Existence of a competitive equilibrium in the Lucas (1988) model without physical capital

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Abstract

This paper considers an endogenous growth model with human capital accumulation. It gives sufficient conditions and a necessary condition for the existence of a unique competitive equilibrium with externalities. These conditions are more stringent than those which prevail for the existence of an equilibrium defined as the solution to a fixed-point problem.

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1 Introduction

In his well-known 1988 model of endogenous growth with external effects, Lucas defines the equilibrium as the solution to a fixed-point problem. Initiated by Arrow (1965) and Romer (1986), this definition is now widely used in macroeconomics. However, the equilibrium that is generated by such a definition does not necessarily coincide with a competitive equilibrium with

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externalities. In this paper, we illustrate this idea using a simple human capital driven endogenous growth model to show that the family of "competitive equilibria" is contained in the family of "equilibria".

The model we use is a modified version of Uzawa (1965) and Lucas (1988) with no physical capital nor population growth. By reducing the economy to a single sector we drastically increase the tractability of the model. Nevertheless, since such a framework has been barely used in the literature [2], a natural extension of our work would be to consider physical capital accumulation. Another key characteristic of our model hinges on the human capital accumulation specification that is supposed to be an increasing and concave function of the schooling effort. Uzawa (1965) makes the same assumption (see also Caballé and Santos, 1993 or Chamley, 1993) while Lucas (1988) assumes it is simply linear.

Considering this simple economy with human capital externalities, we first characterize the equilibrium defined as the solution to a fixed point problem and obtain the following results: (*i*) if the exogenous maximal growth rate is greater than the discount rate, the equilibrium growth rate is strictly positive for at least some time. (*ii*) A path is an equilibrium if and only if it satisfies the Euler equation and a transversality condition of the optimal growth problem. Notably, the transversality condition is shown to be necessary using the assumption on parameters that guaranties that the optimization problem is restricted to a set of consumption paths that yield a bounded intertemporal utility. (*iii*) There exists a unique equilibrium and along this equilibrium, the growth rate is constant. Remark there is also a continuum of solutions to the Euler equation, along which the growth rate converges to the maximal growth rate, but we show they violate the transversality condition. As a consequence, the human capital driven endogenous growth model we consider has a rather simple dynamic behavior and does not generate indeterminacies: this features distinguishes it from its multi-sector counterpart (See Boldrin and Rustichini, 1994, Benhabib and Perli, 1994, Xie, 1994, and Alonso-Carrera and Freire-Serén, 2004).

Then, we characterize the competitive equilibrium demonstrating the following results: (*iv*) any C^2-competitive equilibrium is an equilibrium. Remark that Gomez (2004) obtains the same result for the Uzawa (1965) - Lucas (1988) model with human and physical capitals. Then, we derive our main result: (*v*) an equilibrium is a C^2-competitive equilibrium if the production function is globally not too convex and only if the human capital technology is a strictly concave function of the schooling effort. The first condition is similar as in Le Van, Morhaim and Dimaria (2002) for the Romer (1986) model. The second

[2] One of the exceptions is Lahiri (2001) who analyzes the impact of capital mobility on indeterminacy.

condition ensures the intertemporal labor income to be finite when we use equilibrium wages. It is satisfied in Le Van, Morhaim and Dimaria (2002) who assume that the marginal productivity of the knowledge technology is infinite at the origin. (*vi*) Using results (*iii*), (*iv*) and (*v*), we conclude that, upon existence, the competitive equilibrium is unique. Finally, we point out that when there is no externalities, and that the equilibrium consequently coincides with the optimum, result (*v*) still hold. Hence, (*vii*) the optimum is not a C^2-competitive equilibrium if the human capital technology is a linear function of the schooling effort.

The rest of the paper is organized as follows. Section 2 presents the model and the assumptions. Section 3 and Section 4 respectively study the fixed-point problem and the competitive equilibrium. Section 5 concludes.

2 The model

Consider a human capital driven endogenous growth model. Let c_t, u_t and h_t respectively denote the consumption, the labor supply and the human capital at time $t \geq 0$ of an infinitely-lived individual. The problem writes:

$$\max_{(c,u)} \int_0^\infty \frac{c_t^\mu}{\mu} e^{-\rho t} dt$$
$$s.t. \begin{cases} 0 \leq \dot{h}_t = \lambda \Phi (1 - u_t) h_t, \; h_0 > 0 \, \text{given} \\ 0 \leq c_t = \bar{h}_t^\gamma (u_t h_t)^\alpha \\ 0 \leq u_t \leq 1 \end{cases} \tag{P1}$$

where $\bar{h}_t$ is an externality and where $\dot{h}_t$ denotes the first difference with respect to time. Parameters satisfy the following restrictions: $\mu \in (0,1)$, $\lambda > 0$, $\rho > 0$, $\alpha \in (0,1)$, $\gamma > 0$. Finally, Φ is a function whose properties are discussed below.

