EUROPEAN CENTRAL BANK
EUROSYSTEM

WORKING PAPER SERIES
NO 709 / JANUARY 2007

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In 2007 all ECB publications feature a motif taken from the €20 banknote.

1 We acknowledge financial support for this research from the ESRC, project no. RES-000-23-1126 and from the European Central Bank's Research Visitors Programme for Paul Levine. Thanks are owing to the ECB for this hospitality and to numerous resident and visiting researchers for stimulating discussions. Views expressed in this paper do not necessarily reflect those of the ECB. We are grateful for comments from an anonymous ECB working paper referee and from the editor Frank Smets. We also acknowledge feedback from seminar participants at City University, the University of Surrey, the Bank of England, the ECB and the 12th Conference on Computing in Economics and Finance in Limassol, Cyprus, June 2006. We are particularly grateful for discussions or correspondence with Roberto Billi, Gregory DeWalque, Albert Marcet, Stephan Sauer, Peter Welz and Raf Wouters.

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ISSN 1561-0810 (print)
ISSN 1725-2806 (online)

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Abstract

The objectives of this paper are : first, to quantify the stabilization welfare gains from commitment; second, to examine how commitment to an optimal rule can be sustained as an equilibrium and third, to find a simple interest rate rule that closely approximates the optimal commitment one. We utilize an influential empirical micro-founded DSGE model, the euro area model of Smets and Wouters (2003), and a quadratic approximation of the representative household's utility as the welfare criterion. Importantly, we impose the effect of a nominal interest rate zero lower bound. In contrast with previous studies, we find significant stabilization gains from commitment: our central estimate is a $0.4 - 0.5\%$ equivalent permanent increase in consumption, but in a variant with a higher degree of price stickiness, gains of over 2% are found. We also find that a simple optimized commitment rule with the nominal interest rate responding to current inflation and the real wage closely mimics the optimal rule.

JEL Classification: E52, E37, E58

Keywords: Monetary rules, commitment, discretion, welfare gains.

Non-technical Summary

This paper has three principle objectives. First, to quantify the stabilization gains from commitment in terms of household welfare. Second, to examine how commitment to an optimal or approximately optimal rule can be sustained as an equilibrium in which reneging hardly ever occurs. And finally, to find a simple interest rate rule that closely approximates the optimal commitment (and complex) rule.

We utilize an influential empirical micro-founded dynamic stochastic general equilibrium (DSGE) model of the euro area in which there are four sources of time-inconsistency: from forward-looking pricing, consumption, investment and wage setting. In the absence of commitment, following a shock which diverts the economy from its steady state and given expectations of inflation, the opportunist policy-maker can increase or decrease output by reducing or increasing the interest rate which increases or decreases inflation. This results in a higher variability of inflation and the nominal interest rate under discretion. The latter means that the interest rate zero lower bound constraint is tighter under discretion and its presence increases the stabilization gains from commitment. The constraint can be relaxed by increasing the steady state inflation rate, but at a cost of an increase in the deterministic component of the welfare loss.

In terms of methodology our welfare-based loss function uses the 'small distortions' quadratic approximation to the consumer's utility which is accurate if the steady state is close to the social optimum. In assessing this condition we highlight a neglected aspect of typical New Keynesian models: external habit in consumption tends to make labour supply and the natural rate of output too high compared with the social optimum. If the habit effect is sufficiently high and labour market and product market distortions are not too big then, with a sufficiently small tax wedge, the natural rate can actually be above the social optimum. This would then render the long-run 'inflationary bias' negative.

Whilst the validity of an inflationary bias arising from the pursuit of an ambitious output target above its natural rate has been criticized, our analysis suggests a rather different form of bias arising from the interest rate zero lower bound. We find that the optimal steady state inflation rate necessary to avoid the lower bound is far lower under commitment than under discretion, so there is a new sense in which there is a long-run inflationary bias which is really an integral part of the stabilization bias.

Our exercises, suggest that the stabilization gains from commitment rise considerably if the lower bound effect is taken into account. Using empirical estimates from the core model and the preferred variant of the model without indexation, we find an average consumption and inflation-equivalent gains of gain 0.4-0.5% and 0.6-0.7% respectively, the latter on a quarterly basis. For the variant of the model with lower price stickiness, these rise considerably.

Given these large gains from commitment, the incentive for central banks to avoid a loss of reputation for commitment is substantial. Consequently, unless the policymaker is implausibly myopic, a commitment rule can be sustained as a perfect Bayesian equilibrium in which deviation from commitment hardly ever happens despite the possibility of large exogenous shocks.

Finally, we find that simple interest rate rules should respond to labour-market conditions as well as inflation. The optimal commitment rule can be closely approximated in terms of its good stabilization properties by an interest rate rule that responds positively to current inflation and to the current real wage.

1 Introduction

Following the pioneering contributions of Kydland and Prescott (1977) and Barro and Gordon (1983), the credibility problem associated with monetary policy has stimulated a huge academic literature that has been influential with policymakers. The central message underlying these contributions is the existence of significant macroeconomic gains, in some sense, from 'enhancing credibility' through formal commitment to a policy rule or through institutional arrangements for central banks such as independence, transparency, and forward-looking inflation targets, that achieve the same outcome.

In the essentially static model used in those seminal papers and in much of the huge literature they inspired, the loss associated with a lack of credibility takes the form of a long-run inflationary bias. For a dynamic models of the New Keynesian genre, such as the DSGE euro area model employed in this paper, the influential review of Clarida *et al.* (1999) emphasizes the *stabilization gains* from commitment which exist whether or not there is a long-run inflationary bias. But what are the size of these stabilization gains from commitment? If they are small then the credibility problem is solely concerned with the credibility of long-run low inflation.

The first objective of the paper is to quantify the stabilization gains from commitment in terms of household welfare. Previous work has addressed this question (see, for example, Currie and Levine (1993), Vestin (2001), Ehrmann and Smets (2003), McCallum and Nelson (2004), Dieppe *et al.* (2005) and Dennis and Söderström (2006)), but only in the context of econometric models without micro-foundations and using an ad hoc loss function, or both, or for rudimentary New Keynesian models. The credibility issue only arises because the decisions of consumers and firms are forward looking and depends on expectations of future policy. In the earlier generation of econometric models lacking micro-foundations, many aspects of such forward-looking behaviour were lacking and therefore important sources of time-inconsistency were missing. Although for simple New Keynesian models a quadratic approximation of the representative consumer's utility coincides with the standard ad hoc loss that penalizes variances of the output gap and inflation, in more developed DSGE models this is far from the case. By utilizing an influential empirical micro-founded DSGE model, the euro area model of Smets and Wouters (2003), and using a quadratic approximation of the representative household's utility as the welfare criterion, we remedy these deficiencies of earlier estimates of commitment gains.

An further important consideration when addressing the gains from commitment, and missing from these earlier studies, is the existence of a nominal interest rate zero lower bound. A number of papers have studied optimal commitment policy with this constraint (for example, Coenen and Wieland (2003), Eggertsson and Woodford (2003), Woodford (2003), chapter 6). In an important contribution to the credibility literature, Adam and Billi (2006) show that ignoring the zero lower bound constraint for the setting of the nominal interest rate can result in considerably underestimating the stabilization gain from commitment. The reason for this is that under discretion the monetary authority cannot make credible promises about future policy. For a given setting of future interest rates the volatility of inflation is driven up by the expectations of the private sector that the monetary authority will re-optimize in the future. This means that to achieve a given low volatility of inflation the lower bound is reached more often under discretion than under commitment. These authors study a simple New Keynesaian model and are able to employ non-linear techniques. Since we employ a more developed model, we choose a more tractable linear-quadratic (LQ) framework.[1] We follow Woodford (2003) in approximating the effects of a zero interest rate lower bound by imposing the requirement that the interest rate volatility in a commitment are discretionary equilibria are small enough to ensure that the violations of the zero lower bound are very infrequent.

Our second objective is to examine how commitment to an optimal or approximately optimal rule can be sustained as an 'reputational' equilibrium in which reneging hardly ever occurs. We extend the incomplete information[2] of Barro (1986) to a stochastic setting and to a model with structural dynamics. Our final objective is to search for a simple interest rate rule that closely approximates the optimal commitment (and complex) rule. This particular part of the paper closely follows Levin *et al.* (2006) but unlike these authors incorporates a interest rate lower bound into the design of the rule.[3]

The rest of the paper is organized as follows. Section 2 begins by using a simple New Keynesian model to show analytically how a stabilization bias arises in models with structural dynamics. It goes on to generalize the treatment to any linear DSGE model with a quadratic loss function and also to take into account the interest rate lower bound.

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[1] A LQ framework is convenient for a number of reasons: it allows closed-form expressions for the welfare loss under optimal commitment, discretion and simple commitment rules that enable us to study the incentives to renege on commitment. Bayesian estimation methods use a linearized form of the dynamic model. A linear framework further allows us to characterize saddle-path stability and the possible indeterminacy of simple rules. Last but not least, the implementation of the numerical methods utilized by Adam and Billi (2006) for a simple New Keynesian model with only 2 state variables would fall foul of the "curse of dimensionality" (Judd (1998), chapter 7) in our model with 11 state variables.

[2] This avoids well-established problems of trigger strategies use in Barro and Gordon (1983)–see al-Nowaihi and Levine (1994) and Persson and Tabellini (1994).

[3] See Primiceri (2006) for a discussion of the importance of imposing the zero lower bound in the design of monetary rules.

We derive closed-form expressions for welfare under optimal commitment, discretion and simple commitment rules and use these to derive a 'no-deviation condition' for commitment to exist as an equilibrium in which reneging on commitment takes place very infrequently.

Section 3 sets out a version of the Smets-Wouters model (henceforth SW) with one additional feature: the addition of a tax wedge in the steady state. A linearization of the model about a zero-interest steady state and a quadratic approximation of the representative household's utility (provided in section 5) sets up the optimization problem facing the monetary authority in the required LQ framework. Section 4 estimates the SW model and variants where the indexing of prices and/or wages is suppressed, and a price contract of 4 quarters is imposed.

