{\Xi} \int_{\Lambda} \phi\left[g_{t+1}\left(x(a, \widetilde{e}, \varepsilon) - c, \widetilde{e}\nu', \varepsilon'\right)\right] dL_{t+1}\left(\nu'\right) dG_{t+1}\left(\varepsilon'\right), \tag{12}$$

where $\phi(\cdot)$ is the marginal utility function. We now show that $\mathcal{H}_t(\varepsilon)$ and $\mathcal{G}_t(\varepsilon)$ are equivalent. Fix $\varepsilon \in \Xi$. For any $(c, a, \widetilde{e}) \in \mathcal{H}_t(\varepsilon)$, it must be the case that $c \in \mathcal{B}_t(a, \widetilde{e}, \varepsilon)$ and $x(a, \widetilde{e}, \varepsilon) - c \geq x(a, \widetilde{e}, \varepsilon) - g_t(a, \widetilde{e}, \varepsilon)$. Since $\phi[g_{t+1}(a', \widetilde{e}', \varepsilon')]$ is strictly decreasing in a', we have

$$\phi\left[g_{t+1}\left(x(a, \widetilde{e}, \varepsilon) - g_t(a, \widetilde{e}, \varepsilon), \widetilde{e}\nu', \varepsilon'\right)\right] \geq \phi\left[g_{t+1}\left(x(a, \widetilde{e}, \varepsilon) - c, \widetilde{e}\nu', \varepsilon'\right)\right], \tag{13}$$

for all $(\nu', \varepsilon') \in \Lambda \times \Xi$. It follows that

$$\begin{aligned}
\phi(c) &\geq \phi\left[g_t(a, \widetilde{e}, \varepsilon)\right] \\
&\geq \beta(1+r) \int_{\Xi} \int_{\Lambda} \phi\left[g_{t+1}\left(x(a, \widetilde{e}, \varepsilon) - g_t(a, \widetilde{e}, \varepsilon), \widetilde{e}\nu', \varepsilon'\right)\right] dL_{t+1}\left(\nu'\right) dG_{t+1}\left(\varepsilon'\right) \\
&\geq \beta(1+r) \int_{\Xi} \int_{\Lambda} \phi\left[g_{t+1}\left(x(a, \widetilde{e}, \varepsilon) - c, \widetilde{e}\nu', \varepsilon'\right)\right] dL_{t+1}\left(\nu'\right) dG_{t+1}\left(\varepsilon'\right).
\end{aligned}$$

The second inequality uses the Euler equation and the third inequality follows from (13). This shows that $\mathcal{H}_t(\varepsilon) \subseteq \mathcal{G}_t(\varepsilon)$. Next, pick any $(c, a, \widetilde{e})$ in $\mathcal{G}_t(\varepsilon)$ and suppose the contrary that $c > g_t(a, \widetilde{e}, \varepsilon)$. If $g_t(a, \widetilde{e}, \varepsilon) = x(a, \widetilde{e}, \varepsilon) + \underline{a}_{t+1}$, then any feasible consumption must be no greater than $g_t(a, \widetilde{e}, \varepsilon)$ and hence there is a contradiction. Consider the case when $x(a, \widetilde{e}, \varepsilon) + \underline{a}_{t+1} \geq c > g_t(a, \widetilde{e}, \varepsilon)$. This has two implications: (i) $h_t(a, \widetilde{e}, \varepsilon) > -\underline{a}_{t+1}$, and (ii) $h_t(a, \widetilde{e}, \varepsilon) > x(a, \widetilde{e}, \varepsilon) - c$. The first inequality implies

[24] The same step is also used in the proof of Lemma 1 in Huggett (2004). It is, however, necessary to include the details in here because the model specifications in the two work are not identical.

that the Euler equation holds with equality under $g_t(a, \widetilde{e}, \varepsilon)$. Thus, we have

$$\begin{aligned}\phi(c) &< \phi[g_t(a, \widetilde{e}, \varepsilon)] = \beta(1+r) \int_{\Xi} \int_{\Lambda} \phi\left[g_{t+1}\left(h_t(a, \widetilde{e}, \varepsilon), \widetilde{e}\nu', \varepsilon'\right)\right] dL_{t+1}(\nu')\, dG_{t+1}(\varepsilon') \\ &< \beta(1+r) \int_{\Xi} \int_{\Lambda} \phi\left[g_{t+1}\left(x(a, \widetilde{e}, \varepsilon) - c, \widetilde{e}\nu', \varepsilon'\right)\right] dL_{t+1}(\nu')\, dG_{t+1}(\varepsilon').\end{aligned}$$

This means $(c, a, \widetilde{e}) \notin \mathcal{G}_t(\varepsilon)$ which gives rise to a contradiction. Hence, $\mathcal{G}_t(\varepsilon) \subseteq \mathcal{H}_t(\varepsilon)$. This establishes the equivalence between $\mathcal{H}_t(\varepsilon)$ and $\mathcal{G}_t(\varepsilon)$.

Since $\phi(\cdot)$ is strictly decreasing, the inequality in (12) is equivalent to

$$c \leq \phi^{-1}\left\{\beta(1+r) \int_{\Xi} \int_{\Lambda} \phi\left[g_{t+1}\left(x(a, \widetilde{e}, \varepsilon) - c, \widetilde{e}\nu', \varepsilon'\right)\right] dL_{t+1}(\nu')\, dG_{t+1}(\varepsilon')\right\}.$$

Define a function $\Psi_{t+1} : A_{t+1} \times \Delta_t \to \mathcal{D}$ according to

$$\Psi_{t+1}(a', \widetilde{e}) \equiv \phi^{-1}\left\{\beta(1+r) \int_{\Xi} \int_{\Lambda} \phi\left[g_{t+1}\left(a', \widetilde{e}\nu', \varepsilon'\right)\right] dL_{t+1}(\nu')\, dG_{t+1}(\varepsilon')\right\}. \tag{14}$$

Then the set $\mathcal{H}_t(\varepsilon)$ can be rewritten as

$$\mathcal{H}_t(\varepsilon) \equiv \left\{(c, a, \widetilde{e}) \in \mathcal{D} \times A_t \times \Delta_t : c \in \mathcal{B}_t(a, \widetilde{e}, \varepsilon) \text{ and } c \leq \Psi_{t+1}(x(a, \widetilde{e}, \varepsilon) - c, \widetilde{e})\right\}.$$

This set is convex if $\Psi_{t+1}(a', \widetilde{e})$ is jointly concave in $(a', \widetilde{e})$. To see this, pick any $(c_1, a_1, \widetilde{e}_1)$ and $(c_2, a_2, \widetilde{e}_2)$ in $\mathcal{H}_t(\varepsilon)$. Define $c_\delta \equiv \delta c_1 + (1-\delta) c_2$ for any $\delta \in [0, 1]$. Similarly define a_δ and $\widetilde{e}_\delta$. Since $\mathcal{D} \times A_t \times \Delta_t$ is a convex set, we have $(c_\delta, a_\delta, \widetilde{e}_\delta) \in \mathcal{D} \times A_t \times \Delta_t$. Also, we have $c_\delta \in \mathcal{B}_t(a_\delta, \widetilde{e}_\delta, \varepsilon)$. If $\Psi_{t+1}(\cdot)$ is concave, then

$$\begin{aligned}\Psi_{t+1}(x(a_\delta, \widetilde{e}_\delta, \varepsilon) - c_\delta, \widetilde{e}_\delta) &\geq \delta \Psi_{t+1}(x(a_1, \widetilde{e}_1, \varepsilon) - c_1, \widetilde{e}_1) + (1-\delta) \Psi_{t+1}(x(a_2, \widetilde{e}_2, \varepsilon) - c_2, \widetilde{e}_2) \\ &\geq \delta c_1 + (1-\delta) c_2 \equiv c_\delta.\end{aligned}$$

This means $(c_\delta, a_\delta, \widetilde{e}_\delta) \in \mathcal{H}_t(\varepsilon)$. Hence, if $\Psi_{t+1}(\cdot)$ is concave, then $g_t(a, \widetilde{e}, \varepsilon)$ is also concave in $(a, \widetilde{e})$. The converse, however, is not necessarily true.

To establish the result in part (ii), we first define the hypograph of $g_t(a, \widetilde{e}, \varepsilon)$ for a given $\widetilde{e} \in \Delta_t$. Using the same procedure, we can derive an alternate expression for this hypograph, which involves the same function $\Psi_{t+1}(a', \widetilde{e})$ as defined in (14). If $\Psi_{t+1}(a', \widetilde{e})$ is concave in a' for each given $\widetilde{e} \in \Delta_t$, then the consumption function is concave in (a, ε). Note that concavity of $\Psi_{t+1}(\cdot)$ implies concavity of $\Psi_{t+1}(\cdot, \widetilde{e})$ for a given $\widetilde{e}$, but the converse is not true in general. Hence, concavity in $(a, \widetilde{e})$ implies concavity in (a, ε), but not vice versa.

Case 1: Quadratic Utility

Suppose $u'''(c) = 0$ for all $c \in \mathcal{D}$. Then the marginal utility function can be expressed as $\phi(c) = \vartheta_1 + \vartheta_2 c$, with $\vartheta_2 < 0$ and $\vartheta_1 + \vartheta_2 \underline{c} > 0$. It follows that

$$\Psi_{t+1}\left(a', \widetilde{e}\right) \equiv \frac{\left[\beta(1+r) - 1\right]\vartheta_1}{\vartheta_2} + \beta(1+r)\int_{\Xi}\int_{\Lambda} g_{t+1}\left(a', \widetilde{e}\nu', \varepsilon'\right) dL_{t+1}\left(\nu'\right) dG_{t+1}\left(\varepsilon'\right).$$

Concavity of $\Psi_{t+1}(\cdot)$ follows immediately from an inductive argument. In the terminal period, the policy function is $g_T(a, \widetilde{e}, \varepsilon) \equiv w\widetilde{e}\varepsilon + (1+r)a$, which is linear in $(a, \widetilde{e})$ for all $\varepsilon \in \Xi$. Suppose $g_{t+1}\left(a', \widetilde{e}', \varepsilon'\right)$ is concave in $\left(a', \widetilde{e}'\right)$ for any given $\varepsilon' \in \Xi$. Since concavity is preserved under integration, it follows that $\Psi_{t+1}\left(a', \widetilde{e}\right)$ is also concave in $\left(a', \widetilde{e}\right)$. Hence $\mathcal{H}_t(\varepsilon)$ is a convex set and $g_t(a, \widetilde{e}, \varepsilon)$ is concave in $(a, \widetilde{e})$ for all $\varepsilon \in \Xi$.

Case 2: Utility with Strictly Positive Third Derivative

Suppose now $u'''(c) > 0$ for all $c \in \mathcal{D}$. Again we use an inductive argument to establish the concavity of $\Psi_{t+1}(\cdot)$. Suppose $g_{t+1}\left(a', \widetilde{e}', \varepsilon'\right)$ is concave in $\left(a', \widetilde{e}'\right)$ for all $\varepsilon' \in \Xi$ and for some $t+1 \leq T$. We first establish the concavity of $\Psi_{t+1}(\cdot)$ for the case when both $G_{t+1}(\cdot)$ and $L_{t+1}(\cdot)$ are discrete distributions defined on some finite point sets. We then extend this result to continuous distributions.

Suppose $L_{t+1}(\cdot)$ is a discrete distribution with positive masses over a set of real numbers $\{\overline{\nu}_1, ..., \overline{\nu}_J\}$, with $\overline{\nu}_j \in \Lambda$ for all j. Similarly, suppose $G_{t+1}(\cdot)$ is a discrete distribution with positive masses over a set of real numbers $\{\overline{\varepsilon}_1, ..., \overline{\varepsilon}_K\}$, with $\overline{\varepsilon}_k \in \Xi$ for all k. Both J and K are finite. Define the probability $\mathcal{P}_{t+1}(j, k) \equiv \Pr\left\{(\nu_{t+1}, \varepsilon_{t+1}) = (\overline{\nu}_j, \overline{\varepsilon}_k)\right\}$ for each pair $(\overline{\nu}_j, \overline{\varepsilon}_k)$. The function $\Psi_{t+1}(\cdot)$ defined in (14) is now given by

$$\Psi_{t+1}\left(a', \widetilde{e}\right) \equiv \phi^{-1}\left\{\beta(1+r)\sum_{j,k}\mathcal{P}_{t+1}(j, k)\,\phi\left[g_{t+1}\left(a', \widetilde{e}\overline{\nu}_j, \overline{\varepsilon}_k\right)\right]\right\}.$$

Pick any $\left(a'_1, \widetilde{e}_1\right)$ and $\left(a'_2, \widetilde{e}_2\right)$ in $A_{t+1} \times \Delta_t$. Define $a'_\delta \equiv \delta a'_1 + (1-\delta)a'_2$ for any $\delta \in (0, 1)$. Similarly define $\widetilde{e}_\delta$. Since $g_{t+1}(a', \widetilde{e}', \varepsilon')$ is concave in $(a', \widetilde{e}')$ and $\phi(\cdot)$ is strictly decreasing, we have

$$\phi\left[g_{t+1}\left(a'_\delta, \widetilde{e}_\delta \overline{\nu}_j, \overline{\varepsilon}_k\right)\right] \leq \phi\left[\delta g_{t+1}\left(a'_1, \widetilde{e}_1 \overline{\nu}_j, \overline{\varepsilon}_k\right) + (1-\delta) g_{t+1}\left(a'_2, \widetilde{e}_2 \overline{\nu}_j, \overline{\varepsilon}_k\right)\right],$$

for all possible $(\overline{\nu}_j, \overline{\varepsilon}_k)$. Taking the expectation over all possible $(\overline{\nu}_j, \overline{\varepsilon}_k)$ gives

$$\begin{aligned} & \beta(1+r)\sum_{j,k}\mathcal{P}_{t+1}(j,k)\,\phi\left[g_{t+1}\left(a'_\delta, \widetilde{e}_\delta \overline{\nu}_j, \overline{\varepsilon}_k\right)\right] \\ \leq\ & \beta(1+r)\sum_{j,k}\mathcal{P}_{t+1}(j,k)\,\phi\left[\delta g_{t+1}\left(a'_1, \widetilde{e}_1 \overline{\nu}_j, \overline{\varepsilon}_k\right) + (1-\delta) g_{t+1}\left(a'_2, \widetilde{e}_2 \overline{\nu}_j, \overline{\varepsilon}_k\right)\right]. \end{aligned}$$

Since the inverse function $\phi^{-1}(\cdot)$ is strictly decreasing, we can write

$$\begin{aligned} \Psi_{t+1}\left(a'_\delta, \widetilde{e}_\delta\right) &\equiv \phi^{-1}\left\{\beta(1+r)\sum_{j,k}\mathcal{P}_{t+1}(j,k)\,\phi\left[g_{t+1}\left(a'_\delta, \widetilde{e}_\delta \overline{\nu}_j, \overline{\varepsilon}_k\right)\right]\right\} \\ &\geq \phi^{-1}\left\{\beta(1+r)\sum_{j,k}\mathcal{P}_{t+1}(j,k)\,\phi\left[\delta g_{t+1}\left(a'_1, \widetilde{e}_1 \overline{\nu}_j, \overline{\varepsilon}_k\right) + (1-\delta) g_{t+1}\left(a'_2, \widetilde{e}_2 \overline{\nu}_j, \overline{\varepsilon}_k\right)\right]\right\}. \end{aligned}$$

To express this more succinctly, define an index $n \equiv (j-1)\times K + k$ for any pair (j,k). Set $N \equiv J \times K$. Using the index n, we can reduce the double summation to a single one. Define two sets of positive real numbers $\{x_n\}_{n=1}^N$ and $\{y_n\}_{n=1}^N$ according to $x_n \equiv g_{t+1}(a'_1, \widetilde{e}_1 \overline{\nu}_j, \overline{\varepsilon}_k)$ and $y_n \equiv g_{t+1}(a'_2, \widetilde{e}_2 \overline{\nu}_j, \overline{\varepsilon}_k)$ for all (j,k). With a slight abuse of notation, we will use $\mathcal{P}_{t+1}(n)$ to replace $\mathcal{P}_{t+1}(j,k)$. Then the above inequality can be expressed as

$$\Psi_{t+1}\left(a'_\delta, \widetilde{e}_\delta\right) \geq \phi^{-1}\left\{\beta(1+r)\sum_{n=1}^N \mathcal{P}_{t+1}(n)\,\phi\left[\delta x_n + (1-\delta) y_n\right]\right\}$$

The function $\Psi_{t+1}(\cdot)$ is concave if

$$\begin{aligned} & \phi^{-1}\left\{\beta(1+r)\sum_{n=1}^N \mathcal{P}_{t+1}(n)\,\phi\left[\delta x_n + (1-\delta) y_n\right]\right\} \\ \geq\ & \delta\Psi_{t+1}\left(a'_1, \widetilde{e}_1\right) + (1-\delta)\Psi_{t+1}\left(a'_2, \widetilde{e}_2\right) \\ \equiv\ & \delta\phi^{-1}\left\{\beta(1+r)\sum_{n=1}^N \mathcal{P}_{t+1}(n)\,\phi(x_n)\right\} + (1-\delta)\phi^{-1}\left\{\beta(1+r)\sum_{n=1}^N \mathcal{P}_{t+1}(n)\,\phi(y_n)\right\}. \end{aligned}$$

In other words, if the function $\theta : (\underline{c}, \infty)^N \to \mathcal{D}$ defined by

$$\theta(\mathbf{y}) \equiv \phi^{-1}\left\{\beta(1+r)\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\phi(y_n)\right\}, \tag{15}$$

is concave, then $\Psi_{t+1}(\cdot)$ is concave. To show that $\theta(\cdot)$ is a concave function, we use the same argument as in Hardy *et al.* (1952) p.85-88.