Observe that our problem could be extend to the case where the utility function is unbounded from below (i.e. for $\mu \leq 0$) by applying to the continuous time environment the argument developed in Alvarez and Stokey (1998) or in Le Van and Morhaim (2002). Moreover, assume:

H1 $\Phi : [0,1] \to [0,1]$, $\Phi(0) = 0$, $\Phi(1) = 1$, $\Phi' > 0$, $\Phi'' \leq 0$.

Assumption H1 implies $\Phi'(0) \geq 1$; moreover[3], $\Phi'(0) = 1$ is equivalent to $\Phi(x) = x$.

[3] Consider the function $Z(x) = \Phi(x) - x$ with $x \in [0,1]$ and suppose $\Phi'(0) = 1$; then $Z'(x) \leq 0$; since $Z(0) = Z(1) = 0$, conclude that $Z(x) = 0$.

H2 $\lambda > \rho$.

Parameter λ measures the maximal possible growth rate. Assumption H2 hence introduces a lower bound to this rate.

H3 $\rho > \lambda\mu\left(\alpha + \gamma\right)$.

Assumption H3 ensures that the optimization problem is restricted to a set of trajectories that yield a finite intertemporal utility.

H4 The function h is piecewise C^1 while functions c and u are piecewise continuous.

It is useful to define $\Psi\left(x\right) := 1 - \Phi^{-1}\left(\frac{x}{\lambda}\right)$ where $\Psi\left(0\right) = 1$, $\Psi\left(\lambda\right) = 0$, $\Psi' < 0$, $\Psi'' \leq 0$. The problem (P1) is consequently equivalent to:

$$\begin{array}{l} \max \int_0^\infty \frac{1}{\mu}\left(\Psi\left(\frac{\dot{h}_t}{h_t}\right)\right)^{\alpha\mu} h_t^{\alpha\mu}\bar{h}_t^{\gamma\mu}e^{-\rho t}dt \\ s.t.\, 0 \leq \dot{h} \leq \lambda h,\ h_0 > 0\,\text{given.} \end{array} \tag{P2}$$

3 The solution to the fixed-point problem

Definition 1 *Let* $\mathbf{h} = \left(h_t\right)_{t\geq 0}$ *be the solution to (P2).* $\mathbf{h}$ *depends on the path* $\bar{\mathbf{h}} = \left(\bar{h}_t\right)_{t\geq 0}$*. Let us posit* $\mathbf{h} = F\left(\bar{\mathbf{h}}\right)$*. An equilibrium is a path* $\hat{\mathbf{h}}$ *such that* $\hat{\mathbf{h}} = F\left(\hat{\mathbf{h}}\right)$*.*

Lemma 2 *Assume H1-H2; an equilibrium path* $\hat{\mathbf{h}}$ *could not be a constant path.*

PROOF. Consider first an equilibrium path $\hat{\mathbf{h}}$ such that $\hat{u}_t = 1$ where $\hat{u}_t$ is the equilibrium value of u. In this case, $\hat{h}_t = h_0$, $\forall t$. Consider another path $\mathbf{h}$ which satisfies the constraints in (P1) and is such that (i) $u_t = 1 - \varepsilon$ for $t \in [0, \tau]$ and $u_t = 1$ for $t > \tau$ and (ii) $\bar{h}_t = h_0$, $\forall t$. Let Δ denotes the difference in utility between the two paths:

$$\begin{aligned} \Delta = & \frac{h_0^{\mu(\alpha+\gamma)}}{\mu}\int_0^\tau e^{-\rho t}\left(\left(1-\varepsilon\right)^{\mu\alpha}e^{\mu\alpha\lambda\Phi(\varepsilon)t} - 1\right)dt \\ & + \frac{h_0^{\mu(\alpha+\gamma)}}{\mu}\left(e^{\mu\alpha\lambda\Phi(\varepsilon)\tau} - 1\right)\int_\tau^\infty e^{-\rho t}dt. \end{aligned}$$

