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Von Neumann-Morgenstern Clutters

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Abstract - A clutter on a set X is a simple hypergraph with pairwise not-comparable hyperedges, hence in particular any set of Von Neumann-Morgenstern (VNM) -stable sets of an irreflexive simple digraph is a clutter. A clutter (X, E) is representable by VNM-stable sets or VNM if there exists an irreflexive simple digraph (X, Δ) such that E is a set of VNM-stable sets of (X, Δ). The class of VNM clutters on a set X is characterized.

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Let $H = (X, \mathbf{E})$ be a *simple hypergraph* i.e. X is a set, and $\mathbf{E} \subseteq \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X: X will also be referred to as the set of vertices of H, and $\mathbf{E}$ as the set of hyperedges of H. Then, H is a *clutter* iff $\mathbf{E}$ is a $\subseteq$-antichain, namely $E \nsubseteq E'$ for any two *distinct* $E, E' \in \mathbf{E}$. In particular the 'boundary' cases with $\mathbf{E} = \{\emptyset\}$, or $\mathbf{E} = \emptyset$ are allowed. [Notice that such a terminology is slightly at variance with that of other authors, who sometimes denote as 'simple hypergraphs', 'Sperner systems' or 'Sperner families' those clutters $(X, \mathbf{E})$ -as defined above- such that $\emptyset \notin \mathbf{E}$ and $\bigcup \mathbf{E} = X$: see e.g. Berge (1989)]. A *simple digraph* is a pair $D = (X, \Delta)$ where X is a set and $\Delta \subseteq X \times X$; D is *irreflexive* -or loopless- iff $(x, x) \notin \Delta$ for any $x \in X$. A *Von Neumann-Morgenstern stable set* (henceforth *VNM-stable set*) of an irreflexive simple digraph (X, Δ) is a set $S \subseteq X$ such that (i) internal stability i.e. $[(x, y) \notin \Delta$ for any $x, y \in S]$, and (ii) external stability i.e. [for any $z \in X \setminus S$ there exists $x \in S$ such that $(x, z) \in \Delta$] hold (see e.g. Von Neumann and Morgenstern (1953), Schmidt and Ströhlein (1985), and Ghoshal, Laskar and Pillone (1998)). Let $\mathcal{S}(X, \Delta)$ denote the set of all VNM-stable sets of (X, Δ). Clearly, $S_1 \nsubseteq S_2$ for any two *distinct* S_1, $S_2 \in \mathcal{S}(X, \Delta)$ (otherwise, properties (i) and (ii) of S_1 turn out to be mutually inconsistent). Thus, for any irreflexive simple digraph (X, Δ), $(X, \mathcal{S}(X, \Delta))$ is a clutter. This observation motivates the following

Definition 1 *(VNM clutters) A clutter $H = (X, \mathbf{E})$ is representable by VNM-stable sets or* VNM *iff there exists an irreflexive simple digraph (X, Δ) such that $\mathbf{E} \subseteq \mathcal{S}(X, \Delta)$.*

In general, a clutter may or may not be VNM, as made clear by the following examples

Example 2 *Consider $X = \{x, y, z\}$ with $x \neq y \neq z \neq x$, and*
$\mathbf{E} = \{\{x, y\}, \{x, z\}, \{y, z\}\}$.

It is easily checked that $(X, \mathbf{E})$ is a clutter, but not *a VNM one. Indeed, suppose to the contrary that there exists an irreflexive simple digraph (X, Δ) such that $\mathbf{E} \subseteq \mathcal{S}(X, \Delta)$. Then, by property (ii) of VNM-stable sets as applied to $\{x, y\}$, $\Delta \cap \{(x, z), (y, z)\} \neq \varnothing$, which contradicts the assumption that both $\{x, z\}$ and $\{y, z\}$ satisfy property (i) of VNM-stable sets.*

Example 3 *Consider now X as defined above in the previous example, $\mathbf{E}' = \{\{x, y\}, \{x, z\}\}$, and $\Delta' = \{(y, z), (z, y)\}$. Clearly, $\mathbf{E}' = \mathcal{S}(X, \Delta')$ i.e. $(X, \mathbf{E}')$ is indeed a VNM clutter.*

Example 4 *Let* $(X, \mathbf{E})$ *be a clutter such that* $E \cap E' = \varnothing$ *for any pair of* distinct $E, E' \in \mathbf{E}$.