Our welfare quadratic approximations assumes that the zero-inflation steady state is close to the social optimum (the 'small distortions case' of Woodford (2003)). In section 5 we therefore assess the quality of this approximation. In doing this we examine a relatively neglected aspect of New Keynesian models that arises with the inclusion of external habit in consumption, namely that the natural rate of output and employment can actually be *above* the social optimum making the inflationary bias negative and the tax wedge, up to a point, welfare-enhancing. In section 6 we address the three central questions in the paper: how big are stabilization gains, how can the fully optimal commitment rule be sustained as an equilibrium given the time-inconsistency problem and can a simple rule mimic the optimal commitment rule? Section 7 concludes the paper.

2 The Time Inconsistency Problem

2.1 The Stabilization Bias in Two Simple DSGE Models

We first demonstrate how a stabilization bias in addition to the better known long-run inflationary bias can arise using two simple and now very standard DSGE models. The first popularized notably by Clarida *et al.* (1999) and Woodford (2003) is 'New Keynesian' and takes the form.

$$\pi_t = \beta E_t \pi_{t+1} + \lambda y_t + u_t \quad (1)$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(r_t - E_t \pi_{t+1}) \quad (2)$$

In (1) and (2), π_t is the inflation rate, β is the private sector's discount factor, $E_t(\cdot)$ is the expectations operator and y_t is output measured relative to its flexi-price value, the 'output gap', which equals consumption measured relative to its flexi-price value in this closed-economy model without capital or government spending. (1) is derived as a linearized form of Calvo staggered price setting about a zero-inflation steady state and (2) is a linearized Euler equation with nominal interest rate r_t and a risk aversion parameter σ. u_t is a zero-mean shock to marginal costs. All variables are expressed as deviations about the steady state, π_t and r_t as absolute deviations, and y_t as a proportional deviation.

The second model simply replaces (1) with a 'New Classical Phillips Curve' (see Woodford (2003), chapter 3):

$$\pi_t = E_{t-1}\pi_t + \lambda y_t + u_t \tag{3}$$

This aggregate supply curve can be derived by assuming some firms fix prices one period in advance and others can adjust immediately.

Kydland and Prescott (1977) and Barro and Gordon (1983) employed the 'New Classical Phillips Curve' (3) and showed that a time-inconsistency or credibility problem in monetary policy arises when the monetary authority at time 0 sets a state-contingent inflation rate π_t to minimize the loss function

$$\Omega_0 = E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t \left[w_y (y_t - k)^2 + \pi_t^2 \right] \right] \tag{4}$$

Having set the inflation rule, the Euler equation (2) then determines the nominal interest rate that will put the economy on a path with the implied interest rate trajectory. The constant k in the loss function arises because the steady state is inefficient owing to imperfect competition and other distortions. For this simple, essentially static model of the economy (it is really SGE rather than DSGE), optimal rules must take the form of a constant deterministic component plus a stochastic shock-contingent component. These rules depend on whether the policymaker can commit, or she exercises discretion and engages in period-by-period optimization. The standard results in these two cases are respectively:

$$\pi_t = \frac{w_y}{w_y + \lambda^2} u_t = \pi^C(u_t) \tag{5}$$

$$\pi_t = \frac{w_y k}{\lambda} + \frac{w_y}{w_y + \lambda^2} u_t = \pi^D(u_t) \tag{6}$$

Thus the optimal inflation rule with commitment, $\pi^C(u_t)$ consists of zero average inflation plus a shock-contingent component which sees inflation raised (i.e., monetary policy relaxed) in the face of a negative supply shock. The discretionary policy, $\pi^D(u_t)$, can be implemented as a rule with the *same* shock-continent component as the ex ante optimal rule. The *only* difference is now that it includes a non-zero average inflation or *inflationary* bias equal to $\frac{w_y k}{\lambda}$ which renders the rule time-consistent. The credibility or 'time-inconsistency' problem, first raised by Kydland and Prescott, was simply how to *eliminate the inflationary bias whilst retaining the flexibility to deal with exogenous shocks.*

We have established that there are no stabilization gains from commitment in a model economy characterized by the New Classical Philips Curve. This is not the case when we move to the New Keynesian Phillips Curve, (1). Then using general optimization procedures described below in section 2.2.2 and in Appendix A, (5) and (6) now become

$$\pi_t^C = \pi_t^C(u_t, u_{t-1}) = \delta\pi_{t-1}^C + \delta(u_t - u_{t-1}) \tag{7}$$

$$\pi_t^D = \pi_t^D(u_t) = \frac{w_y k}{\lambda} + \frac{w_y}{w_y + \lambda^2} u_t \tag{8}$$

where $\delta = \frac{1-\sqrt{1-4\beta w_y^2}}{2b\beta}$.[4] Comparing these two sets of results we see that the discretionary rule is unchanged, but the commitment rule now is a rule responding to past shocks (i.e., is a *rule with memory*) and therefore the stabilization component of the commitment rules now differs from that of the discretionary rule. Since the commitment rule is the ex ante optimal policy it follows that there are also now *stabilization gains from commitment.* The time-inconsistency problem facing the monetary authority in a New Keynesian economic environment now becomes the elimination of the inflationary bias whilst retaining the flexibility to deal with exogenous shocks *in an optimal way.*

2.2 The Stabilization Bias in General DSGE Models

The stabilization bias arose in our simple DSGE model by replacing a Phillips Curve based on one-period ahead price contracts with one based on staggered Calvo-type price setting. In the DSGE model of the euro area presented in the next section there are a number of additional mechanisms that create price, wage and output persistence. The model also incorporates capital accumulation. All these features add *structural dynamics* to the model and these, together will forward-looking behaviour involving consumption, investment, price-setting and wage- setting add further sources of stabilization gains from commitment.

To examine this further, consider a general linear state-space model

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ E_t\mathsf{x}_{t+1} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + Br_t + C\epsilon_{t+1} \tag{9}$$

$$\mathsf{o}_t = E \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{10}$$

where z_t is a $(n-m)\times 1$ vector of predetermined variables at time t with z_0 given, x_t, is a $m \times 1$ vector of non-predetermined variables and o_t is a vector of outputs. All variables are expressed as absolute or proportional deviations about a steady state. A, C and A are

[4]See also Clarida *et al.* (1999)

fixed matrices and ϵ_t as a vector of random zero-mean shocks. Rational expectations are formed assuming an information set $\{z_s, x_s, \epsilon_s\}$, $s \leq t$, the model and the monetary rule. The linearized euro-area model set out in the next section can be expressed in this form where z_t consists of exogenous shocks, lags in non-predetermined and output variables and capital stock; x_t consists of current inflation, the real wage, investment, Tobin's Q, consumption and flexi-price outcomes for the latter two variables, and outputs o_t consist of marginal costs, the marginal rate of substitution between consumption and leisure, the cost of capital, labour supply, output, flexi-price outcomes, the output gap and other target variables for the monetary authority. Let $\mathsf{s}_t = M[\mathsf{z}_t^T \mathsf{x}_t^T]^T$ be the vector of such target variables. For both ad hoc and welfare-based loss function discussed below, the inter-temporal loss function (4) generalizes to

$$\Omega_0 = E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t L_t \right] \tag{11}$$

where the single-period loss function is given by $L_t = \mathsf{s}_t^T Q_1 \mathsf{s}_t = \mathsf{y}_t^T Q \mathsf{y}_t$ where $\mathsf{y}_t = [\mathsf{z}_t^T \mathsf{x}_t^T]^T$ and $Q = M^T Q_1 M$.

2.2.1 Imposing an Interest Rate Zero Lower Bound Constraint

In the absence of a lower bound constraint on the nominal interest rate the policymaker's optimization problem is to minimize (11) subject to (9) and (10). If the variances of shocks are sufficiently large, this will lead to a large nominal interest rate variability and the possibility of the nominal interest rate becoming negative. To rule out this possibility and to remain within the convenient LQ framework of this paper we follow Woodford (2003), chapter 6, and approximate the interest rate lower bound effect by introducing constraints of the form

$$E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t r_t \right] \geq 0 \tag{12}$$

$$E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t r_t^2 \right] \leq K \left[E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t r_t \right] \right]^2 \tag{13}$$

Then Woodford shows that the effect of these extra constraints is to follow the same optimization as before except that the single period loss function is replaced with

$$L_t = \mathsf{y}_t^T Q \mathsf{y}_t + w_r (r_t - r^*)^2 \tag{14}$$

where $w_r > 0$ if (13) binds (which we assume) and r^* is a nominal interest rate target. We linearize around a zero-inflation deterministic steady state with r_t an absolute deviation about its steady state value. Then r^* equals a non-zero steady state inflation target. $r^* > 0$ if monetary transactions frictions are negligible, but $r^* < 0$ is possible otherwise. In what follows we consider a model with no monetary transactions frictions, so $r^* > 0$ is appropriate. The policymaker's optimization problem is now to choose an unconditional distribution for r_t (i.e., the steady state variance) shifted to the right about a new non-zero steady state inflation rate, such that the probability of the interest rate hitting the lower bound is very low. As we demonstrate below in section 6.2, this is achieved by an optimal combination of a sufficiently small unconditional variance and the choice of the new steady state inflation rate.