The function $\theta(\mathbf{y})$ is concave if and only if its Hessian matrix is negative semi-definite. Let $\mathbf{H}(\mathbf{y}) = [\mathbf{h}_{m,n}(\mathbf{y})]$ be the Hessian matrix of $\theta(\cdot)$ evaluated at a point $\mathbf{y}$. Then for any column vector $\boldsymbol{\varpi} \in \mathbb{R}^N$, $\boldsymbol{\varpi}^T \cdot \mathbf{H}(\mathbf{y})\,\boldsymbol{\varpi} \leq 0$ if and only if [25]

$$\frac{\{\phi'[\theta(\mathbf{y})]\}^2}{\phi''[\theta(\mathbf{y})]} \geq \beta(1+r)\frac{\left[\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\varpi_n\phi'(y_n)\right]^2}{\left[\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\varpi_n^2\phi''(y_n)\right]}. \tag{16}$$

Using the definitions in (9) and (10), we can rewrite the left-hand side of this inequality as

$$\begin{aligned}\frac{\{\phi'[\theta(\mathbf{y})]\}^2}{\phi''[\theta(\mathbf{y})]} &\equiv \eta[\theta(\mathbf{y})] = \eta\left[\phi^{-1}\left\{\beta(1+r)\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\phi(y_n)\right\}\right] \\ &= \Phi\left[\beta(1+r)\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\phi(y_n)\right].\end{aligned}$$

Using Assumption A, we can obtain

$$\begin{aligned}\frac{\{\phi'[\theta(\mathbf{y})]\}^2}{\phi''[\theta(\mathbf{y})]} &= \Phi\left[\beta(1+r)\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\phi(y_n)\right] \\ &\geq \beta(1+r)\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\Phi[\phi(y_n)] = \beta(1+r)\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\frac{[\phi'(y_n)]^2}{\phi''(y_n)}.\end{aligned} \tag{17}$$

Finally, we will show that (17) implies (16). Define two sets of real numbers $\{\mathbf{b}_n\}_{n=1}^N$ and $\{\mathbf{d}_n\}_{n=1}^N$ according to $\mathbf{b}_n \equiv \left[\mathcal{P}_{t+1}(n)\,\phi''(y_n)\right]^{\frac{1}{2}}\varpi_n$ and $\mathbf{d}_n \equiv \left\{\mathcal{P}_{t+1}(n)\left[\phi'(y_n)\right]^2/\phi''(y_n)\right\}^{\frac{1}{2}}$. By the Cauchy-

[25] The mathematical derivation of this expression can be found in Appendix B.

Schwartz inequality,

$$\begin{aligned}\left(\sum_{n=1}^{N}\mathbf{b}_n^2\right)\left(\sum_{n=1}^{N}\mathbf{d}_n^2\right) &= \left[\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\varpi_n^2\phi''(y_n)\right]\left[\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\frac{\left[\phi'(y_n)\right]^2}{\phi''(y_n)}\right] \\ &\geq \left(\sum_{n=1}^{N}\mathbf{b}_n\mathbf{d}_n\right)^2 \\ &= \left[\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\varpi_n\phi'(y_n)\right]^2.\end{aligned}$$

Since the marginal utility function is strictly convex, i.e., $\phi''(\cdot) > 0$, this yields

$$\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\frac{\left[\phi'(y_n)\right]^2}{\phi''(y_n)} \geq \frac{\left[\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\varpi_n\phi'(y_n)\right]^2}{\left[\sum_{n=1}^{N}\mathcal{P}_{t+1}(n)\,\varpi_n^2\phi''(y_n)\right]},$$

for any $\boldsymbol{\varpi} \in \mathbb{R}^N$. Hence (17) implies (16). This establishes the concavity of $\theta(\mathbf{y})$ which implies that $\Psi_{t+1}(\cdot)$ is concave. As a result, the hypograph of $g_t(a, \widetilde{e}, \varepsilon)$ is a convex set for each fixed $\varepsilon \in \Xi$. This proves the desired result for the case when both $G_{t+1}(\cdot)$ and $L_{t+1}(\cdot)$ are discrete distributions defined on some finite point sets.

Suppose now both $G_{t+1}(\cdot)$ and $L_{t+1}(\cdot)$ are continuous distributions defined on the compact intervals $\Xi \equiv [\underline{\varepsilon}, \overline{\varepsilon}]$ and $\Lambda \equiv [\underline{\nu}, \overline{\nu}]$, respectively. Fix $(a', \widetilde{e}) \in A_{t+1} \times \Delta_t$. Let J and K be two positive integers. Let $\{\overline{\nu}_0, ..., \overline{\nu}_J\}$ be an arbitrary partition of Λ so that $\underline{\nu} = \overline{\nu}_0 \leq ... \leq \overline{\nu}_J = \overline{\nu}$. Define a set of real numbers $\{p_1, ..., p_J\}$ according to $p_j \equiv L_{t+1}(\overline{\nu}_j) - L_{t+1}(\overline{\nu}_{j-1})$ for each $j \geq 1$, and a step function

$$\widetilde{L}_J(\nu') \equiv \sum_{j=1}^{J}\chi_j(\nu')\,L_{t+1}(\overline{\nu}_{j-1}),$$

where $\chi_j(\nu')$ equals one if $\nu' \in [\overline{\nu}_{j-1}, \overline{\nu}_j]$ and zero otherwise. This step function converges pointwise to $L_{t+1}(\cdot)$ when J is sufficiently large. Similarly, let $\{\overline{\varepsilon}_0, ..., \overline{\varepsilon}_K\}$ be an arbitrary partition of Ξ so that $\underline{\varepsilon} = \overline{\varepsilon}_0 \leq ... \leq \overline{\varepsilon}_K = \overline{\varepsilon}$. Define a set of positive real numbers $\{q_1, ..., q_K\}$ so that $q_k \equiv G_{t+1}(\overline{\varepsilon}_k) - G_{t+1}(\overline{\varepsilon}_{k-1})$ for each $k \geq 1$. Define the step function

$$\widetilde{G}_K(\varepsilon') \equiv \sum_{k=1}^{K}\widetilde{\chi}_k(\varepsilon')\,G_{t+1}(\overline{\varepsilon}_{k-1}),$$

where $\widetilde{\chi}_k(\varepsilon')$ equals one if $\varepsilon' \in [\overline{\varepsilon}_{k-1}, \overline{\varepsilon}_k]$ and zero otherwise. This step function converges pointwise to

$G_{t+1}(\cdot)$ when K is sufficiently large. These conditions are sufficient to ensure that

$$\sum_{j,k} p_j q_k \phi \left[g_{t+1} \left(a', \widetilde{e}\overline{\nu}_j, \overline{\varepsilon}_k \right) \right] \to \int_{\Xi} \int_{\Lambda} \phi \left[g_{t+1} \left(a', \widetilde{e}\nu', \varepsilon' \right) \right] dL_{t+1} \left(\nu' \right) dG_{t+1} \left(\varepsilon' \right),$$

for any given $(a', \widetilde{e}) \in A_{t+1} \times \Delta_t$, when J and K are sufficiently large. Set $N = J \times K$ and define a function $\Psi_{t+1}^N (a', \widetilde{e})$ according to

$$\Psi_{t+1}^N \left(a', \widetilde{e} \right) \equiv \phi^{-1} \left\{ \beta (1+r) \sum_{j,k} p_j q_k \phi \left[g_{t+1} \left(a', \widetilde{e}\overline{\nu}_j, \overline{\varepsilon}_k \right) \right] \right\}.$$

Our earlier result shows that $\Psi_{t+1}^N (a', \widetilde{e})$ is jointly concave in its arguments for any positive integer N. By the continuity of $\phi^{-1}(\cdot)$, $\Psi_{t+1}^N(\cdot)$ converges to the function in (14) pointwise. Hence, $\left\{ \Psi_{t+1}^N (\cdot) \right\}$ forms a sequence of finite concave function on $A_{t+1} \times \Delta_t$ that converges pointwise to $\Psi_{t+1}(\cdot)$. By Theorem 10.8 in Rockafellar (1970), the limiting function $\Psi_{t+1}(\cdot)$ is also a concave function on $A_{t+1} \times \Delta_t$. This completes the proof of Theorem 5.

3.3 HARA Utility Functions

We now show that the conditions in Theorem 5 are satisfied by the utility functions considered in Carroll and Kimball (1996). To begin with, a twice continuously differentiable utility function $u : \mathcal{D} \to \mathbb{R}$ is called a HARA utility function if there exists $(\alpha, \gamma) \in \mathbb{R}^2$ such that $\alpha + \gamma c \geq 0$ and

$$-\frac{u''(c)}{u'(c)} = \frac{1}{\alpha + \gamma c}, \qquad \text{for all } c \in \mathcal{D}. \tag{18}$$

The reciprocal of the absolute risk aversion is often referred to as the absolute risk tolerance. Thus, all HARA utility functions exhibit linear absolute risk tolerance. The above definition also implies that all HARA utility functions are at least thrice continuously differentiable in the interior of its domain. The HARA class of utility functions encompasses a number of commonly used utility functions. For instance, the CARA or exponential utility functions correspond to the case when $\alpha > 0$ and $\gamma = 0$. The standard CRRA utility functions correspond to the case when $\underline{c} = 0$, $\alpha = 0$ and $\gamma > 0$. The more general Stone-Geary utility functions $u(c) = (c - \underline{c})^{1-1/\gamma} / (1 - 1/\gamma)$ correspond to the case when

$\underline{c} > 0$, $\alpha = -\underline{c}\gamma$, and $\gamma > 0$. Finally, quadratic utility functions of the form

$$u(c) = \vartheta_0 + \vartheta_1 (c - \underline{c}) + \vartheta_2 (c - \underline{c})^2, \qquad \text{with } \vartheta_2 < 0,$$

correspond to the case when $\alpha = \underline{c} - \vartheta_1/\vartheta_2 > 0$ and $\gamma = -1$. Except for the quadratic utility functions, all the HARA utility functions mentioned above have strictly positive third derivative. However, not all of them satisfy the Inada condition $u'(\underline{c}+) = +\infty$. For instance, $u'(\underline{c}+)$ is finite under the CARA case and the quadratic case.

An alternative characterization of the HARA utility functions can be obtained by differentiating (18) with respect to c, which yields

$$\frac{u'(c)\, u'''(c)}{[u''(c)]^2} = 1 + \gamma, \qquad \text{for all } c \in \mathcal{D}. \tag{19}$$

This, together with (18), implies that the *inverse* of absolute prudence for HARA utility functions is given by

$$I(c) \equiv -\frac{u''(c)}{u'''(c)} = \frac{\alpha + \gamma c}{1 + \gamma}, \qquad \text{for } \gamma \neq -1.$$

Thus, the inverse of absolute prudence for HARA utility functions is again a linear function.

Carroll and Kimball (1996) consider the subclass of HARA utility functions with $\gamma \geq -1$, which implies a nonnegative $u'''(\cdot)$. When $\gamma > -1$, equation (19) implies

$$\eta(c) = \frac{[u''(c)]^2}{u'''(c)} = \frac{u'(c)}{1+\gamma} = \frac{\phi(c)}{1+\gamma},$$

$$\Rightarrow \Phi(m) = \frac{m}{1+\gamma}.$$

In other words, the subclass of HARA utility functions with $\gamma > -1$ corresponds to the case when $\Phi(m) \equiv bm$ for some $b > 0$. *Hence, all HARA utility functions with $\gamma > -1$ satisfy Assumption A whenever $\beta(1+r) > 0$.*

The following corollary summarizes what we have learned about the consumption functions when the utility function is of the HARA class. These results generalize Huggett (2004) Lemma 1 in two ways. First, Huggett only considers serially independent labor income shocks, while we consider both permanent and purely transitory labor income shocks. Second, Huggett proves that the consumption functions are strictly increasing and concave in two particular cases: (i) when the utility function exhibits CRRA [hence $u'(0+) = +\infty$], and (ii) when the utility function exhibits CARA [hence $u'(0+) < +\infty$] and $\beta(1+r) \leq 1$. The following corollary generalizes the first case to any HARA utility functions with $\gamma \geq -1$ and $u'(\underline{c}+) = +\infty$. It also generalizes the second case to any HARA utility functions with $\gamma \geq -1$ and $u'(\underline{c}+) < \infty$, and $\beta(1+r) \leq 1$.

Corollary 6 *Suppose the utility function $u(\cdot)$ is of the HARA class with $\gamma \geq -1$. Suppose either $u'(\underline{c}+) = +\infty$ or $\beta(1+r) \leq 1$. Then the following results hold for all $t \in \{0, 1, ..., T\}$.*

(i) For any $\varepsilon \in \Xi$, the consumption function $g_t(a, \widetilde{e}, \varepsilon)$ is strictly increasing and concave in $(a, \widetilde{e})$.

(ii) For any $\widetilde{e} \in \Delta_t$, the consumption function $g_t(a, \widetilde{e}, \varepsilon)$ is strictly increasing and concave in (a, ε).

3.4 General Utility Functions

In Section 3.3, it is shown that all HARA utility functions with strictly positive third derivative can be captured by the simple linear form $\Phi(m) \equiv bm$. According to Lemma 4, this is only one particular form of $\Phi(\cdot)$ that satisfies Assumption A when $\beta(1+r) \leq 1$. This suggests that Assumption A is consistent with a more general class of utility functions which includes all HARA utility functions with strictly positive third derivative as special cases. The main objective of this subsection is to identify this class of utility functions.

Suppose the utility function $u(\cdot)$ is sufficiently smooth so that the function $\Phi(\cdot)$ is twice differentiable. Then it is straightforward to show that

$$\Phi'(m) = 1 - \frac{d}{dc}\left[-\frac{u''(c)}{u'''(c)}\right] \qquad \text{and} \qquad \Phi''(m) = -\frac{1}{u''(c)}\frac{d^2}{dc^2}\left[-\frac{u''(c)}{u'''(c)}\right],$$

for $m = u'(c)$ and for all $c \in \mathcal{D}$. Hence, $\Phi(\cdot)$ is (strictly) concave if and only if the inverse of absolute prudence $I(\cdot)$ is (strictly) concave. This, combined with Lemma 4, leads to the following result.

Lemma 7 *Suppose $u'''(\cdot) > 0$ and $\beta(1+r) \leq 1$. Suppose the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by (9) and (10) is twice differentiable. Then Assumption A is satisfied if the inverse of absolute prudence $I(c) \equiv -u''(c)/u'''(c)$ is a concave function on $\mathcal{D}$.*

Recall that the inverse of absolute prudence for all HARA utility functions with $\gamma > -1$ is a linear function. Hence, the above lemma also applies to these functions. Theorem 5 and Lemma 7 together imply that, given $\beta(1+r) \leq 1$, the consumption function at every stage of the life cycle exhibits concavity when the utility function has strictly positive third derivative and a globally concave $I(\cdot)$. This result is summarized in Theorem 8.

Theorem 8 *Suppose* $u'''(\cdot) > 0$ *and* $\beta(1+r) \leq 1$. *Suppose the inverse of absolute prudence* $I(\cdot) \equiv -u''(\cdot)/u'''(\cdot)$ *is a concave function. Then the following results hold for all* $t \in \{0, 1, ..., T\}$.

(i) For any $\varepsilon \in \Xi$, *the consumption function* $g_t(a, \widetilde{e}, \varepsilon)$ *is strictly increasing and concave in* $(a, \widetilde{e})$.

(ii) For any $\widetilde{e} \in \Delta_t$, *the consumption function* $g_t(a, \widetilde{e}, \varepsilon)$ *is strictly increasing and concave in* (a, ε).

Theorem 8 generalizes the concavity result in Carroll and Kimball (1996, 2005) and Huggett (2004) to a more general class of utility functions. It also complements the findings in Huggett and Vidon (2002). Using specific numerical examples, these authors show that a strictly positive $u'''(\cdot)$ alone is not enough to generate convex savings functions (or equivalently, concave consumption functions) in a multi-period setting. Huggett and Vidon, however, do not specify the additional conditions needed to generate concave consumption functions. According to our Theorem 8, the additional condition needed is the concavity of $I(\cdot) \equiv -u''(\cdot)/u'''(\cdot)$.

3.5 Strict Concavity of Consumption Function

In this subsection, we focus on HARA utility functions with $\gamma > -1$ and establish strict concavity of the consumption functions.[26] Before proceeding further, we first recall some established results. First, the consumption function in the terminal period is always linear in $(a, \widetilde{e})$ and in (a, ε). Second, the consumption function in any other period is linear in $(a, \widetilde{e})$ and in (a, ε) when the borrowing constraint is binding. Third, when the utility function is quadratic (i.e., $\gamma = -1$), the consumption function is always (piecewise) linear in $(a, \widetilde{e})$ and in (a, ε). Fourth, when the utility function is exponential (i.e., $\gamma = 0$), the consumption function is always (piecewise) linear in (a, ε).[27] Fifth, the consumption function in any period is (piecewise) linear in $(a, \widetilde{e})$ and in (a, ε) when there is no uncertainty in labor

[26] We only consider HARA utility functions in here for the following reason. One limitation of the proof in Section 3.2 is that even if $\{\Psi_{t+1}^N(\cdot)\}$ forms a convergent sequence of *strictly* concave functions, the limiting function $\Psi_{t+1}(\cdot)$ needs not be strictly concave. Strict concavity of $\Psi_{t+1}(\cdot)$, however, is important in establishing the strict concavity of $g_t(\cdot)$ when the borrowing constraint is not binding. This is not an issue for HARA utility functions because we can directly apply Minkowski's inequality for integrals to establish the strict concavity of $\Psi_{t+1}(\cdot)$.