Using l'Hôpital's Rule, tedious computations yield:

$$\lim_{\varepsilon\to 0}\frac{\Delta}{\varepsilon}=\alpha\left(\frac{\lambda\Phi'(0)}{\rho}-1\right)\int_0^{\tau}e^{-\rho t}dt.$$

Hence, assumptions H1 and H2 are sufficient to ensure $\lim_{\varepsilon\to 0}(\Delta/\varepsilon)>0$. That means that, given the externality $\bar{\mathbf{h}}$, $\mathbf{h}$ is not optimal: a contradiction. □

Lemma 3 *Assume H1 and H3-H4; let* $\mathbf{h}$ *be a path which is* C^2 *and satisfies* $\dot{h}_t>0,\forall t$ *and* $u_t<1,\forall t$. *Then,* $\mathbf{h}$ *is an equilibrium if and only if it satisfies the Euler equation:*

$$\frac{d}{dt}\left[h_t^{\mu(\alpha+\gamma)-1}\Psi'\left(\frac{\dot{h}_t}{h_t}\right)\left(\Psi\left(\frac{\dot{h}_t}{h_t}\right)\right)^{\alpha\mu-1}e^{-\rho t}\right]$$
$$=\left[-\frac{\dot{h}_t}{h_t}\Psi'\left(\frac{\dot{h}_t}{h_t}\right)+\Psi\left(\frac{\dot{h}_t}{h_t}\right)\right]\left(\Psi\left(\frac{\dot{h}_t}{h_t}\right)\right)^{\alpha\mu-1}h_t^{(\alpha+\gamma)\mu-1}e^{-\rho t},\tag{1}$$

and the transversality condition:

$$\lim_{t\to+\infty}e^{-\rho t}h_t^{\mu(\alpha+\gamma)}\Psi'\left(\frac{\dot{h}_t}{h_t}\right)\left(\Psi\left(\frac{\dot{h}_t}{h_t}\right)\right)^{\alpha\mu-1}=0.\tag{2}$$

In particular, it satisfies:

$$\dot{\nu}_t=\frac{-\Psi'(\nu_t)\left[\rho-\nu_t\mu(\alpha+\gamma)\right]-\Psi(\nu_t)}{(1-\alpha\mu)\left(\Psi'(\nu_t)\right)^2\left(\Psi(\nu_t)\right)^{-1}-\Psi''(\nu_t)}\tag{3}$$

where $\nu_t=\dot{h}_t/h_t$.

PROOF. Let $\hat{\mathbf{h}}$ be an equilibrium; assume it is C^2. It is easy to check that (i) $\hat{\mathbf{h}}$ satisfies the Euler condition (1), and (ii) the Euler equation (1) and the transversality condition (2) are sufficient conditions for optimality. Let us now prove that the transversality condition (2) is also necessary. Let V_t be the value function at period t:

$$V_t\left(\hat{h}_t\right)=\max\int_t^{\infty}\frac{e^{-\rho(s-t)}}{\mu}\left(\Psi\left(\frac{\dot{h}_s}{h_s}\right)\right)^{\alpha\mu}h_s^{\alpha\mu}\bar{h}_s^{\gamma\mu}ds$$
$$0\leq\dot{h}_s\leq\lambda h_s,\forall s\geq t,$$
$$h_t=\hat{h}_t\text{ given.}$$

Let $\tau=s-t$ and $x_\tau=h_{t+\tau}$, hence V_t rewrites:

$$V_t\left(\hat{h}_t\right) = \max \int_0^\infty \frac{e^{-\rho\tau}}{\mu}\left(\Psi\left(\frac{\dot{x}_\tau}{x_\tau}\right)\right)^{\alpha\mu} x_\tau^{\alpha\mu}\bar{h}_{t+\tau}^{\gamma\mu} d\tau$$
$$0 \leq \dot{x}_\tau \leq \lambda x_\tau, \forall\tau \geq 0,$$
$$x_0 = \hat{h}_t.$$

It is obvious that, $\forall t$, $V_t(0) = 0$ and that V_t is concave. Moreover, V_t is differentiable (see Benveniste and Scheinkman, 1979) and:

$$V_t'\left(\hat{h}_t\right) = -\alpha\Psi'\left(\frac{d\hat{h}_t/dt}{\hat{h}_t}\right)\left(\Psi\left(\frac{d\hat{h}_t/dt}{\hat{h}_t}\right)\right)^{\alpha\mu-1}\hat{h}_t^{\alpha\mu-1}\bar{h}_t^{\gamma\mu}.$$