2.2.2 Commitment Versus Discretion

To derive the ex ante optimal policy with commitment following Currie and Levine (1993) we maximize the Lagrangian

$$\mathcal{L}_0 = E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t \left[(\mathsf{y}_t^T Q \mathsf{y}_t + w_r r_t^2 + \mathsf{p}_{t+1}(A\mathsf{y}_t + Br_t - \mathsf{y}_{t+1}) \right] \right] \tag{15}$$

with respect to $\{r_t\}$, $\{\mathsf{y}_t\}$ and the row vector of costate variables, p_t, given z_0. From Appendix A where more details are provided, this leads to an optimal rule of the form

$$r_t = D \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \tag{16}$$

where

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{p}_{2t+1} \end{bmatrix} = H \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \tag{17}$$

and the optimality condition[5] at time $t = 0$ imposes $\mathsf{p}_{20} = 0$. In (16) and (17) $\mathsf{p}_t^T = \left[\mathsf{p}_{1t}^T\, \mathsf{p}_{2t}^T\right]$ is partitioned so that p_{1t}, the co-state vector associated with the predetermined variables, is of dimension $(n-m)\times 1$ and p_{2t}, the co-state vector associated with the non-predetermined variables, is of dimension $m \times 1$. The (conditional) loss function is given by

$$\Omega_t^{OP} = -(1-\beta)\mathrm{tr}\left(N_{11}\left(Z_t + \frac{\beta}{1-\beta}\Sigma \right) + N_{22}\mathsf{p}_{2t}\mathsf{p}_{2t}^T \right) \tag{18}$$

where $Z_t = \mathsf{z}_t \mathsf{z}_t^T$, $\Sigma = \mathrm{cov}(\epsilon_t)$,

$$N = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{19}$$

[5]Optimality from a 'timeless perspective' imposes a different condition at time $t = 0$ (see Appendix A.1.2), but this has no bearing on the stochastic component of policy.

and S is the solution to the steady-state Ricatti equation. In (19) matrices S and N are partitioned conformably with $\mathsf{y}_t = [\mathsf{z}_t^T \mathsf{x}_t^T]^T$ so that S_{11} for instance has dimensions $(n-m) \times (n-m)$.

Note that in order to achieve optimality the policy-maker sets $\mathsf{p}_{20} = 0$ at time $t = 0$. At time $t > 0$ there exists a gain from reneging by resetting $\mathsf{p}_{2t} = 0$. It can be shown that matrices N_{11} and N_{22} are negative definite, so the the loss in (18) is positive and an incentive to renege exists at all points along the trajectory of the optimal policy by resetting $p_{2t} = 0$. This essentially is the time-inconsistency problem facing stabilization policy in a model with structural dynamics.

To evaluate the discretionary (time-consistent) policy we write the expected loss Ω_t at time t as

$$\Omega_t = E_t \left[(1-\beta) \sum_{\tau=t}^{\infty} \beta^{\tau-t} L_\tau \right] = (1-\beta)(\mathsf{y}_t^T Q \mathsf{y}_t + w_r r_t^2) + \beta \Omega_{t+1} \tag{20}$$

The dynamic programming solution then seeks a stationary solution of the form $r_t = -F z_t$, $\Omega_t = \mathsf{z}^T S \mathsf{z}$ and $\mathsf{x} = -N\mathsf{z}$ where matrices S and N are different matrices from those under commitment (unless there is no forward-looking behviour), now of dimensions $(n-m)\times(n-m)$ and $m\times(n-m)$ respectively. The value function Ω_t is minimized at time t, subject to (9) and (10), in the knowledge that a similar procedure will be used to minimize Ω_{t+1} at time $t+1$.[6] Both the instrument r_t and the forward-looking variables x_t are now proportional to the predetermined component of the state-vector z_t and the equilibrium we seek is therefore *Markov Perfect*. In Appendix A we set out an iterative process for F_t, N_t, and S_t starting with some initial values. If the process converges to stationary values independent of these initial values,[7] F, N and S say, then the time-consistent feedback rule is $r_t = -F z_t$ with loss at time t given by

$$\Omega_t^{TC} = (1-\beta)\mathrm{tr}\left(S \left(Z_t + \frac{\beta}{1-\beta}\Sigma \right) \right) \tag{21}$$

2.2.3 Simple Commitment Rules

We now address a problem with the optimal commitment rule: that in all but very simple models it is extremely complex, with the interest rate feeding back at time t on the full state vector z_t and all past realizations of z_t back to the initiation of the rule at $t = 0$.[8] We therefore seek to mimic the optimal commitment rule with simpler rules of the form

$$r_t = D\mathsf{y}_t = D \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{22}$$

[6] See Currie and Levine (1993) and Söderlind (1999).

[7] Indeed we find this is the case in the results reported in the paper.

[8] See Appendix A.1.1.

where D is constrained to be sparse in some specified way. In Appendix A we show that the loss at time t is given by

$$\Omega_t^{SIM} = (1-\beta)\text{tr}\left(V\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right)\right) \quad (23)$$

where $V = V(D)$ satisfies a Lyapanov equation. Ω_t^{SIM} can now be minimized with respect to D to give an *optimized simple rule* of the form (22) with $D = D^*$. A very important feature of optimized simple rules is that unlike their optimal commitment or optimal discretionary counterparts they are *not certainty equivalent*. In fact if the rule is designed at time $t = 0$ then $D^* = D^*\left(Z_0 + \frac{\beta}{1-\beta}\Sigma\right)$ and so depends on the displacement z_0 at time $t = 0$ and on the covariance matrix of innovations $\Sigma = \text{cov}(\epsilon_\mathsf{t})$. From non-certainty equivalence it follows that if the simple rule were to be re-designed at ant time $t > 0$, since the re-optimized D^* will then depend on Z_t the new rule will differ from that at $t = 0$. This feature is true in models with or without rational forward-looking behaviour and it implies that *simple rules are time-inconsistent even in non-RE models.*

2.3 Sustaining the Commitment Outcome as An Equilibrium

Suppose that there are two types of monetary policymaker, a ‘strong’ type who likes to commit and perceives substantial costs from reneging on any such commitment, and a ‘weak’ type who optimizes in an opportunistic fashion on a period-by-period basis. The ‘strong type’ could be a policymaker with a modified loss function as in Rogoff (1985), Walsh (1995), Svensson and Woodford (2005), though for the case of Rogoff-delegation the outcome is second-best. In a complete information setting, these types would be observed by the public and the strong type would pursue the optimal commitment monetary rule or a simple approximation, and the weak type would pursue the discretionary policy. We assume there is uncertainty about the type of policymaker and the weak type is trying to build a reputation for commitment. The game is now one of incomplete information and we examine the possibility that commitment rules can be sustained as a Perfect Bayesian Equilibrium.

Consider the following strategy profile.

1. A strong type follows an optimal or simple commitment rule.

2. In period t a weak type acts as strong and follows the commitment rule with probability $1 - q_t$, if it has acted strong ($q_t = 0$) in all previous periods. Otherwise it pursues the discretionary rule and reveals its type.

3. Let ρ_t be the probability assigned by the private sector to the event that the policymaker is of the strong type. We can regard ρ_t as a measure of reputation. At

the beginning of period 0 the private sector chooses its prior $\rho_0 > 0$. In period t the private sector receives the 'signal' consisting of the inflation set by the policymaker. At the end of the period it updates the probability ρ_t, using Bayes rule, and then forms expectations of the next period's inflation rate.

In principle there are three types of equilibria to these games. If both strong and weak governments send the same message (i.e. implement the same interest rate) we have a *pooling equilibrium*. If they send different messages this gives a *separating equilibrium*. If one or more players randomizes with a mixed strategy we have a *hybrid equilibrium*. Thus in the above game, $q_t = 0$ gives a pooling equilibrium, $q_t = 1$ a separating equilibrium and $0 < q_t < 1$ a hybrid equilibrium. *If $q_t = 0$ is a Perfect Bayesian Equilibrium to this game, then we have solved the time-inconsistency problem.*

To show that $q_t = 0$, it is sufficient to show that, given beliefs by the private sector, there is no incentive for a weak government to ever deviate from acting strong. To show this we must compare the welfare if the policymaker continues with the optimal commitment policy at time t with that if it reneges, re-optimizes and then suffers a loss of reputation.

Consider the optimal commitment rule first. At time t the single period loss function is $L(\mathsf{z}_t, \mathsf{p}_{2t})$ and the intertemporal loss function can be written

$$\Omega_t^{OP}(\mathsf{z}_t, \mathsf{p}_{2t}) = (1-\beta)L(\mathsf{z}_t, \mathsf{p}_{2t}) + \beta\Omega_{t+1}^{OP}(\mathsf{z}_{t+1}^{OP}, \mathsf{p}_{2,t+1}) \tag{24}$$

where $(\mathsf{z}_{t+1}^{OP}, \mathsf{p}_{2,t+1})$ is given by (17). If the policymaker re-optimizes at time t the corresponding loss is

$$\Omega_t^{R}(\mathsf{z}_t, 0) = (1-\beta)L(\mathsf{z}_t, 0) + \beta\Omega_{t+1}^{TC}(\mathsf{z}_{t+1}^{R}) \tag{25}$$

where from (17) we now have that $\mathsf{z}_{t+1}^{R} = H_{11}\mathsf{z}_t$.

The condition for a perfect Bayesian pooling equilibrium is that for all realizations of shocks to $(\mathsf{z}_t, \mathsf{p}_{2t})$ at every time t the no-deviation condition

$$\Omega_t^{OP}(\mathsf{z}_t, \mathsf{p}_{2t}) < \Omega_t^{R}(\mathsf{z}_t, 0) \tag{26}$$

holds. If this condition holds, then the weak authority always mimics the strong authority and follows the commitment rule thus sustaining average zero inflation coupled with optimal stabilization.

Using (24), (31), (18) and (21) the no-deviation condition (26) can be written as

$$\begin{aligned} L(\mathsf{z}_t, \mathsf{p}_{2t}) - L(\mathsf{z}_t, 0) \quad &- \quad \beta E_t\left[\mathrm{tr}(SZ_{t+1}^{R} + N_{11}Z_{t+1}^{OP} + N_{22}\mathsf{p}_{2,t+1}^{T}\mathsf{p}_{2,t+1})\right] \\ &< \quad \frac{\beta^2}{1-\beta}tr((S+N_{11})\Sigma) \end{aligned} \tag{27}$$

The first term on the left-hand-side of (27) is the *single-period* gain from reneging and putting $p_{2t} = 0$. The second term on the left-hand-side of (27) are the possible *one-off stabilization gains* since the state of the economy after reneging reflected in z_{t+1}^R will be closer to the long-run than that along the commitment policy reflected in $\mathsf{z}_{t+1}^{OP}, \mathsf{p}_{2t+1}$. These two terms together constitute the *temptation* to renege. Since $tr((S + N_{11}) > 0$, the right-hand-side is always positive and constitutes the *penalty* in the shape of the stabilization loss when dealing with future shocks following a loss of reputation.

