Demographic Research a free, expedited, online journal
of peer-reviewed research and commentary
in the population sciences published by the
Max Planck Institute for Demographic Research
Konrad-Zuse Str. 1, D-18057 Rostock · GERMANY
www.demographic-research.org

DEMOGRAPHIC RESEARCH

VOLUME 23, ARTICLE 3, PAGES 63-72
PUBLISHED 09 JULY 2010
http://www.demographic-research.org/Volumes/Vol23/3/
DOI: 10.4054/DemRes.2010.23.3

Formal Relationships 9

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This article is part of the Special Collection "Formal Relationships".
Guest Editors are Joshua R. Goldstein and James W. Vaupel.

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Sensitivity of life disparity with respect to changes in mortality rates

Peter Wagner [1, 2]

Abstract

This article is concerned with sensitivity analysis of life disparity with respect to changes in mortality rates. A relationship is derived that describes the effect on life disparity caused by a perturbation of the force of mortality. Recently Zhang and Vaupel introduced a “threshold age”, before which averting deaths reduces disparity, while averting deaths after that age increases disparity. I provide a refinement to this result by characterizing the ages at which averting deaths has an extremal impact on life disparity. The results are illustrated using data for the female populations of Denmark in 1835, and for the United States in 2005.

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[2] The author thanks Carl Schmertmann for thoughtful recommendations, including the interpretation of the three summands in equation (5), that helped to improve and shorten the article, Joshua Goldstein for valuable suggestions concerning its structure and presentation and Thomas Cassidy for proofreading and further help with its shortening. Useful comments by James Vaupel and also Trifon Missov and Zhen Zhang are gratefully acknowledged.

1. Introduction

In keeping with Keyfitz's idea that everybody dies prematurely, since every death deprives the person involved of the remainder of his expectation of life (Keyfitz 1977:61-68), the measure $e^\dagger$ for the average life expectancy lost due to death has been widely studied. It first appeared in Mitra (1978) and was developed further by Vaupel (1986) and recently in Vaupel and Canudas-Romo (2003), Zhang and Vaupel (2008) and Shkolnikov et al. (2009). Zhang and Vaupel (2009) initiated a new direction of analysis, studying the impact on $e^\dagger$ of a concentrated decrease in mortality at age a.

Life disparity is measured by life expectancy lost due to death

$$e^\dagger = \int_0^\infty e(x)\, d(x)\, dx, \tag{1}$$

where $e(x)$ is the remaining life expectancy at age x,

$$e(x) = \frac{1}{\ell(x)} \int_x^\infty \big(y - x\big)\, d(y)\, dy = \frac{1}{\ell(x)} \int_x^\infty \ell(y)\, dy, \tag{2}$$

$\ell(x) = \exp\left(-H(x)\right)$ is the probability of survival to age x, $H(x) = \int_0^x \mu(y)\, dy$ is the cumulative hazard function and $\mu(x)$ is the age-specific hazard of death. Since nobody lives forever, it is generally assumed that H is strictly increasing, attaining all non-negative real numbers. The function $d(x) = \frac{d}{dx}\big(1 - \ell(x)\big) = \ell(x)\mu(x)$ is the life table distribution of deaths.

Goldman and Lord (1986: equations (6), (14)) and Vaupel (1986: equations (4), (5)) independently showed that life disparity (1) is the product of the life expectancy at birth and the entropy of the life table

$$e^\dagger = e(0) \left(- \frac{\int_0^\infty \ell(x)\, \log \ell(x)\, dx}{\int_0^\infty \ell(x)\, dx} \right). \tag{3}$$

In this article, I will formally derive and discuss a relationship concerning the effect of a change in mortality at some age on life disparity.

2. Relationship

Let $\varphi(a)$ represent the change of $e^\dagger$ caused by a reduction in mortality at age a (with a precise definition thereof via a limit of derivatives given in the proof). The main relationship in this article consists of two equivalent formulae ((4), (5)) for the function φ.

