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Expectations, Animal Spirits, and Evolutionary Dynamics

Angelo Antoci, Massimiliano Landi, Pier Luigi Sacco

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Expectations, Animal Spirits, and Evolutionary Dynamics.

Angelo Antoci
University of Sassari

Massimiliano Landi
Singapore Management University

Pier Luigi Sacco
IUAV, Venice

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Abstract

We consider a (deterministic) evolutionary model where players have dynamic expectations about the strategy distribution. We provide a global analysis of the co-evolution of play and expectations for a generic two–by–two game. Besides the the typical indeterminacy of the evolutionary dynamics, we find some other ones: for any initial strategy configuration the dynamics can converge to any asymptotically stable fixed point, for different initial values of the expectations. Moreover, starting from the same initial pair of strategy configuration and values of expectations, the dynamics may lead to different asymptotically stable fixed points for different parameters of the expectations.

Keywords: evolutionary games; dynamic systems; animal spirits.

JEL classification: C73

Expectations, Animal Spirits, and Evolutionary Dynamics.

1 Introduction

In an evolutionary game players typically have no reason to care about the future rounds of play. There are instances where this assumption is not reasonable. In particular, whenever any sort of commitment occurs (because, for example, of switching costs, or investment decisions), players are very well concerned about the future consequences of their current actions. The previous literature has studied extensively the conditions for stability of behavior when players form rational expectations about future paths of play. (See, for example, Matsui and Matsuyama (1995), Burdzy et al. (2001)). We consider a scenario in which players' expectations are more in line with a bounded rationality assumption. Players play a two–by–two population game while taking into account some estimation of the present value of future payoffs when deciding which action to take. In particular, players follow an adaptive expectation formation mechanism, whose dynamics depend only partly on how players tend to extrapolate the current outcomes into the future. We provide a global analysis of the induced (deterministic) dynamics as a function of the parameters governing the dynamics of expectations. In particular, we observe that the results of standard deterministic evolutionary dynamics hold only when players' tendency to extrapolate is not too strong. In addition, we find that for <u>any</u> initial strategy configuration, the dynamics can converge to any (if more than one) asymptotically stable fixed point, for a suitable choice of the initial value of the payoff expectations. (Theorem 1.) Moreover, starting from the <u>same</u> initial pair of strategy configuration and values of expectations, the dynamics may lead to different (if more than one) asymptotically stable fixed points depending on the values of the parameters that regulate the expectation formation process. (Theorem 2, and Theorem 3.) It is important to stress the meaning of both our results: when our dynamics show the same stability properties as the standard evolutionary dynamics, then one sort of indeterminacy already emerges in games where there are multiple asymptotically stable steady states (as in the case of coordination games). Here the dynamics can converge to any of them, according to the initial distribution of strategies in the population. The type of indeterminacy we refer to, instead, is different, in the sense that, the parameters governing the expectations dynamics determine to which asymptotically stable fixed point the dynamics converge for any initial distribution of strategies and for any initial distributions of expectations.

The distinction between history dependence and role of expectations in selecting equilibria has been studied in the macroeconomics literature, with a focus on rational expectations. (See for example, Matsuyama (1991), Krugman (1991) and Diamond and Fudenberg (1989).)

Evolutionary dynamics with perfect foresight have been analyzed by Matsui and Matsuyama (1995). Players can only change their action with some probability (friction) and take that into account, as well as the nature of the friction, when choosing an action in a two-by-two coordination game. As the friction gets smaller a unique equilibrium is selected, the risk dominant one. In a similar fashion, Burdzy et al. (2001) consider a two-by-two game with strategic complementarity, whose payoffs change over time. They consider the same sort of friction in players' ability to change strategies and find that the risk dominant equilibrium is played at any point in time when the friction is sufficiently small.

A departure from perfect foresight was taken first by Matsui and Rob (1992). They consider a game with stochastic overlapping generations of players whose actions are fixed for the entire life cycle. Players may have heterogeneous beliefs about the future evolution of play, and their individual behavior has to be rationalized by one of them. They find, among other things, that the Pareto efficient equilibrium can be the unique globally absorbing state. More recently, Matsui and Oyama (2005) consider the same setup as in Matsui and Matsuyama (1995) but move away from perfect foresight by assuming rationalizable expectations. They find that, when the level of friction is small enough, and players are playing a generic two–by–two game, the risk dominant equilibrium is the unique stable set of their dynamics. Lagunoff (2000) considers an infinitely repeated common interest game in which players play self-fulfilling equilibria. His model is close to Matsui and Rob (1992) in all the other features. It is shown that the Pareto dominant equilibrium is a globally absorbing state of the dynamics when there are relatively small inertia and discounting.

We follow the interpretation of myopic players, and extend it to a setting in which players observe a public signal about the state variable, and form (the same) adaptive expectations about its evolution. We provide an explicit model of these expectations dynamics, that is given by a combination of extrapolation into the future of current payoffs and an error correction mechanism, and study the (deterministic) dynamic system that is generated accordingly. A very similar approach to ours is provided by Conlisk (2001), who considers an evolutionary (zero–sum) game of competence selection in which players need to choose how much to invest in ability to overplay their opponents, and introduces a modification of the replicator dynamics as follows: the rate of growth of a strategy distribution, in fact, depends on both the differential payoff of that strategy compared to the average and the gradient of such a differential. This last term is intended to take into account a simple extrapolation of the present that allows the dynamics to converge to some equilibrium rather than cycle around it.

Our formalization is very similar to the one previously used in Antoci et al. (1992), who analyze the role of animal spirits in the transition process of an underdeveloped economy into a more advanced one. In such a context, different dynamics in the expectations lead to different levels of growth.

The remainder of the paper is organized as follows: in section 2 the model is introduced; in section 3 and section 4, some preliminary results are presented. Section 5 presents our main results, while section 6 contains the final comments. All the proofs are in the appendix.