If the time-period is small (i.e. $\beta \simeq 1$), then the single-period gains are also relatively small and we can treat the loss of reputation as if it were instantaneous. Then the no deviation condition becomes simply

$$\Omega_t^{OP} < \Omega_t^{TC} \tag{28}$$

for all realizations of exogenous stochastic shocks. From (18) and (21) this becomes

$$\mathrm{tr}((N_{11} + S)(Z_t + \frac{\beta}{1-\beta}\Sigma) > -\mathrm{tr}(N_{22}p_{2t}p_{2t}^T) \tag{29}$$

Note that both $-N_{22}$ and $(N_{11} + S)$ are positive definite (see Currie and Levine (1993), chapter 5 for a continuous-time analysis on which the discrete-time analysis here is based). It follows that both the right-hand-side and the left-hand side are positive, so (29) is not automatically satisfied.[9] Finally we consider the no-deviation condition for a simple rule. Consider the optimized rule set at $t = 0$ which we take to be the steady state. Then $Z_0 = 0$ and $D^* = D^*(\Sigma)$. If the policymaker continues with this policy then in state z_t at time t the welfare loss is given by

$$\Omega_t^{SIM}(\mathsf{z}_t, D^*) = (1-\beta)L(\mathsf{z}_t, D^*) + \beta\Omega_{t+1}^{SIM}(\mathsf{z}_{t+1}^{SIM}, D^*) \tag{30}$$

where $\mathsf{z}_{t+1}^{SIM} = H(D^*)\mathsf{z}_t$ and H is given in Appendix A. If the policy deviates she goes to a re-optimized reneging rule $D^R = D^R((Z_t + \frac{\beta}{1-\beta}\Sigma))$ which now depends on the realization of z_t at Time t. The welfare loss is then

$$\Omega_t^R(\mathsf{z}_t, D^R) = (1-\beta)L(\mathsf{z}_t, D^R) + \beta\Omega_{t+1}^{TC}(\mathsf{z}_{t+1}^R) \tag{31}$$

where $\mathsf{z}_{t+1}^{SIM} = H(D^R)\mathsf{z}_t$. Proceeding as before the no-deviation condition now becomes

$$L(\mathsf{z}_t, D^*) - L(\mathsf{z}_t, D^R) - \beta E_t\left[\mathrm{tr}(SZ_{t+1}^R - VZ_{t+1}^{SIM}\right] < \frac{\beta^2}{1-\beta}tr((S-V)\Sigma) \tag{32}$$

[9] The analysis of this section assumes that the steady state is the same under commitment and discretion. When the interest rate lower bound constraint is introduced this is no longer the case. Let the new steady-state inflation rates for the two cases be $(\pi^*)^{OP}$ and $(\pi^*)^{TC}$ respectively and the increase in the steady state welfare loss arising from an positive inflation rate be $W(\pi^*)$. Then a term $\frac{W((\pi^*)^{TC}) - W((\pi^*)^{OP})}{1-\beta}$ is added to the left-hand-side of (27) and (29).

The intuition behind this condition is very similar to that of (27). In these three no-deviation conditions (27), (29) and (32), since Z_t or p_{2t} are unbounded stochastic variables there will inevitably be *some* realizations for which they are *not* satisfied. In other words the Bayesian equilibrium must be of the mixed-strategy type with $q_t > 0$. What we must now show that q_t is very small so we will only experience very occasional losses of reputation. This we examine in section 6.6.

3 The Model

3.1 The Smets-Wouters Model

The Smets-Wouters (SW) model in an extended version of the standard New-Keynesian DSGE closed-economy model with sticky prices and wages estimated by Bayesian techniques. The model features three types of agents: households, firms and the monetary policy authority. Households maximize a utility function with two arguments (goods and leisure) over an infinite horizon. Consumption appears in the utility function relative to a time-varying external habit-formation variable. Labour is differentiated over households, so that there is some monopoly power over wages, which results in an explicit wage equation and allows for the introduction of sticky nominal Calvo-type wages contracts (Calvo (1983)). Households also rent capital services to firms and decide how much capital to accumulate given certain capital adjustment costs. Firms produce differentiated goods, decide on labour and capital inputs, and set Calvo-type price contracts. Wage and price setting is augmented by the assumption that those prices and wages that can not be freely set are partially indexed to past inflation. Prices are therefore set as a function of current and expected real marginal cost, but are also influenced by past inflation. Real marginal cost depends on wages and the rental rate of capital. The short-term nominal interest rate is the instrument of monetary policy. The stochastic behaviour of the model is driven by ten exogenous shocks: five shocks arising from technology and preferences, three cost-push shocks and two monetary-policy shocks. Consistent with the DSGE set up, potential output is defined as the level of output that would prevail under flexible prices and wages in the absence of cost-push shocks.

We incorporate one important modification to the SW model: the addition of distortionary taxes at the steady state. As we will see this has a bearing on the inefficiency at the steady state, the quadratic approximation of the utility function used for the welfare analysis and the existence of an inflationary bias.

3.2 Households

There are ν households of which a representative household r maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t U_{C,t} \left[\frac{(C_t(r) - H_{C,t})^{1-\sigma}}{1-\sigma} + U_{M,t} \frac{\left(\frac{M_t(r)}{P_t}\right)^{1-\varphi}}{1-\varphi} - U_{L,t} \frac{L_t(r)^{1+\phi}}{1+\phi} + u(G_t) \right] \quad (33)$$

where E_t is the expectations operator indicating expectations formed at time t, β is the household's discount factor, $U_{C,t}$, $U_{M,t}$ and $U_{L,t}$ are preference shocks common to all households, $C_t(r)$ is an index of consumption, $L_t(r)$ are hours worked, $H_{C,t}$ represents the habit in consumption, or desire not to differ too much from other households, and we choose $H_{C,t} = hC_{t-1}$, where $C_t = \frac{1}{\nu} \sum_{r=1}^{\nu} C_t(r)$ is the average consumption index, $h \in [0, 1)$. When $h = 0$, $\sigma > 1$ is the risk aversion parameter (or the inverse of the intertemporal elasticity of substitution) $M_t(r)$ are end-of-period nominal money balances and $u(G_t)$ is the utility from exogenous real government spending G_t. We normalize the household number ν to unity.

The representative household r must obey a budget constraint:

$$(1+T_{C,t})P_t(C_t(r)+I(r))+E_t[D_{t+1}B_{t+1}(r)]+M_t(r) = (1-T_{Y,t})P_tY_t(r)+B_t(r)+M_{t-1}(r)+TR_t \quad (34)$$

where P_t is the GDP price index and $I_t(r)$ is investment. Assuming complete financial markets, $B_{t+1}(r)$ is a random variable denoting the payoff of the portfolio $B_t(r)$, purchased at time t, and D_{t+1} is the stochastic discount factor over the interval $[t, t+1]$ that pays one unit of currency in a particular state of period $t+1$ divided by the probability of an occurrence of that state given information available in period t. The nominal rate of return on bonds (the nominal interest rate), R_t, is then given by the relation $E_t[D_{t+1}] = \frac{1}{1+R_t}$. The tax structure is as follows: TR_t are lump-sum transfers to households by the government net of lump-sum taxes, $T_{C,t}$ and $T_{Y,t}$ are consumption and income tax rates respectively. The income tax rate is paid on total income, $P_tY_t(r)$, given by

$$P_tY_t(r) = W_t(r)L_t(r) + (R_{K,t}Z_t(r) - \Psi(Z_t(r))P_tK_{t-1}(r) + \Gamma_t(r) \quad (35)$$

where $W_t(r)$ is the wage rate, $R_{K,t}$ is the real return on beginning-of period t capital stock, K_{t-1}, owned by households, $Z_t(r) \in [0, 1]$ is the degree of capital utilization with costs $P_t\Psi(Z_t(r))K_{t-1}(r)$ where Ψ', $\Psi'' > 0$, and $\Gamma_t(r)$ is income from dividends derived from the imperfectly competitive intermediate firms plus the net cash inflow from state-contingent securities. We first consider the case of flexible wages and introduce wage stickiness at a later stage.

Capital accumulation is given by

$$K_t(r) = (1-\delta)K_{t-1}(r) + (1 - S(X_t(r)))\, I_t(r) \tag{36}$$

where $X_t(r) = \frac{U_{I,t}I_t(r)}{I_{t-1}(r)}$, $U_{I,t}$ is a shock to investment costs and we assume the investment adjustment cost function, $S(\cdot)$, has the properties $S(1) = S'(1) = 0$.

As set below, intermediate firms employ differentiated labour with a constant CES technology with elasticity of supply η. Then the demand for each consumer's labour is given by

$$L_t(r) = \left(\frac{W_t(r)}{W_t}\right)^{-\eta} L_t \tag{37}$$

where $W_t = \left[\int_0^1 W_t(r)^{1-\eta} dr\right]^{\frac{1}{1-\eta}}$ is an average wage index and $L_t = \left[\int_0^1 L_t(r)^{\frac{\eta-1}{\eta}} dr\right]^{\frac{\eta}{\eta-1}}$ is average employment.