[27] Caballero (1991) and Binder *et al.* (2000) show that when the stochastic labor endowment follows an arithmetic random walk process (i.e., $e_t = e_{t-1} + \varepsilon_t$, where ε_t is a white noise), then the consumption function is linear in a_t and e_t. Weil (1993) shows that this result holds under the more general Krep-Porteus preferences with constant elasticity of intertemporal subsitution and constant absolute risk aversion (i.e., exponential risk preferences). Under the current specification of e_t, the consumption function is only linear in a_t and ε_t, but not in $\widetilde{e}_t$, when the utility function is exponential. Details of this are available from the author upon request.

endowment (which happens when both $L_t(\cdot)$ and $G_t(\cdot)$ are degenerate distributions in every period). In light of these observations, we seek conditions under which the consumption function in any period $t < T$ is strictly concave in $(a, \widetilde{e})$ and in (a, ε) when the borrowing constraint is not binding and the labor endowment is stochastic. The results are summarized in Propositions 9 and 10.

First, we consider the strict concavity of the consumption function in $(a, \widetilde{e})$. For any $\varepsilon \in \Xi$, define the set $\mathcal{A}_t(\varepsilon)$ according to

$$\mathcal{A}_t(\varepsilon) \equiv \left\{(a, \widetilde{e}) \in A_t \times \Delta_t : h_t(a, \widetilde{e}, \varepsilon) > -\underline{a}_{t+1}\right\}.$$

Given that the realization of the transitory shock is ε, the set $\mathcal{A}_t(\varepsilon)$ contains all combinations of $(a, \widetilde{e})$ under which the borrowing constraint is not binding at time t. If $\mathcal{A}_t(\varepsilon)$ is empty for all $\varepsilon \in \Xi$, then the consumption function is linear in $(a, \widetilde{e})$ and in (a, ε). Thus, we focus on the case in which $\mathcal{A}_t(\varepsilon)$ is not empty for some $\varepsilon \in \Xi$. Proposition 9 states that, for any given $\varepsilon \in \Xi$, the consumption function $g_t(a, \widetilde{e}, \varepsilon)$ is strictly concave over $\mathcal{A}_t(\varepsilon)$ if the utility function is a HARA utility function with $\gamma > -1$ and $\gamma \neq 0$.

Proposition 9 *Suppose the utility function $u(\cdot)$ is of the HARA class with $\gamma > -1$ and $\gamma \neq 0$. Suppose either $u'(\underline{c}+) = +\infty$ or $\beta(1+r) \leq 1$. Suppose both $L_t(\cdot)$ and $G_t(\cdot)$ are non-degenerate in every period t. If $\mathcal{A}_t(\varepsilon)$ is a non-empty convex set for some $\varepsilon \in \Xi$, then $g_t(a, \widetilde{e}, \varepsilon)$ is strictly concave over $\mathcal{A}_t(\varepsilon)$.*

One implication of Proposition 9 is that, when the borrowing constraint is not binding, the marginal propensity to consume out of permanent shocks is less than unity and is strictly decreasing in the level of $\widetilde{e}$. This result is consistent with the numerical results in Carroll (2009). On the contrary, empirical studies often assume that unanticipated permanent earnings shock will induce an one-for-one adjustment in consumption.[28] Proposition 9 shows that this assumption is inconsistent with a standard life-cyle model that features HARA utility function.

Strict concavity in (a, ε) can be established in a parallel fashion. For any $\widetilde{e} \in \Delta_t$, define the set $\widetilde{\mathcal{A}}_t(\widetilde{e})$ according to

$$\widetilde{\mathcal{A}}_t(\widetilde{e}) \equiv \left\{(a, \varepsilon) \in A_t \times \Xi : h_t(a, \widetilde{e}, \varepsilon) > -\underline{a}_{t+1}\right\}.$$

Again, we focus on the case in which $\widetilde{\mathcal{A}}_t(\widetilde{e})$ is not empty for some $\widetilde{e} \in \Delta_t$. The strict concavity result is summarized in Proposition 10, which is a direct analogue of Proposition 9.

[28] See, for instance, Blundell and Preston (1998) and Primiceri and Van Rens (2009).

Proposition 10 *Suppose the utility function $u(\cdot)$ is of the HARA class with $\gamma > -1$ and $\gamma \neq 0$. Suppose either $u'(\underline{c}+) = +\infty$ or $\beta(1+r) \leq 1$. Suppose both $L_t(\cdot)$ and $G_t(\cdot)$ are non-degenerate in every period t. If $\widetilde{\mathcal{A}}_t(\widetilde{e})$ is a non-empty convex set for some $\widetilde{e} \in \Delta_t$, then $g_t(a, \widetilde{e}, \varepsilon)$ is strictly concave over $\widetilde{\mathcal{A}}_t(\widetilde{e})$.*

4 Precautionary Wealth Accumulation

We now explore the implications of concave consumption function on aggregate wealth accumulation. Recall the model economy described in Section 2. All consumers in this economy share the same set of consumption functions and savings functions $\{g_t(s), h_t(s)\}_{t=0}^T$, where $s = (a, \widetilde{e}, \varepsilon)$ is a set of individual state variables. The joint distribution of individual state is captured by a set of probability measures $\{\pi_t(\cdot)\}_{t=0}^T$, where $\pi_t : S_t \to [0,1]$ for all t. In particular, $\pi_t(s)$ represents the share of age-t consumers whose current state is s. Since all consumers share the same level of initial asset a_0 and the same initial value $\widetilde{e}_0$, the probability measure $\pi_0(\cdot)$ is completely determined by the distribution $G_0(\cdot)$. The probability measure in all subsequent ages are defined recursively according to

$$\pi_{t+1}(B) = \int_{S_t} P_t(s, B)\, \pi_t(ds), \tag{20}$$

for any Borel set $B \subseteq S_{t+1}$. The stochastic kernel $P_t(s, B)$ is defined as

$$P_t(s, B) \equiv \Pr\{(\nu_{t+1}, \varepsilon_{t+1}) : (h_t(s), \widetilde{e}\nu_{t+1}, \varepsilon_{t+1}) \in B\}. \tag{21}$$

For each age group t, the economy-wide average level of wealth is determined by

$$\mathcal{W}_t \equiv \int_{S_t} h_t(s)\, \pi_t(ds). \tag{22}$$

The sequence $\{\mathcal{W}_t\}_{t=0}^T$ then forms the average life-cycle profile of wealth under a given set of prices.

The main objective of this section is to examine how changes in the *riskiness* of the permanent and transitory income shocks would affect the average life-cycle profile of wealth when prices are held constant. Throughout this section, we use the following criterion to compare the riskiness of two sets of distributions. Let $\mathbf{L}_1 \equiv \{L_{1,t}(\cdot)\}_{t=0}^T$ and $\mathbf{L}_2 \equiv \{L_{2,t}(\cdot)\}_{t=0}^T$ denote two sets of distribution functions for the permanent earnings shocks $\{\nu_t\}$. These two sets of distribution functions are defined on the same compact interval $\Lambda \equiv [\underline{\nu}, \overline{\nu}]$. The distributions in $\mathbf{L}_1$ are said to be *more risky* than those in $\mathbf{L}_2$ if the inequality below holds for all $t \in \{0, 1, ..., T\}$ and for all concave function $f : \Lambda \to \mathbb{R}$,

$$\int_\Lambda f(\nu)\, dL_{1,t}(\nu) \leq \int_\Lambda f(\nu)\, dL_{2,t}(\nu),$$

provided that the integrals exist. As shown in Rothschild and Stiglitz (1970), this definition is equivalent to saying that each $L_{1,t}(\cdot)$ is a mean-preserving spread of $L_{2,t}(\cdot)$. It also means that the variance of $L_{1,t}(\cdot)$ is no less than that of $L_{2,t}(\cdot)$ in every period t. The same criterion is also used to compare the riskiness of any two sets of distributions for the transitory earnings shocks $\{\varepsilon_t\}$.

We first consider a change in the riskiness of the permanent shocks. The results are summarized in Theorem 11. Let $h_{j,t}(s)$ be the savings function at time t obtained under the distributions $\mathbf{L}_j$, for $j \in \{1, 2\}$. Using (20)-(22), define the probability measure $\pi_{j,t}(\cdot)$, the stochastic kernel $P_{j,t}(\cdot)$, and the average level of wealth $\mathcal{W}_{j,t}$ for every period $t \in \{0, 1, ..., T\}$ and for each $j \in \{1, 2\}$. The first part of Theorem 11 states that when consumption function is concave, an increase in the riskiness of the permanent earnings shocks would induce all consumers to accumulate more wealth. Since all other things (including prices) are being held constant, the increase in individual wealth accumulation is a manifestation of the precautionary motive. Intuitively, precautionary saving behavior emerges when the increase in future risks raises the expected marginal utility of future consumption. In a two-period model, future consumption is linear in $(a, \widetilde{e})$. Thus, a strictly convex marginal utility function [i.e., $u'''(\cdot) > 0$] is both necessary and sufficient to ensure that an increase in future risks raises the expected marginal utility of future consumption. In a general multi-period model, future consumption is not linear in $(a, \widetilde{e})$ in general. Thus, additional conditions are needed to ensure that precautionary saving behavior occurs. According to Theorem 11, the additional condition needed is Assumption A.

The second part of Theorem 11 states that a mean-preserving spread of the permanent earnings shocks would raise the expected value of any increasing convex transformation $\Gamma(\cdot)$ of the savings function. This result can be obtained because the function $h_t(a, \widetilde{e}, \varepsilon)$ is convex in $(a, \widetilde{e})$ and convexity is preserved by any increasing convex transformation. Since $\Gamma(x) = x$ is an increasing convex transformation, it follows that an increase in the riskiness of the permanent shocks would raise the average level of wealth at each stage of the life cycle. This result is stated in part (iii) of the theorem.

Theorem 11 *Suppose the conditions in Theorem 5 hold. Suppose the distributions in* $\mathbf{L}_1 \equiv \{L_{1,t}(\cdot)\}_{t=0}^T$ *are more risky than those in* $\mathbf{L}_2 \equiv \{L_{2,t}(\cdot)\}_{t=0}^T$. *Then the following results hold for all* $t \in \{0, 1, ..., T\}$.

(i) Holding other things constant, an increase in the riskiness of the permanent earnings shocks would raise the level of asset holdings for all individuals, i.e., $h_{1,t}(s) \geq h_{2,t}(s)$*, for all* $s \in S_t$*.*

(ii) For every continuous, increasing and convex function $\Gamma : A_{t+1} \to \mathbb{R}$*, we have*

$$\int_{S_t} \Gamma[h_{1,t}(s)] \pi_{1,t}(ds) \geq \int_{S_t} \Gamma[h_{2,t}(s)] \pi_{2,t}(ds).$$

(iii) Holding other things constant, an increase in the riskiness of the permanent earnings shocks would raise the average level of wealth, $\mathcal{W}_{1,t} \geq \mathcal{W}_{2,t}$*.*

In general, the transformation $\Gamma(x) = x^n$ is increasing and convex for any $n \in \{1, 2, ...\}$. Thus, another implication of part (ii) of Theorem 11 is that, holding other things constant, an increase in the riskiness of the permanent shocks would raise all the moments of individual savings.[29] However, we are unable to derive the same result for central moments, which are more appropriate for measuring the dispersion and skewness of the wealth distribution.

Theorem 12 summarizes the results pertaining to an increase in the riskiness of the transitory earnings shocks. As one might expect, the results and their proof are parallel to those in Theorem 11. Hence, the proof is omitted.

Theorem 12 *Suppose the conditions in Theorem 5 hold. Suppose the distributions in* $\mathbf{G}_1 \equiv \{G_{1,t}(\cdot)\}_{t=0}^{T}$ *are more risky than those in* $\mathbf{G}_2 \equiv \{G_{2,t}(\cdot)\}_{t=0}^{T}$*. Then the following results hold for all* $t \in \{0, 1, ..., T\}$*.*

(i) Holding other things constant, an increase in the riskiness of the transitory earnings shocks would raise the level of asset holdings for all individuals, i.e., $h_{1,t}(s) \geq h_{2,t}(s)$*, for all* $s \in S_t$*.*

(ii) For every continuous, increasing and convex function $\Gamma : A_{t+1} \to \mathbb{R}$*, we have*

$$\int_{S_t} \Gamma[h_{1,t}(s)] \pi_{1,t}(ds) \geq \int_{S_t} \Gamma[h_{2,t}(s)] \pi_{2,t}(ds).$$

(iii) Holding other things constant, an increase in the riskiness of the transitory earnings shocks would raise the average level of wealth, i.e., $\mathcal{W}_{1,t} \geq \mathcal{W}_{2,t}$*.*

[29] An alternative way to see this is that, for any $n \in \{1, 2, ...\}$, the moment-generating function $\mathcal{M}(n) \equiv \int_{S_t} \exp[nh_t(s)] \pi_t(ds)$ is the expected value of an increasing convex transformation of the savings function.

5 Numerical Examples

In this section, we provide a set of numerical examples in which the utility function has strictly positive third derivative, but the inverse of absolute prudence $I(c) \equiv -u''(c)/u'''(c)$ is not a globally concave function. Under certain parameter values, the consumption function in certain period is not globally concave in either $(a, \widetilde{e})$ or (a, ε). These examples thus illustrate the importance of the concavity of $I(\cdot)$ in Theorem 5. Using the non-concave consumption functions, we then construct examples in which a mean-preserving spread in the transitory earnings shock would lead to a reduction in aggregate savings. These results illustrate the importance of the concavity of $I(\cdot)$ in Theorem 12.

Suppose now the consumers live only three periods, i.e., $T = 2$. Each consumer solves the optimization problem described in Section 2. The borrowing limits $\underline{a}_t$ are equal to zero in all three periods. In both the second and third periods, the permanent shock ν_t is drawn from a finite set $\{\overline{\nu}_1, ..., \overline{\nu}_J\}$ with probabilities $\{p_1, ..., p_J\}$. In the existing literature, it is typical to assume that ν_t is i.i.d. over time and has a lognormal distribution with mean zero and variance σ_ν^2. Thus, we choose the elements in $\{\overline{\nu}_1, ..., \overline{\nu}_J\}$ and $\{p_1, ..., p_J\}$ so as to approximate such a distribution. First, we truncate a lognormal distribution with mean zero and variance σ_ν^2 by discarding the top 0.5% and the bottom 0.5%. Then, we divide the restricted domain into J evenly-spaced intervals. The probability p_j is the probability of drawing ν_t from the jth interval, and $\overline{\nu}_j$ is the mid-point of that interval. We set $J = 50$. The value of σ_ν^2 is chosen so that the variance of $\ln \nu_t$ from the discrete distribution is 0.0212, which is consistent with the estimate obtained by Gourinchas and Parker (2002). After ν_t is drawn, the permanent component $\widetilde{e}_t$ is updated according to $\widetilde{e}_t = \widetilde{e}_{t-1}\nu_t$, with $\widetilde{e}_0 = 1$.

As for the transitory earnings shock, we assume that ε_t take only two possible values, $\{1 - \varkappa_t, 1 + \varkappa_t\}$, with equal probability in all three periods. We adopt this specification because a mean-preserving spread in ε_t can be obtained simply by increasing $\varkappa_t$. In the benchmark example, we set $\varkappa_t = 0.1$ for all t. To examine the effects of a mean-preserving spread in the transitory shock, we increase the value of $\varkappa_0$, while maintaining $\varkappa_1 = \varkappa_2 = 0.1$.

The main component of this exercise is the utility function $u(\cdot)$, which is assumed to take the form

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} + \frac{c^{1-\theta}}{1-\theta}, \qquad \text{with } \sigma > 0 \text{ and } \theta > 0. \tag{23}$$

This function is thrice continuously differentiable and has strictly positive third derivative. Figure 1 shows the inverse of absolute prudence $I(\cdot)$ implied by this utility function when $\sigma = 10$ and $\theta = 0.1$. The function $I(\cdot)$ is convex when the values of consumption are small and concave when the values of consumption are large. Similar pattern can be obtained for a wide range of values of σ and θ. Huggett and Vidon (2002) consider the sum of two exponential utility functions, i.e.,

$$u(c) = -\frac{1}{\sigma}\exp(-\sigma c) - \frac{1}{\theta}\exp(-\theta c), \qquad \text{with } \sigma > 0 \text{ and } \theta > 0, \tag{24}$$

which also yields a convex-concave form of $I(\cdot)$ when the difference between σ and θ is large. These two specifications, however, have very different implications for the relative risk aversion. For the one

Table 1: Results on Asset Holdings

	$\varkappa_0 = 0.10$	$\varkappa_0 = 0.25$	$\varkappa_0 = 0.50$	$\varkappa_0 = 0.75$
Optimal Savings at $t = 0$				
$h_0(a_0, \widetilde{e}_0, 1 - \varkappa_0)$	1.5222	1.5039	1.4683	1.4250
$h_0(a_0, \widetilde{e}_0, 1 + \varkappa_0)$	1.5440	1.5588	1.5818	1.6035
$\mathcal{W}_1$	1.5331	1.5314	1.5250	1.5142
Optimal Savings at $t = 1$				
$E(a_2 \mid \varepsilon_0 = 1 - \varkappa_0)$	0.7249	0.7192	0.7076	0.6928
$E(a_2 \mid \varepsilon_0 = 1 + \varkappa_0)$	0.7315	0.7359	0.7425	0.7485
$\mathcal{W}_2$	0.7282	0.7275	0.7251	0.7207

in (23), the coefficient of relative risk aversion decreases monotonically from 10 to 0.1 as c increases. For the utility function in (24), the relative risk aversion is a non-monotonic function in consumption. The other parameter values that we used are $\beta = 0.9$, $r = 0.03$, $w = 0.5$, and $a_0 = 2.6$.