2 The expectations augmented evolutionary dynamics

Evolutionary dynamics have been extensively studied. Samuelson (1997), and Weibull (1995) offer, among the others, a comprehensive review of the literature. For a two–by–two game, the standard specification of a dynamic evolutionary process is in terms of a vector field

$$\dot{x}(t) = F(\Delta\pi[x(t)]), \tag{1}$$

where $\dot{x}(t) \equiv dx(t)/dt$, and $\Delta\pi[x(t)]$ is the payoff difference between strategies 1 and 2 at time t. F is generally sign preserving and increasing in its argument. [1] We know that, for a generic specification of the payoff function and the initial conditions $x(0)$, there exists an open set $I_{x(0)}$ such that all the trajectories starting at $I_{x(0)}$ converge to the same asymptotically stable fixed point of (1). If the dynamics represented by (1) possess more than one asymptotically stable fixed point, then convergence to either one is just a matter of differences in the initial conditions $x(0)$.

In this paper we modify the dynamics given by equation (1) in order to take into account the expectations of the players. In particular, we consider a dynamic equation that describes how expectations evolve over time. Moreover, we allow for a variety of instantaneous payoff functions $\pi[x(t)]$,

[1] Here we are following the terminology introduced by Friedman (1991). When F is sign preserving, we usually speak of weak compatible dynamics; if moreover it is increasing in its argument, we speak of order compatible dynamics.

that can account for positive or negative (network) externalities. Therefore $\Delta\pi[x(t)]$ will not in general be a monotone function of $x(t)$.

Our model, which we call expectations-augmented evolutionary dynamics, is constructed as follows. Time is continuous and the population of players is infinite. At any instant $t \in [0, \infty)$ each player participates in a two–by–two population game. Players can choose only a pure strategy $i = 1, 2$. Let $x(t) \in [0, 1]$ denote the proportion of the population of individuals adopting strategy 1 at time t. The instantaneous payoff from adopting strategy i, at time t, given the state $x(t)$, is represented by $\pi_i[x(t)]$. Note that such a function need not necessarily be monotone. If the stage game payoffs (not the expected payoffs) depend on the fraction of players adopting each strategy, and if positive and negative externalities are present, then the function $\pi_i[x(t)]$ may as well have more than one peak.

The present value (evaluated at time t) of the discounted flow of payoffs of playing strategy i from time t to time $t + T$ is

$$V_{i,t}^T \equiv \int_t^{t+T} \{\pi_i [x(s)]\} e^{-\alpha(s-t)} ds, \tag{2}$$

where the parameter $\alpha > 0$ represents the discount rate and T can be taken equal to $+\infty$. Similarly, the present value of the instantaneous payoff difference at the instant t is

$$\Delta V_t^T \equiv \int_t^{t+T} \{\Delta\pi [x(s)]\} e^{-\alpha(s-t)} ds. \tag{3}$$

It is assumed that $x(t)$ is perfectly observable and common knowledge among the players at any t. Nevertheless players need to form some expectations about the evolution of $x(t)$ from t to $t + T$. It is important to stress that T need not be finite. Consequently this model can accommodate for scenarios where players are replaced or can change action only with some (i.i.d.) probability. In that case α is the effective discounted rate, which incorporates the death rate. Let $y(t)$ denote the expectation at t about ΔV_t^T. Players adopt the following (non-autonomous) adaptive expectation formation mechanism

$$y(t) \equiv H\left(t, \Omega^{\Delta V_t^T}\right), \tag{4}$$

where $H(\cdot)$ is a continuously differentiable function and $\Omega^{\Delta V_t^T}$ is a variable used as observable proxy of ΔV_t^T. It follows that the time derivative of $y(t)$ is

$$\dot{y}(t) = H_t + H_{\Omega^{\Delta V_t^T}} \frac{d\Omega^{\Delta V_t^T}}{dt} \tag{5}$$

The following assumptions are made. For any $x(t)$, A1 $\Omega^{\Delta V_t^T}[x(t)] = \int_t^{t+T} \{\Delta\pi[x(t)]\} e^{-\alpha(s-t)} ds = \frac{1}{\alpha}\Delta\pi[x(t)](1 - e^{-\alpha T})$;

A2 $H_{\Omega^{\Delta V_t^T}} \equiv (1+\beta)$, with $\beta > -1$;

A3 $H_t \equiv \gamma \left[\Omega^{\Delta V_t^T}[x(t)] - y(t)\right]$, with $\gamma > 0$.

A1 states that the observable proxy of ΔV_t^T is obtained by considering that the distribution x at time t does not change in the interval $[t, t+T]$. The parameter β in A2 measures how conservatively players tend to adjust their expectations as a consequence of the variability of their proxy. Ceteris paribus, if $\beta < 0$ ($\beta > 0$), changes in the proxy will generate a contracted (amplified) change in the players' expectations, $\dot{y}(t)$. Finally, A3 represents an error correction mechanism; if $x(t)$ becomes almost stationary, then expectations $y(t)$ approach the value of the observable proxy, which in this case is a good approximation of the true value. Otherwise, expectations are fed by the variability of the observable proxy through a simple homeostatic mechanism: if players find out that their guess is too pessimistic on the basis of the available evidence, they revise their expectations upwards, and conversely in the opposite case. The parameter γ measures the strength of this informational feedback.

To sum up, our expectations augmented evolutionary model can be expressed in terms of the following system of equations

$$\dot{x}(t) = g[x(t), y(t)] \tag{6}$$

$$\dot{y}(t) = \gamma\left[\Omega^{\Delta V_t^T}[x(t)] - y(t)\right] + (1+\beta)\frac{d\Omega^{\Delta V_t^T}[x(t)]}{dt}, \tag{7}$$

where

$$\frac{d\Omega^{\Delta V_t^T}[x(t)]}{dt} = \frac{1}{\alpha}(1 - e^{-\alpha T})\frac{d\Delta\pi[x(t)]}{dx(t)} g[x(t), y(t)]. \tag{8}$$

Like in Conlisk (2001), the dynamics of the strategy distribution directly depend on the value of the expectations. In addition, the dynamics of the expectations are not a mere extrapolation of some proxy from current payoffs differentials. The following conditions represent a natural analogue of the order compatible dynamics assumptions for standard evolutionary models

A4 $g(x, y)$ is $C^1$for every $x \in [0,1]$ and $y \in \mathbb{R}$.

A5 $g(x, 0) = 0$, for every $x \in [0,1]$;

A6 $g(x, y)$ is (strictly) increasing in y, for every $x \in (0,1)$ and $y \in \mathbb{R}$;

A7 $g(0,y) > 0$ if $y > 0$ and $g(0,y) = 0$ if $y < 0$; $g(1,y) < 0$ if $y < 0$ and $g(1,y) = 0$ if $y > 0$.

