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Subjective expected utility theory* with “small worlds”

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Abstract

We model the notion of a “small world” as a context dependent state space embedded into the “grand world”. For each situation the decision maker creates a “small world” reflecting the events perceived to be relevant for the act under consideration. The “grand world” is represented by an event space which is a more general construction than a state space. We retain preference axioms similar in spirit to the Savage axioms and obtain, without abandoning linearity of expectations, a subjective expected utility theory which allows for an intuitive distinction between risk and uncertainty. We also obtain separation of subjective probability and utility as in the state space models.

JEL classification: D8 and G12.

Key words: Subjective expected utility, decision making under uncertainty, uncertainty aversion, Ellsberg paradox.

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1 Introduction

In this paper we develop a new and more general theory of subjective expected utility. Similar to Savage, we assume that the decision maker which faces the “grand world”, for each group of related decisions, creates a “small world” consisting of only those events which are considered relevant in the given context. This may be interpreted as a cognitive process, where, before a decision is taken, it is grouped together with other decisions in a small and more manageable world. In the model events belonging to a single “small world” are only risky, while uncertainty is related to the comparison of events across different “small worlds”.

One can view our setup as a way of capturing Savage’s notion of a small world in a way which is flexible enough to allow for the introduction of a notion of uncertainty aversion. The idea that the decision maker creates small worlds from the set of all possible “events in the world” is an integral part of Savage’s decision theory (Savage (1954)). Savage however only considered “small worlds” created by partitioning a state space. Alternatively, one may view our paper as an attempt to generalize existing models of state-dependent utilities where, as is pointed out in Schervish, Seidenfeld, and Kadane (1990), small world acts can be ranked in a consistent manner within the grand world by multiplying by suitable constants.

In each “small world” we rely on the axioms in Fishburn (1973) together with an additional axiom which lead to a Savage-type expected utility description. Fishburn’s generalization of the Savage theory ensures that the decision maker is able to decide taking into account given “objective” probabilities in the same way as suggested by von Neumann and Morgenstern (1947) for game theory and only assign subjective probabilities to the states on which the acts are defined. To obtain a generalized subjective expected utility formulation for the “grand world” we introduce two additional axioms which put relatively mild additional restrictions on preferences across “small worlds”.

A new feature in our model is the introduction of a so called event space to represent the events in the “grand world”. An event space is a set of projections satisfying a number of requirements to be discussed in Section 3. It is convenient to use projections to represent events because sets of projections naturally are equipped with the same lattice operations normally associated with an event structure. We are not losing any generality by applying an event space formalism¹. The usage of an event space formalism leads to an expected subjective utility theory with a linear expectation functional, and a natural separation of risk and uncertainty. The assignment of (possibly subjective) events to projections must satisfy the general rules for the interplay of events. The projection assigned to

¹The notion of an event space was introduced in Hansen (2003), who proved that under very mild conditions any state space with a σ -algebra is isomorphic to an event space.

a smaller event must be contained in the projection assigned to a more comprehensive event, and the projection assigned to the joining (union) of two events must be represented by the minorant (majorant) of the projections assigned to each of the events. These requirements or rules put relatively mild conditions to the assignment of events to projections for coarse experiments. Rich experiments lead to an essential unique assignment and consequently to a unique separation of subjective probability and utility, and a fixed value of the uncertainty aversion, cf. Theorem 2.

Both the classical state space formalism and our construction lead to expected subjective utility theories with separation between subjective probability and utility. But it is important to realize that the separation only applies to models that satisfy the full set of axioms specified in the theory (the state space formalisms may come in somewhat different versions depending on the theoretical foundation). In contrast, if one considers a coarse experiment with only a limited number of acts to consider, then both frameworks allow non-equivalent models consistent with the preferences revealed by the decision maker in the coarse experiment. An implication of this observation is that additional questions added to a coarse experiment may be answered in non-unique ways without compromising consistency. Additional questions may thus lead to non-isomorphic models with different resolutions of subjective probability and utility which still are consistent with the preferences revealed in the initial experiment.

This phenomenon may be even more pronounced in our model as demonstrated in Example 6.1 where the two-color Ellsberg experiment is studied. In this example we present different assignments of projections to events all leading to an accurate representation of the observed preferences, but with different subjective probabilities and different levels of uncertainty aversion. This finding may be attributed to the lack of information provided by a rather coarse experiment that does not reveal all aspects of the decision makers preferences.

The Fishburn model is presented in section 2, and the notion of an event space is introduced in section 3. The preference relations in the “grand world” are listed and discussed in section 3.3, and the main representation result is proved and a measure of uncertainty aversion introduced in section 4. We reconsider the Ellsberg paradox, in our framework, in section 5, and we compare the approach taken in this paper with the literature in the final section 6.

2 The Fishburn model

The standard subjective expected utility model is well-known to most readers, but since the underlying assumptions come in slightly different versions we shall take the effort to specify the axioms underpinning our use of the model. We choose to

rely on Fishburn’s rendition of the Luce-Krantz axioms for two reasons. First, we make sure that a decision maker uses the utility function provided by the subjective expected utility theorem to evaluate also objective lotteries not associated with acts. This is nicely provided for in Fishburn’s setup and is used in our analysis across “small worlds” to be introduced later. Secondly, Fishburn’s setup elucidates the non-uniqueness of the standard model in fairly general situations.

Definition 1 *An act (basic act) is a measurable map $x: \Omega \rightarrow C$ defined on a state space Ω equipped with a σ -algebra \mathcal{E} , where C is the set of consequences. The elements in \mathcal{E} are called events, and the set of non-empty events is denoted by \mathcal{E}' .*

The set of consequences is equipped with an affine structure and is convex.

Definition 2 *We consider for any consequence $c \in C$ and any event A the constant act c defined by setting $c(A) = c$ for every event A .*

Some authors see it as a problem if there are too many constant acts. The reason is that some consequences may be so dire, that it is inconceivable that they may be chosen regardless of the obtaining events. These kind of considerations will be ignored and may at most limit the usage of the theory.

Definition 3 *A convex combination of (basic) acts x_1, \dots, x_n given by*

$$x(s) = \sum_{i=1}^n t_i x_i(s),$$

where $t_i \geq 0$ and $t_1 + \dots + t_n = 1$, is called a mixed act. The factors t_i are sometimes interpreted as probabilities.