To derive the consumption functions in the first and second periods, we solve the Euler equation on a set of 2,501 evenly-spaced gridpoints for current asset holdings over the interval $[0, 2.5]$. Figure 2 plots the consumption function $g_1(a, \widetilde{e}, \varepsilon)$ under various combinations of $\widetilde{e}_1$ and ε_1. In the diagram, Case 1 corresponds to the pair $(\widetilde{e}_1, \varepsilon_1) = (0.6825, 0.9)$, Case 2 corresponds to the pair $(\widetilde{e}_1, \varepsilon_1) = (0.9178, 1.1)$, and Case 3 corresponds to $(\widetilde{e}_1, \varepsilon_1) = (1.2537, 1.1)$. It is clear that the consumption function is convex under certain range of asset values. Since $g_1(a, \widetilde{e}, \varepsilon)$ is locally convex in a for some $(\widetilde{e}, \varepsilon)$, it cannot be globally concave in $(a, \widetilde{e})$ or in (a, ε).

Next, we consider the effects of increasing $\varkappa_0$ when all other parameters (including $\varkappa_1$ and $\varkappa_2$) are held constant. The results are summarized in Table 1. Under the benchmark specification, an increase in $\varkappa_0$ would lower the unconditional expectation of a_1 and a_2, represented by $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively. The intuitions of this result are as follows. Since the savings function in the first period is increasing in ε_0, an increase in $\varkappa_0$ reduces savings when $\varepsilon_0 = 1 - \varkappa_0$ [i.e., $h_0(a_0, \widetilde{e}_0, 1 - \varkappa_0)$] and raises savings when $\varepsilon_0 = 1 + \varkappa_0$ [i.e., $h_0(a_0, \widetilde{e}_0, 1 + \varkappa_0)$]. This essentially widens the dispersion of asset holdings in the second period. In addition, the decline in $h_0(a_0, \widetilde{e}_0, 1 - \varkappa_0)$ is larger than the increase in $h_0(a_0, \widetilde{e}_0, 1 + \varkappa_0)$ in all cases. Thus, the unconditional expectation $\mathcal{W}_1$ falls as $\varkappa_0$ increases. As for the second period, let $E(a_2 | \varepsilon_0 = 1 - \varkappa_0)$ and $E(a_2 | \varepsilon_0 = 1 + \varkappa_0)$ be the expectations of $a_2 = h_1(a_1, \widetilde{e}_1, \varepsilon_1)$ conditional on the realization of ε_0. Table 1 shows that an increase in $\varkappa_0$ also widens the dispersion of these conditional expectations. In particular, the decline in $E(a_2 | \varepsilon_0 = 1 - \varkappa_0)$ is larger than the increase in $E(a_2 | \varepsilon_0 = 1 + \varkappa_0)$. This is due to two factors. First, the consumption function $g_1(a_1, \widetilde{e}_1, \varepsilon_1)$ exhibits local convexity, as depicted in Figure 2. This means the savings function $h_1(a_1, \widetilde{e}_1, \varepsilon_1)$ is locally concave. Second, an increase in $\varkappa_0$ widens the dispersion of $h_0(a_0, \widetilde{e}_0, \varepsilon_0)$. These in turn lead to a reduction in the unconditional expectation $\mathcal{W}_2$ as $\varkappa_0$ increases.

6 Concluding Remarks

In this paper, we explore the theoretical foundations for the concavity of consumption function and precautionary wealth accumulation. This study departs from the existing literature by considering a general class of utility functions. We show that the consumption function at each stage of the life cycle exhibits concavity when the utility function has strictly positive third derivative and the inverse of absolute prudence is a concave function. We also show that when consumption function is concave, a mean-preserving spread in either permanent or transitory earnings shock would encourage wealth accumulation at both the individual and aggregate levels. Finally, our numerical examples show that if the inverse of absolute prudence is not globally concave, then the consumption function may be locally convex and precautionary saving may not occur even when the utility function has strictly positive third derivative.

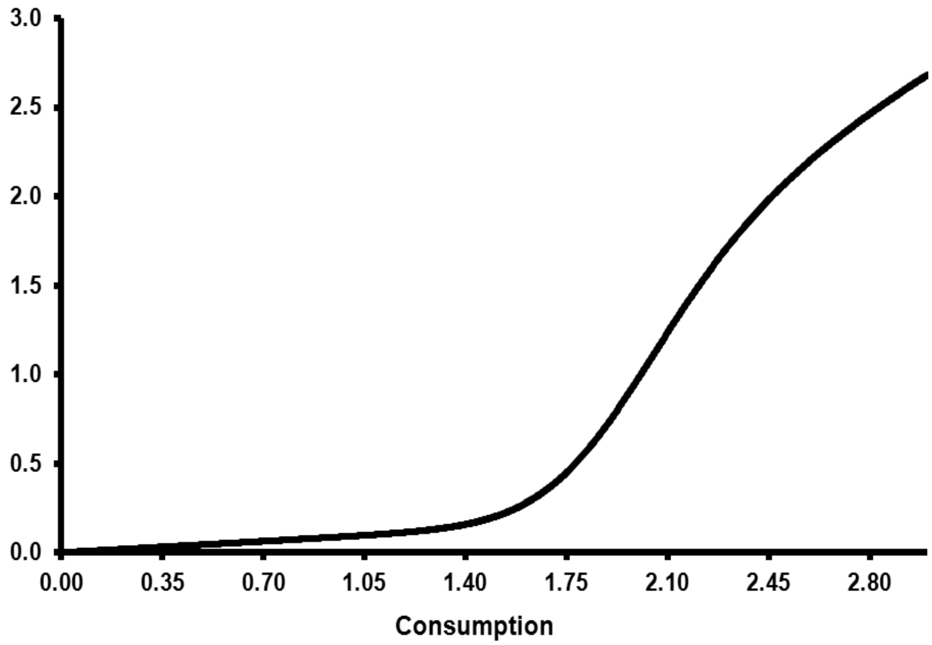

Figure 1: Inverse of Absolute Prudence when $\sigma = 10$ and $\theta = 0.1$.

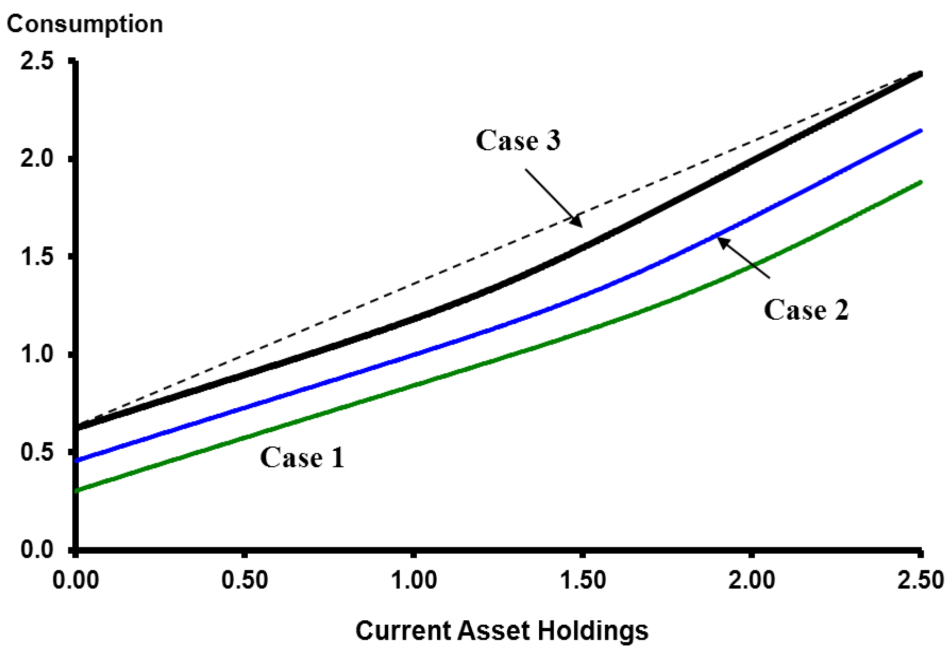

Figure 2: Consumption Functions in the Second Period.

Appendix A

Proof of Theorem 1

The proof of this theorem is divided into two parts. The first part establishes the boundedness and the continuity of the value functions. Once these properties are established, the proofs of strict monotonicity and strict concavity are standard and are thus omitted. The second part of the proof establishes the differentiability of $V_t(\cdot, z)$ for each t and for all $z \in Z_t$. An inductive argument is used in each part. For each $t \in \{0, 1, ..., T\}$, define $d_t \equiv w\underline{e}_t - (1+r)\underline{a}_t + \underline{a}_{t+1}$.

Part 1: Boundedness and Continuity

In the terminal period, the value function is given by

$$V_T(a, z) = u[we(z) + (1+r)a], \qquad \text{for all } (a, z) \in S_T.$$

This function is bounded above by $u[w\overline{e}_T + (1+r)\overline{a}_T] < \infty$, bounded below by

$$u[w\underline{e}_T - (1+r)\underline{a}_T] > u(\underline{c}) \geq -\infty, \tag{25}$$

and continuous on S_T. The first inequality in (25) follows from condition (3).

Suppose $V_{t+1}(a, z)$ is bounded and continuous on S_{t+1} for some $t+1 \leq T$. For each $(a, z) \in S_t$, define the budget correspondence $\mathcal{B}_t$ according to

$$\mathcal{B}_t(a, z) \equiv \left\{c : \underline{c} \leq c \leq x(a, z) + \underline{a}_{t+1}\right\},$$

where $x(a, z) \equiv we(z) + (1+r)a$. Define the objective function at time t as

$$W_t(c; a, z) \equiv u(c) + \beta \int_{Z_{t+1}} V_{t+1}[x(a, z) - c, z'] \, Q_t(z, dz').$$

Since $V_{t+1}(a, z)$ is bounded and continuous on S_{t+1}, the conditional expectation in the above expression is well-defined. Since the transition function Q_t satisfies the Feller property, W_t is continuous whenever it is finite. If $u(\underline{c}) > -\infty$, then the objective function $W_t(c; a, z)$ is bounded and continuous on $\mathcal{B}_t(a, z)$ for all $(a, z) \in S_t$. By the Theorem of the Maximum, the value function V_t is continuous and the optimal policy correspondence g_t defined in (4) is non-empty, compact-valued and upper hemicontinuous. Since $W_t(c; a, z)$ is bounded for all $c \in \mathcal{B}_t(a, z)$ and for all $(a, z) \in S_t$, the value function $V_t(\cdot)$ is also bounded.

Suppose now $u(\underline{c}) = -\infty$. In this case, $W_t(\underline{c}; a, z) = -\infty$ for all $(a, z) \in S_t$. This means for all $(a, z) \in S_t$, the objective function in (P1) is not continuous on the budget set. Consequently, we cannot apply the Theorem of the Maximum directly. However, the same results can be obtained with some additional effort. The following argument is similar to the one used in Alvarez and Stokey (1998) Lemma 2. Given (3), we have $x(a, z) + \underline{a}_{t+1} \geq d_t > \underline{c}$, for all $(a, z) \in S_t$ and for all t. Define a modified budget correspondence $\mathcal{B}_t^*$ according to

$$\mathcal{B}_t^*(a, z) \equiv \left\{c : d_t \leq c \leq x(a, z) + \underline{a}_{t+1}\right\}.$$

This correspondence is non-empty, compact-valued and continuous. Most importantly, the objective function $W_t(c; a, z)$ is finite and continuous on $\mathcal{B}_t^*(a, z)$ for all $(a, z) \in S_t$. Define the set of maximizers of $W_t(c; a, z)$ on $\mathcal{B}_t^*(a, z)$ as

$$g_t^*(a, z) \equiv \underset{c \in \mathcal{B}_t^*(a,z)}{\arg\max} \left\{W_t(c; a, z)\right\}.$$

Then $g_t^*(a, z)$ is non-empty and $W_t(c^*; a, z) > -\infty$ for any $c^* \in g_t^*(a, z)$. Pick any $c^* \in g_t^*(a, z)$. If $W_t(c^*; a, z) \geq W_t(c; a, z)$ for all $c \in [\underline{c}, d_t)$, then $c^* \in g_t(a, z)$. Suppose there exists $\widetilde{c} \in [\underline{c}, d_t)$ such that $W_t(\widetilde{c}; a, z) > W_t(c^*; a, z) > -\infty$, then $\widetilde{c} \in g_t(a, z)$. In either case, the optimal policy correspondence $g_t(a, z)$ is non-empty. Note that in the latter case, $\widetilde{c}$ must be strictly greater than $\underline{c}$ because $W_t(\widetilde{c}; a, z) > -\infty$. It follows that $c > \underline{c}$ whenever $c \in g_t(a, z)$. Hence $V_t(a, z) > -\infty$. Since $W_t(c; a, z)$ is still bounded above for all $c \in \mathcal{B}_t(a, z)$ and for $(a, z) \in S_t$, the value function $V_t(\cdot)$ is also bounded above.

We now establish the continuity of $V_t(\cdot)$ for the case when $u(\underline{c}) = -\infty$. Since $V_t(\cdot)$ is single-valued, it suffice to show that it is upper hemicontinuous. Let $\{(a_n, z_n)\}$ be a sequence in S_t that converges to some $(a, z) \in S_t$. Pick a sequence of consumption $\{c_n\}$ such that $c_n \in g_t(a_n, z_n)$ for each n. Such a sequence can always be drawn because $g_t(a_n, z_n)$ is non-empty for all n. Since $c_n \in \mathcal{B}_t(a_n, z_n)$ and $\mathcal{B}_t$ is compact-valued and upper hemicontinuous, there exists a subsequence of $\{c_n\}$, denoted by $\{c_{n_k}\}$, such that c_{n_k} converges to some $c^{**} \in \mathcal{B}_t(a, z)$. Since $c_{n_k} \in g_t(a_{n_k}, z_{n_k})$, it follows that $c_{n_k} > \underline{c}$ and $W_t(c_{n_k}; a_{n_k}, z_{n_k}) > -\infty$ for all n_k. By the continuity of $W_t(\cdot)$, we have $V_t(a_{n_k}, z_{n_k}) = W_t(c_{n_k}; a_{n_k}, z_{n_k}) \to W_t(c^{**}; a, z)$. If we can show that $W_t(c^{**}; a, z) = V_t(a, z)$, then this will establish (i) the upper hemicontinuity of $V_t(\cdot)$ at (a, z), and (ii) $c^{**} \in g_t(a, z)$ which implies the upper hemicontinuity of g_t.

Suppose the contrary that there exists $\widehat{c} \in \mathcal{B}_t(a, z)$ such that $W_t(\widehat{c}; a, z) > W_t(c^{**}; a, z) > -\infty$. This implies $\widehat{c} > \underline{c}$. Since $\mathcal{B}_t$ is lower hemicontinuous and (a_{n_k}, z_{n_k}) converges to (a, z), there exists a sequence $\{\widehat{c}_{n_k}\}$ such that $\widehat{c}_{n_k} \in \mathcal{B}_t(a_{n_k}, z_{n_k})$ for all n_k and $\widehat{c}_{n_k}$ converges to $\widehat{c}$. Since $\widehat{c} > \underline{c}$, it follows that $\widehat{c}_{n_k} > \underline{c}$ when n_k is sufficiently large. Then by the continuity of $W_t(\cdot)$,

$$\lim_{n_k \to \infty} W_t(\widehat{c}_{n_k}; a_{n_k}, z_{n_k}) = W_t(\widehat{c}; a, z) > W_t(c^{**}; a, z) = \lim_{n_k \to \infty} W_t(c_{n_k}; a_{n_k}, z_{n_k}).$$

This means when n_k is sufficiently large, we have $W_t(\widehat{c}_{n_k}; a_{n_k}, z_{n_k}) > W_t(c_{n_k}; a_{n_k}, z_{n_k})$ which contradicts the fact that $c_{n_k} \in g_t(a_{n_k}, z_{n_k})$. Hence $W_t(c^{**}; a, z) = V_t(a, z)$ for any $(a, z) \in S_t$.

From (25), it is obvious that $V_T(\cdot, z)$ is strictly increasing and strictly concave for all $z \in Z_T$. An inductive argument can be used to establish these properties for all $t \leq T-1$. Given the strict concavity of $V_t(\cdot, z)$, both $g_t(\cdot, z)$ and $h_t(\cdot, z)$ are single-valued continuous functions for all t. If $u(\underline{c}) = -\infty$, then the above argument shows that it is never optimal to choose $c_t = \underline{c}$. Hence $g_t(a, z) > \underline{c}$ for all $(a, z) \in S_t$. When $u(\underline{c}) > -\infty$, it is still possible to have $g_t(a, z) = \underline{c}$ for some (a, z).

Part 2: Differentiability

Fix $z \in Z_t$. Let $V_t^+(a, z)$ be the right-hand derivative of $V_t(\cdot, z)$ at any $a \in [-\underline{a}_t, \overline{a}_t)$ and $V_t^-(a, z)$ be the left-hand derivative of $V_t(\cdot, z)$ at $a \in (-\underline{a}_t, \overline{a}_t)$. Since $V_t(\cdot, z)$ is strictly concave, both $V_t^+(a, z)$ and $V_t^-(a, z)$ exist and are finite for all $a \in (-\underline{a}_t, \overline{a}_t)$. To show that $V_t(a, z)$ is differentiable on $[-\underline{a}_t, \overline{a}_t)$, we need to establish two properties: (i) it is differentiable in the interior of A_t, and (ii) $V_t^+(-\underline{a}_t, z)$ exists and is finite. To establish the first property, we will appeal to Theorem 25.1 in Rockafellar (1970) which states that if the set of supergradients of $V_t(\cdot, z)$ at point a is a singleton, then $V_t(\cdot, z)$ is differentiable at a. Recall that a real number $\lambda(a)$ is a supergradient of $V_t(\cdot, z)$ at $a \in A_t$ if it satisfies the following condition

$$V_t(\widetilde{a}, z) - V_t(a, z) \leq \lambda(a) \cdot (\widetilde{a} - a), \qquad \text{for every } \widetilde{a} \in A_t.$$

Both $V_t^+(a, z)$ and $V_t^-(a, z)$ are supergradients at a. Any supergradient at $a \in (-\underline{a}_t, \overline{a}_t)$ must also satisfy $V_t^+(a, z) \leq \lambda(a) \leq V_t^-(a, z) < \infty$.