A5 states that when the expected payoff difference is zero, i.e. $y = 0$, none of the players has an incentive to change action. Therefore any strategy distribution x is invariant for the population dynamics. A6 amounts to postulating that the proportion of a given strategy across the population is increasing (with a growth rate which depends positively on y) if and only if its payoff is expected to be higher than that of the rival strategy. Finally, A7 represents a boundary condition that allows the variable x to remain in the interval $[0,1]$.

Remark 1 *Assumptions A4-A7 pose implicitly some restriction on the partial derivatives of $g(x,y)$. In particular it must be the case that*

a) $g_x(x,0) = 0$, *for every* $x \in [0,1]$.

b) $g_y(0,y) = 0$, *for every* $y \leq 0$; $g_y(1,y) = 0$, *for every* $y \geq 0$.

c) $g_x(0,y) < 0$, *for every* $y < 0$; $g_x(1,y) < 0$, *for every* $y > 0$.

Remark 2 *Equation (7) may be alternatively written as*

$$\frac{d}{dt}\left[y(t) - \Omega^{\Delta V_t^T}[x(t)]\right] = -\gamma\left[y(t) - \Omega^{\Delta V_t^T}[x(t)]\right] + \beta\frac{d\Omega^{\Delta V_t^T}[x(t)]}{dt} \tag{9}$$

Thus, if $\beta = 0$, expectations dynamics are entirely characterized by the general solution

$$y(t) - \Omega^{\Delta V_t^T}[x(t)] = [y(0) - \Omega^{\Delta V_t^T}[x(0)]]e^{-\gamma t} \tag{10}$$

Notice that if $y(0) = \Omega^{\Delta V_t^T}[x(0)]$, then $y(t) = \Omega^{\Delta V_t^T}[x(t)]$ for every future time.

The aim of the following sections is to study the dynamic properties of the system (6)-(7), and to derive some preliminary results.

3 Local stability

Definition 1 *A fixed point of the system of equations (6)-(7) is a pair (x,y) such that $\dot{x} = \dot{y} = 0$.*

The following result characterizes the set of fixed points of the expectations augmented dynamics.

Proposition 1 *The set of fixed points of (6)-(7) is given by*

1. *Any* $(x, 0) : x \in (0, 1)$ *and* $\Omega^{\Delta V_t^T}[x(t)] = 0$ *(i.e.* $\Delta\pi[x(t)] = 0$*)*
2. $(0, \Omega^{\Delta V_t^T}[0])$*, if* $\Omega^{\Delta V_t^T}[0] \leq 0$ *and* $(1, \Omega^{\Delta V_t^T}[1])$*, if* $\Omega^{\Delta V_t^T}[1] \geq 0$.

Definition 2 *We call mixed population fixed point any fixed point of (6)-(7) of the first type, and pure population fixed point any fixed point of the second type.*

Observe that $x = 0$ and $x = 1$ are attractive fixed points of (1) (and therefore are symmetric Nash equilibria of the underlying game) if and only if $(x, y) = (0, \Omega^{\Delta V_t^T}[0])$ and $(x, y) = (1, \Omega^{\Delta V_t^T}[1]))$ are pure population fixed points of (6)-(7). Moreover, $\overline{x} \in (0, 1)$ is a fixed point of (1), if and only if $(\overline{x}, 0)$ is a mixed population fixed point of of (6)-(7).

The following result concerns the local stability of fixed points.

Proposition 2 *Suppose that* $\Omega^{\Delta V_t^T}[x(t)]$ *always intersects the* x*-axis transversely, then we have that*

1. *Pure population fixed points are locally attractive.*
2. *A mixed population fixed point* $(\hat{x}, 0)$ *is a saddle point for the dynamics (6)-(7) if* $\left.\frac{d\Omega^{\Delta V_t^T}[x(t)]}{dx(t)}\right|_{(\hat{x},0)} > 0$ *(i.e.* $\left.\frac{d\Delta\pi[x(t)]}{dx(t)}\right|_{(\hat{x},0)} > 0$*); it is an attractive fixed point if the opposite inequality holds.*

Proposition 2 means that the stability properties of any fixed point of (6)-(7) are the same as those of the corresponding fixed point of (1). Therefore, the dynamics (6)-(7) preserve the stability properties of the dynamics (1).

Observe that to require a transversal intersection of $\Omega^{\Delta V_t^T}[x(t)]$ with the x-axis is equivalent to consider games with generic payoffs. Therefore there is no loss of generality in introducing such a condition, as we will be doing in the remainder of the paper.

We can now move to the analysis of the global dynamics of the system (6)-(7) that is the topic of the next two sections.

4 Global analysis: preliminary results

In order to make a global analysis of the expectations augmented dynamics, some preliminary results are needed. Consider the following sets

$$\Phi \equiv \{(x, y) : x \in [0, 1),\ y \in (0, +\infty)\}, \tag{11}$$

and

$$\Lambda \equiv \{(x,y) : x \in (0,1],\ y \in (-\infty, 0)\}\,. \tag{12}$$

We have that $\dot{x} > 0$ for any $(x,y) \in \Phi$ and $\dot{x} < 0$ for any $(x,y) \in \Lambda$. Therefore, in both Φ and Λ, $x(t)$ is strictly monotonic, and every trajectory in these sets can be considered as the graph of a function of x, $y = Y(x)$, that must satisfy the differential equation

$$\frac{dy}{dx} = \gamma \frac{\Omega^{\Delta V_t^T}[x] - y}{g(x,y)} + (1+\beta)\frac{d\Omega^{\Delta V_t^T}[x]}{dx}, \tag{13}$$

where $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$.