Household r chooses $\{C_t(r)\}$, $\{M_t(r)\}$, $\{K_t(r)\}$, $\{Z(r)\}$ and $\{L_t(r)\}$ (or $\{W_t(r)\}$) to maximize (33) subject to (34)–(37), taking external habit $H_{C,t}$, $R_{K,t}$ and prices and as given. The insurance provided by state-contingency securities (the complete financial markets assumption) enables us to impose symmetry on households (so that $C_t(r) = C_t$, etc). Then by the standard Lagrangian method we have the first-order necessary conditions:

$$\begin{aligned}
1 &= \beta(1+R_t)E_t\left[\frac{MU^C_{t+1}}{MU^C_t}\frac{P_t}{P_{t+1}}\right] \\
&= \beta(1+R_t)E_t\left[\left(\frac{U_{C,t+1}(C_{t+1}-H_{C,t+1})^{-\sigma}}{U_{C,t}(C_t-H_{C,t})^{-\sigma}}\right)\frac{P_t}{P_{t+1}}\right] && (38)\\
U_{M,t}\left(\frac{M_t}{P_t}\right)^{-\varphi} &= \frac{(C_t-H_{C,t})^{-\sigma}}{\chi P_t}\left[\frac{R_t}{1+R_t}\right] && (39)\\
Q_t &= E_t\left[\beta\left(\frac{(C_{t+1}-H_{C,t+1})}{(C_t-H_{C,t})}\right)^{-\sigma}(Q_{t+1}(1-\delta)\right. \\
&+ \left. R_{K,t+1}Z_t - \Psi(Z_{t+1}))\right] && (40)\\
1 &= Q_t[1-(1-S(X_t)-S'(X_t)X_t)] \\
&+ \beta E_t Q_{t+1}\left(\frac{(C_{t+1}-H_{C,t+1})}{(C_t-H_{C,t})}\right)^{-\sigma} S'(X_t)\frac{U_{I,t+1}I^2_{t+1}}{I^2_t} && (41)\\
R_{K,t} &= \Psi'(Z_t) && (42)\\
\frac{W_t(1-T_{Y,t})}{(1+T_{C,t})P_t} &= -\frac{1}{(1-\frac{1}{\eta})}\frac{MU^L_t}{MU^C_t} \equiv \frac{1}{(1-\frac{1}{\eta})}MRS_t = \frac{U_{L,t}}{(1-\frac{1}{\eta})}L^{\phi}_t(C_t-H_{C,t})^{\sigma} && (43)
\end{aligned}$$

where MU^C_t and MU^L_t are the marginal utilities of consumption and work respectively. (38) is the familiar Keynes-Ramsey rule adapted to take into account habit in consumption.

In (39), the demand for money balances depends positively on consumption relative to habit and negatively on the nominal interest rate. Given the central bank's setting of the latter, (39) is completely recursive to the rest of the system describing our macro-model. In (40) and (41), Q_t is the real value of capital (Tobin's Q) and these conditions describe optimal investment behaviour. (42) describes optimal capacity utilization and (43) equates the real disposable wage with the marginal rate of substitution (MRS_t) between consumption and leisure and reflects the market power of households arising from their monopolistic supply of a differentiated factor input with elasticity η.

3.3 Firms

Competitive final goods firms use a continuum of intermediate goods according to a constant returns CES technology to produce aggregate output

$$Y_t = \left(\int_0^1 Y_t(f)^{(\zeta-1)/\zeta} df \right)^{\zeta/(\zeta-1)} \tag{44}$$

where ζ is the elasticity of substitution. This implies a set of demand equations for each intermediate good f with price $P_t(f)$ of the form

$$Y_t(f) = \left(\frac{P_t(f)}{P_t} \right)^{-\zeta} Y_t \tag{45}$$

where $P_t = \left[\int_0^1 P_t(f)^{1-\zeta} df \right]^{\frac{1}{1-\zeta}}$. P_t is an aggregate intermediate price index, but since final goods firms are competitive and the only inputs are intermediate goods, it is also the GDP price level.

In the intermediate goods sector each good f is produced by a single firm f using differentiated labour and capital with a Cobb-Douglas technology:

$$Y_t(f) = A_t(Z_t(f)K_{t-1}(f))^{\alpha} L_t(f)^{1-\alpha} - F \tag{46}$$

where F are fixed costs of production and

$$L_t(f) = \left(\int_0^1 L_t(r,f)^{(\eta-1)/\eta} dr \right)^{\eta/(\eta-1)} \tag{47}$$

is an index of differentiated labour types used by the firm, where $L_t(r,f)$ is the labour input of type r by firm f, and A_t is an exogenous shock capturing shifts to trend total factor productivity (TFP) in this sector. The cost of labour to the firm is $(1+T_{L,t})W_t$ where $T_{L,t}$ is a pay-roll tax paid by the firm. Minimizing costs $P_t R_{K,t} Z_t(f) K_{t-1}(f) + (1+T_{L,t}) W_t L_t(f)$ gives

$$\frac{(1+T_{L,t})W_t L_t(f)}{Z_t P_t R_{K,t} K_{t-1}(f)} = \frac{1-\alpha}{\alpha} \tag{48}$$

Then aggregating over firms and denoting $\int_0^1 L_t(r,f)df = L_t(r)$ leads to the demand for labour as shown in (37). In an equilibrium of equal households and firms, all wages adjust to the same level W_t and it follows that $Y_t = A_t(Z_tK_{t-1})^{\alpha}L_t^{1-\alpha}$. For later analysis we need the firm's minimum real marginal cost:

$$MC_t = \frac{\left(\frac{(1+T_{L,t})W_t}{P_t}\right)^{1-\alpha} R_K^{\alpha}}{A_t}\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} \tag{49}$$

Turning to price-setting, we assume there is a probability of $1-\xi_p$ at each period that the price of each good f is set optimally to $P_t^0(f)$. If the price is not re-optimized, then it is indexed to last period's aggregate producer price inflation.[10] With indexation parameter $\gamma_p \geq 0$, this implies that successive prices with no re-optimization are given by $P_t^0(f),\ P_t^0(f)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p},\ P_t^0(f)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_p}, \ldots$. For each producer firm f the objective is at time t to choose $P_t^0(f)$ to maximize discounted profits

$$E_t\sum_{k=0}^{\infty}\xi_H^k D_{t+k}Y_{t+k}(f)\left[P_t^0(f)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_p} - P_{t+k}\mathrm{MC}_{t+k}\right] \tag{50}$$

where D_{t+k} is the stochastic discount factor over the interval $[t, t+k]$, subject to a common downward sloping demand given by (45). The solution to this is

$$E_t\sum_{k=0}^{\infty}\xi_p^k D_{t+k}Y_{t+k}(f)\left[P_t^0(f)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_p} - \frac{\zeta}{(\zeta-1)}P_{t+k}\mathrm{MC}_{t+k}\right] = 0 \tag{51}$$

and by the law of large numbers the evolution of the price index is given by

$$P_{t+1}^{1-\zeta} = \xi_p\left(P_t\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p}\right)^{1-\zeta} + (1-\xi_p)(P_{t+1}^0(f))^{1-\zeta} \tag{52}$$

3.4 Staggered Wage-Setting

We introduce wage stickiness in an analogous way. There is a probability $1-\xi_w$ that the wage rate of a household of type r is set optimally at $W_t^0(r)$. If the wage is not re-optimized then it is indexed to last period's GDP inflation. With a wage indexation parameter γ_w, the wage rate trajectory with no re-optimization is given by $W_t^0(r)$, $W_t^0(r)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_w}$, $W_t^0(r)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_w}, \cdots$. The household of type r at time t then chooses $W_t^0(r)$ to maximize

$$E_t\sum_{k=0}^{\infty}(\xi_w\beta)^k\left[W_t^0(r)(1-T_{Y,t+k})\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_w}L_{t+k}(r)\Lambda_{t+k}(r) - U_{L,t+k}\frac{(L_{t+k}(r))^{1+\phi}}{1+\phi}\right] \tag{53}$$

[10] Thus we can interpret $\frac{1}{1-\xi_p}$ as the average duration for which prices are left unchanged.

where $\Lambda_t(r) = \frac{MU_t^C(r)}{P_t}$ is the real marginal utility of consumption income and $L_t(r)$ is given by (37). The first-order condition for this problem is

$$E_t \sum_{k=0}^{\infty} (\xi_w \beta)^k \, W_{t+k}^{\eta} \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{-\gamma_w \eta} L_{t+k} \Lambda_{t+k}(r) \Big[W_t^0(r)(1 - T_{Y,t+k}) \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma_w} - \frac{1}{(1-\frac{1}{\eta})} P_{t+k} MRS_{t+k}(r) \Big] = 0 \quad (54)$$

Note that as $\xi_w \rightarrow 0$ and wages become perfectly flexible, only the first term in the summation in (53) counts and we then have the result (43) obtained previously. By analogy with (52), by the law of large numbers the evolution of the wage index is given by

$$W_{t+1}^{1-\eta} = \xi_w \left(W_t \left(\frac{P_t}{P_{t-1}} \right)^{\gamma_w} \right)^{1-\eta} + (1-\xi_w)(W_{t+1}^0(r))^{1-\eta} \quad (55)$$

3.5 Equilibrium

In equilibrium, goods markets, money markets and the bond market all clear. Equating the supply and demand of the consumer good we obtain

$$Y_t \;=\; A_t (Z_t K_{t-1})^{\alpha} L_t^{1-\alpha} - F = C_t + G_t + I_t + \Psi(Z_t) K_{t-1} \quad (56)$$

We examine the dynamic behaviour in the vicinity of a steady state in which the government budget constraint is in balance; i.e.,

$$TR_t + P_t G_t = (T_{Y,t} + T_{C,t}) P_t Y_t + T_{L,t} W_t L_t + M_t - M_{t-1} \quad (57)$$

As in Coenen *et al.* (2005) we further assume that changes in government spending are financed exclusively by changes in lump-sum taxes with tax rates $T_{Y,t}$, $T_{C,t}$ and $T_{L,t}$ held constant at their steady-state values.

Given the interest rate R_t (expressed later in terms of an optimal or IFB rule) the money supply is fixed by the central banks to accommodate money demand. By Walras' Law we can dispense with the bond market equilibrium condition and therefore the household constraint. Then the equilibrium is defined at $t = 0$ by stochastic processes C_t, B_t, I_t, P_t, M_t, L_t, K_t, Z_t, $R_{K,t}$, W_t, Y_t, given past price indices and exogenous shocks and government spending processes.

In what follows we will assume a 'cashless economy' version of the model in which both seigniorage in (57) and the utility contribution of money balances in (33) are negligible. Then given the nominal interest rate, our chosen monetary instrument, we can dispense altogether with the money demand relationship (39). For estimation purposes the model is closed with a "empirical Taylor rule" specified in section 4.