An inductive argument is used in the following proof. In the terminal period, we have

$$g_T(a,z) = we(z) + (1+r)a \geq d_T > \underline{c},$$

and $V_T(a,z) = u[we(z) + (1+r)a]$, for all $(a,z) \in S_T$. Under the stated assumptions for $u(\cdot)$, $V_T(\cdot,z)$ is continuously differentiable in the interior of A_T and the derivative is given by $p_T(a,z) = (1+r)u'[g_T(a,z)]$. Also, the right-hand derivative of $V_T(\cdot,z)$ at $a = -\underline{a}_T$ exists and is given by $(1+r)u'[g_T(-\underline{a}_T,z)]$ which is finite. Suppose the desired result is true for some $t+1 \leq T$ and $g_{t+1}(a',z') > \underline{c}$ for all $(a',z') \in S_{t+1}$. The remaining proof is divided into several steps. Steps 1-4 essentially establish all the results in Theorem 2.

Step 1 First, we show that if $g_t(a,z) > \underline{c}$, then $g_t(a,z)$ and $h_t(a,z)$ satisfies

$$u'[g_t(a,z)] \geq \beta \int_{Z_{t+1}} p_{t+1}\left[h_t(a,z), z'\right] Q_t\left(z, dz'\right). \tag{26}$$

If in addition $h_t(a,z) > -\underline{a}_{t+1}$, then (26) holds with equality.

Fix $(a,z) \in S_t$. Define $\widetilde{c} = g_t(a,z)$ and $\widetilde{a}' = h_t(a,z)$. If $\widetilde{c} > \underline{c}$, then $\widetilde{a}' < \overline{a}_{t+1}$. Suppose now we increase $\widetilde{a}'$ by $\epsilon > 0$, reduce $\widetilde{c}$ by $\epsilon > 0$ but maintain $\widetilde{c} - \epsilon > \underline{c}$. The utility loss generated by this is $u(\widetilde{c}) - u(\widetilde{c} - \epsilon)$. The utility gain generated by this is

$$\beta \int_{Z_{t+1}} \left[V_{t+1}\left(\widetilde{a}' + \epsilon, z'\right) - V_{t+1}\left(\widetilde{a}', z'\right)\right] Q_t\left(z, dz'\right).$$

If the borrowing constraint is binding originally, i.e., $\widetilde{a}' = -\underline{a}_{t+1}$, then any reduction in consumption would lower the value of the objective function. This means the loss in utility is no less than the gain so that

$$\frac{u(\widetilde{c}) - u(\widetilde{c} - \epsilon)}{\epsilon} \geq \beta \int_{Z_{t+1}} \left[\frac{V_{t+1}(\widetilde{a}' + \epsilon, z') - V_{t+1}(\widetilde{a}', z')}{\epsilon}\right] Q_t\left(z, dz'\right).$$

By taking the limit $\epsilon \to 0+$, we get

$$u'(\widetilde{c}) \geq \lim_{\epsilon \to 0+} \left\{\beta \int_{Z_{t+1}} \left[\frac{V_{t+1}(\widetilde{a}' + \epsilon, z') - V_{t+1}(\widetilde{a}', z')}{\epsilon}\right] Q_t\left(z, dz'\right)\right\}. \tag{27}$$

Since $V_{t+1}(a', z')$ is strictly concave in a', the function

$$\Omega\left(\epsilon; \widetilde{a}', z'\right) \equiv \frac{V_{t+1}(\widetilde{a}' + \epsilon, z') - V_{t+1}(\widetilde{a}', z')}{\epsilon} > 0$$

is strictly decreasing in ϵ. Hence, it is bounded above by $p_{t+1}(\widetilde{a}', z')$ which is finite as $V_{t+1}(\cdot, z')$ is differentiable on $\left[-\underline{a}_{t+1}, \overline{a}_{t+1}\right)$. By the Lebesgue Convergence Theorem, the limit in (27) can be expressed as

$$\beta \int_{Z_{t+1}} \lim_{\epsilon \to 0+} \left[\frac{V_{t+1}(\widetilde{a}' + \epsilon, z') - V_{t+1}(\widetilde{a}', z')}{\epsilon}\right] Q_t\left(z, dz'\right) = \beta \int_{Z_{t+1}} p_{t+1}\left(\widetilde{a}', z'\right) Q_t\left(z, dz'\right).$$

Substituting this into (27) gives (26) for the case when $\widetilde{a}' = h_t(a, z) = -\underline{a}_{t+1}$. If $\widetilde{a}' = h_t(a, z) > -\underline{a}_{t+1}$, then any infinitesimal change in consumption would not affect the maximized value of the objective. This means (27) will hold with equality. It follows from the above argument that (26) holds with equality when $\widetilde{a}' = h_t(a, z) > -\underline{a}_{t+1}$.

Step 2 Using a similar perturbation argument, we can show that if $g_t(a, z) = \underline{c}$, then the following condition must be satisfied

$$u'(\underline{c}+) \leq \beta \int_{Z_{t+1}} p_{t+1}\left[h_t(a, z), z'\right] Q_t\left(z, dz'\right), \tag{28}$$

where $u'(\underline{c}+)$ denote the right-hand derivative of $u(\cdot)$ at $\underline{c}$. Suppose $\widetilde{c} = g_t(a, z) = \underline{c}$ for some $(a, z) \in S_t$. Suppose now we increase $\widetilde{c}$ by $\epsilon > 0$, reduce $\widetilde{a}' = h_t(a, z)$ by $\epsilon > 0$, but maintain $\widetilde{a}' - \epsilon \geq -\underline{a}_{t+1}$. If it is optimal to consume $\underline{c}$, then any infinitesimal increase in consumption would either lower or have no effect on the value of the objective function. Hence,

$$\lim_{\epsilon \to 0+} \left[\frac{u(\underline{c} + \epsilon) - u(\underline{c})}{\epsilon}\right] = u'(\underline{c}+) \leq \beta \int_{Z_{t+1}} \lim_{\epsilon \to 0+} \left[\frac{V_{t+1}(\widetilde{a}', z') - V_{t+1}(\widetilde{a}' - \epsilon, z')}{\epsilon}\right] Q_t\left(z, dz'\right).$$

Since $V_{t+1}(\cdot, z')$ is differentiable on $\left[-\underline{a}_{t+1}, \overline{a}_{t+1}\right)$, the left-hand derivative in the above expression exists and is given by $p_{t+1}\left[h_t(a, z), z'\right]$.

Step 3 We now show that $g_t(a, z) > \underline{c}$ if either $u'(\underline{c}+) = +\infty$ or $\beta(1 + r) \leq 1$ holds. If $u'(\underline{c}+) = +\infty$, then (28) cannot be satisfied and hence it is never optimal to consume the minimum level $\underline{c}$. Consider the case when $u'(\underline{c}+) < +\infty$ and $\beta(1 + r) \leq 1$. By the induction hypothesis, we have $g_{t+1}(a', z') > \underline{c}$

and $p_{t+1}(a',z') = (1+r)\,u'[g_{t+1}(a',z')]$ for all $(a',z') \in S_{t+1}$. Fix $(a,z) \in S_t$. Suppose the contrary that $g_t(a,z) = \underline{c}$. Then according to Step 2, it must be the case that

$$\begin{aligned} u'(\underline{c}+) &\leq \beta(1+r)\int_{Z_{t+1}} u'\left[g_{t+1}\left(h_t(a,z),z'\right)\right] Q_t\left(z,dz'\right) \\ &\leq \int_{Z_{t+1}} u'\left[g_{t+1}\left(h_t(a,z),z'\right)\right] Q_t\left(z,dz'\right) \\ &< u'(\underline{c}+). \end{aligned}$$

The second inequality uses the assumption that $\beta(1+r) \leq 1$. The third inequality uses the fact that $g_{t+1}(a',z') > \underline{c}$ for all $(a',z') \in S_{t+1}$. This gives rise to a contradiction. Hence $g_t(a,z) > \underline{c}$ for all $(a,z) \in S_t$. This means the optimal policy functions would only satisfy (26), but not (28).

Step 4 We now show that $g_t(a,\widetilde{e},\varepsilon)$ is strictly increasing for all $(a,\widetilde{e},\varepsilon) \in S_t$. This is true for $g_T(a,\widetilde{e},\varepsilon)$. Suppose $g_{t+1}(a',\widetilde{e}',\varepsilon')$ is strictly increasing for all $(a',\widetilde{e}',\varepsilon') \in S_{t+1}$. Pick any $(a_2,\widetilde{e}_2,\varepsilon_2)$ and $(a_1,\widetilde{e}_1,\varepsilon_1)$ from S_t such that $a_2 \geq a_1$, $\widetilde{e}_2 \geq \widetilde{e}_1$ and $\varepsilon_2 \geq \varepsilon_1$ with strict inequality holds for at least one variable. Suppose the contrary that $g_t(a_2,\widetilde{e}_2,\varepsilon_2) \leq g_t(a_1,\widetilde{e}_1,\varepsilon_1)$, which implies $h_t(a_2,\widetilde{e}_2,\varepsilon_2) > h_t(a_1,\widetilde{e}_1,\varepsilon_1) \geq -\underline{a}_{t+1}$. Since $g_{t+1}(a',\widetilde{e}',\varepsilon')$ is strictly increasing, we have

$$g_{t+1}\left(h_t(a_2,\widetilde{e}_2,\varepsilon_2),\widetilde{e}_2\nu',\varepsilon'\right) > g_{t+1}\left(h_t(a_1,\widetilde{e}_1,\varepsilon_2),\widetilde{e}_1\nu',\varepsilon'\right)$$

$$\Rightarrow u'\left[g_{t+1}\left(h_t(a_2,\widetilde{e}_2,\varepsilon_2),\widetilde{e}_2\nu',\varepsilon'\right)\right] < u'\left[g_{t+1}\left(h_t(a_1,\widetilde{e}_1,\varepsilon_2),\widetilde{e}_1\nu',\varepsilon'\right)\right],$$

for all $(\nu',\varepsilon') \in \Lambda \times \Xi$. Integrating this over all possible values of (ν',ε') gives

$$\begin{aligned} u'[g_t(a_2,\widetilde{e}_2,\varepsilon_2)] &= \beta(1+r)\int_{\Xi}\int_{\Lambda} u'\left[g_{t+1}\left(h_t(a_2,\widetilde{e}_2,\varepsilon_2),\widetilde{e}_2\nu',\varepsilon'\right)\right] dL_{t+1}\left(\nu'\right) dG_{t+1}\left(\varepsilon'\right) \\ &< \beta(1+r)\int_{\Xi}\int_{\Lambda} u'\left[g_{t+1}\left(h_t(a_1,\widetilde{e}_1,\varepsilon_1),\widetilde{e}_1\nu',\varepsilon'\right)\right] dL_{t+1}\left(\nu'\right) dG_{t+1}\left(\varepsilon'\right) \\ &\leq u'[g_t(a_1,\widetilde{e}_1,\varepsilon_1)]. \end{aligned}$$

The first line uses the Euler equation and the fact that $h_t(a_2,\widetilde{e}_2,\varepsilon_2) > -\underline{a}_{t+1}$. The above result contradicts $g_t(a_2,\widetilde{e}_2,\varepsilon_2) \leq g_t(a_1,\widetilde{e}_1,\varepsilon_1)$. Hence $g_t(a,\widetilde{e},\varepsilon)$ is a strictly increasing function. A similar argument can be used to show that, for any given $\widetilde{e} \in \Delta_t$, the savings function $h_t(a,\widetilde{e},\varepsilon)$ is nondecreasing in (a,ε).

Step 5 We now show that $V_t(\cdot,z)$ is differentiable in the interior of A_t and $p_t(\cdot,z) = (1+r)u'[g_t(\cdot,z)]$ for each $z \in Z_t$. Fix $z \in Z_t$. Let $\lambda_t(a,z)$ be a supergradient of $V_t(a,z)$ at $a \in (-\underline{a}_t, \overline{a}_t)$. Since $V_t(\cdot,z)$ is strictly increasing and strictly concave on A_t, $\lambda_t(a,z)$ is strictly positive and finite. The main idea of the proof is to show that for any $a \in (-\underline{a}_t, \overline{a}_t)$, there exists a neighborhood $\mathcal{M}(a,z)$ of $g_t(a,z)$ such that $g_t(a,z)$ is an interior solution of the following problem

$$\max_{c \in \mathcal{M}(a,z)} \{(1+r)u(c) - \lambda_t(a,z)c\}. \tag{P4}$$

This problem is well-posed as $\lambda_t(a,z)$ is finite and the objective function is strictly concave. If $g_t(a,z)$ is an interior solution of (P4), then it must satisfy the first-order condition

$$\lambda_t(a,z) = (1+r)u'[g_t(a,z)].$$

Since this is true for *any* supergradient $\lambda_t(a,z)$, this means $(1+r)u'[g_t(a,z)]$ must be the unique supergradient at $a \in (-\underline{a}_t, \overline{a}_t)$ and so $V_t(\cdot,z)$ is differentiable at a. We now establish the key steps of this argument. Fix $a \in (-\underline{a}_t, \overline{a}_t)$. Since $g_t(a,z) > \underline{c} \geq 0$ and $(1+r) > 0$, we can find an $\epsilon > 0$ such that

$$a - \epsilon \geq \max\left\{-\underline{a}_t, a - \frac{g_t(a,z) - \underline{c}}{1+r}\right\} \qquad \text{and} \qquad a + \epsilon < \overline{a}_t.$$

For any $\widetilde{a} \in (a-\epsilon, a+\epsilon)$, define $c(\widetilde{a})$ according to

$$c(\widetilde{a}) = we(z) + (1+r)\widetilde{a} - h_t(a,z) = (1+r)(\widetilde{a} - a) + g_t(a,z). \tag{29}$$

which is strictly greater than $\underline{c}$ as $\widetilde{a} > a - \epsilon \geq a - [g_t(a,z) - \underline{c}]/(1+r)$. In other words, the combination of $c(\widetilde{a})$ and $h_t(a,z)$ is feasible when the current state is $(\widetilde{a}, z)$. In addition, $\widetilde{a} \in (a-\epsilon, a+\epsilon)$ implies $c(\widetilde{a})$ is within a certain neighborhood of $g_t(a,z)$ given by

$$\mathcal{M}_\epsilon(a,z) = \{c : g_t(a,z) - (1+r)\epsilon < c < g_t(a,z) + (1+r)\epsilon\}.$$

The rest of the proof is similar to the proof of Lemma 1.1 in Schechtman (1976). Let $\lambda_t(a,z)$ be a supergradient of $V_t(a,z)$ at $a \in (-\underline{a}_t, \overline{a}_t)$. Then for any $\widetilde{a} \in A_t$, we have

$$V_t(\widetilde{a},z) - V_t(a,z) \leq \lambda_t(a,z) \cdot (\widetilde{a} - a).$$

Since $\lambda_t(a,z)$ is finite, we can rewrite this as

$$V_t(\widetilde{a},z) - \lambda_t(a,z)\,\widetilde{a} \leq V_t(a,z) - \lambda_t(a,z)\,a. \tag{30}$$

This inequality has to be true for any $\widetilde{a} \in A_t$. So pick $\widetilde{a} \in (a-\epsilon, a+\epsilon)$ and define $c(\widetilde{a})$ as in (29). Since $c(\widetilde{a})$ and $h_t(a,z)$ are feasible when the current state is $(\widetilde{a},z)$, it follows from the definition of the value function and (30) that

$$\begin{aligned}
& u[c(\widetilde{a})] + \beta \int_{Z_{t+1}} V_{t+1}[h_t(a,z), z'] \, Q_t(z, dz') - \lambda_t(a,z) \cdot \left[\frac{h_t(a,z) + c(\widetilde{a}) - we(z)}{1+r}\right] \\
\leq \; & V_t(\widetilde{a},z) - \lambda_t(a,z) \cdot \left[\frac{h_t(a,z) + c(\widetilde{a}) - we(z)}{1+r}\right] \\
\leq \; & V_t(a,z) - \lambda_t(a,z) \cdot \left[\frac{h_t(a,z) + g_t(a,z) - we(z)}{1+r}\right] \\
= \; & u[g_t(a,z)] + \beta \int_{Z_{t+1}} V_{t+1}[h_t(a,z), z'] \, Q_t(z, dz') - \lambda_t(a,z) \cdot \left[\frac{h_t(a,z) + g_t(a,z) - we(z)}{1+r}\right].
\end{aligned}$$

This can be simplified to become

$$(1+r)\,u[c(\widetilde{a})] - \lambda_t(a,z)\,c(\widetilde{a}) \leq (1+r)\,u[g_t(a,z)] - \lambda_t(a,z)\,g_t(a,z),$$

which is true for all $c(\widetilde{a}) \in \mathcal{M}_\epsilon(a,z)$. In other words, $g_t(a,z)$ is an interior solution of the problem (P4). This establishes the desired results.