Lemma 1 *Equation (13) shows that*

1. *if $\beta = 0$, the curve $y = \Omega^{\Delta V_t^T}[x]$ is an invariant set.*
2. *if $\beta > 0$, then in Φ the trajectories cross the curve $y = \Omega^{\Delta V_t^T}[x]$ from below when $\frac{d\Omega^{\Delta V_t^T}[x]}{dx} > 0$, and from above when $\frac{d\Omega^{\Delta V_t^T}[x]}{dx} < 0$;*
3. *if $\beta < 0$, then in Φ the trajectories cross the curve $y = \Omega^{\Delta V_t^T}[x]$ from above when $\frac{d\Omega^{\Delta V_t^T}[x]}{dx} > 0$, and from below when $\frac{d\Omega^{\Delta V_t^T}[x]}{dx} < 0$;*
4. *A symmetric configuration holds in Λ.*

From proposition 2, we know that mixed population fixed points are saddles (resp. attractive) if the curve $y = \Omega^{\Delta V_t^T}[x]$ at the fixed point crosses the x-axis from below (resp. above). Thus saddles alternate to attractive fixed points. When $\beta = 0$, the outset (unstable branch) of each saddle connects the saddle with the nearest attractive fixed points. Figure 1 shows a possible case. The horizontal line underneath the (x,y) axes represents the phase portrait in the case of the corresponding dynamics (1), and is drawn to allow an easy comparison. (Remember proposition 2.)

Insert figure 1 about here

The central part of this section concerns the variations of the outset and the variations of the inset (stable branch) of each saddle with respect to changes of the parameters. The analysis highlights some interesting properties of the shape of the basins of attraction that will be useful in characterizing our results.

We focus on the case in which there exist at least two mixed population fixed points, as the results can easily be extended to the remaining cases.

In particular, it is important to remember that when only pure population fixed points exist, the predictions by dynamics (1) and (6)-(7) do coincide. [2]

Definition 3 *Two mixed population fixed points are consecutive if between them (along the x axis) there are no other fixed points. Let $(x_n, 0)$ and $(x_{n+1}, 0)$ be two consecutive fixed points with $0 < x_n < x_{n+1} < 1$.*

Without loss of generality, we let $\Omega^{\Delta V_t^T}[x] > 0$ in the interval (x_n, x_{n+1}). [3]

Lemma 2 *If $\beta \geq 0$, the region of the phase space between the x-axis and the curve $y = (1+\beta)\Omega^{\Delta V_t^T}[x]$ is a positively invariant set for the trajectories of the system (6)-(7)*

From the previous lemma, the following result follows.

Lemma 3 *if $\beta \geq 0$, then for $x > x_n$ the outset of $(x_n, 0)$ lies in the region delimitated by the curve $y = (1+\beta)\Omega^{\Delta V_t^T}[x]$ and the x-axis; therefore along it the fixed point $(x_{n+1}, 0)$ is reached.*

Lemma 3 also applies to the case in which $(x_n, 0)$ is attracting whereas $(x_{n+1}, 0)$ is a saddle. In this context, the result concerns the outset of $(x_{n+1}, 0)$ which, for $x \in (x_n, x_{n+1})$, connects these two fixed points. Therefore, for $\beta \geq 0$, the outset of any saddle point links each fixed point to its consecutive.

5 Global analysis: main results

We are now in the position to establish our results.

Theorem 1 *Suppose that $g(x, y)$ is an infinite of the same order as y, and that $\beta \geq 0$. Then for any given $x(0) \in (0, 1)$, and any given attractive fixed point (x^*, y^*) of the dynamics (6)-(7), there exists an open set of initial values of the expectations $y(0)$ such that the trajectory starting at $(x(0), y(0))$ converges to (x^*, y^*).*

[2] See proposition 2 and the preceding paragraph.

[3] In this case $(x_n, 0)$ is a saddle and $(x_{n+1}, 0)$ is attractive. The context in which $\Omega^{\Delta V_t^T}[x] < 0$ in (x_n, x_{n+1}) can be analyzed in the same way by considering the distribution of strategies across the population in terms of the variable $(1 - x)$ instead of x.

Theorem 1 not only states that whatever the initial distribution of strategies in the population, the trajectories of the dynamics (6)-(7) converge to any attractive fixed point by suitably choosing the initial values of the expectations. It also provides us with a parametrization of the attractive fixed points by open sets, for given $x(0)$. As $y(0)$ varies, the asymptotic behavior of the expectations augmented dynamics jumps from one fixed point to another. Figures 2 and 3 give a graphical representation of theorem 1.

Insert figure 2 about here

Insert figure 3 about here

Observe that in the example of figure 2, for a given initial value $x_0 \in (0,1)$, the dynamics (6)-(7) converge to A, B, or C, if the initial value of the expectations, $y(0)$, is equal to, respectively, y^1, y^2, or y^3.

Figure 3 shows instead the case of a coordination game, with two strict Nash equlibria (which are attractive fixed points) and one mixed Nash equilibrium (which is a saddle point). In the region below the stable branch of the saddle point (C), the dynamics (6)-(7) converge to the fixed point A. In the region above the stable branch of the saddle point, they converge to the fixed point B.

As a result, while for the dynamics (1), initial strategy distributions that are sufficiently close converge to the same attractive fixed point, in the expectations augmented dynamics, they can converge to different fixed points. This result is due to the fact that the expectations play a role in the determination of the basins of attraction of the fixed points. Therefore a clear prediction on the evolution of the system can be done only after we know both the initial strategy distribution and the initial value of the expectations. We suggest that this feature of the expectations-augmented dynamics can be interpreted as a representation of the action of the Keynesian animal spirits.

A description of the proof of theorem 1 may help to better appreciate its significance and its limitations. The condition on $g(x,y)$ (which basically means that x needs to be sufficiently reactive to expectations y) implies that all the trajectories, included those that generate the stable branches of each saddle, do not have vertical asymptotes at any $\bar{x} \in (0,1)$. [4] Moreover, $\beta \geq 0$ implies, by lemma 3, that the stable branches of each saddle lie,

[4]Possible specifications of $\dot{x}$ satisfying this condition are $g(x,y) = x(1-x)y$ as well as $g(x,y) = p(1-x)y$ if $y > 0$ and $g(x,y) = -pxy$ if $y < 0$, where p can be interpreted as the replacement rate of players as in Matsui and Matsuyama (1995).

respectively, one entirely in the region of the plane where $y > 0$, and the other in $y < 0$. Therefore, for any $x(0)$, the vertical line passing through that point has a non-empty intersection with all the basins of attractions of all the fixed points. Unfortunately nothing can be said, analytically, about the robustness of this result to some restrictions about the initial values of the expectations, one natural candidate being $y(0) \in [\frac{\alpha \min \Delta\pi(x)}{(1-e^{-\alpha T})}, \frac{\alpha \max \Delta\pi(x)}{(1-e^{-\alpha T})}]$. Such a restriction about initial beliefs may not be too stringent in the case where players do not know the payoffs of the game. [5]

Notice also that, for $\beta \geq 0$, along the system's trajectories the sign of $\dot{x}$ can change at most once; thus we cannot observe persistent oscillations of the distribution of strategies. On the other hand, this may not be the case if $\beta < 0$.