Convex combinations of mixed acts are again naturally interpreted as mixed acts. The set of mixed acts is a mixture set in the sense of Herstein and Milnor (1953). A basic act x may be thought of as a mixed act that assigns probability 1 to x .

Definition 4 *A mixed conditional act $x|_A$ is the restriction $x: A \rightarrow C$ of a mixed act x to an event $A \in \mathcal{E}'$.*

Let X denote a non-empty convex set of mixed acts². The primary datum in Fishburn’s version of the standard model is a binary preference relation \succeq over $L = \{x|_A \mid x \in X, A \in \mathcal{E}'\}$ that satisfies the following axioms:

- (i) **Totality:** For all $x|_A$ and $y|_B$ we have either $x|_A \succeq y|_B$ or $y|_B \succeq x|_A$.

²We are here slightly simplifying Fishburn’s model.

(ii) **Transitivity:** If $x|_A \succeq y|_B$ and $y|_B \succeq z|_C$ then $x|_A \succeq z|_C$.

A total and transitive order relation is also called a *weak ordering*.

(iii) **Archimedean continuity:** The sets

$$\{t \in [0, 1] \mid (tx + (1-t)y)|_A \succeq z|_B\}, \{t \in [0, 1] \mid z|_B \succeq (tx + (1-t)y)|_A\}$$

are closed for arbitrary $x, y, z \in X$ and $A, B \in \mathcal{E}'$.

(iv) **Mixture indifference:** If $x|_A \sim z|_B$ and $y|_A \sim w|_B$, then

$$\frac{1}{2}x|_A + \frac{1}{2}y|_A \sim \frac{1}{2}z|_B + \frac{1}{2}w|_B$$

for arbitrary $x, y, z, w \in X$ and $A, B \in \mathcal{E}'$.

(v) **Averaging condition:** If $A \cap B = \emptyset$ and $x|_A \succeq x|_B$ then

$$x|_A \succeq x|_{A \cup B} \succeq x|_B$$

for $x \in X$ and $A, B \in \mathcal{E}'$.

(vi) **Non-degeneracy:** There exist $x, y \in X$ such that $x \succ y$.

(vii) **Weak act richness:** If $A \cap B = \emptyset$ then

$$x|_A \succ x|_B \quad \text{and} \quad y|_B \succ y|_A$$

for some acts x and y .

(viii) **Strong act richness:** If A, B and C are mutually disjoint, and if there is an act $x \in X$ such that $x|_A \sim x|_B$ then there is an act $y \in X$ such that exactly two of the acts $y|_A, y|_B$ and $y|_C$ are equivalent.

It is a main feature of the model that the decision maker only need to have preferences over a rather restricted set X of mixed acts, for example a set generated by only a few basic acts, and their restrictions to the non-empty events. The state space may be finite and the set of events \mathcal{E} may be a “small” σ -algebra on the state space.

Theorem 1 (Fishburn 1973) *Assume that the axioms (i) through (viii) are satisfied. Then there exists a map $u: L \times \mathcal{E}' \rightarrow \mathbf{R}$ and for each $A \in \mathcal{E}'$ a finitely additive probability measure P_A on $\{A \cap B \mid B \in \mathcal{E}\}$ such that*

$$(i) \quad x|_A \succ y|_B \quad \text{if and only if} \quad u(x|_A) > u(y|_B)$$

for all acts $x|_A$ and $y|_B$ in $L \times \mathcal{E}'$.

(ii) $x \rightarrow u(x|_A)$ is a linear function on X for each $A \in \mathcal{E}'$.

(iii) $P_C(A) = P_C(B)P_B(A)$,

whenever $A \subseteq B \subseteq C$ for $A \in \mathcal{E}$ and $B, C \in \mathcal{E}'$.

(iv) $u(x|_{A \cup B}) = P_{A \cup B}(A)u(x|_A) + P_{A \cup B}(B)u(x|_B)$

whenever $x \in X$, $A, B \in \mathcal{E}'$ and $A \cap B = \emptyset$.

The map u is uniquely defined up to an increasing affine transformation, and the probability measures P_A are uniquely defined for each $A \in \mathcal{E}'$.

The statement in (iv) is extended by induction to

$$u(x) = u(x|\Omega) = \sum_{j=1}^n u(x|_{A_j})P(A_j)$$

for a (mixed) act x and a finite partition A_1, \dots, A_n of Ω with each $A_j \in \mathcal{E}'$. If in addition $x = \sum_{i=1}^m \lambda_i x_i$ we obtain from (ii) the formula

$$u\left(\sum_{i=1}^m \lambda_i x_i\right) = \sum_{i=1}^m \lambda_i \sum_{j=1}^n u(x_i|_{A_j})P(A_j). \quad (2.1)$$

This is more flexible than in Savage's theory. If for example $c \in X$ is the constant act with consequence $c \in C$ then

$$u(c) = \sum_{j=1}^n u(c|_{A_j})P(A_j). \quad (2.2)$$

We can therefore model that the constant act of getting an umbrella is more utile when it is raining than otherwise. But we retain the attractive property, to be used later, that the utility of an unconditional constant act is the subjectively weighted average of utilities of the corresponding conditional constant acts. We will eventually add two more axioms to Fishburn's list. The first is straightforward although controversial in some settings.

(ix) **Richness of constant acts:** The set of acts X contains the constant act c associated with each consequence $c \in C$.

Fishburn's model allows different acts to be subjectively indistinguishable to the decision maker. Consider an (unconditional) act x with finite many consequences c_1, \dots, c_n and set $A_j = \{s \in \Omega \mid x(s) = c_j\}$ for $j = 1, \dots, n$. The corresponding constant acts (also denoted by c_1, \dots, c_n) are in X by axiom (ix), hence the mixed act

$$\tilde{x} = \sum_{j=1}^n P(A_j)c_j$$

is also in X , since X is a convex set. The two acts x and \tilde{x} are subjectively indistinguishable to the decision maker. Indeed, \tilde{x} is an objective lottery between the consequences c_1, \dots, c_n with probabilities $P(A_1), \dots, P(A_n)$, and this is exactly how x is perceived by the decision maker who subjectively assigns the same probabilities $P(A_1), \dots, P(A_n)$ to the events A_1, \dots, A_n with outcomes c_1, \dots, c_n . It seems natural to assume that the decision maker is indifferent between two acts which are subjectively indistinguishable.

(x) **Equivalence:** Subjectively indistinguishable acts are equivalent.