Now observe, using the budget constraints, that $\forall\tau$, $\bar{h}_{t+\tau} \leq \hat{h}_t e^{\lambda\tau}$ and $x_\tau \leq x_0 e^{\lambda\tau} = \hat{h}_t e^{\lambda\tau}$ and, using H1, that $\Psi \leq 1$; moreover, $\mu \leq 1$; then, $\forall t$:

$$0 \leq V_t\left(\hat{h}_t\right) \leq \frac{\hat{h}_t^{(\alpha+\gamma)\mu}}{\mu}\int_0^\infty e^{[\lambda\mu(\alpha+\gamma)-\rho]\tau} d\tau,$$

and consequently:

$$V_t\left(\hat{h}_t\right) \leq \frac{h_0^{(\alpha+\gamma)\mu} e^{\lambda(\alpha+\gamma)\mu t}}{\mu}\int_0^\infty e^{[\lambda\mu(\alpha+\gamma)-\rho]\tau} d\tau.$$

Therefore, using H3, conclude that $\lim_{t\to\infty} e^{-\rho t} V_t\left(\hat{h}_t\right) = 0$.

Then, since $V_t(h_t) = V_t(h_t) - V_t(0) \geq V_t'(h_t)\, h_t$, it yields that:

$$e^{-\rho t} V_t(h_t) \geq -\alpha e^{-\rho t}\Psi'\left(\frac{\dot{h}_t}{h_t}\right)\left(\Psi\left(\frac{\dot{h}_t}{h_t}\right)\right)^{\alpha\mu-1} h_t^{\alpha\mu}\bar{h}_t^{\gamma\mu}.$$

We hence obtain the transversality condition (2) by replacing $\bar{h}_t$ by h_t. Finally, differentiating (1) and rearranging using $h_t = h_0 e^{\int_0^t \nu_u du}$ yields (3). □

Theorem 4 *Assume H1-H4; there exists a unique C^2-equilibrium path $\hat{\mathbf{h}}$; it grows at a constant rate $\hat{\nu}$ such that:*

$$\hat{\nu} = \frac{\lambda - \rho}{1 - \mu(\alpha + \gamma)} \quad \text{if} \quad \Phi'(0) = 1, \tag{4}$$
$$\hat{\nu} > \frac{\lambda - \rho}{1 - \mu(\alpha + \gamma)} \quad \text{if} \quad \Phi'(0) > 1. \tag{5}$$

PROOF. We proceed showing that (a) there exists a unique equilibrium path that grows at a constant rate $\hat{\nu} \in (0, \lambda)$; (b) there is no equilibrium path whose growth rate is not constant.

(a) Let us show that there exist two steady-states to (3): a first one denoted $\hat{\nu}$ that belongs to $(0, \lambda)$ and which is unstable and an other one, λ, which is stable. With (3), define function G such that

$$G(\nu) = \frac{-\Psi'(\nu)[\rho - \nu\mu(\alpha+\gamma)] - \Psi(\nu)}{(1-\alpha\mu)(\Psi'(\nu))^2(\Psi(\nu))^{-1} - \Psi''(\nu)}.$$

Hence, $\hat{\nu}$ satisfies $G(\hat{\nu}) = 0$. Since, the denominator of G is strictly positive, $\hat{\nu}$ solves the equation:

$$\frac{\Psi(\hat{\nu})}{\Psi'(\hat{\nu})} - \hat{\nu} = -\rho - \hat{\nu}[1 - \mu(\alpha+\gamma)].$$

When $\hat{\nu}$ goes from 0 to λ, using H1, the first member is non decreasing from $-\lambda\Phi'(0)$ to $-\lambda$, while, using H2-H3, the second decreases from $-\rho$ to $-\rho - \lambda[1-\mu(\alpha+\gamma)]$. Hence, there exists a unique solution $\hat{\nu}$ for all $\nu \in (0, \lambda)$. It satisfies $\hat{\nu} \geq (\lambda - \rho) / [1 - \mu(\alpha+\gamma)] \Leftrightarrow \Phi'(0) \geq 1$.

Moreover, observe that, since $\Psi(\lambda) = 0$, then $G(\lambda) = 0$ but remark that the solution such that $\nu_t = \lambda$ for all t does not correspond to an optimal path since the associated consumptions equals zero for all t.

Finally, some computations yields to:

$$G'(\hat{\nu}) = \frac{-\Psi''(\hat{\nu})[\rho - \hat{\nu}\mu(\alpha+\gamma)] - \Psi'(\hat{\nu})[1-\mu(\alpha+\gamma)]}{(1-\alpha\mu)(\Psi'(\hat{\nu}))^2(\Psi(\hat{\nu}))^{-1} - \Psi''(\hat{\nu})}.$$

Using H1-H3, concludes that $G'(\hat{\nu}) > 0$. Consequently $\hat{\nu}$ is unstable while λ is stable.