Theorem 1
The function $\varphi(a)$, representing the change of $e^\dagger$ caused by a reduction in mortality at age a, satisfies the relationship

$$\varphi(a) = -\int_a^\infty \ell(x)\Big(1 + \log \ell(x)\Big)\,dx \tag{4}$$

and, equivalently,

$$\varphi(a) = \ell(a)\Big(H(a)\,e(a) - e(a) + e^\dagger(a)\Big), \tag{5}$$

where $e^\dagger(a)$ denotes life expectancy lost due to death among people surviving to age a,

$$e^\dagger(a) = \frac{1}{\ell(a)}\int_a^\infty e(x)\,d(x)\,dx. \tag{6}$$

Furthermore, φ has the following three properties:

(i) **Monotonicity**
Let $\tilde{a}$, the age of cumulative hazard unity, be defined via $H(\tilde{a}) = 1$. Then φ is strictly increasing on $[0, \tilde{a}]$ and strictly decreasing and strictly positive on $[\tilde{a}, \infty)$, having a global maximum of $\varphi(\tilde{a}) = \exp(-1)e^\dagger(\tilde{a})$ at $a = \tilde{a}$ and a local minimum of $\varphi(0) = e^\dagger(0) - e(0)$ at $a = 0$. More precisely,

$$\frac{d}{da}\,\varphi(a) = \ell(a)\big(1 - H(a)\big). \tag{7}$$

(ii) **Curvature**
Let a^* be defined by $H(a^*) = 2$. Then φ is strictly concave on $[0, a^*]$ and strictly convex on $[a^*, \infty)$. More precisely,

$$\frac{d^2}{da^2}\,\varphi(a) = d(a)\big(H(a) - 2\big).$$

(iii) **Asymptotic Behaviour**

$$\lim_{a\to\infty} \varphi(a) = 0.$$

Zhang and Vaupel (2009) showed that, if the life table entropy (cf. (3)) satisfies $e^\dagger/e(0) < 1$, then there exists a positive "threshold age" $a^\dagger$ between the regions of "early" ages $a < a^\dagger$, in which the effect of averting deaths on life disparity is negative, and "late" ages $a > a^\dagger$, in which this effect is positive. If $e^\dagger/e(0) = 1$, then $a^\dagger = 0$, and the effect is positive at all ages other than zero. Finally, if $e^\dagger/e(0) > 1$, then $a^\dagger$ does not exist, and the effect is positive everywhere. Refining these results, Theorem 1(i) highlights some helpful monotonicity properties and draws attention to $\tilde{a}$ and 0, the ages of extremal effect on life disparity caused by averting deaths.

Remark 1
Recall definition (1) and think of $e^\dagger$ as the sum of three integrals, by partitioning the range of integration into the ages below, at and above a. It turns out that the three summands in equation (5) correspond to the effects of a mortality reduction at age a on the three integrals, which I shall refer to as the "early", "instant" and "late" effect, respectively:

- **Early**: At ages $x < a$, deaths $d(x)$ are unchanged, and life expectancies of survivors $e(x)$ increase, which *increases* disparity by $\ell(a)\, H(a)\, e(a)$.
- **Instant**: At age $x = a$, deaths $d(x)$ decrease, and life expectancy of survivors $e(x)$ is unchanged, which *decreases* disparity by $\ell(a)\, e(a)$.
- **Late**: At ages $x > a$, deaths $d(x)$ increase, and life expectancies of survivors $e(x)$ are unchanged, which *increases* disparity by $\ell(a)\, e^\dagger(a)$.

3. Proof

Theorem 1, the relationship

In the general case, the force of mortality μ changes by some function $\Delta\mu$, with a small step s:

$$\mu(x,s) = \mu(x) + s\Delta\mu(x),$$
$$\mu_s(x,s) = \frac{\partial}{\partial s}\,\mu(x,s) = \Delta\mu(x). \tag{8}$$

Similarly, for the s-dependent survival function,

$$\ell(x,s) = \exp\left(-\int_0^x \mu(y,s)\,dy\right),$$
$$\ell_s(x,s) = \left(-\int_0^x \Delta\mu(y)\,dy\right)\,\ell(x,s)$$

Noting $e^\dagger = e^\dagger(0)$ (cf. (1), (6)), by equation (3),

$$e^\dagger(0,s) = -\int_0^\infty \ell(x,s)\,\log\,\ell(x,s)\,dx.$$

Hence

$$\begin{aligned} e_s^\dagger(0,s) &= -\int_0^\infty \ell_s(x,s)\big(1+\log\,\ell(x,s)\big)\,dx \\ &= -\int_0^\infty \left(-\int_0^x \Delta\mu(y)\,dy\right)\ell(x,s)\big(1+\log\,\ell(x,s)\big)\,dx \\ &= \int_0^\infty \Delta\mu(y)\left(\int_y^\infty \ell(x,s)\big(1+\log\,\ell(x,s)\big)\,dx\right)\,dy. \end{aligned}$$