It is worthwhile to discuss whether such a condition is behavioral or functional. We would argue that it is behavioral since the decision maker knows by his own perceptions whether two given acts are indistinguishable. It is only an outside observer that need to calculate probabilities before it can be established analytically whether two acts are subjectively indistinguishable to the decision maker.

The equivalence axiom (x) states that the two acts x and \tilde{x} considered above are equivalent. The utility of x is given by

$$u(x) = \sum_{j=1}^n u(x|_{A_j})P(A_j)$$

according to (2.1), and the utility of \tilde{x} is given by

$$\begin{aligned} \tilde{u}(x) &= \sum_{j=1}^n u(\tilde{x}|_{A_j})P(A_j) \\ &= \sum_{j=1}^n u\left(\sum_{i=1}^n P(A_i)c_i \Big|_{A_j}\right)P(A_j) \\ &= \sum_{j=1}^n \sum_{i=1}^n P(A_i)u(c_i|_{A_j})P(A_j) \\ &= \sum_{i=1}^n u(c_i)P(A_i), \end{aligned}$$

where we first used the linearity (ii) and then (2.2). The equivalence axiom thus leads to the formula

$$u(x) = \sum_{i=1}^n u(c_i)P(A_i). \quad (2.3)$$

But this is exactly Savage's expected utility function where the (state independent) utility of consequences are weighted with the subjective probabilities of the events leading to the consequences.³ We finally introduce the following axiom that only serves to facilitate subsequent proofs.

- (xi) **Certainty equivalent:** To each act in X there is an equivalent constant act.

3 A model of the “grand world”

We have studied the Fishburn model which we shall take as sample model of a small world. The problem is how to define a model of the “grand world” and into this fit numerous “small worlds” each enjoying the characteristics found in the Fishburn model.

The “grand world” is represented by an event space, which we now are going to introduce. The notion generalizes the notion of a state space with a σ -algebra. We shall use projections to represent events and begin by demonstrating that ensembles of projections are equipped with exactly the properties we naturally associate with the hierarchy and logical rules for the interplay of events.

3.1 An event space model of the “grand world”

Definition 5 (Event space) An event space is a pair (\mathcal{F}, H) of a (separable) Hilbert space H and a family \mathcal{F} of projections on H satisfying:

- (i) The zero projection on H (denoted 0) and the identity projection on H (denoted 1) are both in \mathcal{F} .
- (ii) $1 - P \in \mathcal{F}$ for arbitrary $P \in \mathcal{F}$.
- (iii) The minorant projection $P \wedge Q \in \mathcal{F}$ for arbitrary $P, Q \in \mathcal{F}$.
- (iv) $\sum_{i \in I} P_i \in \mathcal{F}$ for any family $(P_i)_{i \in I}$ of mutually orthogonal projections in \mathcal{F} .

³Fishburn (1973) shows that the subjective probabilities are not necessarily uniquely determined if the strong act richness axiom (viii) is dropped from the list.

We begin by listing some comments directly pertinent to the definition of an event space.

- The family \mathcal{F} inherits the natural (partial) order relation $P \leq Q$ for projections on a Hilbert space. Notice that $0 \leq P \leq 1$ for arbitrary events $P \in \mathcal{F}$.
- We define a bijective mapping $P \rightarrow P^\perp$ of \mathcal{F} onto itself by setting $P^\perp = 1 - P$. The event P^\perp is called the event complementary to P .
- The minorant projection $P \wedge Q$ is the projection on the intersection of the ranges of P and Q . It has the property that $R \leq P \wedge Q$ for any event $R \in \mathcal{F}$ such that both $R \leq P$ and $R \leq Q$.

The majorant projection $P \vee Q$ is the projection on the closure of the sum of the ranges of P and Q . It has the property that $P \vee Q \leq R$ for any event $R \in \mathcal{F}$ with $P \leq R$ and $Q \leq R$. Since

$$P \vee Q = 1 - (1 - P) \wedge (1 - Q)$$

it follows that \mathcal{F} is closed also under majorant formation.

- Condition (iv) in the definition is a technical requirement⁴ ensuring that \mathcal{F} is closed under arbitrary formation of minorants or majorants. Thus to any family $(P_i)_{i \in I}$ of events in \mathcal{F} there is a minorant event $\bigwedge_{i \in I} P_i$ and a majorant event $\bigvee_{i \in I} P_i$ both contained in \mathcal{F} .

An event space has a number of properties which are natural for the representation of events.

- An event space contains the projections 0 and 1 corresponding respectively to the vacuous (empty) event and the universal (sure) event.
- There is a partial order relation \leq defined in \mathcal{F} such that any event $P \in \mathcal{F}$ is placed between the vacuous and the universal events, that is $0 \leq P \leq 1$. More generally, for two events P and Q in \mathcal{F} we consider Q to be a larger, more comprehensive event than P if $P \leq Q$.⁵ The interpretation is that we know for sure that the event Q occurs (obtains) if P occurs.
- The joining of two events P and Q in \mathcal{F} is represented by $P \wedge Q$ and the union is represented by $P \vee Q$, and these are both included in the event space \mathcal{F} .⁶ It follows from (iv) that \mathcal{F} is even closed under the joining or union of arbitrary families of events.

⁴The condition corresponds to the requirement that a σ -algebra is complete.

⁵It corresponds to the statement $A \subseteq B$ for measurable subsets A and B of a state space.

⁶We express this by saying that \mathcal{F} is a lattice.

The bijective mapping $P \rightarrow P^\perp = 1 - P$ of \mathcal{F} which associates an event with its complimentary event has the following natural properties:

- More comprehensive events have smaller complementary events, ie $P \leq Q \Rightarrow Q^\perp \leq P^\perp$ for all $P, Q \in \mathcal{F}$.
- The joining between an event and its complementary event is the empty event, ie. $P \wedge P^\perp = 0$ for all $P \in \mathcal{F}$.
- The union between an event and its complementary event is the sure event, ie. $P \vee P^\perp = 1$ for all $P \in \mathcal{F}$.
- The complementary event to the complementary event to an event is the event itself, ie. $P^{\perp\perp} = P$ for all $P \in \mathcal{F}$.