Step 6 We now show that $V_t(a,z)$ is right-hand differentiable at $a = -\underline{a}_t$ and the right-hand derivative is given by $(1+r)\,u'[g_t(-\underline{a}_t, z)]$. Fix $z \in Z_t$ and $a \in (-\underline{a}_t, \overline{a}_t)$. By the concavity of $u(\cdot)$, we have

$$u[g_t(a,z)] - u[g_t(-\underline{a}_t, z)] \leq u'[g_t(-\underline{a}_t, z)]\,[g_t(a,z) - g_t(-\underline{a}_t, z)].$$

The result in Step 3 implies $u'[g_t(-\underline{a}_t, z)] < u'(\underline{c}) \leq +\infty$. Similarly, by the concavity and differentiability of $V_{t+1}(\cdot, z')$ on $[-\underline{a}_{t+1}, \overline{a}_{t+1})$, we have

$$V_{t+1}(h_t(a,z), z') - V_{t+1}(h_t(-\underline{a}_t, z), z') \leq p_{t+1}(h_t(-\underline{a}_t, z), z') \cdot [h_t(a,z) - h_t(-\underline{a}_t, z)],$$

for all $z' \in Z_{t+1}$. Using these, we can write

$$
\begin{aligned}
& V_t(a,z) - V_t(-\underline{a}_t, z) \\
= \; & u[g_t(a,z)] - u[g_t(-\underline{a}_t, z)] \\
& +\beta \int_{Z_{t+1}} \left[V_{t+1}\left(h_t(a,z), z'\right) - V_{t+1}(h_t(-\underline{a}_t, z), z')\right] Q_t\left(z, dz'\right) \\
\leq \; & u'[g_t(-\underline{a}_t, z)][g_t(a,z) - g_t(-\underline{a}_t, z)] \\
& +\beta \left[\int_{Z_{t+1}} p_{t+1}\left(h_t(-\underline{a}_t, z), z'\right) Q_t\left(z, dz'\right)\right] [h_t(a,z) - h_t(-\underline{a}_t, z)] \\
\leq \; & u'[g_t(-\underline{a}_t, z)][g_t(a,z) + h_t(a,z) - g_t(-\underline{a}_t, z) - h_t(-\underline{a}_t, z)] = u'[g_t(-\underline{a}_t, z)](1+r)(a + \underline{a}_t).
\end{aligned}
$$

The second line follows from the definition of $V_t(a,z)$ and $V_t(-\underline{a}_t, z)$. The fourth line is obtained by using condition (26) and the fact that $h_t(a,z)$ is non-decreasing in a. The last equality uses the consumer's budget constraint. Thus, we have

$$
\frac{V_t(a,z) - V_t(-\underline{a}_t, z)}{a - (-\underline{a}_t)} \leq (1+r) u'[g_t(-\underline{a}_t, z)] < +\infty.
$$

By taking the limit $a \to -\underline{a}_t+$, we can establish that $V_t^+(-\underline{a}_t, z)$ exists and is bounded above by $(1+r) u'[g_t(-\underline{a}_t, z)]$. Hence $V_t^+(-\underline{a}_t, z)$ is finite. Since $V_t^+(a,z)$ is strictly decreasing in a, we have

$$
(1+r) u'[g_t(a,z)] = V_t^+(a,z) < V_t^+(-\underline{a}_t, z) \leq (1+r) u'[g_t(-\underline{a}_t, z)],
$$

for all $a \in (-\underline{a}_t, \overline{a}_t)$. Since both $u'(c)$ and $g_t(\cdot, z)$ are continuous, we have $u'[g_t(a,z)] \to u'[g_t(-\underline{a}_t, z)]$ as $a \to -\underline{a}_t$. Hence $V_t^+(-\underline{a}_t, z) = (1+r) u'[g_t(-\underline{a}_t, z)]$. This completes the proof of Theorem 1.

Proof of Theorem 2

Part (i) of this theorem is proved in Step 3 of the second part of the above proof. Part (ii) of Theorem 2 is established in Step 1 of that proof. Part (iii) is established in Step 4 of that proof.

Proof of Theorem 3

For each $z \in Z_t$ and for each $t \in \{0, 1, ..., T\}$, let $\mathcal{H}_t(z; u)$ be the hypograph of the policy function $g_t(\cdot, z; u)$. Formally, $\mathcal{H}_t(z; u) \equiv \{(c, a) \in \mathcal{D} \times A_t : c \leq g_t(a, z; u)\}$. Using the same step as in the proof of Theorem 5, we can express $\mathcal{H}_t(z;u)$ as

$$\mathcal{H}_t(z;u) \equiv \{(c,a) \in \mathcal{D} \times A_t : c \in \mathcal{B}_t(a,z) \text{ and } c \leq \Psi_{t+1}(x(a,z)-c;z)\},$$

where $\mathcal{B}_t(a,z)$ is the budget set at state (a,z) in period t, and

$$\Psi_{t+1}(a';z) \equiv \phi^{-1}\left\{\beta(1+r)\int_{Z_{t+1}} \phi\left[g_{t+1}(a',z';u)\right] Q_t(z,dz')\right\}, \tag{31}$$

for $t \in \{0,1,...,T-1\}$, with $\phi(c) \equiv u'(c)$. Similarly, define the function

$$\Theta_{t+1}(a';z) \equiv \varphi^{-1}\left\{\beta(1+r)\int_{Z_{t+1}} \varphi\left[g_{t+1}(a',z';v)\right] Q_t(z,dz')\right\},$$

for each $t \in \{0,1,...,T-1\}$, with $\varphi(c) \equiv v'(c)$. Then the hypograph of $g_t(\cdot,z;v)$ is given by

$$\mathcal{H}_t(z;v) \equiv \{(c,a) \in \mathcal{D} \times A_t : c \in \mathcal{B}_t(a,z) \text{ and } c \leq \Theta_{t+1}(x(a,z)-c;z)\}.$$

Suppose the following inequality holds for all $z \in Z$,

$$\Theta_{t+1}(\cdot;z) \leq \Psi_{t+1}(\cdot;z). \tag{32}$$

Then we have $\mathcal{H}_t(z;v) \subseteq \mathcal{H}_t(z;u)$ for all $z \in Z$, and it follows that $g_t(a,z;v) \leq g_t(a,z;u)$ for all $(a,z) \in S_t$. Hence, it suffice to show that (32) holds for all $z \in Z_t$ and $t \in \{0,1,...,T-1\}$. This is achieved in two steps.

Step 1 We first derive two useful properties of $\Upsilon(\cdot)$. Since $\Upsilon(\cdot)$ is convex and $\Upsilon(0) \geq 0$, we have $\Upsilon(\delta m) \leq \delta\Upsilon(m)$ for any $m \geq 0$ and $\delta \in [0,1]$. Let m_1 and m_2 be two strictly positive real numbers such that $m_1 > m_2 > 0$. Then there exists $\delta \in (0,1)$ such that $m_2 = \delta m_1$. Hence,

$$\Upsilon(m_2) = \Upsilon(\delta m_1) \leq \delta\Upsilon(m_1) = \frac{m_2}{m_1}\Upsilon(m_1)$$

$$\Rightarrow \frac{\Upsilon(m_2)}{m_2} \leq \frac{\Upsilon(m_1)}{m_1}.$$

This means $m^{-1}\Upsilon(m)$ is a nondecreasing function in $m > 0$. If $\Upsilon(\cdot)$ is *strictly* convex, then $m^{-1}\Upsilon(m)$ is a *strictly* increasing function in $m > 0$. Next, define $m \equiv \phi(c) > 0$ for any $c \in \mathcal{D}$. Substituting $c = \phi^{-1}(m)$ into $\varphi(c) \equiv \Upsilon[\phi(c)]$ gives

$$\varphi\left[\phi^{-1}(m)\right] = \Upsilon(m). \tag{33}$$

Step 2 We now use an inductive argument to show that (32) holds for all $t \in \{0, 1, ..., T-1\}$. In the terminal period, we have $g_T(a, z; v) = g_T(a, z; u) = we(z) + (1+r)a$ for all $(a, z) \in S_T$. Substituting this into (31) gives

$$\begin{aligned}
\Psi_T\left(a'; z\right) &\equiv \phi^{-1}\left\{\beta(1+r)\int_{Z_T}\phi\left[g_T\left(a', z'; u\right)\right]Q_{T-1}\left(z, dz'\right)\right\} \\
&= \phi^{-1}\left\{\beta(1+r)\int_{Z_T}\phi\left[g_T\left(a', z'; v\right)\right]Q_{T-1}\left(z, dz'\right)\right\}
\end{aligned}$$

Using (33), we can get

$$\varphi\left[\Psi_T\left(a'; z\right)\right] = \Upsilon\left\{\beta(1+r)\int_{Z_T}\phi\left[g_T\left(a', z'; v\right)\right]Q_{T-1}\left(z, dz'\right)\right\}. \tag{34}$$

Since $m^{-1}\Upsilon(m)$ is a nondecreasing function in $m > 0$, and $\beta(1+r) \leq 1$, we have

$$\frac{\Upsilon\left\{\beta(1+r)\int_{Z_T}\phi\left[g_T(a', z'; v)\right]Q_{T-1}(z, dz')\right\}}{\beta(1+r)\int_{Z_T}\phi\left[g_T(a', z'; v)\right]Q_{T-1}(z, dz')} \leq \frac{\Upsilon\left\{\int_{Z_T}\phi\left[g_T(a', z'; v)\right]Q_{T-1}(z, dz')\right\}}{\int_{Z_T}\phi\left[g_T(a', z'; v)\right]Q_{T-1}(z, dz')}, \tag{35}$$

which implies

$$\begin{aligned}
\varphi\left[\Psi_T\left(a'; z\right)\right] &\leq \beta(1+r)\Upsilon\left\{\int_{Z_T}\phi\left[g_T\left(a', z'; v\right)\right]Q_{T-1}\left(z, dz'\right)\right\} \\
&\leq \beta(1+r)\int_{Z_T}\Upsilon\left\{\phi\left[g_T\left(a', z'; v\right)\right]\right\}Q_{T-1}\left(z, dz'\right) \\
&= \beta(1+r)\int_{Z_T}\varphi\left[g_T\left(a', z'; v\right)\right]Q_{T-1}\left(z, dz'\right).
\end{aligned} \tag{36}$$

The second line uses Jensen's inequality. The third line uses the fact that $\varphi(c) \equiv \Upsilon[\phi(c)]$ for any $c \in \mathcal{D}$. Since $\varphi^{-1}(\cdot)$ is nonincreasing, we have

$$\Psi_T\left(a'; z\right) \geq \varphi^{-1}\left\{\beta(1+r)\int_{Z_T}\varphi\left[g_T\left(a', z'; v\right)\right]Q_{T-1}\left(z, dz'\right)\right\} \equiv \Theta_T\left(a'; z\right).$$

As explained above, this implies $g_{T-1}(a,z;v) \leq g_{T-1}(a,z;u)$ for all $(a,z) \in S_{T-1}$. Note that if $\Upsilon(\cdot)$ is strictly convex, then strict inequality holds in both (35) and (36). These in turn imply $g_{T-1}(a,z;v) < g_{T-1}(a,z;u)$ for all $(a,z) \in S_{T-1}$.

Suppose $g_{t+1}(a,z;v) \leq g_{t+1}(a,z;u)$ for all $(a,z) \in S_{t+1}$ and for some $t+1 \leq T-1$. Using a similar argument as in (34)-(36), we can get

$$\begin{aligned} \Psi_{t+1}(a';z) &\equiv \phi^{-1}\left\{\beta(1+r)\int_{Z_{t+1}} \phi\left[g_{t+1}(a',z';u)\right] Q_t(z,dz')\right\} \\ &\geq \varphi^{-1}\left\{\beta(1+r)\int_{Z_{t+1}} \varphi\left[g_{t+1}(a',z';u)\right] Q_t(z,dz')\right\}. \end{aligned} \tag{37}$$

What remains is to show that

$$\begin{aligned} &\varphi^{-1}\left\{\beta(1+r)\int_{Z_{t+1}} \varphi\left[g_{t+1}(a',z';u)\right] Q_t(z,dz')\right\} \\ \geq\ &\varphi^{-1}\left\{\beta(1+r)\int_{Z_{t+1}} \varphi\left[g_{t+1}(a',z';v)\right] Q_t(z,dz')\right\} \equiv \Theta_{t+1}(a';z). \end{aligned} \tag{38}$$

By the induction hypothesis, we have $g_{t+1}(a,z;v) \leq g_{t+1}(a,z;u)$ for all $(a,z) \in S_{t+1}$. Since $\varphi(\cdot)$ is nonincreasing and $\beta(1+r) > 0$, we have

$$\beta(1+r)\int_{Z_{t+1}} \varphi\left[g_{t+1}(a',z';u)\right] Q_t(z,dz') \leq \beta(1+r)\int_{Z_{t+1}} \varphi\left[g_{t+1}(a',z';v)\right] Q_t(z,dz').$$

Since $\varphi^{-1}(\cdot)$ is also nonincreasing, this implies (38). This establishes (32) for all $t \in \{0,1,...,T-1\}$. If $\Upsilon(\cdot)$ is strictly convex, then strict inequality holds in (37). It follows that $g_t(a,z;v) < g_t(a,z;u)$ for all $(a,z) \in S_t$. This completes the proof of Theorem 3.

Note that the above argument is valid even if $\{z_t\}$ is a deterministic process. Suppose the variable z follows a deterministic time path $\{\overline{z}_0, \overline{z}_1, ..., \overline{z}_T\}$ such that $e(\overline{z}_t) \in [\underline{e}_t, \overline{e}_t]$ for all t. Then (31) becomes

$$\Psi_{t+1}(a';\overline{z}_t) \equiv \phi^{-1}\left\{\beta(1+r)\phi\left[g_{t+1}(a',\overline{z}_{t+1};u)\right]\right\}.$$

The expression for $\Theta_{t+1}(a';z)$ can be modified in a similar fashion. Using the same argument, we can show that $\Theta_{t+1}(\cdot;\overline{z}_t) \leq \Psi_{t+1}(\cdot;\overline{z}_t)$ for all t, which implies $g_t(a,\overline{z}_t;v) \leq g_t(a,\overline{z}_t;u)$ for all $a \in A_t$ and for all t.

Proof of Lemma 4

As mentioned in the text, the inequality in (11) becomes Jensen's inequality when $\beta(1+r) = 1$. Hence, Assumption A is satisfied by any concave function $\Phi(\cdot)$ when $\beta(1+r) = 1$. Consider the case when $\beta(1+r) \in (0,1)$. Since $u'''(c) > 0$ for all $c \in \mathcal{D}$, we have $\Phi(m) \geq 0$ for all $m \geq 0$. This, together with the concavity of $\Phi(\cdot)$, implies $\Phi(\delta m) \geq \delta\Phi(m)$ for all $m \geq 0$ and $\delta \in [0,1]$. Using a similar argument as in Step 1 of the proof of Theorem 3, one can show that $m^{-1}\Phi(m)$ is a nonincreasing function in $m > 0$.

Let $N \geq 1$ be a positive integer. Let $\boldsymbol{\mu}$ be a discrete probability measure with masses $(\mu_1, ..., \mu_N)$ on a set of points $(\psi_1, ..., \psi_N) \in \mathbb{R}_+^N$. First consider the case when $\sum_{n=1}^N \mu_n\psi_n = 0$. Since $\psi_i \geq 0$ for all i, this can only happen when either (i) $\psi_n = 0$ for all n, or (ii) $\psi_n > 0$ for some n but $\mu_n = 0$. In both cases, we have

$$\Phi\left[\beta(1+r)\sum_{n=1}^N \mu_n\psi_n\right] = \Phi(0) > \beta(1+r)\sum_{n=1}^N \mu_n\Phi(\psi_n) = \beta(1+r)\Phi(0),$$

as $\beta(1+r) \in (0,1)$. Hence, Assumption A is satisfied. Next, consider the case when $\sum_{n=1}^N \mu_n\psi_n > 0$. Since $m^{-1}\Phi(m)$ is a nonincreasing function in $m > 0$, we have

$$\frac{\Phi\left[\beta(1+r)\sum_{n=1}^N \mu_n\psi_n\right]}{\beta(1+r)\sum_{n=1}^N \mu_n\psi_n} \geq \frac{\Phi\left[\sum_{n=1}^N \mu_n\psi_n\right]}{\sum_{n=1}^N \mu_n\psi_n} \geq \frac{\sum_{n=1}^N \mu_n\Phi(\psi_n)}{\sum_{n=1}^N \mu_n\psi_n}.$$

The second inequality follows from the concavity of $\Phi(\cdot)$. Condition (11) can be obtained by rearranging terms. This completes the proof of Lemma 4.