Thus the functional derivative of disparity with respect to mortality change $\Delta\mu$ is

$$e_s^\dagger(0,0) = \int_0^\infty \Delta\mu(y)\left(\int_y^\infty \ell(x)\big(1+\log\,\ell(x)\big)\,dx\right)\,dy.$$

Keeping mortality *reductions at age* a in mind, when $\Delta\mu$ is a negative step function in a small neighbourhood around age a, such that its integral equals minus unity; for example, and from now on, $\Delta\mu(y) = -1/\varepsilon$ for $y \in [a, a+\varepsilon]$ and zero elsewhere, then

$$e_s^\dagger(0,0) = \frac{1}{\varepsilon}\int_a^{a+\varepsilon}\left(-\int_y^\infty \ell(x)\big(1+\log\,\ell(x)\big)\,dx\right)\,dy \;\sim\; -\int_a^\infty \ell(x)\big(1+\log\,\ell(x)\big)\,dx$$

or, more precisely, and hereby defining $\varphi(a)$ as a limit of derivatives,

$$\varphi(a) = \lim_{\varepsilon\to 0} e_s^\dagger(0,0) = -\int_a^\infty \ell(x)\big(1+\log\,\ell(x)\big)\,dx.$$

Finally, note that

$$\begin{aligned} -\int_a^\infty \ell(x)(1+\log\,\ell(x))\,dx &= \ell(a)\bigg(-\int_a^\infty \frac{\ell(x)}{\ell(a)}\Big(\log\,\ell(a)+1+\log\frac{\ell(x)}{\ell(a)}\Big)\,dx\bigg) \\ &= \ell(a)\Big(H(a)\,e(a) - e(a) + e^\dagger(a)\Big). \end{aligned}$$

Theorem 1, part (i)

By equation (5), φ is strictly positive on $[\tilde{a}, \infty)$. Equation (7) follows from equation (4), so that the first derivative of φ is strictly positive on $[0, \tilde{a})$ and strictly negative on $(\tilde{a}, \infty)$. Thus, φ is strictly increasing on $[0, \tilde{a}]$ and strictly decreasing on $[\tilde{a}, \infty)$. Consequently, φ has a global maximum at $a = \tilde{a}$ and a local minimum at $a = 0$ with

$$\varphi(\tilde{a}) = \ell(\tilde{a})\, e^{\dagger}(\tilde{a}) = \exp(-1)\, e^{\dagger}(\tilde{a}) \quad \text{and} \quad \varphi(0) = e^{\dagger}(0) - e(0).$$

Theorem 1, part (ii)

Differentiating equation (7) with respect to a,

$$\frac{d^2}{da^2}\, \varphi(a) = -d(a)\,\bigl(1 - H(a)\bigr) - \ell(a)\mu(a) = d(a)\,\bigl(H(a) - 2\bigr).$$

Hence the second derivative of φ is strictly negative on $[0, a^*)$ and strictly positive on (a^*, ∞). Thus, φ is strictly concave on $[0, a^*]$ and strictly convex on $[a^*, \infty)$.

Theorem 1, part (iii)

Clearly, both the “instant” and “late” effect (cf. (2), (6)) approach zero, that is,

$$\lim_{a\to\infty}\Bigl(-\ell(a)\, e(a)\Bigr) = 0 \quad \text{and} \quad \lim_{a\to\infty}\Bigl(l(a)\, e^{\dagger}(a)\Bigr) = 0,$$

so that, by equation (5), it remains to show

$$\lim_{a\to\infty}\Bigl(\ell(a)\, H(a)\, e(a)\Bigr) = 0$$

for the “early” effect $\ell(a)\, H(a)\, e(a)$. Using integration by parts,

$$\begin{aligned} e^{\dagger} &= \int_0^{\infty} e(x)\, d(x)\, dx = \int_0^{\infty} \mu(x)\Bigl(\ell(x)\, e(x)\Bigr)\, dx \\ &= \Bigl(\lim_{x\to\infty}\Bigl(H(x)\, \ell(x)\, e(x)\Bigr) - 0\Bigr) + \int_0^{\infty} H(x)\, \ell(x)\, dx, \end{aligned}$$