Suppose that the complementary event to a given event Q is more comprehensive than another event P , meaning that if P obtains then so does the complement to Q . If the events are represented by projections (here also denoted P and Q) on a Hilbert space H , then the condition is equivalent to the requirement $P \leq 1 - Q = Q^\perp$ which means that the ranges of P and Q are orthogonal subspaces of H . For this reason it becomes natural to say that such events are orthogonal.

Definition 6 *We say that events P and Q in \mathcal{F} are mutually exclusive if the minorant $P \wedge Q = 0$, and we say that P and Q are orthogonal⁷ if $P \leq Q^\perp$.*

It readily follows that orthogonal events are mutually exclusive. However, it may happen that mutually exclusive events are not orthogonal, and it is exactly because of this possibility that an event space generally differs from a state space. Hansen (2003) demonstrates that every state space with a σ -algebra is (under very mild conditions) isomorphic to an event space. Furthermore, an event space is isomorphic to a state space with a σ -algebra (satisfying the same mild conditions as in the first result), if and only if each pair of mutually exclusive events are orthogonal. Hansen (2003) also proves that if mutually exclusive projections are orthogonal then they necessarily commute⁸, cf. Hansen (2003, Theorem 4.3). Therefore, if an event space only contains commuting projections then it is isomorphic to a state space with a σ -algebra. On the other hand, if an event space contains non-commuting projections then it cannot be associated with a state space.

In the remainder of the paper we assume, to avoid unnecessary technical difficulties, that the Hilbert space H is of finite dimension. This corresponds to assuming a finite state space in the standard model.

⁷Note that the definition is symmetric in P and Q , ie. $P \leq Q^\perp$ if and only if $Q \leq P^\perp$.

⁸Two projections P and Q commute if $PQ = QP$. Note that the multiplicative structure plays no direct role in the theory.

3.2 An event space with embedded “small worlds”

Given an event space representing the “grand world,” a “small world” is simply a subdivision of the sure event into those risky events which are pertinent for a particular set of acts. In what follows it will be clear that a “small world” basically functions like a local state space. As such it fits neatly into Savage’s concept of “neglecting some distinctions between states”.

Definition 7 A “small world” is a set $\{P_1, \dots, P_n\}$ of projections in \mathcal{F} with sum $P_1 + \dots + P_n = 1$, where 1 denotes the identity projection representing the sure event⁹. The set of small worlds is denoted by $P(H)$.

The events in a “small world” are mutually exclusive and their majorant event is the sure event. Therefore exactly one of the events obtains. This is why a “small world” essentially acts like a local state space. The events in the “small world” function as (local) states and the obtaining event as the “true state of nature”. The set of small worlds $P(H)$ thus becomes a set of state spaces, each describing a certain part of the “grand world” represented by the event space (\mathcal{F}, H) .

3.3 Preferences in the “grand world”

Let C be a common set of consequences. We consider, for each “small world” $\alpha \in P(H)$, a set L_α of (local) acts defined in α with consequences in C . The set of (global) acts is defined by setting

$$L = \bigcup_{\alpha \in P(H)} L_\alpha$$

and the “grand world” preferences are specified by a binary preference relation \succeq over L . Note that every act is local in the sense that it belongs to a specific “small world” but that the preference relation is given over L .

We assume that the restriction of \succeq to each set of (local) acts L_α satisfies the axioms (i) through (xi).

We indicate that a constant act corresponding to a consequence $c \in C$ is defined relative to a local state space $\alpha \in P(H)$ by writing (α, c) . Note that one can consider an objective lottery with consequences $c = (c_1, \dots, c_n)$ and probabilities $p = (p_1, \dots, p_n)$ as a mixed act in α and write $(\alpha, (c, p))$.

We introduce two new axioms for the preferences in the “grand world”.

(xii) **Indifference:** Let $(\alpha, (c, p))$ and $(\alpha, (d, q))$ be lotteries between constant acts in a “small world” $\alpha \in P(H)$. Then

$$(\alpha, (c, p)) \succeq (\alpha, (d, q)) \quad \Rightarrow \quad (\beta, (c, p)) \succeq (\beta, (d, q))$$

⁹Note that the projections P_1, \dots, P_n automatically are mutually orthogonal.

for any other “small world” $\beta \in P(H)$.

The axiom states that the ordering of lotteries of constant acts does not depend on the “small world” in which they are considered. It may be interpreted as the requirement that an objective lottery should be equally attractive, independent of the context in which it is available.

- (xiii) **Separation:** Let $\alpha, \beta \in P(H)$ be “small worlds” with a common event $P \in \alpha \cap \beta$. There exist equivalent actions (α, f) and (β, g) in L and non-equivalent consequences $a, b \in C$ such that

$$f(P) \sim g(P) \sim a \quad \text{and} \quad f(Q) \sim g(R) \sim b.$$

for every $Q \in \alpha \setminus \{P\}$ and $R \in \beta \setminus \{P\}$.

In the axiom the common event P functions as a local state in both α and β . The equivalent actions (α, f) and (β, g) may be interpreted as two bets, one in each of the two “small worlds”, on the local state P . If P obtains then both bets have outcomes equivalent to consequence a . If P does not obtain then both bets have outcomes equivalent to consequence b .

4 The subjective expected utility model

Before we state and prove the main result, we proceed by demonstrating that the introduced axioms give rise to a common utility function for all small worlds. We also demonstrate that the decision maker assigns subjective probability in a consistent way across “small worlds”.

4.1 Common utility

We first note that for each small world $\alpha \in P(H)$, the preference relation on L induces a preference relation \succeq_α on C by setting

$$c \succeq_\alpha d \quad \text{if} \quad (\alpha, c) \succeq (\alpha, d)$$

for consequences c and d in C . It follows from the indifference axiom that all the order relations \succeq_α induced on C for $\alpha \in P(H)$ are equivalent. We may therefore suppress the subscript in \succeq_α and just write

$$c \succeq d \quad \text{if} \quad (\alpha, c) \succeq (\beta, d)$$

for consequences c and d in C , and small worlds $\alpha, \beta \in P(H)$.

Since axioms (i) through (xi) are assumed there exist, for each “small world” $\alpha \in P(H)$, a subjective probability measure E_α and a (local) utility function u_α such that the preferences in L_α are represented by the (local) subjective expected utility function

$$U_\alpha(\alpha, f) = \sum_{i=1}^n E_\alpha(P_i)u_\alpha(f(P_i)),$$

cf. equation (2.3).