Proof of Proposition 9

To establish this result, it suffice to show that $\Psi_{t+1}(\cdot)$ is a strictly concave function for any given period t. To see this, choose $\varepsilon \in \Xi$ such that $\mathcal{A}_t(\varepsilon)$ is non-empty and convex. Suppose the contrary that $g_t(a, \widetilde{e}, \varepsilon)$ has a linear portion over the set $\mathcal{A}_t(\varepsilon)$. This means there exists $(a_1, \widetilde{e}_1)$ and $(a_2, \widetilde{e}_2)$ in $\mathcal{A}_t(\varepsilon)$ such that for any $\delta \in (0,1)$, we have

$$g_t(a_\delta, \widetilde{e}_\delta, \varepsilon) = \delta g_t(a_1, \widetilde{e}_1, \varepsilon) + (1-\delta)g_t(a_2, \widetilde{e}_2, \varepsilon), \tag{39}$$

where $a_\delta \equiv \delta a_1 + (1-\delta) a_2$ and $\widetilde{e}_\delta \equiv \delta \widetilde{e}_1 + (1-\delta)\widetilde{e}_2$. This implies $h_t(a_\delta, \widetilde{e}_\delta, \varepsilon) = \delta h_t(a_1, \widetilde{e}_1, \varepsilon) + (1-\delta) h_t(a_2, \widetilde{e}_2, \varepsilon)$. By the strict concavity of $\Psi_{t+1}(\cdot)$, we have

$$\Psi_{t+1}[h_t(a_\delta, \widetilde{e}_\delta, \varepsilon), \widetilde{e}_\delta] > \delta \Psi_{t+1}[h_t(a_1, \widetilde{e}_1, \varepsilon), \widetilde{e}_1] + (1-\delta)\Psi_{t+1}[h_t(a_2, \widetilde{e}_2, \varepsilon), \widetilde{e}_2].$$

Since $\mathcal{A}_t(\varepsilon)$ is a convex set, we have $h_t(a_\delta, \widetilde{e}_\delta, \varepsilon) > -\underline{a}_{t+1}$. Thus, the Euler equation holds with strict equality under all three states: $(a_1, \widetilde{e}_1, \varepsilon)$, $(a_2, \widetilde{e}_2, \varepsilon)$ and $(a_\delta, \widetilde{e}_\delta, \varepsilon)$. This implies

$$\begin{aligned} g_t(a_\delta, \widetilde{e}_\delta, \varepsilon) &= \Psi_{t+1}[h_t(a_\delta, \widetilde{e}_\delta, \varepsilon), \widetilde{e}_\delta] \\ &> \delta \Psi_{t+1}[h_t(a_1, \widetilde{e}_1, \varepsilon), \widetilde{e}_1] + (1-\delta)\Psi_{t+1}[h_t(a_2, \widetilde{e}_2, \varepsilon), \widetilde{e}_2] \\ &= \delta g_t(a_1, \widetilde{e}_1, \varepsilon) + (1-\delta) g_t(a_2, \widetilde{e}_2, \varepsilon), \end{aligned}$$

which contradicts (39). Hence, $g_t(a, \widetilde{e}, \varepsilon)$ must be strictly concave over $\mathcal{A}_t(\varepsilon)$.

Suppose $u(\cdot)$ is a HARA utility function with $\gamma > -1$ and $\gamma \neq 0$. Then the marginal utility function is given by $\phi(c) = \varphi(\alpha + \gamma c)^{-\frac{1}{\gamma}}$, for some $\varphi > 0$, and $\alpha + \gamma c > 0$ for all $c \in \mathcal{D}$. Then for any $t \in \{0, 1, ..., T-1\}$, the function $\Psi_{t+1}(a', \widetilde{e})$ can be expressed as

$$\begin{aligned} \Psi_{t+1}(a', \widetilde{e}) &\equiv \phi^{-1}\left\{\beta(1+r)\int_\Xi \int_\Lambda \phi\left[g_{t+1}(a', \widetilde{e}\nu', \varepsilon')\right] dL_{t+1}(\nu')\, dG_{t+1}(\varepsilon')\right\} \\ &= -\frac{\alpha}{\gamma} + \frac{[\beta(1+r)]^{-\gamma}}{\gamma}\left\{\int_\Xi \int_\Lambda \left[g_{t+1}(a', \widetilde{e}\nu', \varepsilon')\right]^{-\frac{1}{\gamma}} dL_{t+1}(\nu')\, dG_{t+1}(\varepsilon')\right\}^{-\gamma}. \end{aligned} \quad (40)$$

Pick any $(a_1', \widetilde{e}_1)$ and $(a_2', \widetilde{e}_2)$ from $A_{t+1} \times \Delta_t$. For any $\delta \in (0,1)$, define the convex combination $(a_\delta, \widetilde{e}_\delta)$. By Theorem 5, we know that $g_{t+1}(a', \widetilde{e}', \varepsilon')$ is joint concave in its first two arguments when ε' is held constant. This, together with the assumption that $\phi(\cdot)$ is strictly decreasing, gives

$$\phi\left[g_{t+1}(a_\delta', \widetilde{e}_\delta \nu', \varepsilon')\right] \leq \phi\left[\delta g_{t+1}(a_1', \widetilde{e}_1 \nu', \varepsilon') + (1-\delta) g_{t+1}(a_2', \widetilde{e}_2 \nu', \varepsilon')\right],$$

for all $(\nu', \varepsilon') \in \Lambda \times \Xi$. Since $\phi^{-1}(\cdot)$ is also strictly decreasing, we have

$$\begin{aligned} &\Psi_{t+1}(a_\delta', \widetilde{e}_\delta) \\ \geq\ &\phi^{-1}\left\{\beta(1+r)\int_\Xi \int_\Lambda \phi\left[\delta g_{t+1}(a_1', \widetilde{e}_1 \nu', \varepsilon') + (1-\delta) g_{t+1}(a_2', \widetilde{e}_2 \nu', \varepsilon')\right] dL_{t+1}(\nu')\, dG_{t+1}(\varepsilon')\right\}. \end{aligned}$$

By Minkowski's inequality for integrals,[30] we can get

$$\begin{aligned}
&\left\{\int_{\Xi}\int_{\Lambda}\left[\delta g_{t+1}\left(a_1',\widetilde{e}_1\nu',\varepsilon'\right)+(1-\delta)\,g_{t+1}\left(a_2',\widetilde{e}_2\nu',\varepsilon'\right)\right]^{-\frac{1}{\gamma}}dL_{t+1}\left(\nu'\right)dG_{t+1}\left(\varepsilon'\right)\right\}^{-\gamma}\\
>\;&\left\{\int_{\Xi}\int_{\Lambda}\left[\delta g_{t+1}\left(a_1',\widetilde{e}_1\nu',\varepsilon'\right)\right]^{-\frac{1}{\gamma}}dL_{t+1}\left(\nu'\right)dG_{t+1}\left(\varepsilon'\right)\right\}^{-\gamma}\\
&+\left\{\int_{\Xi}\int_{\Lambda}\left[(1-\delta)\,g_{t+1}\left(a_2',\widetilde{e}_2\nu',\varepsilon'\right)\right]^{-\frac{1}{\gamma}}dL_{t+1}\left(\nu'\right)dG_{t+1}\left(\varepsilon'\right)\right\}^{-\gamma}\\
=\;&\delta\left\{\int_{\Xi}\int_{\Lambda}\left[g_{t+1}\left(a_1',\widetilde{e}_1\nu',\varepsilon'\right)\right]^{-\frac{1}{\gamma}}dL_{t+1}\left(\nu'\right)dG_{t+1}\left(\varepsilon'\right)\right\}^{-\gamma}\\
&+(1-\delta)\left\{\int_{\Xi}\int_{\Lambda}\left[g_{t+1}\left(a_2',\widetilde{e}_2\nu',\varepsilon'\right)\right]^{-\frac{1}{\gamma}}dL_{t+1}\left(\nu'\right)dG_{t+1}\left(\varepsilon'\right)\right\}^{-\gamma}.
\end{aligned}$$

Multiplying both sides by $\left[\beta\left(1+r\right)\right]^{-\gamma}/\gamma>0$ and combining it with (40) gives

$$\Psi_{t+1}\left(a_\delta',\widetilde{e}_\delta\right)>\delta\Psi_{t+1}\left(a_1',\widetilde{e}_1\right)+(1-\delta)\,\Psi_{t+1}\left(a_2',\widetilde{e}_2\right).$$

This completes the proof of Proposition 9.

Proof of Proposition 10

The proof is almost identical to that of Proposition 9. Hence, we only outline the main ideas of the proof. First, using the same line of argument as in the proof of Proposition 9, one can show the following: if $\Psi_{t+1}\left(\cdot,\widetilde{e}\right)$ is a strictly concave function for any $\widetilde{e}\in\Delta_t$, then $g_t\left(a,\widetilde{e},\varepsilon\right)$ must be strictly concave over $\widetilde{\mathcal{A}}_t\left(\widetilde{e}\right)$. As shown in the proof of Proposition 9, the function $\Psi_{t+1}\left(\cdot\right)$ is strictly concave when the utility function is HARA with $\gamma>-1$ and $\gamma\neq0$. This implies $\Psi_{t+1}\left(\cdot,\widetilde{e}\right)$ is a strictly concave function for any $\widetilde{e}\in\Delta_t$. Hence, the desired result follows.

Proof of Theorem 11

Part (i) Start from age $T-1$. Suppose the contrary that $h_{2,T-1}\left(a,\widetilde{e},\varepsilon\right)>h_{1,T-1}\left(a,\widetilde{e},\varepsilon\right)\geq-\underline{a}_T$, for some $\left(a,\widetilde{e},\varepsilon\right)\in S_{T-1}$. Then for any $\left(\nu',\varepsilon'\right)\in\Lambda\times\Xi$, we have

$$w\widetilde{e}\nu'\varepsilon'+(1+r)\,h_{2,T-1}\left(a,\widetilde{e},\varepsilon\right)>w\widetilde{e}\nu'\varepsilon'+(1+r)\,h_{1,T-1}\left(a,\widetilde{e},\varepsilon\right).$$

[30] For a formal statement of Minkowski's inequality for integrals, see Hardy *et al.* (1952) p.146.

Since $\phi(\cdot)$ is strictly decreasing, we have

$$\phi\left[w\widetilde{e}\nu'\varepsilon' + (1+r)\, h_{2,T-1}(a,\widetilde{e},\varepsilon)\right] < \phi\left[w\widetilde{e}\nu'\varepsilon' + (1+r)\, h_{1,T-1}(a,\widetilde{e},\varepsilon)\right], \tag{41}$$

for all (ν',ε'). Since $\phi(\cdot)$ is also strictly convex, the expression $\phi[w\widetilde{e}\nu'\varepsilon' + (1+r)\, h_{2,T-1}(a,\widetilde{e},\varepsilon)]$ is strictly convex in ν' when $(a,\widetilde{e},\varepsilon,\varepsilon')$ are held fixed. This, together with the assumption that $L_{1,T}(\cdot)$ is more risky than $L_{2,T}(\cdot)$, implies

$$\int_{\Lambda} \phi\left[w\widetilde{e}\nu'\varepsilon' + (1+r)\, h_{2,T-1}(a,\widetilde{e},\varepsilon)\right] dL_{1,T}\left(\nu'\right) > \int_{\Lambda} \phi\left[w\widetilde{e}\nu'\varepsilon' + (1+r)\, h_{2,T-1}(a,\widetilde{e},\varepsilon)\right] dL_{2,T}\left(\nu'\right), \tag{42}$$

for any given $\varepsilon' \in \Xi$. Using the Euler equation, we can get

$$\begin{aligned}
\phi\left[g_{1,T-1}(a,\widetilde{e},\varepsilon)\right] &\geq \beta(1+r)\int_{\Xi}\int_{\Lambda} \phi\left[w\widetilde{e}\nu'\varepsilon' + (1+r)\, h_{1,T-1}(a,\widetilde{e},\varepsilon)\right] dL_{1,T}\left(\nu'\right) dG_T\left(\varepsilon'\right) \\
&> \beta(1+r)\int_{\Xi}\int_{\Lambda} \phi\left[w\widetilde{e}\nu'\varepsilon' + (1+r)\, h_{2,T-1}(a,\widetilde{e},\varepsilon)\right] dL_{1,T}\left(\nu'\right) dG_T\left(\varepsilon'\right) \\
&\geq \beta(1+r)\int_{\Xi}\int_{\Lambda} \phi\left[w\widetilde{e}\nu'\varepsilon' + (1+r)\, h_{2,T-1}(a,\widetilde{e},\varepsilon)\right] dL_{2,T}\left(\nu'\right) dG_T\left(\varepsilon'\right) \\
&= \phi\left[g_{2,T-1}(a,\widetilde{e},\varepsilon)\right].
\end{aligned}$$

The second line uses (41) while the third line uses (42). The last line follows from the assumption that $h_{2,T-1}(a,\widetilde{e},\varepsilon) > -\underline{a}_T$. The above result implies $g_{2,T-1}(a,\widetilde{e},\varepsilon) > g_{1,T-1}(a,\widetilde{e},\varepsilon)$ which contradicts $h_{2,T-1}(a,\widetilde{e},\varepsilon) > h_{1,T-1}(a,\widetilde{e},\varepsilon)$. Hence, $h_{2,T-1}(a,\widetilde{e},\varepsilon) \leq h_{1,T-1}(a,\widetilde{e},\varepsilon)$ for all $(a,\widetilde{e},\varepsilon) \in S_{T-1}$.

Suppose $h_{2,t+1}(a',\widetilde{e}',\varepsilon') \leq h_{1,t+1}(a',\widetilde{e}',\varepsilon')$ for all $(a',\widetilde{e}',\varepsilon') \in S_{t+1}$ and for some $t+1 \leq T-1$. This means $g_{2,t+1}(a',\widetilde{e}',\varepsilon') \geq g_{1,t+1}(a',\widetilde{e}',\varepsilon')$ for all $(a',\widetilde{e}',\varepsilon') \in S_{t+1}$. Suppose the contrary that $h_{2,t}(a,\widetilde{e},\varepsilon) > h_{1,t}(a,\widetilde{e},\varepsilon) \geq -\underline{a}_{t+1}$, for some $(a,\widetilde{e},\varepsilon) \in S_t$. Then for any $(\nu',\varepsilon') \in \Lambda \times \Xi$, we have

$$\begin{aligned}
g_{1,t+1}\left(h_{1,t}(a,\widetilde{e},\varepsilon), \widetilde{e}\nu', \varepsilon'\right) &\leq g_{2,t+1}\left(h_{1,t}(a,\widetilde{e},\varepsilon), \widetilde{e}\nu', \varepsilon'\right) \\
&< g_{2,t+1}\left(h_{2,t}(a,\widetilde{e},\varepsilon), \widetilde{e}\nu', \varepsilon'\right).
\end{aligned}$$

The first inequality follows from the induction hypothesis. The second inequality follows our earlier result that $g_{2,t+1}(a',\widetilde{e}',\varepsilon')$ is strictly increasing in a'. Since $\phi(\cdot)$ is strictly decreasing, we have

$$\phi\left[g_{2,t+1}\left(h_{2,t}(a,\widetilde{e},\varepsilon), \widetilde{e}\nu', \varepsilon'\right)\right] < \phi\left[g_{1,t+1}\left(h_{1,t}(a,\widetilde{e},\varepsilon), \widetilde{e}\nu', \varepsilon'\right)\right],$$

which is analogous to (41). If we can show that $\phi\left[g_{2,t+1}\left(h_{2,t}\left(a,\widetilde{e},\varepsilon\right),\widetilde{e}\nu',\varepsilon'\right)\right]$ is convex in ν', then a contradiction can be obtained by using the same argument. Pick any ν_1' and ν_2' in Λ. For any $\delta \in (0,1)$, define $\nu_\delta' \equiv \delta\nu_1' + (1-\delta)\nu_2'$. Since $g_{2,t+1}\left(a',\widetilde{e}',\varepsilon'\right)$ is concave in $\widetilde{e}'$, we have

$$
\begin{aligned}
& g_{2,t+1}\left(h_{2,t}\left(a,\widetilde{e},\varepsilon\right),\widetilde{e}\nu_\delta',\varepsilon'\right) \\
\geq\ & \delta g_{2,t+1}\left(h_{2,t}\left(a,\widetilde{e},\varepsilon\right),\widetilde{e}\nu_1',\varepsilon'\right) + (1-\delta)\, g_{2,t+1}\left(h_{2,t}\left(a,\widetilde{e},\varepsilon\right),\widetilde{e}\nu_2',\varepsilon'\right).
\end{aligned}
$$

Since $\phi(\cdot)$ is strictly decreasing and strictly convex, we have

$$
\begin{aligned}
& \phi\left[g_{2,t+1}\left(h_{2,t}\left(a,\widetilde{e},\varepsilon\right),\widetilde{e}\nu_\delta',\varepsilon'\right)\right] \\
\leq\ & \phi\left[\delta g_{2,t+1}\left(h_{2,t}\left(a,\widetilde{e},\varepsilon\right),\widetilde{e}\nu_1',\varepsilon'\right) + (1-\delta)\, g_{2,t+1}\left(h_{2,t}\left(a,\widetilde{e},\varepsilon\right),\widetilde{e}\nu_2',\varepsilon'\right)\right] \\
\leq\ & \delta\phi\left[g_{2,t+1}\left(h_{2,t}\left(a,\widetilde{e},\varepsilon\right),\widetilde{e}\nu_1',\varepsilon'\right)\right] + (1-\delta)\,\phi\left[g_{2,t+1}\left(h_{2,t}\left(a,\widetilde{e},\varepsilon\right),\widetilde{e}\nu_2',\varepsilon'\right)\right].
\end{aligned}
$$

We can now use the same argument as in the terminal period to show that $g_{2,t}\left(a,\widetilde{e},\varepsilon\right) > g_{1,t}\left(a,\widetilde{e},\varepsilon\right)$ which is inconsistent with $h_{2,t}\left(a,\widetilde{e},\varepsilon\right) > h_{1,t}\left(a,\widetilde{e},\varepsilon\right)$. Hence, $h_{2,t}\left(a,\widetilde{e},\varepsilon\right) \leq h_{1,t}\left(a,\widetilde{e},\varepsilon\right)$ for all $\left(a,\widetilde{e},\varepsilon\right) \in S_t$. This completes the proof of part (i).

Part (ii) Unlike Huggett (2004), which uses the Markov operator to establish his results, we use the sequential approach in the following proof. The sequential approach is particularly useful when dealing with multiple shocks. In particular, it allows us to fully exploit the assumption that $\{\varepsilon_0,\varepsilon_1,...,\varepsilon_t,\nu_1,...,\nu_t\}$ is a set of independent random variables. This will become clear very soon.