Then, define $\Delta = \{(x, y) \in X \times X\colon \{x, y\} \nsubseteq E \text{ for all } E \in \mathbf{E}\}$.

Clearly, for any $E \in \mathbf{E}$ *and any* $x, y \in E$, $(x, y) \notin \Delta$ *by definition. Moreover, suppose there exists* $z \in X \setminus E$ *such that* $(x, z) \notin \Delta$ *for all* $x \in E$. *Then, for any* $x \in E$ *there exists* $E' \in \mathbf{E}$ *such that* $\{x, z\} \subseteq E'$. *Since by construction* $E \neq E'$ *and* $E \cap E' \neq \varnothing$, *the existence of such an* E' *contradicts our starting hypothesis. It follows that* E *is indeed a VNM-stable set of* (X, Δ). *Therefore,* $(X, \mathbf{E})$ *is a VNM clutter.*

Remark 5 *(Kernel-representable clutters) Let* (X, Δ) *be an irreflexive simple digraph, and* $\Delta^{-1} \subseteq X \times X$ *the inverse of* Δ, *namely, for any* $x, y \in X$, $(x, y) \in \Delta^{-1}$ *iff* $(y, x) \in \Delta$. *Of course,* (X, Δ^{-1}) *is also an irreflexive simple digraph. A subset of vertices* $K \subseteq X$ *is a* kernel *of* (X, Δ) *- written* $K \in \mathcal{K}(X, \Delta)$ *- iff* $K \in \mathcal{S}(X, \Delta^{-1})$. *Now, consider any clutter* $(X, \mathbf{E})$ *and declare it* kernel-representable (KR) *iff there exists an irreflexive simple digraph such that* $\mathbf{E} \subseteq \mathcal{K}(X, \Delta)$. *It is immediately checked that, in view of the foregoing observations, a clutter is KR iff it is VNM.*

The foregoing observations and examples raise the issue of identifying VNM clutters (in the same vein, a representation problem concerning finite lattices and stable matchings is addressed by Blair (1984)).

The aim of this note is in fact to provide a simple characterization of VNM clutters. In order to state our result in a most concise manner, let us first introduce some further auxiliary notions.

Definition 6 *(*Conjugation relation of a simple hypergraph*) Let* $H = (X, \mathbf{E})$ *be a simple hypergraph. Then the* conjugation *relation* $I_H \subseteq X \times X$ *of* H *is defined as follows: for any* $x, y \in X$, $(x, y) \in I_H$ *iff there exists an* $E \in \mathbf{E}$ *such that* $\{x, y\} \subseteq E$.

Definition 7 *(*Complete bigraph*) Let* Y, Z *be two nonempty sets. A (simple, nontrivial)* bigraph *on* (Y, Z) *is a triple* (Y, Z, R) *where* $R \subseteq Y \times Z$. *A bigraph* (Y, Z, R) *is* complete *if* $R = Y \times Z$.

Definition 8 *Let* (X, Δ) *be a simple digraph. A* Δ-basis of a complete bigraph *is a* $B \subseteq X$ *such that* $(B, X \setminus B, \Delta \cup \Delta^{-1})$ *is a complete bigraph.*

Definition 9 *(*Complete bigraph generator (CBG)*) An hyperedge $E \in \mathbf{E}$ of a clutter $H = (X, \mathbf{E})$ is a* complete bigraph generator *iff it is an I_H-basis of a complete bigraph.*

A clutter $H = (X, \mathbf{E})$ is *CBG-free* iff no $E \in \mathbf{E}$ is a complete bigraph generator. Moreover, clutter H is *conjugation-saturated* iff it is CBG-free and $I_H \neq I_{H'}$ for any CBG-free clutter $H' = (X, \mathbf{E}')$ such that $\mathbf{E} \subset \mathbf{E}'$.

Then, we have the following

Theorem 10 *Let $H = (X, \mathbf{E})$ be a clutter. Then, H is VNM iff it is CBG-free.*

Proof. Observe that, by definition, H is not CBG iff for any $E \in \mathbf{E}$ and any $z \in X \smallsetminus E$ there exists $x_z \in E$ such that $(z, x_z) \notin I_H$.