Lemma 1 *There exists a common utility function $u : C \rightarrow \mathbf{R}$, unique up to an increasing affine transformation, such that*

$$c \succ d \quad \text{if and only if} \quad u(c) > u(d)$$

for consequences $c, d \in C$. For each $\alpha \in P(H)$ the local utility function u_α is an increasing affine transformation of the common utility function u .

The proof may be found in Appendix A.1.

4.2 Subjective probabilities across “small worlds”

It is essential for the theory that a decision maker assigns subjective probability to an event independent of the “small world” in which it is considered.

Lemma 2 *If two “small worlds” $\alpha, \beta \in P(H)$ share a common event $P \in \alpha \cap \beta$, then necessarily $E_\alpha(P) = E_\beta(P)$, where E_α and E_β are the subjective probability measures, derived from the decision maker’s preferences, in each of the two small worlds.*

Proof: Consider two small worlds $\alpha, \beta \in P(H)$ with a common event $P \in \alpha \cap \beta$. We may write the small worlds on the form

$$\alpha = \{P, Q_1, \dots, Q_n\} \quad \text{and} \quad \beta = \{P, R_1, \dots, R_m\}.$$

By the separation axiom there exist equivalent actions (α, f) and (β, g) in L and non-equivalent consequences $a, b \in C$ such that

$$f(P) \sim g(P) \sim a \quad \text{and} \quad f(Q_i) \sim g(R_j) \sim b$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. The certainty equivalent axiom (vii) ensures the existence of a constant acts (α, c) and (β, d) such that

$$\begin{aligned} u(c) &= U_\alpha(\alpha, c) = U_\alpha(\alpha, f) \\ &= E_\alpha(P)u(f(P)) + \sum_{i=1}^n E_\alpha(Q_i)u(f(Q_i)) \\ &= E_\alpha(P)u(a) + (1 - E_\alpha(P))u(b) \end{aligned}$$

and similarly

$$u(d) = U_\beta(\beta, d) = U_\beta(\beta, g) = E_\beta(P)u(a) + (1 - E_\beta(P))u(b).$$

Since the constant acts (α, c) and (β, d) are equivalent by the transitivity axiom, we conclude that $u(c) = u(d)$. We have thus written $u(c)$ as two convex combinations of $u(a)$ and $u(b)$. Since $u(a) \neq u(b)$ we conclude that $E_\alpha(P) = E_\beta(P)$. **QED**

4.3 Main theorem

Lemma 2 ensures that we can unambiguously define a function

$$E : \mathcal{F} \rightarrow [0, 1]$$

by setting $E(P) = E_\alpha(P)$ for any small world $\alpha \in P(H)$ containing P . This function has the property

$$E(P_1) + \dots + E(P_n) = 1$$

for any sequence P_1, \dots, P_n of projections in \mathcal{F} with sum $P_1 + \dots + P_n = 1$. A function with this property is called a frame function, and such functions were studied by Mackey (1957), Gleason (1957), Varadarajan (1968), Piron (1976) and others. The following remarkable result was conjectured by Mackey and proved by Gleason.

Gleason's theorem *Let \mathcal{F} be the event space of projections on a (real or complex) separable Hilbert space H of dimension greater than or equal to three, and let $F : \mathcal{F} \rightarrow [0, 1]$ be a frame function. Then there exists a uniquely defined positive semi-definite trace class operator h on H with unit trace such that*

$$F(P) = \text{Tr}(hP)$$

for any $P \in \mathcal{F}$.¹⁰

We are now ready to state and prove the main result.

Theorem 2 *Let (\mathcal{F}, H) be the event space consisting of all projections on a (real or complex) Hilbert space of finite dimension greater than or equal to three, let C be a common set of consequences equipped with an affine structure, and let L be a set of actions. The primitive datum of the utility theory is a binary preference relation \succeq over the set L satisfying the axioms (i) through (xiii). Then there exists*

¹⁰Note that a frame function automatically is continuous by Gleason's theorem.

a map $u: C \rightarrow \mathbf{R}$, unique up to an increasing affine transformation, and a positive semi-definite operator h on H with unit trace such that

$$(\alpha, f) \succ (\beta, g) \quad \text{if and only if} \quad U(\alpha, f) > U(\beta, g)$$

for arbitrary acts (α, f) and (β, g) in L , where the expected utility function U is defined by setting

$$U(\alpha, f) = \sum_{i=1}^n \text{Tr}(hP_i)u(f(P_i))$$

for any act $(\alpha, f) \in L$ with small world $\alpha = \{P_1, \dots, P_n\}$.

Proof: The acts (α, f) and (β, g) in L are by the certainty equivalent axiom (xi) equivalent to constant acts (α, c) and (β, d) respectively, therefore we obtain

$$U(\alpha, f) = U(\alpha, c) = u(c) \quad \text{and} \quad U(\beta, g) = U(\beta, d) = u(d),$$

where U is defined as in the statement of the theorem. Since

$$(\alpha, f) \succ (\beta, g) \Leftrightarrow (\alpha, c) \succ (\beta, d) \Leftrightarrow u(c) > u(d),$$

the statement follows. **QED**

Note that the statement in the main result entails that the indifference axiom for preferences across small worlds is satisfied. The implication is that this axiom must be satisfied in any expected utility formulation of the given form.

4.4 Measuring uncertainty aversion

In this subsection we introduce a numerical measure of uncertainty aversion. With this purpose in mind, consider two events P and Q in an event space (\mathcal{F}, H) and a decision maker with preferences as given in Theorem 2. If the number

$$\nu(P, Q) = E(P \vee Q) - (E(P) + E(Q))$$

is positive, this is interpreted as a reflection of the decision maker's uncertainty aversion. We may think of an experiment in which a ball is drawn from an urn with an unknown distribution of red and black balls. The event P represents the drawing of a red ball while the event Q represents the drawing of a black ball. The union (majorant) of the two events $P \vee Q$ is the sure event so $E(P \vee Q) = 1$. The decision maker may assign so low probabilities to the individual events that their sum is less than the probability of the union, and hereby exhibit uncertainty aversion.

Let now P_1, \dots, P_n be events in \mathcal{F} with no further assumptions and consider the number

$$\nu(P_1, \dots, P_n) = E(P_1 \vee \dots \vee P_n) - \sum_{i=1}^n E(P_i).$$

This number is obviously less or equal to one and it may be negative. But if the events are part of a small world, then $P_1 \vee \dots \vee P_n = P_1 + \dots + P_n$ and $\nu(P_1, \dots, P_n) = 0$.