(b) Consider any solution (ν_t) to (3) which is not constant over time. Since $\hat{\nu}$ is unstable, (ν_t) either converges to 0 if $\nu_0 < \hat{\nu}$ or converges to λ if $\nu_0 > \hat{\nu}$. Since 0 is not a steady-state, it cannot exist a solution such that $\nu_0 < \hat{\nu}$. We now show that solutions that converge to λ violate (2). Define first function J such that:

$$J(t) = e^{\left(-\rho t + \mu(\alpha+\gamma)\int_0^t v_u du\right)} (\Psi(\nu_t))^{\alpha\mu-1} \geq 0.$$

Hence, condition (2) is equivalent to $\lim_{t\to\infty} J(t) = 0$. Now, observe that $\Psi(\nu_t) \sim -\Psi'(\lambda)\varepsilon_t$ with $\varepsilon_t = \lambda - \nu_t$. Hence, replace to obtain

$$J(t) \sim e^{-(\rho-\lambda\mu(\alpha+\gamma))t} e^{-\mu(\alpha+\gamma)\int_0^t \varepsilon_u du} (-\Psi'(\lambda)\varepsilon_t)^{\alpha\mu-1}.$$

Equivalently, with (3), one has: $\dot{\nu}_t \sim (\nu_t - \lambda)(\rho - \lambda\mu(\alpha+\gamma)) / (\alpha\mu - 1)$; solving this equation, it yields that $-\Psi'(\lambda)\varepsilon_t \sim Ae^{\frac{(\rho-\lambda\mu(\alpha+\gamma))t}{(\alpha\mu-1)}}$ with $A > 0$. Then,

$$J(t) \sim e^{\frac{\mu(\alpha+\gamma)(1-\alpha\mu)A}{(\rho-\mu\lambda(\alpha+\gamma))}\left(e^{-\frac{\rho-\lambda\mu(\alpha+\gamma)}{(1-\alpha\mu)}t}-1\right)} A^{\alpha\mu-1}.$$

Consequently, using H3, $\lim_{t\to\infty} J(t) = e^{-\frac{\mu(\alpha+\gamma)(1-\alpha\mu)A}{(\rho-\mu\lambda(\alpha+\gamma))}} A^{\alpha\mu-1} > 0$. □

4 The competitive equilibrium

A competitive equilibrium is well-defined for some proper spaces of prices, wages, consumption, human capital and labor. Since $0 \leq \dot{h}_t \leq \lambda\Phi(1-u_t)h_t$, a feasible human capital path $\mathbf{h}$ belongs to the set L_h^∞ such that:

$$L_h^\infty = \left\{\mathbf{h} : \sup |h_t| e^{-\lambda t} < \infty\right\}. \tag{6}$$

Similarly, a feasible consumption path $\mathbf{c}$ satisfies $0 \leq c_t \leq (h_t)^{\alpha+\gamma}$. Hence $\mathbf{c} \in L_c^\infty$ with:

$$L_c^\infty = \left\{\mathbf{c} : \sup_t |c_t| e^{-\lambda(\alpha+\gamma)t} < \infty\right\}. \tag{7}$$

Moreover, the price path $\mathbf{p}$ must be such that $\int_0^\infty p_t c_t dt < \infty$ for any $\mathbf{c} \in L_c^\infty$. Hence, $\mathbf{p} \in L_p^1$ with:

$$L_p^1 = \left\{\mathbf{p} : \int_0^\infty |p_t| e^{\lambda(\alpha+\gamma)t} dt < \infty\right\}. \tag{8}$$

Firms have the information that labor supply is uniformly bounded. They thus choose a labor path $\mathbf{n}$ that belongs to L_n^∞ with:

$$L_n^\infty = \left\{\mathbf{n} : \sup_t |n_t| < \infty\right\}. \tag{9}$$

Consequently, the wage path must be such that $\int_0^\infty w_t n_t h_t dt < \infty$ for any $\mathbf{n} \in L_n^\infty$ and any $\mathbf{h} \in L_h^\infty$. Therefore, a feasible wage path is such that $\mathbf{w} \in L_w^1$ with:

$$L_w^1 = \left\{\mathbf{w} : \int_0^\infty |w_t| e^{\lambda t} dt < \infty\right\}. \tag{10}$$