Then, suppose $H = (X, \mathbf{E})$ is VNM, and let (X, Δ) be an irreflexive simple digraph such that $\mathbf{E} \subseteq \mathcal{S}(X, \Delta)$. Now, assume that there exist an hyperedge $E \in \mathbf{E}$ and a vertex $z \in X \smallsetminus E$ such that $(z, x) \in I_H$ for all $x \in E$. Thus, for any $x \in E$ there exists an $E_{zx} \in \mathbf{E}$ such that $\{x, z\} \subseteq E_{zx}$ whence, by internal stability of E_{zx}, $(x, z\ ) \notin \Delta$: but then, external stability of E is violated. It follows that H is not VNM, a contradiction.

Conversely, suppose that $H = (X, \mathbf{E})$ is such that for any $E \in \mathbf{E}$ and any $z \in X \smallsetminus E$ there exists $x_z \in E$ with $(z, x_z) \notin I_H$. Then, define $\Delta^H \subseteq X \times X$ by the following rule: for any $x, y \in X$, $\{(x, y), (y, x)\} \subseteq \Delta^H$ if $(x, y) \notin I_H$, and $\{(x, y), (y, x)\} \cap \Delta^H = \varnothing$ if $(x, y) \in I_H$: notice that (X, Δ^H) is irreflexive. Therefore, for any $E \in \mathbf{E}$ and any $x, y \in E$, $(x, y) \notin I_H$ hence $(x, y) \notin \Delta^H$, and E satisfies internal stability with respect to (X, Δ^H). Moreover, by hypothesis, for any $z \in X \smallsetminus E$ there exists $x_z \in E$ with $(z, x_z) \notin I_H$ i.e. in particular $(x_z, z) \in \Delta^H$, by definition of Δ, hence E also satisfies external stability with respect to (X, Δ^H). Thus, $\mathbf{E} \subseteq \mathcal{S}(X, \Delta^H)$ as required. ∎

Remark 11 *Notice that, as it is easily checked, the trivial clutters $(\emptyset, \{\emptyset\})$ and $(X, \{\emptyset\})$ with a nonempty X are both VNM. Moreover, it is worth emphasizing that the foregoing Theorem also implies that clutter $(X, \mathbf{E})$ of Example 2 is not VNM because it is clearly not CBG-free (indeed,* each *of its hyperedges is a complete bigraph generator). On the contrary, its* odd-ciclycity *is* not *key to its being* not *VNM. To see this, consider clutter $(\{x, y, z, u, v\}, \mathbf{E})$ with $\mathbf{E} = \{\{x, y\}, \{y, z\}, \{z, u\}, \{u, v\}, \{v, x\}\}$,*

which is also odd-cyclic *(namely, there exist a positive integer k, and $2k + 1$ distinct hyperedges $E_i \in \mathbf{E}$ and vertices x_i, $i = 1, ..., 2k + 1$ such* *that* $\{x_i, x_{i+1}\} \subseteq E_i$*,* $i = 1, ..., 2k$ *and* $\{x_1, x_{2k+1}\} \subseteq E_{2k+1}$*). However,* $(\{x, y, z, u, v\}, \mathbf{E})$ *is CBG-free hence by Theorem 10 is indeed a VNM clutter: to confirm the latter statement it is only to be checked that there is no* $E_i \in \mathbf{E}$ *comprising one of the following pairs:* $\{x, z\}, \{x, u\}, \{y, v\}, \{y, u\}, \{z, v\}$*.*

Remark 12 *In view of Theorem 10 it is easily checked that a remarkable class of clutters which are not VNM is provided by (nontrivial)* Steiner triple systems *i.e. clutters* $H = (X, \mathbf{E})$ *such that* $\#X \geq 4$*,* $\mathbf{E} \subseteq \{Y \subseteq X : \#Y = 3\}$ *and for any two distinct* $x, y \in X$ *there exists precisely* one $E \in \mathbf{E}$ *with* $\{x, y\} \subseteq E$*. Indeed, any hyperedge* E *of such a clutter is a CBG: to check the latter statement, take any* $E \in \mathbf{E}$ *and observe that there exist* $x \in X \smallsetminus E$ *and* $y \in E$*, and for any such* x, y *there exists by assumption an* $E' \in \mathbf{E}$ *with* $\{x, y\} \subseteq E'$*.*