where, by reversing the order of integration,

$$\begin{aligned} \int_0^{\infty} H(x)\, \ell(x)\, dx &= \int_0^{\infty} \left(\int_0^{x} \mu(y)\, dy\right) \ell(x)\, dx = \int_0^{\infty} \left(\int_y^{\infty} \ell(x)\, dx\right) \mu(y)\, dy \\ &= \int_0^{\infty} \ell(y)\, e(y)\, \mu(y)\, dy = \int_0^{\infty} e(y)\, d(y)\, dy = e^{\dagger}. \end{aligned}$$

4. History and related results

For a concise introduction to functional derivatives see, for example, Arthur (1984) and Frigyik, Srivastava, and Gupta (2008), with applications of the mathematical techniques used here to demography and subjects such as engineering, respectively.

Recall that the perturbation (8) for my specific choice of $\Delta\mu$ represents an *absolute* reduction of the death rate on the age interval $[a, a+\varepsilon]$. A reduction of the death rate on the same interval *relative* to its value at age a is given by

$$\tilde{\mu}(x,s) = \mu(x) + s\,\mu(a)\,\Delta\mu(x) = \mu(x, s\,\mu(a)).$$

Then

$$\begin{aligned} g(a) &= \lim_{\varepsilon\to 0} \tilde{e}_s^\dagger(0,0) = \lim_{\varepsilon\to 0}\left(\frac{\partial}{\partial s}\,\tilde{e}^\dagger(0,s)\bigg|_{s=0}\right) = \lim_{\varepsilon\to 0}\left(\frac{\partial}{\partial s}\,e^\dagger(0,s\mu(a))\bigg|_{s=0}\right) \\ &= \lim_{\varepsilon\to 0}\left(\mu(a)\,\frac{\partial}{\partial s}\,e^\dagger(0,s)\bigg|_{s=0}\right) = \mu(a)\lim_{\varepsilon\to 0} e_s^\dagger(0,0) = \mu(a)\,\varphi(a) \\ &= d(a)\Big(H(a)e(a) - e(a) + e^\dagger(a)\Big), \end{aligned}$$

which is equation (2) in Zhang and Vaupel (2008). Letting

$$k(a) = H(a)\,e(a) - e(a) + e^\dagger(a)$$

(Zhang and Vaupel 2008: equation (3)), motivated by data for Japanese females in 1950, 1970 and 1990, Zhang and Vaupel discuss implications of the existence of a unique root of k, a so-called “threshold age”, for g, the “age-specific impact of survival improvement on lifespan disparity”.

In Zhang and Vaupel (2009), a formal proof for the existence of at most one root of k is given. Their derivation of the relationship for k (Zhang and Vaupel 2009: equation (1)) is based on an absolute “concentrated decrease in mortality at age a”, in my notation corresponding to

$$\frac{\partial}{\partial s}\left(\lim_{\varepsilon\to 0} e^\dagger(0,s)\right)\bigg|_{s=0},$$

which turns out to equal

$$\lim_{\varepsilon\to 0}\left(\frac{\partial}{\partial s}\,e^\dagger(0,s)\bigg|_{s=0}\right) = \varphi(a) = \ell(a)\,k(a).$$

Although in this case, the order of the two limiting processes does not matter, their demographic interpretations differ. First differentiating and then letting ε tend to zero describes the impact of an absolute mortality decrease over a narrower and narrower age range starting at age a. However, first letting ε tend to zero and then differentiating raises the challenge of having to interpret cumulative hazard functions with a downward step,

$$H_{a,s}(x) = H(x) - s \cdot \mathbb{1}_{[a,\infty)}(x)$$

(where $\mathbb{1}_{[a,\infty)}(x)$ is one for $x \geq a$ and zero otherwise), and negative death rates, possibly by (rather hypothetically) resurrecting some of a cohort's decedents. The latter approach of Zhang and Vaupel (2009) was also taken in Wagner (2010), where the new value of life disparity (corresponding to the $H_{a,s}(x)$ above) was shown to satisfy

$$e^{\dagger}_{a,s} = e^{\dagger} + \big(\exp(s) - 1\big)\,\ell(a)\Big(H(a)\,e(a) - e(a) + e^{\dagger}(a)\Big),$$

from which it follows immediately that

$$\frac{\partial}{\partial s}\, e^{\dagger}_{a,s}\bigg|_{s=0} = \ell(a)\Big(H(a)\,e(a) - e(a) + e^{\dagger}(a)\Big) = \varphi(a).$$