3.6 Zero-Inflation Steady State

For the cashless economy, deterministic zero-inflation steady state, denoted by variables without the time subscripts, $E_{t-1}(U_{C,t}) = 1$ and $E_{t-1}(U_{L,t}) = \kappa$ is given by

$$1 = \beta(1+R) \tag{58}$$

$$Q = \beta(Q(1-\delta) + R_K Z - \Psi(Z)) \tag{59}$$

$$R_K = \Psi'(Z) \tag{60}$$

$$Q = 1 \tag{61}$$

$$\frac{W(1-T_Y)}{P(1+T_C)} = \frac{\kappa(1-h)^\sigma}{1-\frac{1}{\eta}} L^\phi C^\sigma \tag{62}$$

$$Y = A(KZ)^\alpha L^{1-\alpha} - F \tag{63}$$

$$\frac{W(1+T_L)L}{PZR_K K} = \frac{1-\alpha}{\alpha} \tag{64}$$

$$1 = \frac{P^0}{P} = \frac{\text{MC}}{\left(1-\frac{1}{\zeta}\right)} \tag{65}$$

$$\text{MC} = \frac{\left(\frac{W(1+T_L)}{P}\right)^{1-\alpha} R_K^\alpha}{A} \tag{66}$$

$$Y = C + (\delta + \Psi(Z))K + G \tag{67}$$

$$TR + PG = (T_Y + T_C)PY + T_L WL \tag{68}$$

giving us 11 equations to determine R, Z, Q, $\frac{W}{P}$, L, K, R_K, MC, C, Y and possible tax structures, (TR, T_Y, T_C), given G. In this cashless economy the price level is indeterminate.

The solution for steady state values decomposes into a number of independent calculations. First from (58) the natural rate of interest is given by

$$R = \frac{1}{\beta} - 1 \tag{69}$$

which is therefore pinned down by the household's discount factor. Equations (59) to (61) give

$$1 = \beta[1 - \delta + Z\Psi'(Z) - \Psi(Z)] \tag{70}$$

which determines steady state capacity utilization. As in SW we assume that $Z = 1$ and $\Psi(1) = 0$ so that (70) and (60) imply that $R_K = \Psi'(Z) = \frac{1}{\beta} - 1 + \delta = R + \delta$ meaning that perfect capital market conditions apply in the deterministic steady state.[11]

[11] As we shall see later Z is socially efficient thus justifying the assumption $Z = 1$.

From (64) to (66) a little algebra yields the capital-labour ratio and the real wage $\frac{W}{P}$:

$$\frac{K}{L} = \left[A\left(1-\frac{1}{\zeta}\right)\frac{\alpha}{R_K}\right]^{\frac{1}{1-\alpha}} \tag{71}$$

$$\frac{W}{P} = \frac{(1-\alpha)R_K}{(1+T_L)\alpha}\frac{K}{L} \tag{72}$$

Denote the *total tax wedge* by T between the real effective wage income of households (the purchasing power of the post-tax wage) and the real effective labour cost of firms. Then

$$T \equiv 1 - \frac{1-T_Y}{(1+T_C)(1+T_L)} \simeq T_Y + T_C + T_L \tag{73}$$

Then combining (62), (63) and (67) and substituting for R_K from (71) we arrive at

$$\left(1+\frac{F}{Y}\right)^{\phi} Y^{\phi+\sigma}\left(1-\frac{\delta}{A}\left(\frac{K}{L}\right)^{1-\alpha} - \frac{G+\frac{\delta\alpha}{R_K}F}{Y}\right)$$

$$= \frac{(1-\alpha)(1-T)\left(1-\frac{1}{\eta}\right)\left(1-\frac{1}{\zeta}\right)A^{1+\phi}\left(\frac{K}{L}\right)^{\alpha(1+\phi)}}{\alpha\kappa(1-h)^{\sigma}} \tag{74}$$

Equations (74), with $\frac{K}{L}$ defined by (71), and $R_K = \frac{1}{\beta} - 1 + \delta$ define the natural rate of output in terms of underlying parameters and the tax wedge T. Thus given government spending as a proportion of GDP, the natural rate of output falls as market power in output and labour markets increases (with decreases in ζ and η respectively) and the tax wedge T increases. However external habit in consumption causes households to supply more labour thus increasing the natural rate of output. Market power, taxes and external habit are all sources of inefficiency, but as we shall see in section 5, they do not impact on efficiency in the same direction.

3.7 Linearization about the Zero-Inflation Steady State

We now linearize about the deterministic zero-inflation steady state. Define all lower case variables as proportional deviations from this baseline steady state except for rates of change which are absolute deviations.[12] Then the linearization takes the form:

$$
\begin{aligned}
c_t &= \frac{h}{1+h}c_{t-1} + \frac{1}{1+h}E_t c_{t+1} \\
&- \frac{1-h}{(1+h)\sigma}(r_t - E_t\pi_{t+1} + E_t u_{C,t+1} - u_{C,t}) \\
q_t &= \beta(1-\delta)E_t q_{t+1} - (r_t - E_t\pi_{t+1}) + \beta Z E_t r_{K,t+1} + \epsilon_{Q,t+1} \\
z_t &= \frac{r_{K,t}}{Z\Psi''(Z)} = \frac{\psi}{R_K} r_{K,t} \quad \text{where } \psi = \frac{\Psi'(Z)}{Z\Psi''(Z)} \\
i_t &= \frac{1}{1+\beta}i_{t-1} + \frac{\beta}{1+\beta}E_t i_{t+1} + \frac{1}{S''(1)(1+\beta)}q_t + \frac{\beta u_{I,t+1} - u_{I,t}}{1+\beta} \\
\pi_t &= \frac{\beta}{1+\beta\gamma_p}E_t\pi_{t+1} + \frac{\gamma_p}{1+\beta\gamma_p}\pi_{t-1} + \frac{(1-\beta\xi_p)(1-\xi_p)}{(1+\beta\gamma_p)\xi_p}mc_t \\
k_t &= (1-\delta)k_{t-1} + \delta i_t
\end{aligned}
$$

$$
\begin{aligned}
mc_t &= (1-\alpha)wr_t + \frac{\alpha}{R_K}r_{K,t} - a_t + \epsilon_{P,t+1} \\
wr_t &= \frac{\beta}{1+\beta}E_t wr_{t+1} + \frac{1}{1+\beta}wr_{t-1} + \frac{\beta}{1+\beta}E_t\pi_{t+1} - \frac{1+\beta\gamma_w}{1+\beta}\pi_t + \frac{\gamma_w}{1+\beta}\pi_{t-1} \\
&+ \frac{(1-\beta\xi_w)(1-\xi_w)}{(1+\beta)\xi_w(1+\eta\phi)}(mrs_t - wr_t) \\
mrs_t &= \frac{\sigma}{1-h}(c_t - hc_{t-1}) + \phi l_t + u_{L,t} + \epsilon_{W,t+1} \\
l_t &= k_{t-1} + \frac{1}{R_K}(1+\psi)r_{K,t} - wr_t \\
y_t &= c_y c_t + g_y g_t + i_y i_t + k_y \psi r_{K,t} \\
y_t &= \phi_F[a_t + \alpha(\frac{\psi}{R_K}r_{K,t} + k_{t-1}) + (1-\alpha)l_t] \quad \text{where } \phi_F = 1 + \frac{F}{Y} \\
u_{C,t+1} &= \rho_C u_{C,t} + \epsilon_{C,t+1} \\
u_{L,t+1} &= \rho_L u_{L,t} + \epsilon_{L,t+1} \\
u_{I,t+1} &= \rho_I u_{I,t} + \epsilon_{I,t+1} \\
g_{t+1} &= \rho_g g_t + \epsilon_{g,t+1} \\
a_{t+1} &= \rho_a a_t + \epsilon_{a,t+1}
\end{aligned}
$$

where 'inefficient cost-push' shocks $\epsilon_{Q,t+1}$, $\epsilon_{P,t+1}$ and $\epsilon_{W,t+1}$ have been added to the value of capital, marginal cost and marginal rate of substitution equations respectively. Variables y_t, c_t, mc_t, $u_{C,t}$, $u_{N,t}$, a_t, g_t are proportional deviations about the steady state. $[\epsilon_{C,t}, \epsilon_{N,t}, \epsilon_{g,t}, \epsilon_{a,t}]$ are i.i.d. disturbances. π_t, $r_{K,t}$ and r_t are absolute deviations about the steady state.[13]

[12]That is, for a typical variable X_t, $x_t = \frac{X_t - X}{X} \simeq \log\left(\frac{X_t}{X}\right)$ where X is the baseline steady state. For variables expressing a rate of change over time such as i_t, $x_t = X_t - X$.

[13]In the SW model they define $\hat{r}_{K,t} = \frac{r_{K,t}}{R_K}$. Then $z_t = \frac{\Psi'(Z)}{Z\Psi''(Z)}\hat{r}_{K,t} = \psi\hat{r}_{K,t}$. In our set-up $z_t = \frac{\psi}{R_K}r_{K,t}$ has been eliminated. (75) has a term in $r_{K,t}$ omitted in SW, a mistake also corrected in Levin *et al.* (2006).

For later use we require the *output gap* the difference between output for the sticky price model obtained above and output when prices and wages are flexible, $\hat{y}_t$ say. Following SW we also eliminate the inefficient shocks from this target level of output. The latter, obtained by setting $\xi_p = \xi_w = \epsilon_{Q,t+1} = \epsilon_{P,t+1} = \epsilon_{W,t+1} = 0$ in the sticky-price linearizarion above, is given by[14]

$$\begin{aligned}
\hat{c}_t &= \frac{h}{1+h}\hat{c}_{t-1} + \frac{1}{1+h}E_t\hat{c}_{t+1} \\
&- \frac{1-h}{(1+h)\sigma}(\hat{r}_t - E_t\hat{\pi}_{t+1} + E_t u_{C,t+1} - u_{C,t}) \\
\hat{q}_t &= \beta(1-\delta)E_t\hat{q}_{t+1} - (\hat{r}_t - E_t\hat{\pi}_{t+1}) + \beta Z E_t\hat{r}_{K,t+1} \\
\hat{i}_t &= \frac{1}{1+\beta}\hat{i}_{t-1} + \frac{\beta}{1+\beta}E_t\hat{i}_{t+1} + \frac{1}{S''(1)(1+\beta)}\hat{q}_t + \frac{\beta u_{I,t+1} + u_{I,t}}{1+\beta} \\
\hat{k}_t &= (1-\delta)\hat{k}_{t-1} + \delta\hat{i}_t
\end{aligned}$$

$$\begin{aligned}
\widehat{mc}_t &= 0 = (1-\alpha)\hat{wr}_t + \frac{\alpha}{R_K}\hat{r}_{K,t} - a_t \\
\widehat{mrs}_t &= \widehat{wr}_t = \frac{\sigma}{1-h}(\hat{c}_t - h\hat{c}_{t-1}) + \phi\hat{l}_t + u_{L,t} \\
\hat{l}_t &= \hat{k}_{t-1} + \frac{1}{R_K}(1+\psi)\hat{r}_{K,t} - \hat{wr}_t \\
\hat{y}_t &= c_y\hat{c}_t + g_y g_t + i_y\hat{i}_t + k_y\psi\hat{r}_{K,t} \\
\hat{y}_t &= \phi_F[a_t + \alpha(\frac{\psi}{R_K}\hat{r}_{K,t} + \hat{k}_{t-1}) + (1-\alpha)\hat{l}_t]
\end{aligned}$$

Table 1 provides a summary of our notation.