The last thing to check is how the basins of attraction change with respect to the parameters. Consider two successive fixed points $(x_n, 0)$ and $(x_{n+1}, 0)$, $x_{n+1} > x_n$, where the former is a saddle and the latter is attractive.

Theorem 2 *Assume that $x(0) \in (x_n, x_{n+1})$, $y(0) \geq 0$, and $\Omega^{\Delta E_t^T}[x(0)] > 0$. Then there exists an interval $[\beta^*, +\infty)$ such that, if $\beta \in [\beta^*, +\infty)$, the trajectory starting at $[x(0), y(0)]$ approaches $(x_{n+1}, 0)$. The same (symmetric) result holds for $y(0) \leq 0$ when $(x_{n+1}, 0)$ is a saddle and $(x_n, 0)$ is attractive.*

The above result says that, when we have two consecutive fixed points $(x_{n+1}, 0)$ and $(x_n, 0)$, if $x_n < x(0) < x_{n+1}$ and

$$y(0) \geq 0,\ \Omega^{\Delta E_t^T}[x(0)] > 0,\ \text{or}\ y(0) \leq 0,\ \Omega^{\Delta E_t^T}[x(0)] < 0, \tag{14}$$

the trajectory starting from $(x(0), y(0))$ converges to the attracting fixed point $(x_{n+1}, 0)$ if β is big enough. [6] Figure 4 shows a graphical representation of the theorem.

Insert figure 4 about here

It is important to stress that this result holds even if there are many stable fixed points.

Condition (14) requires that both $y(0)$ and the proxy have the same sign. If this holds, theorem 2 states that for any initial level of the expectations, a sufficiently high level of β allows the system to converge to $(x_{n+1}, 0)$. Both dynamics (1) and (6)-(7) converge to the same strategy distribution. Note

[5]It is also important to stress that all the other results in the paper do not depend neither on $y(0)$ nor on the restriction on $g(x, y)$.

[6]Remember that the parameter β represents the reactivity of expectations to changes of the observable proxy $\Omega^{\Delta V_t^T}[x(t)]$.

also that a large β is not sufficient to generate such a qualitative equivalence between the two dynamics. We also need that the signs of the initial value of the expectations and of the proxy be the same. The next theorem takes into account the case in which β is large but the proxy and the initial value of the expectations have different signs. As we will see, the dynamics (1) and (6)-(7) converge to different strategy distributions.

Assume first that there exist only three mixed population fixed points: $(x_1, 0)$, $(x_2, 0)$ and $(x_3, 0)$, $0 < x_1 < x_2 < x_3 < 1$, where the middle one is a saddle and the other two are attracting. Thus $\Omega^{\Delta V_t^T}[x] > 0$ for $x \in (x_2, x_3)$ and $\Omega^{\Delta E_t^T}[x] < 0$ for $x \in (x_1, x_2)$. Theorem 2 concerns the case in which the initial value of expectations $y(0)$ has the same sign of the value of the signal $\Omega^{\Delta E_t^T}[x(0)]$. [See (14).] When this is not the case, we have the following result.

Theorem 3 *Assume that* $x(0) \in (x_1, x_2)$ *and* $\Omega^{\Delta V_t^T}[x(0)] < 0 < y(0)$. *If* $\Omega^{\Delta V_t^T}[x]$ *is increasing in* $[x(0), x_2]$ *and if in* $\{(x, y) : x \in [x(0), x_2],\ y > 0\}$ *the following inequality is satisfied*

$$\left[\frac{\Omega^{\Delta V_t^T}[x] - y}{g(x,y)}\right]_y > 0, \ \textit{i.e.}\ g_y(x,y) > \frac{g(x,y)}{y - \Omega^{\Delta V_t^T}[x]} \tag{15}$$

then there exists an interval $[\beta^{**}, +\infty)$ *such that the trajectory starting from* $[x(0), y(0)]$ *converges to* $(x_3, 0)$ *if* $\beta \in [\beta^{**}, +\infty)$.

Proposition 3 *The symmetric case holds if* $x(0) \in (x_2, x_3)$ *and* $y(0) < 0 < \Omega^{\Delta V_t^T}[x(0)]$.

Condition (15) simply requires that the reactivity of $\dot{x}$ with respect to the expectations be high enough. Theorem 3 says that when the initial value of the expectations has a different sign from that of the observable proxy (animal spirits) and β is big enough, a trajectory starting with a strategy distribution between two successive fixed points, converges to the third fixed point of the system. Figure 5 provides a graphical interpretation of the theorem.

Insert figure 5 about here

In figure 5, the trajectory starting from $(x(0), y(0))$, with $y(0) > 0$ and $x(0) \in (x_*, B)$, converges to point C when β is sufficiently large. On the other hand, under the dynamics (1), the trajectory starting at $x = x(0)$ converges to a point whose coordinate on the x axis is the same as A.

Figure 6 represents theorem 3 in the case of a coordination game, where $x = 0$ and $x = 1$ are the only two attractive fixed points. In this case, since the proxy is always increasing in x, theorem 3 holds for any $x(0) \in [0, 1]$.

Insert figure 6 about here

For any $x(0) < B$, we have that strategy 1's current payoff is smaller than strategy 2's. Nevertheless, the former grows faster as x increases. As x gets bigger than B, strategy 1's current payoff becomes bigger than strategy 2's. Therefore, if β is big enough, the dynamics of the expectations give greater importance to the fact that strategy 1 has a steeper gradient than that of strategy 2, irrespective of their current payoffs. As a result, the dynamics (6)-(7) converge to point C, where everybody adopts strategy 1. On the other hand, dynamics (1) converge to $x = 0$ if $x(0) < x_s$ and to $x = 1$ if $x(0) > x_s$.

Clearly, theorem 3 also applies if the system admits more than three mixed population fixed points. In this case the trajectory converges to one of the non-successive fixed points.