Let us denote a clutter $H = (X, \mathbf{E})$ as (strictly) *VNM-complete* if it is representable as the set of *all* VNM-stable sets of (X, Δ^H) i.e. $\mathbf{E} = \mathcal{S}(X, \Delta^H)$, where $\Delta^H = (\bigcup \mathbf{E} \times \bigcup \mathbf{E}) \smallsetminus I_H$ as defined above in the proof of Theorem 10. Then, we have the following straightforward corollary of the previous theorem, namely

Corollary 13 *A clutter* $H = (X, \mathbf{E})$ *is VNM-complete iff it is conjugation-saturated. In particular, that requires* $\bigcup \mathbf{E} = X$*.*

Proof. Assume that $H = (X, \mathbf{E})$ is VNM-complete and there exists $\mathbf{E}' \supset \mathbf{E}$ such that $H' = (X, \mathbf{E}')$ is CBG-free and $I_H = I_{H'}$. But then, by definition $\mathcal{S}(X, \Delta^{H'}) = \mathcal{S}(X, \Delta^H)$, and by Theorem 10, $\mathbf{E}' \subseteq \mathcal{S}(X, \Delta^{H'}) = \mathcal{S}(X, \Delta^H)$, a contradiction since by hypothesis $\mathbf{E} = \mathcal{S}(X, \Delta^H)$.

Conversely, let $H = (X, \mathbf{E})$ be conjugation-saturated, and suppose that it is not VNM-complete, namely $\mathbf{E} \subset \mathcal{S}(X, \Delta^H)$. Then, by Theorem 10 there exists a CBG-free $H' = (X, \mathbf{E}')$ such that $\mathbf{E} \subset \mathbf{E}' = \mathcal{S}(X, \Delta^H)$. Since H is conjugation-saturated, $I_H \neq I_{H'}$ hence by definition $I_H \subset I_{H'}$. But then, there exist $E' \in \mathcal{S}(X, \Delta^H) \smallsetminus \mathbf{E}$ and $x, y \in E' \smallsetminus \bigcup \mathbf{E}$ whence -by definition of Δ^H- $x \Delta^H y$ which contradicts internal stability of E' with respect to (X, Δ^H).

Finally, observe that if $X \smallsetminus \bigcup \mathbf{E} \neq \emptyset$ then $(x, y) \notin I_H$ for any $x \in X \smallsetminus \bigcup \mathbf{E}$ and $y \in X \smallsetminus \{x\}$ whence by definition $\{x\} \in \mathcal{S}(X, \Delta^H) \smallsetminus \mathbf{E}$ and thus $(X, \mathbf{E})$ is not VNM-complete. ■

As an example, observe that clutter $H = (\{x,y,z\}, \{\{x,y\}\}\ )$ where $x \neq y \neq z \neq x$ is clearly CBG-free but *not* conjugation-saturated since clutter $H' = (\{x,y,z\}, \{\{x,y\}, \{z\}\})$ is such that $I_{H'} = \{(x,y),(y,x)\} = I_H$. Thus, the foregoing Corollary entails that H' is not VNM-complete.

On the contrary, for any set X with $\#X \geq 3$ and any $x \in X$ and positive integer $1 < r < \#X$ take a 'minimal' r-uniform star-clutter with centre x, i.e. a clutter $H_x^* = (X, \mathbf{E})$ such that $\bigcup \mathbf{E} = X$, $\#E = r$ for every $E \in \mathbf{E}$, and for any two distinct $E, E' \in \mathbf{E}$, $E \cap E' = \{x\}$: by construction, H_x^* is CBG-free since for any two distinct $E, E' \in \mathbf{E}$ and any $y \in E \smallsetminus \{x\}$, $z \in E' \smallsetminus \{x\}$ it turns out that $\{(y,z),(z,y)\} \cap I_{H_x^*} = \emptyset$. Moreover, H_x^* is also a conjugation-saturated clutter since for any CBG-free clutter $H' = (X, \mathbf{E}')$ with $\mathbf{E}' \supset \mathbf{E}$, and any $E' \in \mathbf{E}' \smallsetminus \mathbf{E}$ there exist two distinct $E_1, E_2 \in \mathbf{E}$ and $y \in E_1 \smallsetminus E_2$, $z \in E_2 \smallsetminus E_1$ such that $\{y,z\} \subseteq E'$, hence $\{(y,z),(z,y)\} \subseteq I_{H'}$ while $\{(y,z),(z,y)\} \cap I_{H_x^*} = \emptyset$ by construction of H_x^*. Therefore, Corollary 13 implies that H_x^* is indeed VNM-complete.

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