Definition 8 The number

$$\nu = \sup\{\nu(P_1, \dots, P_n) \mid P_1, \dots, P_n \in \mathcal{F}, n = 1, 2, \dots\}.$$

is defined as the decision maker's uncertainty aversion.

Note that the decision maker's uncertainty aversion $\nu \in [0, 1]$. It is determined as the largest possible difference between the weight attached to the union and the sum of the weights of the individual events. Note that by focusing on the "worst possible" situation the introduced measure of uncertainty aversion is linked to that of Schmeidler (1989).

Proposition 1 *Suppose that \mathcal{F} is the event space of all projections on a Hilbert space H , and let h be the positive semi-definite operator (matrix) on H with unit trace such that $E(P) = \text{Tr}(hP)$ for any event $P \in \mathcal{F}$. Then*

$$\nu = 1 - \lambda_{\min} \cdot \dim H,$$

where $\dim H$ is the finite dimension of the Hilbert space H and λ_{\min} is the minimal eigenvalue of the operator h .

Proof: Consider the expression $\nu(P_1, \dots, P_n)$ for events P_1, \dots, P_n . Since E is additive we may without loss of generality assume the majorant event $P_1 \vee \dots \vee P_n = 1$ and that all the constituent projections are one-dimensional. We may then discard events until all remaining events are needed to maintain the sure event as majorant. In this situation $n = \dim H$ and the remaining events are necessarily projections on a set of basis vectors in H . The supremum is then obtained by choosing a sequence of bases of H with each basis vector converging to an eigenvector for the minimal eigenvalue of h . **QED**

If the decision maker's uncertainty aversion $\nu = 0$, then the proposition entails that h is the identity operator on H (the identity matrix) divided by $\dim H$, hence

$$E(P) = \frac{\dim R(P)}{\dim H} \quad P \in \mathcal{F},$$

where $R(P)$ denotes the range of P . An uncertainty neutral ($\nu = 0$) decision maker is thus assigning likelihood to an event solely according to the dimension of the representing projection.

5 The Ellsberg paradox

Below we model Ellsberg’s experiment using our framework. We consider two versions, a two-color variation taken from Mas-Colell, Whinston, and Green (1995) as well as the original three-color thought experiment in Ellsberg (1961).¹¹

5.1 Two-color variation

There are two urns, denoted urn 1 and urn 2. Each urn contains 100 balls that are either white or black. Urn 1 contains 49 white balls and 51 black balls while Urn 2 contains an unspecified assortment of white and black balls. A ball has been picked randomly from each urn; we call them the 1-ball and the 2-ball, respectively. The colors of the chosen balls have not been disclosed. Now we consider two consecutive choice situations or experiments in which the decision maker must choose either the 1-ball or the 2-ball. After both choices have been made, the color will be disclosed. In the first choice situation, a prize is won if the chosen ball is black. In the second choice situation, the same prize is won if the ball is white.

With this information, most people will chose the 1-ball in the first experiment where the objective probability of winning is 0.51. There is no information available concerning the proportion of balls in urn 2, hence there is objectively complete symmetry between the two colors, white and black. One might therefore expect that most people would choose the 2-ball in the second experiment since the likelihood that the 1-ball is white is less than half. However, it turns out that this does not happen overwhelmingly in actual experiments. The decision maker understands that by choosing the 1-ball, he only has a 49 percent chance of winning. This chance however, is “safe” and well understood. The uncertainties incurred are much less clear if the 2-ball is chosen.

The combined likelihood of the two possible outcomes of drawing a ball from urn 2 is considered to be less than one although the two outcomes are mutually exclusive.

We may model this behavior by assigning the event “the 1-ball is black” to the projection P and the event “the 1-ball is white” to the projection $1 - P$, where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

¹¹This variation was mentioned already by Keynes (1921).

The two events are thus understood to be complementary. The matrices

$$Q_a = \begin{pmatrix} a & 0 & a^{1/2}(1-a)^{1/2} \\ 0 & 0 & 0 \\ a^{1/2}(1-a)^{1/2} & 0 & 1-a \end{pmatrix}$$

$$Q_b = \begin{pmatrix} b & 0 & b^{1/2}(1-b)^{1/2} \\ 0 & 1 & 0 \\ b^{1/2}(1-b)^{1/2} & 0 & 1-b \end{pmatrix}$$

are projections for $0 \leq a, b \leq 1$. We assign the “2-ball is black” event to Q_a and the “2-ball is white” event to Q_b for a, b with $0 < a, b < 1$ and $a \neq b$. With these assignments the joined event is vacuous and the union event is the sure event if $a \neq b$.

Note that Q_a and Q_b are mutually exclusive but not complementary events. In addition, none of the four projections introduced above are related by inclusion. We are therefore not forcing the decision maker to assume that the result of one experiment determines the outcome of the other.

Since the pay-offs are equal in the two experiments the subjective utility is proportional to the subjective likelihood of the outcomes in both experiments.

As already discussed, the likelihood $E(X)$ is calculated by $E(X) = \text{Tr}(hX)$, where h is determined by the decision makers preferences. We use in the example the positive semi-definite unit trace matrix h defined by

$$h = \begin{pmatrix} 0.49 & 0 & -0.2 \\ 0 & 0.25 & 0 \\ -0.2 & 0 & 0.26 \end{pmatrix}.$$

We have $E(P) = 0.49$ and $E(1 - P) = 0.51$ as anticipated. In addition, we calculate

$$E(Q_a) = 0.26 + 0.23a - 0.4a^{1/2}(1-a)^{1/2}$$

$$E(Q_b) = 0.51 + 0.23b - 0.4b^{1/2}(1-b)^{1/2}.$$

If we choose $0 < a < 1/2 \leq b < 1$, then by an elementary calculation we obtain

$$E[P] > E[Q_a] \quad \text{and} \quad E[1 - P] > E[Q_b].$$

The “1-ball is white” is thus preferred to the “2-ball is white” and the “1-ball is black” is preferred to the “2-ball is black” events as in the experiment. This phenomenon is not possible with a state space description. Different choices of the parameter values a and b (corresponding to different assignments of events to projections) lead in general to non-isomorphic models. This is most easily realized

by calculating the uncertainty aversion ν associated with drawing a ball from urn 2 which is given by

$$\nu = \max\{0, E(Q_a \vee Q_b) - E(Q_a) - E(Q_b)\}.$$

A small calculation shows that ν for $0 < a < 1/2 \leq b < 1$ may take any value in the interval $[0, c_0]$ where approximately $c_0 = 0.430705$.¹² It demonstrates that decision makers with different subjective probabilities and different degrees of uncertainty aversion may well make identical choices in the rather limited two-color experiment that cannot possibly reveal all aspects of the decision makers behaviour.