To sum up, the dynamics (1) and (6)-(7) generate the same prediction about the asymptotic strategy distribution only when the proxy and the initial value of the expectations have the same sign, with β sufficiently big (theorem 2). In fact, when β is large but the proxy and the initial value of the expectations have opposite sign, the two dynamics generate different predictions (theorem 3).

These results, together with the basic indeterminacy from theorem 1, imply that for large β the expectations matter more than the initial strategy distribution.

6 Conclusion

In this paper we have characterized the stability properties of an expectations augmented evolutionary dynamics and compared them to standard evolutionary dynamics. We found that new, more radical kinds of indeterminacies arise in addition to the already well know ones. First, for fixed parameters, β and γ, of the expectation mechanism and for any initial condition on the strategy profile, $x(0)$, it is possible to converge to any of the attractive fixed points of the expectations augmented dynamics in correspondence of different initial values of the expectations, $y(0)$. Second, for any initial values $x(0)$ and $y(0)$ it is possible to converge to different attractive fixed points of the expectations augmented dynamics in correspondence of different parameter values of the expectation mechanism. A necessary condition for our indeterminacy results is that β, the measure of the reactivity of the expectations to changes of the observable proxy, be large enough. Notice that nothing has been said about the parameter γ. The reason is simple: a large γ implies that the expectation augmented evolutionary dynamics generate the same qualitative result as the standard evolutionary dynamics. We can interpret this setup as a formalization of the Keynesian animal spirits.

A Proofs of the results

In this section we provide the proofs of our results.

Proof of Lemma 1. The first point has already been shown in remark 2. Observe that at $y = \Omega^{\Delta V_t^T}[x]$ equation (13) shows that $\frac{dy}{dx}$ has the same sign as $\frac{d\Omega^{\Delta V_t^T}[x]}{dx}$, for any $\beta > -1$. This implies that, for $y > 0$, when $\beta > 0$ ($\beta < 0$), the trajectories are steeper (flatter) than $y = \Omega^{\Delta V_t^T}[x]$. This shows points 2 and 3. The last point can be proved in a similar way. ■

Proof of Lemma 2. The slope of the trajectories evaluated along the curve $y = (1+\beta)\Omega^{\Delta E_t^T}[x]$ can be written as

$$\left.\frac{dy}{dx}\right|_{y=(1+\beta)\Omega^{\Delta V_t^T}[x]} = \gamma\frac{\Omega^{\Delta V_t^T}[x] - (1+\beta)\Omega^{\Delta V_t^T}[x]}{g(x,y)} + (1+\beta)\frac{d\Omega^{\Delta V_t^T}[x]}{dx}$$
$$= -\beta\gamma\frac{\Omega^{\Delta V_t^T}[x]}{g(x,y)} + (1+\beta)\frac{d\Omega^{\Delta V_t^T}[x]}{dx}, \quad (16)$$

whereas the slope of $y = (1+\beta)\Omega^{\Delta V_t^T}[x]$ is $(1+\beta)\frac{d\Omega^{\Delta V_t^T}[x]}{dx}$. Since along $y = (1+\beta)\Omega^{\Delta V_t^T}[x]$ we have that $sign\left\{(1+\beta)\Omega^{\Delta V_t^T}[x]\right\} = sign\{g(x,y)\}$, the lemma is proved. ■

Proof of Lemma 3. Let $\bar{Y}^u(x)$ denote the outset of x. The case with $\beta = 0$ follows simply by noticing that for $x_n < x < x_{n+1}$ it holds $\bar{Y}^u(x_n) \equiv \Omega^{\Delta V_t^T}[x]$. Consider now the case with $\beta > 0$ and suppose that for x close to x_n, $\bar{Y}^u(x)$ lies above the curve $y = (1+\beta)\Omega^{\Delta V_t^T}[x]$. We shall show that this generates a contradiction. Every trajectory $Y(x)$ of the system (6)-(7) satisfies the following integral equation

$$Y(x^*) = Y(x_*) + \gamma \int_{x_*}^{x^*} \frac{\Omega^{\Delta V_t^T}[x] - Y(x)}{g(x, Y(x))} dx + (1+\beta)\left[\Omega^{\Delta V_t^T}[x^*] - \Omega^{\Delta V_t^T}[x_*]\right], \quad (17)$$

where $x^* > x_*$.

For $x_* \to x_n$ and $x^* > x_n$, if $\bar{Y}^u(x) > (1+\beta)\Omega^{\Delta V_t^T}[x]$ in $(x_n, x^*]$, then it must be the case that

$$\bar{Y}^u(x^*) - (1+\beta)\Omega^{\Delta V_t^T}[x] = \bar{Y}^u(x_n) + \gamma \int_{x_n}^{x^*} \frac{\Omega^{\Delta V_t^T}[x] - \bar{Y}^u(x)}{g\left(x, \bar{Y}^u(x)\right)} dx + (1+\beta)\left[\Omega^{\Delta V_t^T}[x^*] - \Omega^{\Delta V_t^T}[x_n]\right] - (1+\beta)\Omega^{\Delta V_t^T}[x] = \gamma \int_{x_n}^{x^*} \frac{\Omega^{\Delta V_t^T}[x] - \bar{Y}^u(x)}{g\left(x, \bar{Y}^u(x)\right)} dx > 0. \quad (18)$$

This generates the desired contradiction since, by assumption, $\frac{\Omega^{\Delta V_t^T}[x] - \bar{Y}^u(x)}{g\left(x, \bar{Y}^u(x)\right)} < 0$ when $\bar{Y}^u(x^*) > (1+\beta)\Omega^{\Delta V_t^T}[x] > \Omega^{\Delta V_t^T}[x]$. ■

Proof of Theorem 2. For $0 \leq y(0) \leq \Omega^{\Delta V_t^T}[x(0)]$, the interval is $[0, +\infty)$; in fact, the curve $y = \Omega^{\Delta V_t^T}[x]$ is invariant for $\beta = 0$. If $y(0) > \Omega^{\Delta V_t^T}[x(0)]$, we choose β^* to be the solution of the equation $y(0) = (1+\beta^*)\Omega^{\Delta V_t^T}[x(0)]$. The proof for the case in which $y(0) \geq 0$, $(x_{n+1}, 0)$ is attractive, and $(x_n, 0)$ is a saddle is completed by just applying theorem 1. The proof of the symmetric case follows the same steps of the previous one. ■

Proof of Theorem 3. First observe that the partial derivative of $\frac{dy}{dx}$ with respect to β in (13) is given by

$$\left(\frac{dy}{dx}\right)_\beta = \frac{d\Omega^{\Delta V_t^T}[x]}{dx}. \quad (19)$$

This means that the slope $(\frac{dy}{dx})$ of trajectories in $\Phi \cup \Lambda$ increases with β if $\frac{d\Omega^{\Delta V_t^T}[x]}{dx} > 0$ and decreases with β if $\frac{d\Omega^{\Delta V_t^T}[x]}{dx} < 0$.