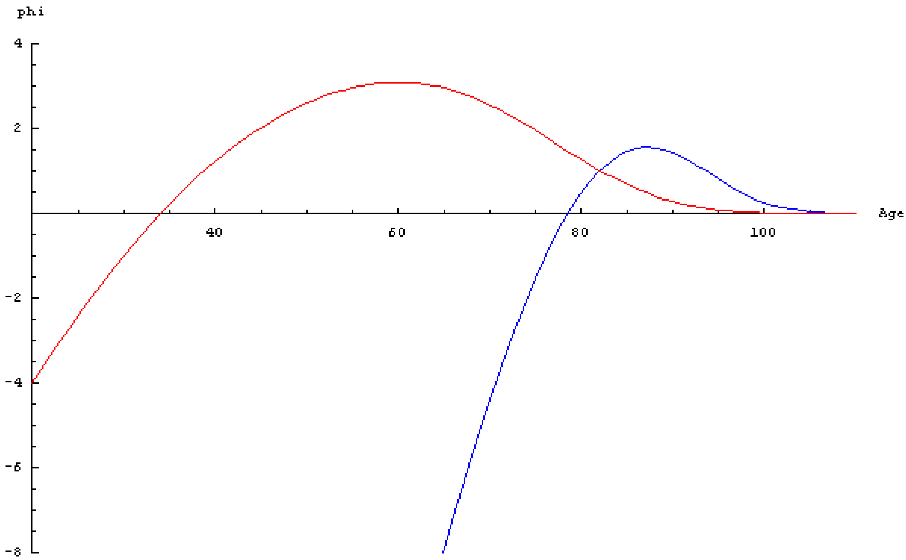
Regardless of the origin of the simple relation $\varphi = \ell \cdot k$, implying that φ and k always have the same sign, it is striking how much more information can be obtained for φ, in terms of monotonicity, curvature, and asymptotic behaviour, as summarised in Theorem 1.

5. Applications

To illustrate the theoretical results, I have computed several relevant quantities for life tables from the Human Mortality Database (2010). Since the function φ is very similar for populations of one and the same era, essentially sharing the same kind of mortality schedule, I concentrate on displaying φ for a contemporary against a historical table. In my Figure 1, the blue curve represents φ for the female population of the United States in 2005, where $a^{\dagger} \sim 78.59$ and $\tilde{a} \sim 87.42$, whereas the red curve represents φ for the population of Danish females in 1835, where $a^{\dagger} \sim 34.02$ and $\tilde{a} \sim 60.36$. Note that my Figure 1 concurs with Figures 1 and 2 of Zhang and Vaupel (2009).

Regardless of the country or era of a particular mortality schedule, the shape of the function φ, representing the effect of a mortality reduction at some age on life disparity, and which can be interpreted as the sum of the "early", "instant" and "late" effect on disparity, is always as governed by Theorem 1. Indeed, the derivative of the sum of

Figure 1: Change in life disparity (expressed via function (5)) as a function of the age at which the mortality rate is reduced. The blue curve (right) represents data from the Human Mortality Database 2010 for US-females in 2005, while the red curve (left) corresponds to Danish females in 1835.



the positive "early" and "late" effect equals $-H(a)\,\ell(a)$, while the derivative of the negative "instant" effect is $\ell(a)$. So while initially the increase of the "instant" effect dominates the decrease of the sum of the other two effects, implying an increase of φ, those roles are reversed at the age $\tilde{a}$ of cumulative hazard unity, from which φ decreases and asymptotically approaches zero. Interestingly, $\tilde{a}$ is both the age when φ reaches its maximum and when the "early" and "instant" effect cancel each other. At age zero, there is no "early" effect. So provided that the negative "instant" effect dominates the positive "late" effect (which is equivalent to the entropy being less than one), $\varphi(0)$ is negative, and there is a unique age $a^\dagger$, when all three effects cancel each other; that is, $\varphi(a^\dagger) = 0$.

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