Definition 5 *The list* $(\mathbf{c}^*, \mathbf{u}^*, \mathbf{p}^*, \mathbf{w}^*, \mathbf{h}^*, \mathbf{n}^*) \in L_c^\infty \times [0,1]^\infty \times L_p^1 \times L_w^1 \times L_h^\infty \times L_n^\infty$ *is a competitive equilibrium with externalities if:*

1. The paths $(\mathbf{c}^*, \mathbf{u}^*, \mathbf{h}^*)$ *are solutions to the consumer problem:*

$$\max_{(c,u)} \int_0^\infty \frac{c_t^\mu}{\mu} e^{-\rho t} dt$$
$$s.t. \begin{cases} \int_0^\infty p_t^* c_t dt \leq \int_0^\infty w_t^* u_t h_t dt + \Pi^* \\ \dot{h}_t = \lambda\Phi(1-u_t)h_t,\ h_0 > 0 \text{ given} \\ c_t \geq 0, 0 \leq u_t \leq 1 \end{cases} \tag{11}$$

2. The path $(\mathbf{n}^*)$ *is solution to the firm problem*[4] *:*

$$\Pi^* = \max\left\{\int_0^\infty p_t^* \left(h_t^*\right)^\gamma \left(n_t h_t^*\right)^\alpha dt - \int_0^\infty w_t^* n_t h_t^* dt\right\} \tag{12}$$

3. Markets clear:

$$c_t^* = \left(h_t^*\right)^\gamma \left(u_t^* h_t^*\right)^\alpha \tag{13}$$

$$n_t^* = u_t^* \tag{14}$$

Lemma 6 *A* C^2-*competitive equilibrium is an equilibrium.*

PROOF. We show that the solution to the competitive equilibrium attains the max in problem (P1) and that the constraints and the equilibrium condition are satisfied. Remark first that (i) the equilibrium condition is necessarily satisfied in the competitive equilibrium, that (ii) constraints on $\dot{h}_t$ and u_t are satisfied by construction, and that (iii) the constraint on c_t is satisfied by the market clearing condition on the competitive equilibrium.

Let $\mathbf{c}^*$ be the path such that $c_t^* = \left(u_t^*\right)^\alpha \left(h_t^*\right)^{\alpha+\gamma}$ and $\mathbf{c}$ be a feasible path such that $c_t = \left(u_t h_t\right)^\alpha \left(h_t^*\right)^\gamma$. Consider the program (11). It can be easily shown that there exists $\zeta > 0$ such that $\zeta p_t^* = e^{-\rho t}\left(c_t^*\right)^{\mu-1}$. We now show that: $\int_0^\infty \frac{1}{\mu} c_t^{*\mu} e^{-\rho t} dt \geq \int_0^\infty \frac{1}{\mu} c_t^\mu e^{-\rho t} dt$. Using the concavity of the utility function, it is equivalent to: $\int_0^\infty c_t^{*\mu-1}\left[c_t^* - c_t\right] e^{-\rho t} dt \geq 0$ and consequently, up to a positive constant, to: $\int_0^\infty p_t^*\left[c_t^* - c_t\right] dt \geq 0$. Using the constraint (11), it turns to be equivalent to: $\int_0^\infty w_t^*\left(u_t^* h_t^* - u_t h_t\right) dt \geq 0$. It is then sufficient to show that: $\int_0^\infty w_t^* u_t^* h_t^* dt = \max \int_0^\infty w_t^* u_t h_t dt$, where the max is taken over all (u,h) which satisfy the first constraint in (P1). Suppose it is not true and that there are (u_t, h_t) such that: $\int_0^\infty w_t^* u_t^* h_t^* dt < \int_0^\infty w_t^* u_t h_t dt$ or equivalently such that: $\int_0^\infty p_t^* c_t^* dt < \int_0^\infty w_t^* u_t h_t dt + \Pi^*$. Then it should exist a function ε, which is positive on a set with positive Lebesgue measure, such that: $\int_0^\infty p_t^*\left(c_t^* + \varepsilon\right) dt < \int_0^\infty w_t^* u_t h_t dt + \Pi^*$. This should imply: $\int_0^\infty \frac{\left(c_t^*+\varepsilon\right)^\mu}{\mu} e^{-\rho t} dt < \int_0^\infty \frac{\left(c_t^*\right)^\mu}{\mu} e^{-\rho t} dt$ which is impossible. □

Lemma 7 *Assume H1-H4.*

(i) Let $\Phi'(0) > 1$*, then there exists* $\varepsilon > 0$ *such that if* $\alpha + \gamma \leq \left(1 \wedge \frac{\rho}{\lambda\mu}\right) + \varepsilon$*, the equilibrium is a* C^2-*competitive equilibrium.*

(ii) If $\Phi'(0) = 1$*, there is no* C^2-*competitive equilibrium.*

[4] Remark that it is equivalent to choose the path $(\mathbf{nh})^*$.