Consider now the case: $x(0) \in (x_1, x_2)$ and $\Omega^{\Delta V_t^T}[x(0)] < 0 < y(0)$. Take an $x^* \in (x(0), x_2)$; the trajectory $Y(x)$ passing through $[x(0), y(0)]$

must satisfy the integral equation

$$Y(x^*) = Y(0) + \gamma \int_{x(0)}^{x^*} \frac{\Omega^{\Delta V_t^T}[x] - Y(x)}{g(x, Y(x))} \, dx \qquad (20)$$
$$+(1+\beta)\left[\Omega^{\Delta V_t^T}[x^*] - \Omega^{\Delta V_t^T}[x(0)]\right]$$

Notice that

1. by (19), if we take two values of β, $\beta^1 < \beta^2$, then in the interval $[x(0), x_2]$ the trajectory passing through $[x(0), y(0)]$ when $\beta = \beta^2$ lies above the one passing through $[x(0), y(0)]$ when $\beta = \beta^1$.

2. By (19), in the interval $[x(0), x_2]$ the inset $\bar{Y}^s(x)$ of $(x_2, 0)$ for $\beta = \beta^2$ lies below the inset $\bar{Y}^s(x)$ of $(x_2, 0)$ for $\beta = \beta^1$.

3. Since, by assumption, $\Omega^{\Delta V_t^T}[x]$ is increasing in $[x(0), x_2]$, it follows that $\left[\Omega^{\Delta V_t^T}[x^*] - \Omega^{\Delta V_t^T}[x(0)]\right] > 0$.

4. By (15), the (negative) value of $\int_{x(0)}^{x^*} \frac{\Omega^{\Delta V_t^T}[x] - Y(x)}{g(x, Y(x))} \, dx$ increases if β increases.

Therefore, from (20) we have that, as β increases, $Y(x^*)$ becomes arbitrarily big and there must exist a β^{**} such that, for every $\beta \in [\beta^{**}, +\infty)$, $Y(x) > \bar{Y}^s(x)$ for every $x \in [x(0), x_2]$. This implies that the trajectory starting from $[x(0), y(0)]$ converges to $(x_3, 0)$. By the same token we can prove the symmetric case of the theorem. ■

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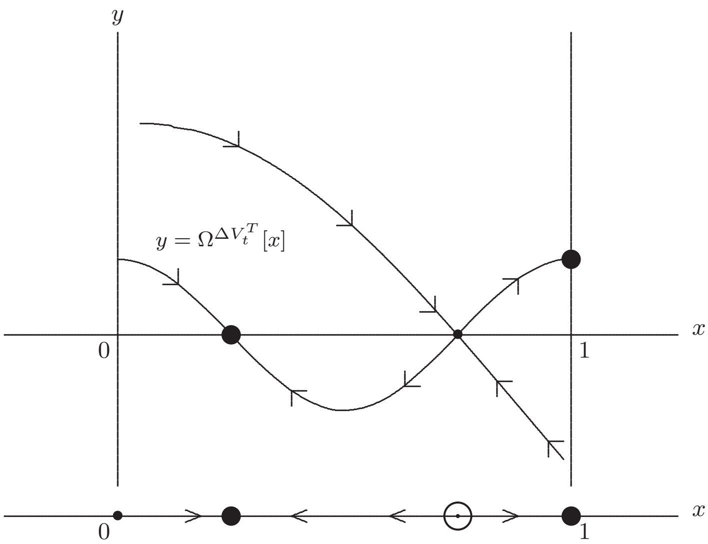

Figure 1: A possible phase portrait for the expectations augmented dynamics. Attractive fixed points are represented by filled circles, repulsive fixed points by dotted circles, and saddle points by dots.

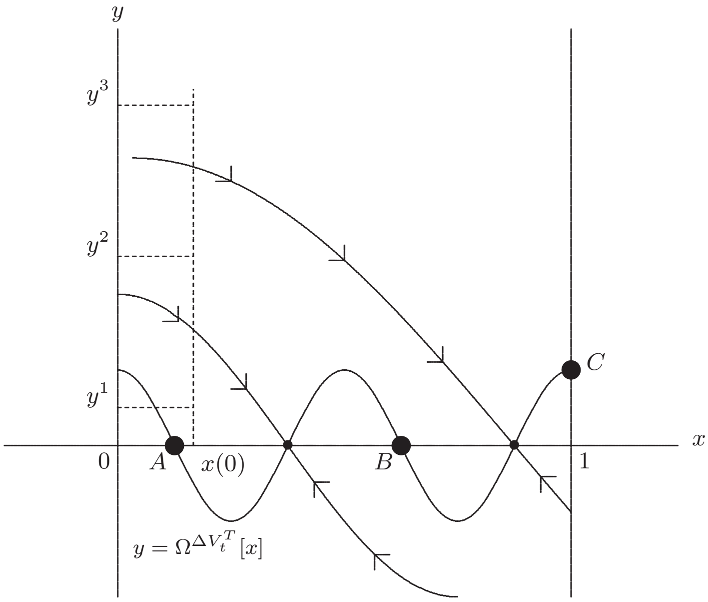

Figure 2: A graphical representation of theorem 1. Attractive fixed points are represented by filled circles, repulsive fixed points by dotted circles, and saddle points by dots.