[14] Note that the zero-inflation steady states of the sticky and flexi-price steady states are the same.

π_t	producer price inflation over interval $[t-1,t]$
r_t	nominal interest rate over interval $[t,t+1]$
$wr_t = w_t - p_t$	real wage
mc_t	marginal cost
mrs	marginal rate of substitution between work and consumption
l_t	employment
z_t	capacity utilization
k_t	end-of-period t capital stock
i_t	investment
$r_{K,t}$	return on capital
q_t	Tobin's Q
c_t	consumption
y_t, $\hat{y}_t$	output with sticky prices and flexi-prices
$o_t = \hat{y}_t - y_t$	output gap
$u_{i,t+1} = \rho_a u_{i,t} + \epsilon_{i,t+1}$	AR(1) processes for utility preference shocks, $u_{i,t}$, $i = C,\ L,\ I$
$a_{t+1} = \rho_a a_t + \epsilon_{a,t+1}$	AR(1) process for factor productivity shock, a_t
$g_{t+1} = \rho_g g_t + \epsilon_{g,t+1}$	AR(1) process government spending shock, g_t
β	discount parameter
γ_p, γ_w	indexation parameters
h	habit parameter
$1-\xi_p$, $1-\xi_w$	probability of a price, wage re-optimization
σ	risk-aversion parameter
ϕ	disutility of labour supply parameter
φ	$\frac{1}{S''(1)}$
ϕ_F	$1 + \frac{F}{Y}$

Table 1. Summary of Notation (Variables in Deviation Form).

4 Estimation

As in SW we estimate the model using Bayesian techniques. We close the model with an empirical linearized Taylor rule of the form

$$r_t = \rho r_{t-1} + (1-\rho)[\bar{\pi}_t + \theta_\pi E_t(\pi_{t+j} - \bar{\pi}_{t+j}) + \theta_y o_t] + \theta_{\Delta\pi}(\pi_t - \pi_{t-1}) + \theta_{\Delta y}(o_t - o_{t-1})$$

The Bayesian approach itself combines the prior distributions for the individual parameters with the likelihood function to form the posterior density. This posterior density can then be optimized with respect to the model parameters through the use of the

Monte-Carlo Markov Chain sampling methods. The model is estimated using the Dynare software, Juillard (2004).[15] Table 2 reports the posterior mean and the 5th and 95th percentiles of the posterior distribution obtained through the Metropolis-Hasting (MH) sampling algorithm (using 100,000 draws from the posterior and an average "acceptance rate" of around 0.25) for the various model variants as well as the marginal likelihood (LL). Note, in re-estimating we use identical priors to those used in SW.

In the table we report results for five models: the core SW model, then the SW model without indexing in wages, $\gamma_w = 0$, without indexing in prices, $\gamma_p = 0$, with neither and finally the SW model with a 4-quarter average price contract, $\xi_p = 0.75$, imposed. From the LL values and the model posterior probabilities (with equal priors) we can see that the model without any indexing performs the best, followed by the core SW mode, followed by the model with only price indexing with $\xi_p = 0.75$ massively behind the others. The latter variant is therefore only empirically supported if the priors are very strongly in its favour.[16] In Section 6, we provide results for the core SW model and the $\gamma_p = \gamma_w = 0$ and $\xi_p = 0.75$ variants.

[15] We are grateful to Gregory De Walque and Raf Wouters for providing the SW model in Dynare code.

[16] As discussed in Geweke (1999), the Bayesian approach to estimation allows a formal comparison of different models based on their marginal likelihoods. The marginal likelihood of Model M_i is given by,

$$p(Y \mid M_i) = \int_{\Xi} p(\xi \mid M_i)\, p(Y \mid \xi, M_i)\, d\xi$$

where $p(\xi \mid M_i)$ is the prior density for model M_i and $p(Y \mid \xi, M_i)$ is the data density for model M_i given the parameter vector ξ and the data vector Y. Then the posterior odds ratio is given by

$$PO_{ij} \equiv \frac{p(M_i|Y)}{p(M_j|Y)} = \frac{p(Y|M_i)p(M_i)}{p(Y|M_j)p(M_j)} = \frac{p(Y|M_i)}{p(Y|M_j)}$$

assuming equal prior model probabilities ($p(M_i) = p(M_j)$). The posterior model probabilities are reported in Table 3.

	Core	$\gamma_w = 0$	$\gamma_p = 0$	$\gamma_w = \gamma_p = 0$	$\xi_p = 0.75$
ρ_a	0.89 [0.81:0.96]	0.89 [0.82:0.97]	0.87 [0.80:0.95]	0.88 [0.81:0.96]	0.91 [0.84:0.99]
ρ_{pb}	0.84 [0.68:0.99]	0.84 [0.68:0.99]	0.86 [0.71:0.99]	0.86 [0.70:0.99]	0.84 [0.70:0.99]
ρ_b	0.83 [0.77:0.89]	0.83 [0.77:0.90]	0.84 [0.78:0.90]	0.84 [0.77:0.89]	0.82 [0.74:0.92]
ρ_g	0.95 [0.90:0.99]	0.95 [0.91:0.99]	0.95 [0.91:0.99]	0.95 [0.91:0.99]	0.95 [0.91:0.99]
ρ_l	0.91 [0.84:0.97]	0.93 [0.89:0.98]	0.92 [0.88:0.98]	0.93 [0.89:0.98]	0.87 [0.78:0.97]
ρ_i	0.91 [0.87:0.97]	0.92 [0.86:0.97]	0.92 [0.86:0.97]	0.92 [0.87:0.98]	0.90 [0.84:0.98]
φ^{-1}	6.79 [5.08:8.55]	6.70 [5.04:8.44]	6.77 [4.96:8.50]	6.78 [5.13:8.65]	6.12 [4.31:8.07]
σ	1.40 [0.94:1.86]	1.44 [0.96:1.88]	1.43 [0.97:1.91]	1.45 [0.97:1.90]	1.36 [0.93:1.76]
h	0.57 [0.45:0.68]	0.57 [0.45:0.68]	0.57 [0.45:0.68]	0.56 [0.45:0.68]	0.54 [0.41:0:68]
ξ_w	0.73 [0.66:0.81]	0.71 [0.64:0.77]	0.74 [0.66:0.81]	0.71 [0.65:0.78]	0.76 [0.67:0.84]
ϕ	2.40 [1.37:3.35]	2.31 [1.29:3.23]	2.39 [1.44:3.39]	2.38 [1.39:3.33]	2.17 [1.28:3.21]
ξ_p	0.91 [0.89:0.92]	0.91 [0.89:0.92]	0.89 [0.87:0.91]	0.90 [0.88:0.92]	0.75 [-]
γ_w	0.69 [0.44:0.94]	-	0.66 [0.40:0.93]	-	0.44 [0.19:0.67]
γ_p	0.44 [0.26:0.60]	0.42 [0.25:0.59]	-	-	0.46 [0.30:0.63]
ψ^{-1}	0.32 [0.21:0.42]	0.32 [0.21:0.42]	0.32 [0.21:0.42]	0.32 [0.21:0.42]	0.34 [0.24:0:45]
ϕ_F	1.56 [1.39:1.73]	1.57 [1.40:1.74]	1.55 [1.37:1.72]	1.55 [1.39:1.72]	1.59 [1.42:1.76]
θ_π	1.69 [1.54:1.86]	1.70 [1.53:1.86]	1.69 [1.53:1.84]	1.69 [1.52:1.85]	1.69 [1.53:1.87]
$\theta_{\Delta\pi}$	0.15 [0.07:0.23]	0.17 [0.09:0.24]	0.17 [0.08:0.25]	0.17 [0.09:0.26]	0.14 [0.06:0.23]
ρ	0.96 [0.94:0.98]	0.96 [0.94:0.98]	0.96 [0.95:0.98]	0.96 [0.95:0.98]	0.96 [0.94:0:98]
θ_y	0.11 [0.04:0.18]	0.10 [0.03:0.17]	0.11 [0.04:0.19]	0.11 [0.04:0.18]	0.12 [0.05:0.19]
$\theta_{\Delta y}$	0.15 [0.11:0.19]	0.15 [0.12:0.19]	0.15 [0.12:0.19]	0.16 [0.13:0.20]	0.16 [0.12:0.19]
$sd(\epsilon_a)$	0.50 [0.39:0.59]	0.49 [0.38:0.58]	0.50 [0.38:0.61]	0.49 [0.39:0.59]	0.47 [0.37:0.57]
$sd(\epsilon_{\bar{\pi}})$	0.01 [0.00:0.06]	0.02 [0.00:0.02]	0.02 [0.00:0.03]	0.05 [0.00:0.03]	0.02 [0.01:0.03]
$sd(\epsilon_C)$	0.38 [0.20:0.56]	0.38 [0.20:0.54]	0.38 [0.19:0.56]	0.37 [0.21:0.54]	0.32 [0.13:0.54]
$sd(\epsilon_g)$	1.99 [1.73:2.26]	1.99 [1.74:2.26]	1.98 [1.73:2.25]	1.97 [1.73:2.23]	2.00 [1.72:2.24]
$sd(\epsilon_L)$	3.33 [1.80:4.88]	2.92 [1.58:4.17]	3.22 [1.93:4.55]	3.01 [1.77:4.13]	3.10 [1.65:4.44]
$sd(\epsilon_I)$	0.07 [0.03:0.10]	0.07 [0.03:0.10]	0.07 [0.03:0.11]	0.07 [0.03:0.10]	0.06 [0.03:0.10]
$sd(\epsilon_R)$	0.08 [0.04:0.11]	0.09 [0.06:0.13]	0.08 [0.04:0.11]	0.08 [0.05:0.12]	0.08 [0.04:0.12]
$sd(\epsilon_Q)$	0.61 [0.50:0.70]	0.61 [0.50:0.70]	0.61 [0.52:0.72]	0.61 [0.52:0.73]	0.60 [0.50:0.69]
$sd(\epsilon_P)$	0.16 [0.13:0.18]	0.16 [0.14:0.19]	0.21 [0.18:0.25]	0.22 [0.18:0.26]	0.34 [0.29:0.39]
$sd(\epsilon_W)$	0.29 [0.24:0.33]	0.27 [0.23:0.31]	0.29 [0.25:0.34]	0.27 [0.23:0.31]	0.33 [0.26:0.39]
LL	-298.72	-298.96	-299.02	-298.17	-348.82
prob	0.235	0.185	0.174	0.407	0.000