5.2 Three-color variation

A decision maker is presented with an urn containing 90 balls. He is told that 30 of the balls are red and that the remaining 60 balls are either black or yellow, but he is given no information about the distribution of the black and yellow balls. The decision maker is first asked to state his preferences between three bets, each on the exact color of a single drawn ball. For example, the bet on a “red ball” is an act where the “small world” or local state space only contains the pertinent events “red ball” and “the ball is not red”.

The outcomes are such that the consequence is 1 if one wins the bet and 0 otherwise. To simplify further the utility function is chosen such that the expected utility of a bet on the “red ball” becomes

$$E(R) \cdot 1 + E(1 - R) \cdot 0 = E(R)$$

which is simply the expected likelihood $E(R)$ of the associated event R . The same approach is taken to the five other bets.

The decision maker is asked to state his preferences between three bets in which he is given a choice between two colors of a single drawn ball. All six bets pay out the same amounts, conditional on the outcome of the draw. In the first choice situation the decision maker is found to prefer a bet on a “red ball” to a bet on a “black ball” and is indifferent between a bet on a “black ball” and a bet on a “yellow ball”, that is

$$\text{Bet}(R) \succ \text{Bet}(B) \sim \text{Bet}(Y). \quad (5.4)$$

In the second choice situation the decision maker is found to prefer a bet on a “black or yellow ball” to a bet on either a “red or yellow ball” or a bet on a “black or red ball”, and is indifferent between these two last bets, that is

$$\text{Bet}(B \vee Y) \succ \text{Bet}(R \vee Y) \sim \text{Bet}(B \vee R). \quad (5.5)$$

¹²The maximum value c_0 is obtained in approximately $a = 0.250764$ and $b = 1/2$ with the corresponding subjective probabilities $E(Q_a) = 0.144295$ and $E(Q_b) = 0.425$.

These preferences display uncertainty aversion in the sense that uncertain events or bets are seen as less attractive.

To model these preferences, we choose an event space with three projections R , B and Y (corresponding to balls of color red, black or yellow respectively) and a likelihood function E such that

$$E(R) > E(B) = E(Y) \quad \text{and} \quad E(B \vee Y) > E(R \vee Y) = E(B \vee R).$$

As the decision maker has exact information about the fraction of the red balls, he considers a bet on the red ball to be a simple lottery described by a probability distribution given the weight $1/3$ to the event “the ball is red” and the weight $2/3$ to the event “the ball is not red”, and this last event is recognized to be the same event as “the ball is either black or yellow”. This may be modeled by letting the event “red ball” be represented by the projection R and the event “the ball is not red” or “black or yellow ball” be represented by the projection $1 - R$.

As in the two-color variation, the decision maker is, in the absence of further information, not able to subdivide the “black or yellow ball” event into two single-color events with a probability distribution; they belong to different small worlds. We capture this by assigning non-orthogonal projections B and Y to the two events.¹³ Note that the three single color events have minorant 0, that is

$$R \wedge B = 0, \quad B \wedge Y = 0, \quad R \wedge Y = 0,$$

and the majorant event $B \vee Y = 1 - R$. The three two-color events $B \vee Y$, $R \vee B$ and $R \vee Y$ are thus endogenously given by the lattice operations once the one-color events R , B and Y are specified. We can choose h such that

$$\begin{aligned} E(R) &= \frac{1}{3} & E(B) &= \frac{1}{6} & E(Y) &= \frac{1}{6} \\ E(B \vee Y) &= \frac{2}{3} & E(R \vee B) &= \frac{1}{2} & E(R \vee Y) &= \frac{1}{2}. \end{aligned}$$

Now, the relations

$$E(R) > E(B) = E(Y) \quad \text{and} \quad E(B \vee Y) > E(R \vee Y) = E(B \vee R)$$

accurately reflects the preferences in Ellsberg’s paradox.

6 Concluding remarks

The Ellsberg paradox has inspired a substantial literature in axiomatic models of decision making. This literature contains alternative suggestions as to how one

¹³See Appendix A.4 for a set of projections which may be used.

can model the appropriate subjective conditions that characterize self-contained small world domains of events, such that the decision maker's preferences over acts restricted to any one domain exhibit probabilistic sophistication. Focus has been on modeling decision making under uncertainty, while at the same time allowing for a clear distinction between risk and uncertainty in the spirit of Knight (1921).¹⁴ In general, this literature has weakened the Savage/Anscombe-Aumann axioms. Some authors have chosen to abandon the Savage axioms - in the case of Vind (2003) the notion of totality of preferences - to construct more flexible expected utility models.

6.1 State space models with non-additive probabilities

Our paper is obviously related to the influential contribution of Schmeidler (1989) which models uncertainty and uncertainty aversion in a state space formalism by assuming that decision makers assign non-additive probabilities to some events as a reflection of uncertainty aversion. By imposing slightly weaker versions of the Anscombe-Aumann axioms on preferences, it is possible to capture preferences towards uncertainty and risk aversion in an expected utility formulation. Clearly, this work demonstrates that it is possible to formulate expected utility theories which capture a notion of uncertainty-aversion while still relying on the use of a state space. Several researchers have applied this type of framework to analyze economic situations.¹⁵

The primitive datum in Schmeidler's theory consists of the space space, the acts, and the preferences, and it is the modelers task to specify this datum in such a way that it adequately reflects the problem at hand. The direct analog in our theory to the states are the projections used to model events. They are taken from an infinite source of projections in an event space, but only a few that adequately corresponds to the problem at hand will be considered by the modeler. The acts and the preferences will then by Gleason's theorem, when applicable, provide a unique representation of the likelihood function and define the measure of uncertainty aversion. The basic problem of choosing states in a state space or choosing events (projections) in an event space are similar in nature. One may nevertheless argue that our approach is less intuitive and offers unwanted additional flexibility. Our approach does however have the advantage that it provides an intuitive representation of uncertainty aversion. Secondly, it retains linearity of the subjective expected utility function - even across small worlds. Finally, our framework allows for easy generalizations of a given model by adding additional small worlds.