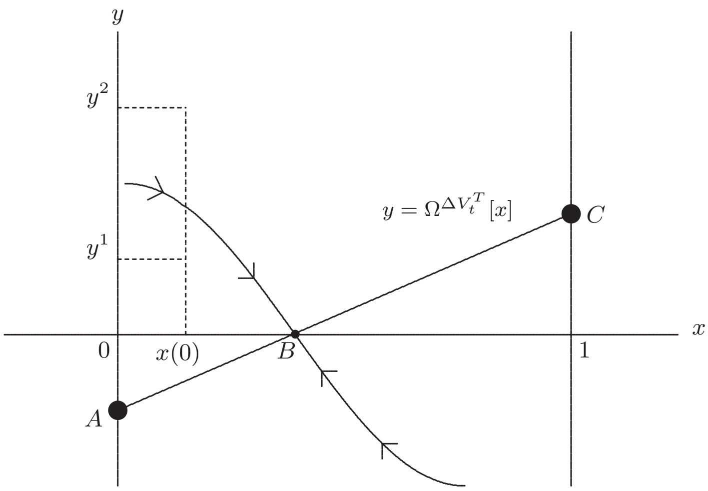

Figure 3: A possible phase portrait for a coordination game. Attractive fixed points are represented by filled circles, repulsive fixed points by dotted circles, and saddle points by dots.

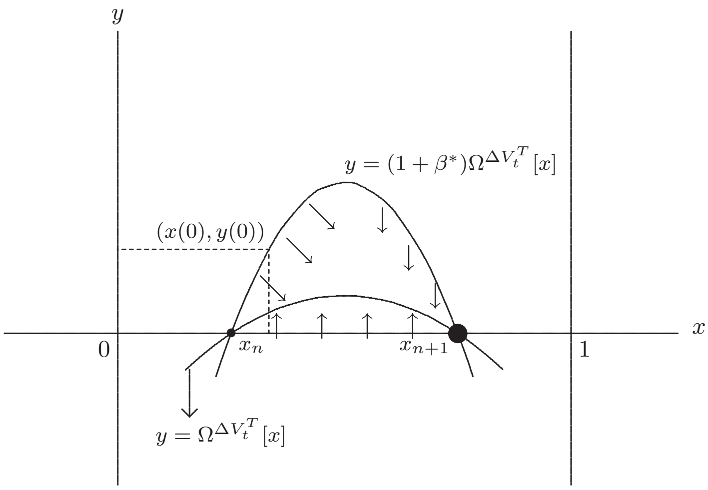

Figure 4: A graphical representation of theorem 2. Attractive fixed points are represented by filled circles, repulsive fixed points by dotted circles, and saddle points by dots.

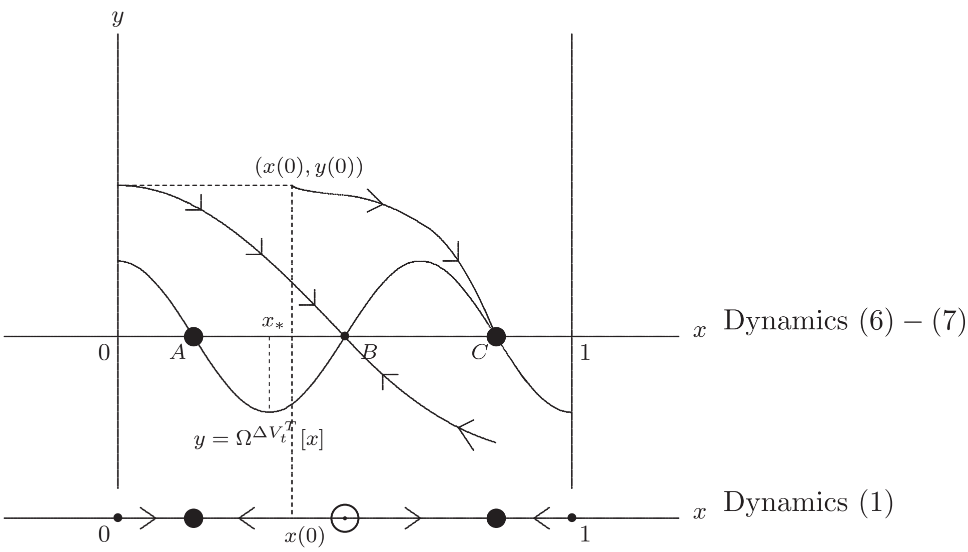

Figure 5: A graphical representation of theorem 3. Attractive fixed points are represented by filled circles, repulsive fixed points by dotted circles, and saddle points by dots.

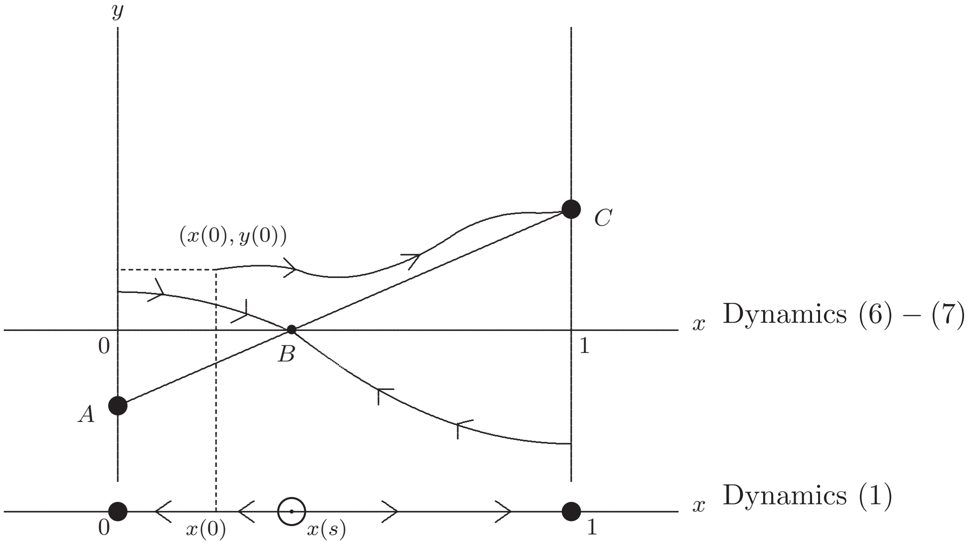

Figure 6: A graphical representation of theorem 3 in the case of a coordination game. Attractive fixed points are represented by filled circles, repulsive fixed points by dotted circles, and saddle points by dots.