PROOF. At the equilibrium, one has, up to a positive scalar, $\hat{p}_t = e^{-\rho t} (\hat{c}_t)^{\mu-1}$, $\hat{w}_t = \alpha \hat{p}_t (\hat{n}_t)^{\alpha-1} \left(\hat{h}_t\right)^{\alpha+\gamma-1}$, and $\hat{c}_t = \left(\hat{h}_t\right)^{\alpha+\gamma} (\hat{n}_t)^{\alpha}$. We proceed showing that, at the equilibrium: *(a)* the consumer satisfaction is maximized, *(b)* the firm profit is maximized, *(c)* the wage path belongs to L^1_w only if $\Phi'(0) > 1$, and *(d)* the price path belongs to L^1_p if $\Phi'(0) > 1$ and $\alpha+\gamma \leq 1+\varepsilon$ for some $\varepsilon > 0$.

(a) With (11), the utility is maximized if $\Delta \geq 0$ with:

$$\Delta \equiv \int_0^\infty \frac{\hat{c}_t^\mu}{\mu} e^{-\rho t} dt - \int_0^\infty \frac{c_t^\mu}{\mu} e^{-\rho t} dt,$$

for any feasible path c_t. The concavity of the instantaneous utility function implies $\left(\hat{c}_t^\mu - c_t^\mu\right)/\mu \geq \hat{c}_t^{\mu-1} (\hat{c}_t - c_t)$, which yields:

$$\Delta \geq \int_0^\infty \hat{p}_t (\hat{c}_t - c_t)\, dt.$$

Then, using the consumer's budget constraint and recalling that $u_t = \Psi(\nu)$ where $\nu = (dh_t/dt)/h_t$, one has:

$$\int_0^\infty \hat{p}_t (\hat{c}_t - c_t)\, dt = \int_0^\infty \hat{w}_t \left[\Psi(\hat{\nu}) \hat{h}_t - \Psi(\nu) h_t\right] dt.$$

By concavity, one has:

$$\begin{aligned} \Delta \geq & \int_0^\infty \hat{w}_t \left[\Psi(\hat{\nu}) - \Psi'(\hat{\nu}) \hat{\nu}\right] \left(\hat{h}_t - h_t\right) dt \\ & + \int_0^\infty \hat{w}_t \Psi'(\hat{\nu}) \left(\frac{d\hat{h}_t}{dt} - \frac{dh_t}{dt}\right) dt. \end{aligned}$$

Observing that $\hat{w}_t = \alpha e^{-\rho t} (\hat{n}_t)^{\alpha\mu-1} \left(\hat{h}_t\right)^{(\alpha+\gamma)\mu-1}$ and replacing yields:

$$\begin{aligned} \Delta \geq & \int_0^\infty \alpha e^{-\rho t} (\Psi(\hat{\nu}))^{\alpha\mu-1} \left(\hat{h}_t\right)^{(\alpha+\gamma)\mu-1} \left[\Psi(\hat{\nu}) - \Psi'(\hat{\nu}) \hat{\nu}\right] \left(\hat{h}_t - h_t\right) dt \\ & + \int_0^\infty \alpha e^{-\rho t} (\Psi(\hat{\nu}))^{\alpha\mu-1} \left(\hat{h}_t\right)^{(\alpha+\gamma)\mu-1} \Psi'(\hat{\nu}) \left(\frac{d\hat{h}_t}{dt} - \frac{dh_t}{dt}\right) dt. \end{aligned}$$

Now, replacing the Euler condition for problem (P2) in the first integral of the right hand side and integrating by parts, one obtains:

$$\Delta \geq \left[\alpha \left(e^{-\rho t} (\Psi(\hat{\nu}))^{\alpha\mu-1} \left(\hat{h}_t\right)^{(\alpha+\gamma)\mu-1} \Psi'(\hat{\nu})\right) \left(\hat{h}_t - h_t\right)\right]_0^\infty$$

and since $\hat{h}_0 = h_0$, one has:

$$\Delta \geq \lim_{t\to\infty} \alpha e^{-\rho t} (\Psi(\hat{\nu}))^{\alpha\mu-1} \left(\hat{h}_t\right)^{(\alpha+\gamma)\mu} \Psi'(\hat{\nu})$$

Using the transversality condition for problem (P2), conclude that $\Delta \geq 0$.