The first step of the proof is to derive an alternate expression for the expectation of $\Gamma\left[h_{j,t}\left(s_t\right)\right]$ using the distributions $\{G_1,...,G_t,L_{j,1},...,L_{j,t}\}$. By Theorem 8.3 of Stokey, Lucas and Prescott (1989), we can obtain

$$
\int_{S_t}\Gamma\left[h_{j,t}\left(s_t\right)\right]\pi_{j,t}\left(ds_t\right) = \int_{S_{t-1}}\left[\int_{S_t}\Gamma\left[h_{j,t}\left(s_t\right)\right]P_{j,t-1}\left(s_{t-1},ds_t\right)\right]\pi_{j,t-1}\left(ds_{t-1}\right),
$$

where

$$
\int_{S_t}\Gamma\left[h_{j,t}\left(s_t\right)\right]P_{j,t-1}\left(s_{t-1},ds_t\right) = \int_{\Xi}\int_{\Lambda}\Gamma\left[h_{j,t}\left(h_{j,t-1}\left(s_{t-1}\right),\widetilde{e}_{t-1}\nu_t,\varepsilon_t\right)\right]dL_{j,t}\left(\nu_t\right)dG_t\left(\varepsilon_t\right) \equiv F_{j,t}\left(s_{t-1}\right),
$$

for all $s_{t-1} \in S_{t-1}$. Applying the same theorem, we can obtain

$$\int_{S_{t-1}} F_{j,t}(s_{t-1})\,\pi_{j,t-1}(ds_{t-1}) = \int_{S_{t-2}} \left[\int_{S_{t-1}} F_{j,t}(s_{t-1})\,P_{j,t-2}(s_{t-2}, ds_{t-1})\right] \pi_{j,t-2}(ds_{t-2}),$$

where

$$\int_{S_{t-1}} F_{j,t}(s_{t-1})\,P_{j,t-2}(s_{t-2}, ds_{t-1}) = \int_{\Xi}\int_{\Lambda} F_{j,t}\left[h_{j,t-2}(s_{t-2}), \widetilde{e}_{t-2}\nu_{t-1}, \varepsilon_{t-1}\right] dL_{j,t-1}(\nu_{t-1})\,dG_{t-1}(\varepsilon_{t-1}),$$

for all $s_{t-2} \in S_{t-2}$. By repeating the same procedure, we can obtain

$$\begin{aligned} & \int_{S_t} \Gamma\left[h_{j,t}(s_t)\right] \pi_{j,t}(ds_t) \\ = & \int_{\Xi} \cdots \int_{\Lambda} \Gamma\left[h_{j,t}(\cdots h_{j,1}(h_{j,0}(a_0, \widetilde{e}_0, \varepsilon_0), \widetilde{e}_0\nu_1, \varepsilon_1)\cdots, \widetilde{e}_{t-1}\nu_t, \varepsilon_t)\right] dL_{j,t}(\nu_t)\ldots dL_{j,1}(\nu_1)\,dG_t(\varepsilon_t)\ldots dG_0(\varepsilon_0), \end{aligned} \tag{43}$$

where $\widetilde{e}_{t-1}\nu_t = \widetilde{e}_0\nu_1\ldots\nu_t$.

Let $\boldsymbol{\varepsilon}^t = \{\varepsilon_0, ..., \varepsilon_t\}$ denote a history of transitory earnings shocks up to age t, and $\boldsymbol{\nu}^t = \{\nu_1, ..., \nu_t\}$ denote a history of permanent earnings shocks up to age t. For $j \in \{1, 2\}$ and for $t \in \{0, 1, ..., T\}$, define a function $f_{j,t}$ according to

$$f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right) \equiv h_{j,t}(\cdots h_{j,1}(h_{j,0}(a_0, \widetilde{e}_0, \varepsilon_0), \widetilde{e}_0\nu_1, \varepsilon_1)\cdots, \widetilde{e}_{t-1}\nu_t, \varepsilon_t).$$

The second step of the proof is to show that, for any history of earnings shocks, the function $f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right)$ is convex in each single ν_τ, $\tau \le t$, when all other arguments $(\varepsilon_0, ..., \varepsilon_t, \nu_1, ..., \nu_{\tau-1}, \nu_{\tau+1}, ...\nu_t)$ are held constant. An induction argument is used to establish this result. When $t = 1$, we have

$$f_{j,1}(\varepsilon_0, \varepsilon_1, \nu_1) \equiv h_{j,1}(h_{j,0}(a_0, \widetilde{e}_0, \varepsilon_0), \widetilde{e}_0\nu_1, \varepsilon_1).$$

By Theorem 5, $h_{j,1}(\cdot)$ is convex in its second argument when the other arguments are held constant. Thus, $f_{j,1}(\varepsilon_0, \varepsilon_1, \nu_1)$ is convex in ν_1 for any $(\varepsilon_0, \varepsilon_1)$. Suppose the desired result is true for $f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right)$. In the next period,

$$f_{j,t+1}\left(\boldsymbol{\varepsilon}^{t+1}, \boldsymbol{\nu}^{t+1}\right) \equiv h_{j,t+1}\left(f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right), \widetilde{e}_0\nu_1\ldots\nu_{t+1}, \varepsilon_{t+1}\right).$$

Since $h_{j,t+1}(\cdot)$ is convex in its second argument, $f_{j,t+1}\left(\boldsymbol{\varepsilon}^{t+1}, \boldsymbol{\nu}^{t+1}\right)$ is convex in ν_{t+1} when all other arguments are held constant. Fix $\tau \leq t$. Pick any $\nu_{1,\tau}$ and $\nu_{2,\tau}$ from Λ. For any $\delta \in (0,1)$, define $\nu_{\delta,\tau} \equiv \delta\nu_{1,\tau} + (1-\delta)\nu_{2,\tau}$. Define two histories of permanent earnings shocks which differ only in terms of ν_τ, i.e., $\boldsymbol{\nu}_i^t \equiv \{\nu_1, ..., \nu_{i,\tau}, ...\nu_t\}$ for $i \in \{1,2\}$. Define $\boldsymbol{\nu}_\delta^t \equiv \delta\boldsymbol{\nu}_1^t + (1-\delta)\boldsymbol{\nu}_2^t$. Similarly, define $\widetilde{e}_{i,t} \equiv \widetilde{e}_0 \times \nu_1 \times \ldots \times \nu_{i,\tau} \times \ldots \times \nu_t$ for $i \in \{1,2\}$. Then we have $\widetilde{e}_{\delta,t} = \widetilde{e}_0 \times \nu_1 \times \ldots \times \nu_{\delta,\tau} \times \ldots \times \nu_t = \delta\widetilde{e}_{1,t} + (1-\delta)\widetilde{e}_{2,t}$. By the induction hypothesis, we have

$$f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}_\delta^t\right) \leq \delta f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}_1^t\right) + (1-\delta) f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}_2^t\right).$$

Since $h_{j,t+1}(\cdot)$ is increasing in its first argument and joint convex in its first two arguments, we have

$$\begin{aligned}
& h_{j,t+1}\left(f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}_\delta^t\right), \widetilde{e}_{\delta,t}\nu_{t+1}, \varepsilon_{t+1}\right) \\
\leq\; & h_{j,t+1}\left(\delta f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}_1^t\right) + (1-\delta) f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}_2^t\right), \widetilde{e}_{\delta,t}\nu_{t+1}, \varepsilon_{t+1}\right) \\
\leq\; & \delta h_{j,t+1}\left(f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}_1^t\right), \widetilde{e}_{1,t}\nu_{t+1}, \varepsilon_{t+1}\right) + (1-\delta) h_{j,t+1}\left(f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}_2^t\right), \widetilde{e}_{2,t}\nu_{t+1}, \varepsilon_{t+1}\right).
\end{aligned}$$

This establishes the convexity of $f_{j,t+1}\left(\boldsymbol{\varepsilon}^{t+1}, \boldsymbol{\nu}^{t+1}\right)$ in each single ν_τ, for $\tau \leq t+1$. Since convexity is preserved by any increasing convex transformation, it follows that $\Gamma\left[f_{j,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right)\right]$ is also convex in each single ν_τ, $\tau \leq t$, when all other arguments $(\varepsilon_0, ..., \varepsilon_t, \nu_1, ..., \nu_{\tau-1}, \nu_{\tau+1}, ...\nu_t)$ are held constant.

The third step of the proof is to show that $f_{1,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right) \geq f_{2,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right)$ for any possible history $\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right)$ and for all t. Fix $\left(\boldsymbol{\varepsilon}^T, \boldsymbol{\nu}^T\right)$. By the result in part (i), we have $h_{1,0}(a_0, \widetilde{e}_0, \varepsilon_0) \geq h_{2,0}(a_0, \widetilde{e}_0, \varepsilon_0)$. When $t = 1$, we have

$$\begin{aligned}
f_{1,1}(\varepsilon_0, \varepsilon_1, \nu_1) &\equiv h_{1,1}(h_{1,0}(a_0, \widetilde{e}_0, \varepsilon_0), \widetilde{e}_0\nu_1, \varepsilon_1) \\
&\geq h_{2,1}(h_{1,0}(a_0, \widetilde{e}_0, \varepsilon_0), \widetilde{e}_0\nu_1, \varepsilon_1) \\
&\geq h_{2,1}(h_{2,0}(a_0, \widetilde{e}_0, \varepsilon_0), \widetilde{e}_0\nu_1, \varepsilon_1) \equiv f_{2,1}(\varepsilon_0, \varepsilon_1, \nu_1).
\end{aligned}$$

The second line again uses the result in part (i). The third line uses the fact that $h_{2,1}(\cdot)$ is increasing in its first argument. Suppose $f_{1,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right) \geq f_{2,t}\left(\boldsymbol{\varepsilon}^t, \boldsymbol{\nu}^t\right)$ for some $t \geq 1$. Using the same line of argument, we can obtain

$$\begin{aligned} f_{1,t+1}\left(\boldsymbol{\varepsilon}^{t+1},\boldsymbol{\nu}^{t+1}\right) &\equiv h_{1,t+1}\left(f_{1,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right),\widetilde{e}_t\nu_{t+1},\varepsilon_{t+1}\right) \\ &\geq h_{2,t+1}\left(f_{1,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right),\widetilde{e}_t\nu_{t+1},\varepsilon_{t+1}\right) \\ &\geq h_{2,t+1}\left(f_{2,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right),\widetilde{e}_t\nu_{t+1},\varepsilon_{t+1}\right) \equiv f_{2,t+1}\left(\boldsymbol{\varepsilon}^{t+1},\boldsymbol{\nu}^{t+1}\right). \end{aligned}$$

This establishes the desired result. Since $\Gamma(\cdot)$ is an increasing function, we have $\Gamma\left[f_{1,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)\right] \geq \Gamma\left[f_{2,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)\right]$ for any possible history $\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)$ and for all t.

Finally, we will show that the expected value of $\Gamma\left[f_{1,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)\right] \equiv \Gamma\left[h_{1,t}(s_t)\right]$ is no less than the expected value of $\Gamma\left[f_{2,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)\right] \equiv \Gamma\left[h_{2,t}(s_t)\right]$. Fix $\boldsymbol{\varepsilon}^{t}$. Then we have

$$\begin{aligned} & \int_{\Lambda}\cdots\int_{\Lambda}\Gamma\left[f_{1,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)\right]dL_{1,t}(\nu_t)\,dL_{1,t-1}(\nu_{t-1})\ldots dL_{1,1}(\nu_1) \\ \geq\; & \int_{\Lambda}\cdots\int_{\Lambda}\Gamma\left[f_{2,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)\right]dL_{1,t}(\nu_t)\,dL_{1,t-1}(\nu_{t-1})\ldots dL_{1,1}(\nu_1) \\ \geq\; & \int_{\Lambda}\cdots\int_{\Lambda}\Gamma\left[f_{2,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)\right]dL_{2,t}(\nu_t)\,dL_{1,t-1}(\nu_{t-1})\ldots dL_{1,1}(\nu_1) \\ & \cdots \\ \geq\; & \int_{\Lambda}\cdots\int_{\Lambda}\Gamma\left[f_{2,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)\right]dL_{2,t}(\nu_t)\,dL_{2,t-1}(\nu_{t-1})\ldots dL_{2,1}(\nu_1). \end{aligned}$$

The first inequality follows from the result in step 3. The second inequality uses the following facts: (i) $\Gamma\left[f_{2,t}\left(\boldsymbol{\varepsilon}^{t},\boldsymbol{\nu}^{t}\right)\right]$ is convex in ν_t when all other components are held constant, and (ii) $L_{1,t}(\cdot)$ is more risky than $L_{2,t}(\cdot)$. The last line can be obtained by repeating the same argument for all preceding periods. This procedure is valid because $\{\nu_1,...,\nu_t\}$ is a set of independent random variables. Since this ordering is true for any given history $\boldsymbol{\varepsilon}^{t}$, the desired result follows by taking the expectation over all possible $\boldsymbol{\varepsilon}^{t}$. This completes the proof of part (ii). Part (iii) follows immediately by using $\Gamma(x) = x$. This completes the proof of this theorem.

Appendix B

This section contains the technical details on how to derive the Hessian matrix $\mathbf{H}(\mathbf{y})$ and the expression of $\boldsymbol{\varpi}^T \cdot \mathbf{H}(\mathbf{y})\,\boldsymbol{\varpi}$ for any column vector $\boldsymbol{\varpi} \in \mathbb{R}^N$. First, rewrite (15) as

$$\phi[\theta(\mathbf{y})] = \beta(1+r)\sum_{n=1}^{N}\mathcal{P}_t(n)\,\phi(y_n).$$

Differentiating this with respect to y_n gives

$$\phi'[\theta(\mathbf{y})]\,\mathbf{h}_n(\mathbf{y}) = \beta(1+r)\,\mathcal{P}_t(n)\,\phi'(y_n), \tag{44}$$

$$\Rightarrow \mathbf{h}_n(\mathbf{y}) = \beta(1+r)\,\mathcal{P}_t(n)\,\frac{\phi'(y_n)}{\phi'[\theta(\mathbf{y})]}, \tag{45}$$

where $\mathbf{h}_n(\mathbf{y}) \equiv \partial\theta(\mathbf{y})/\partial y_n$. Differentiating (44) with respect to y_m gives

$$\phi''[\theta(\mathbf{y})]\,\mathbf{h}_m(\mathbf{y})\,\mathbf{h}_n(\mathbf{y}) + \phi'[\theta(\mathbf{y})]\,\mathbf{h}_{m,n}(\mathbf{y}) = 0, \tag{46}$$

if $m \neq n$, and

$$\phi''[\theta(\mathbf{y})]\,[\mathbf{h}_n(\mathbf{y})]^2 + \phi'[\theta(\mathbf{y})]\,\mathbf{h}_{n,n}(\mathbf{y}) = \beta(1+r)\,\mathcal{P}_t(n)\,\phi''(y_n), \tag{47}$$

if $m = n$. Combining (45) and (46) gives

$$\begin{aligned}\mathbf{h}_{m,n}(\mathbf{y}) &= -\frac{\phi''[\theta(\mathbf{y})]}{\phi'[\theta(\mathbf{y})]}\mathbf{h}_m(\mathbf{y})\,\mathbf{h}_n(\mathbf{y}) \\ &= -[\beta(1+r)]^2\,\mathcal{P}_t(m)\,\mathcal{P}_t(n)\,\phi'(y_m)\,\phi'(y_n)\,\frac{\phi''[\theta(\mathbf{y})]}{\{\phi'[\theta(\mathbf{y})]\}^3},\end{aligned}$$

for $m \neq n$. Similarly, combining (45) and (47) gives

$$\mathbf{h}_{n,n}(\mathbf{y}) = \beta(1+r)\,\mathcal{P}_t(n)\,\frac{\phi''(y_n)}{\phi'[\theta(\mathbf{y})]} - \left[\beta(1+r)\,\mathcal{P}_t(n)\,\phi'(y_n)\right]^2\frac{\phi''[\theta(\mathbf{y})]}{\{\phi'[\theta(\mathbf{y})]\}^3}.$$

For any $\boldsymbol{\varpi} \in \mathbb{R}^N$, we have

$$
\begin{aligned}
& \boldsymbol{\varpi}^T \cdot \mathbf{H}(\mathbf{y})\, \boldsymbol{\varpi} \\
= \;& \beta(1+r) \frac{\sum_{n=1}^N \mathcal{P}_t(n)\, \varpi_n^2 \phi''(y_n)}{\phi'[\theta(\mathbf{y})]} - [\beta(1+r)]^2 \left[\sum_{n=1}^N \mathcal{P}_t(n)\, \phi'(y_n)\right]^2 \frac{\phi''[\theta(\mathbf{y})]}{\left\{\phi'[\theta(\mathbf{y})]\right\}^3} \\
= \;& \beta(1+r) \frac{\phi''[\theta(\mathbf{y})]}{\left\{\phi'[\theta(\mathbf{y})]\right\}^3} \left[\sum_{n=1}^N \mathcal{P}_t(n)\, \varpi_n^2 \phi''(y_n)\right] \left\{ \frac{\left\{\phi'[\theta(\mathbf{y})]\right\}^2}{\phi''[\theta(\mathbf{y})]} - \beta(1+r) \frac{\left[\sum_{n=1}^N \mathcal{P}_t(n)\, \varpi_n \phi'(y_n)\right]^2}{\left[\sum_{n=1}^N \mathcal{P}_t(n)\, \varpi_n^2 \phi''(y_n)\right]} \right\}.
\end{aligned}
$$

Since $\beta(1+r) > 0$ and $\phi'(\cdot) < 0$, $\boldsymbol{\varpi}^T \cdot \mathbf{H}(\mathbf{y})\, \boldsymbol{\varpi} \leq 0$ if and only if (16) holds for all $\boldsymbol{\varpi} \in \mathbb{R}^N$.

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