¹⁴Karni and Schmeidler (1991) and more recently Wakker (2004) provide comprehensive surveys of this literature. Early contributions include Fellner (1961) and Quiggin (1982).

¹⁵See Mukerji and Tallon (2004) for a survey of this literature.

Our paper is also related to Wakker (2005), who relaxes, in a preference axiomatization state space model with non-linear capacities, earlier conditions of richness of the state space (Gilboa (1987)) and richness of the outcome space (Wakker (1989), Wakker (1993)). It is argued that structural restrictions are not mere technical but add content of an unknown nature to models that most naturally have a small finite number of states and outcomes. In our framework the set of consequences is convex and therefore naturally rich. Thus, even though the lattice of events may be small, the preferences must be extendable to the full lattice of projections in order to accommodate Gleason’s theorem, which at present has only been considered for the full lattice of projections. As such the method proposed in this paper is more closely related to those suggested by Schmeidler and Gilboa than that of Wakker. We suspect however, that the different approaches are genuine alternatives, ie it seems unlikely that different approaches are isomorphic in any precise meaning of the word.

6.2 State-dependent utilities

One may interpret our paper as an attempt to generalize models of state-dependent utilities where, as is pointed out in Schervish, Seidenfeld, and Kadane (1990), one can ensure that small world acts are ranked in a consistent manner within the grand world by multiplying by suitable constants. In this paper we propose a set of axioms which result in a grand likelihood functional that provides probability distributions in every small world and simultaneously ensures that the ranking of acts is consistent in the grand world.

A more recent paper by Hill (2008) also discusses preferences that cannot be expressed by state-independent utilities. He motivates his approach by considering the example of a man who would rather bet on his wife’s survival than on her death, even when the probabilities and the pay-offs are the same in the two situations. Axioms are proposed that allow for a situation dependent factor γ that modifies the state independent utilities without compromising the elucidation of subjective probabilities. Hill (2008)’s example cannot readily be described by the event space formalism we rely on in Theorem 2. The reason is that the likelihood functional E is a subjective probability when restricted to a “small world”. One can however reconcile our approach with the one in Hill (2008) by modeling the survival or death of the wife as events taking place in different small worlds rather than being complementary events. This would allow for a description, similar to our description of the two urn variation of Ellsberg’s paradox, where the total subjective probability of either death or survival is less than one due to uncertainty aversion. One might then introduce a functional γ , resembling the factor in Brian Hill’s theory, as a renormalization of probability across small worlds. Alternatively, one can try to retain the flexibility in the Fishburn (1973) setup by restricting the

application of the equivalence axiom and hence the description in (2.3) to acts where it is meaningful (or acceptable) to the decision maker to separate the consequences from the events leading to the consequences. This is accounted for in Fishburn’s setup where the utility of consequences may be conditioned on an event. Note that in our setup this flexibility is offset by our equivalence axiom (x) to the extent that this axiom results in the decision maker being indifferent between acts with the same set of consequences and associated probabilities. However, the benefit of axiom (x) is that it allows for a Savage expected utility representation (2.3) which only involves the utility of unconditioned consequences.

6.3 Models that do not rely on a state space

To our knowledge, there are only a few papers that do not rely on the explicit presence of a state space. Gilboa and Schmeidler (2001) model subjective distributions without relying on a state space. Instead, they model preferences over acts conditional on bets and assume the existence of an outcome-independent linear utility on bets. Subjective probabilities on outcomes, consistent with expected value maximizing behaviour, are then derived. Karni (2004) develops an axiomatic theory of decision making under uncertainty that dispenses with the Savage state space. A subjective expected utility theory, which does not invoke the notion of states of the world to resolve uncertainty, is formulated. Importantly, this approach does not rule out that decision makers may mentally construct a state space to help organize their thoughts - but it does not require that they do. Thus, the traditional approach may be embedded also into this framework.

Chew and Sagi (2008) assumes a Savage state space, but the authors provide a set of axioms which allow for domains of events that arise endogenously according to the preferences of the decision maker and the manner in which sources of uncertainty are treated. The authors also show, given weak assumptions, that preferences restricted to a domain exhibit probabilistic sophistication. This allows for an endogenous formulation of a two-stage approach and a distinction between risk and uncertainty in a setting with a Savage state space. However, as opposed to Savage’s formulation, the approach taken is to model decisions as generally taking place at the “small world” level, hence leaving the question of consistent extension of decision making across “small worlds” unanswered.

Finally, our work also has links to discussions of the foundation of quantum physics, in particular quantum mechanical derivations of probability closely related to the classical notions. See Wallace (2003a) and Wallace (2003b) for a discussion of how decision theory may be applied in quantum mechanics.

A Proofs

A.1 Lemma 1

Proof: Consider acts $(\alpha, f), (\alpha, g) \in L_\alpha$ for a small world $\alpha = (P_1, \dots, P_n)$. Since the subjective expected utility is given by

$$U_\alpha(\alpha, f) = \sum_{i=1}^n E_\alpha(P_i)u_\alpha(f(P_i)),$$

we may also consider $U_\alpha(\alpha, f)$ as the expected utility of a lottery with probabilities $E_\alpha(P_1), \dots, E_\alpha(P_n)$ between constant acts

$$(\alpha, f(P_1)), \dots, (\alpha, f(P_n)).$$

Since such a lottery is equally attractive in any other context we derive that

$$U_\alpha(\alpha, f) \geq U_\alpha(\alpha, g) \Leftrightarrow \sum_{i=1}^n E_\alpha(P_i)u_\beta(f(P_i)) \geq \sum_{i=1}^n E_\alpha(P_i)u_\beta(g(P_i))$$

for any $\beta \in P(H)$. This means that the function

$$V(\alpha, f) = \sum_{i=1}^n E_\alpha(P_i)u_\beta(f(P_i))$$

also represents the ordering in L_α . Accordingly, u_β is an increasing affine transformation of u_α and we may replace u_β with u_α without changing the ordering in L_β .

QED

A.2 Projections for model of Ellsberg paradox

$$\begin{aligned} R &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 1 - R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \\ B &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}. \\ R \vee B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R \vee Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}. \\ h &= \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & -1/6 \\ 0 & -1/6 & 1/2 \end{pmatrix}, \end{aligned}$$

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