(b) With (12), the profit is maximized if:

$$\alpha \hat{p}_t \left(\hat{h}_t\right)^{\gamma+\alpha} (\hat{n}_t)^{\alpha-1} = \hat{w}_t \hat{h}_t.$$

By replacing $\hat{p}_t$ and $\hat{w}_t$ conclude that it is true.

(c) The equilibrium wage path belongs to L^1_w, only if $\Phi'(0) > 1$. Replacing $\hat{w}_t$, one has:

$$\int_0^\infty |w_t| e^{\lambda t} dt = \alpha (\Psi(\hat{\nu}))^{\mu\alpha-1} h_0^{\mu(\alpha+\gamma)-1} \int_0^\infty \left(e^{(\lambda-\rho)t} e^{\hat{\nu}(\mu(\alpha+\gamma)-1)t}\right) dt.$$

Observe with Theorem 1 that this integral is finite when $\Phi'(0) > 1$ because $\hat{\nu} > (\lambda - \rho) / [1 - \mu(\alpha+\gamma)]$ and that, when $\Phi'(0) = 1$, the integral is infinite.

(d) The equilibrium price path belongs to L^1_p if $\Phi'(0) > 1$ and $\alpha + \gamma \leq 1 + \varepsilon$ for some $\varepsilon > 0$. Using $\hat{p}_t$, one indeed has:

$$\int_0^\infty |p_t| e^{\lambda(\alpha+\gamma)t} dt = h_0^{(\mu-1)(\alpha+\gamma)} (\Psi(\hat{\nu}))^{(\mu-1)\alpha} \int_0^\infty e^{-[\rho-\lambda(\alpha+\gamma)]t} e^{\hat{\nu}(\mu-1)(\alpha+\gamma)t} dt.$$

The integral is finite if $\hat{\nu} > [\lambda(\alpha+\gamma) - \rho] / [(1-\mu)(\alpha+\gamma)]$. Deduce from Theorem 1, that the latter inequality is true for $\alpha + \gamma \leq 1$. Since $\hat{\nu}$ is continuous with respect to $(\alpha+\gamma)$ and since the function $\zeta \in]0, +\infty[\to [\lambda\zeta - \rho] / [(1-\mu)\zeta]$ is increasing, then, for $\varepsilon > 0$, sufficiently small, if $0 < \alpha + \gamma \leq 1 + \varepsilon$, then $\hat{\nu} > [\lambda(\alpha+\gamma) - \rho] / [(1-\mu)(\alpha+\gamma)]$. Moreover, since $\alpha + \gamma < \rho/\lambda\mu$, we obtain that $\alpha + \gamma < \min\{1, \rho/\lambda\mu\} + \varepsilon$ implies that $\hat{\nu} > [\lambda(\alpha+\gamma) - \rho] / [(1-\mu)(\alpha+\gamma)]$. □

As a corollary of lemmas 3 and 4, we have:

Theorem 8 *(i) Assume H1-H4 and $\Phi'(0) > 1$; there exists $\varepsilon > 0$ such that if $\alpha + \gamma \leq \left(1 \wedge \frac{\rho}{\lambda\mu}\right) + \varepsilon$, then there exists a unique C^2-competitive equilibrium.*

(ii) Assume H1-H4 and $\Phi'(0) = 1$; there is no competitive equilibrium.

PROOF. Given the previous results, it is immediate. □

Corollary 9 *When there is no externalities (i.e. for $\gamma = 0$), the optimal solution is a C^2-competitive equilibrium if $\Phi'(0) > 1$ while it is not if $\Phi'(0) = 1$.*

5 Conclusion

In this paper, we have developed a simple economic model in which human capital accumulation yields endogenous growth. We use this framework to show that equilibria defined as the solution to a fixed-point problem, aren't always competitive equilibria with externalities. This result hinges on the existence conditions which are more stringent for a competitive equilibrium. In a nutshell, the equilibrium may exist provided that the optimization problem is restricted to a set of trajectories that yield a finite utility while the existence of the competitive equilibrium requires further restrictions on the spaces of prices and wages. Consequently, there is little doubt that our result extend to more relevent economic models that should include, for instance, physical capital. Nevertheless, the formal demonstration would not be easy since it requires to prove the existence of a solution to the fixed-point problem.

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