Table 2. Bayesian Estimation of Parameters

5 Is There a Long-Run Inflationary Bias?

As we have seen a long-run inflationary bias under discretion arises only if the steady state associated with zero inflation, about which we have linearized, is inefficient. To examine the inefficiency of the steady state we consider the social planner's problem for the deterministic case obtained by maximizing

$$\Omega_0 = \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t - hC_{t-1})^{1-\sigma}}{1-\sigma} - \kappa \frac{L_t^{1+\phi}}{(1+\phi)} \right] \tag{75}$$

with respect to $\{C_t\}$, $\{K_t\}$, $\{L_t\}$ and $\{Z_t\}$ subject to the resource constraint

$$Y_t = A_t (Z_t K_{t-1})^{\alpha} L_t^{1-\alpha} - F = C_t + G_t + K_t - (1-\delta)K_{t-1} + \Psi(Z_t)K_{t-1} \tag{76}$$

To solve this optimization problem define the Lagrangian

$$\mathcal{L} = \Omega_0 + \sum_{t=0}^{\infty} \beta^t \mu_t \left[A_t (Z_t K_{t-1})^{\alpha} L_t^{1-\alpha} - C_t - G_t - K_t + (1-\delta)K_{t-1} - \Psi(Z_t)K_{t-1} \right]$$

First order conditions are:

$$\begin{aligned}
C_t : &\quad (C_t - hC_{t-1})^{-\sigma} - \beta h (C_{t+1} - hC_t)^{-\sigma} - \mu_t = 0 \\
K_t : &\quad -\mu_t + \left[(1-\delta)\beta + \alpha\beta A_t Z_{t+1} \left(\frac{L_{t+1}}{Z_{t+1}K_t} \right)^{1-\alpha} - \beta\Psi(Z_{t+1}) \right] \mu_{t+1} = 0 \\
L_t : &\quad -\kappa L_t^{\phi} + (1-\alpha)A_t \left(\frac{Z_t K_{t-1}}{L_t} \right)^{\alpha} \mu_t = 0 \\
Z_t : &\quad \Psi'(Z_t) - \alpha A_t \left(\frac{L_t}{Z_t K_{t-1}} \right)^{1-\alpha} = 0
\end{aligned}$$

The efficient steady-state levels of output $Y_{t+1} = Y_t = Y_{t-1} = Y^*$, say, is therefore found by solving the system:

$$[(1-h)C]^{-\sigma} (1-\beta h) - \mu = 0 \tag{77}$$

$$-1 + (1-\delta)\beta + \alpha\beta A Z \left(\frac{L}{ZK} \right)^{1-\alpha} - \beta\Psi(Z) = 0 \tag{78}$$

$$-\kappa L^{\phi} + (1-\alpha)A \left(\frac{ZK}{L} \right)^{\alpha} \mu = 0 \tag{79}$$

$$\Psi'(Z) - \alpha A \left(\frac{L}{ZK} \right)^{1-\alpha} = 0 \tag{80}$$

Solving as we did for the natural rate and denoting the social optimum by Z^*, Y^* etc we arrive at

$$1 = \beta[1 - \delta + Z^*\Psi'(Z^*) - \Psi(Z^*)] \tag{81}$$

Hence comparing (81) and (70) it can be seen that $Z^* = Z = 1$. Thus the *natural rate of capacity utilization is efficient.* However since

$$\frac{K^*}{L^*} = \left[\frac{A\alpha}{\Psi'(Z^*)}\right]^{\frac{1}{1-\alpha}} > \frac{K}{L} = \left[\frac{A\left(1-\frac{1}{\zeta}\right)\alpha}{\Psi'(Z)}\right]^{\frac{1}{1-\alpha}} \tag{82}$$

it follows that the *natural capital-labour ratio is below the social optimum.* The socially optimal level of output is now found from

$$\left(1+\frac{F^*}{Y^*}\right)^{\phi}(Y^*)^{\phi+\sigma}\left(1-\frac{\delta}{A}\left(\frac{K^*}{L^*}\right)^{1-\alpha} - \frac{\left(G^*+\frac{\delta\alpha}{R_K}F^*\right)}{Y^*}\right) = \frac{(1-\alpha)A^{1+\phi}\left(\frac{K^*}{L^*}\right)^{\alpha(1+\phi)}(1-h\beta)}{\alpha\kappa(1-h)^{\sigma}} \tag{83}$$

The inefficiency of the natural rate of output can now be found by comparing (74) with (83). Since $Y^{\phi+\delta}$ is an increasing function of Y, we arrive at[17]

Proposition

The natural level of output, Y, is below the efficient level, Y^*, if and only if the following conditions are satisfied:

$$(1-T)\left(1-\frac{1}{\eta}\right)\left(1-\frac{1}{\zeta}\right)^{1+\alpha\phi} < (1-h\beta)\,\Theta \tag{84}$$

where

$$\Theta = \frac{\left(1-\delta\left[\left(1-\frac{1}{\zeta}\right)\frac{\alpha}{R_K}\right]^{\frac{1}{1-\alpha}} - \frac{\left(G+\frac{\delta\alpha}{R_K}F\right)}{Y}\right)\left(1+\frac{F}{Y}\right)^{\phi}}{\left(1-\delta\left[\frac{\alpha}{R_K}\right]^{\frac{1}{1-\alpha}} - \frac{\left(G^*+\frac{\delta\alpha}{R_K}F^*\right)}{Y^*}\right)\left(1+\frac{F^*}{Y^*}\right)^{\phi}}$$

where

$$R_K = \frac{1}{\beta} - 1 + \delta$$

Thus $\Phi_y \equiv (1-h\beta)\Theta - (1-T)\left(1-\frac{1}{\zeta}\right)\left(1-\frac{1}{\eta}\right)^{1+\alpha\phi}$ summarizes the overall distortion in the steady state natural level of output as a result of four elements: taxes, market power in the output and labour markets and external habit.[18] Assume government spending is adjusted so that $\frac{G}{Y} = \frac{G^*}{Y^*}$. Since there are reasons from the IO literature for assuming

[17]This generalizes the result in Choudhary and Levine (2006) which considered the same model, but without capital.

[18]This generalizes Woodford (2003), page 394, to include capital, labour-market power and habit.

Ramsey fixed investment may be excessive as well as too little compared with the social optimum, we also put $\frac{F}{Y} = \frac{F^*}{Y^*}$. It then follows that $\Theta > 1$. In the case where there is no habit persistence ($h = 0$), then $\Phi_y > 0$ and (84) always holds. Then tax distortions and market power in the output and labour markets, captured by the elasticities $\eta \in (0, \infty)$ and $\zeta \in (0, \infty)$ respectively, drive the natural rate of output below the efficient level. If $h = T = 0$ and $\eta = \zeta = \infty$, tax distortions and market power both disappear, $\Phi_y = 0$ and the natural rate is efficient. But if $h > 0$, this leads to the possibility that $\Phi_y < 0$ and then the natural rate of output is actually *above* the efficient level (see Choudhary and Levine (2006)).

The intuition behind this result is that external habit ensures that each household's consumption is a *negative externality* that reduces the welfare of others. The greater is h the greater is this externality. In the efficient case the social planner internalizes this externality and, given the other distortions, chooses less consumption and more leisure than the decentralized households in the consumption/leisure trade-off. Consequently in the absence of other distortions output is *lower* in the efficient case. On the other hand, distortions in the output and labour markets, captured by low ζ and η and a high tax wedge T, tend to *raise* the social planner's choice of output relative to the natural rate. Condition (84) shows the inter-play between these opposing effects.

An interesting implication of our results is that there may exist a socially optimal *positive* tax wedge in the steady state for high values of the habit parameter, h. This value can be found by equating $Y = Y(T)$ and Y^* and solving for T. Figure 1 plots the optimal rate of tax wedge at the steady state for various values of the habit parameter h and the parameter η that captures market power in the labour market. The figure suggests that the tax wedge can be *corrective* rather than *distortionary* (as argued by Layard (2005)), if habit is strong and the labour markets is competitive, with η is high. In fact in the core SW model where the unidentified parameter η is set at $\eta = 3$ and $h = 0.57$ the optimal tax wedge is clearly negative (implying a subsidy) which contrasts with an average tax wedge of $T = 0.64$ for the euro area in 2004 reported in Coenen *et al.* (2005).[19]

[19] Apart from the estimated parameter values for the unaltered SW model, we choose $\beta = 0.99$ and $\zeta = 7$, the latter corresponding to a 15% mark-up of the price over marginal cost. Note that an examination of the linearized form of the model reveals the fact that η and ζ are not identified.

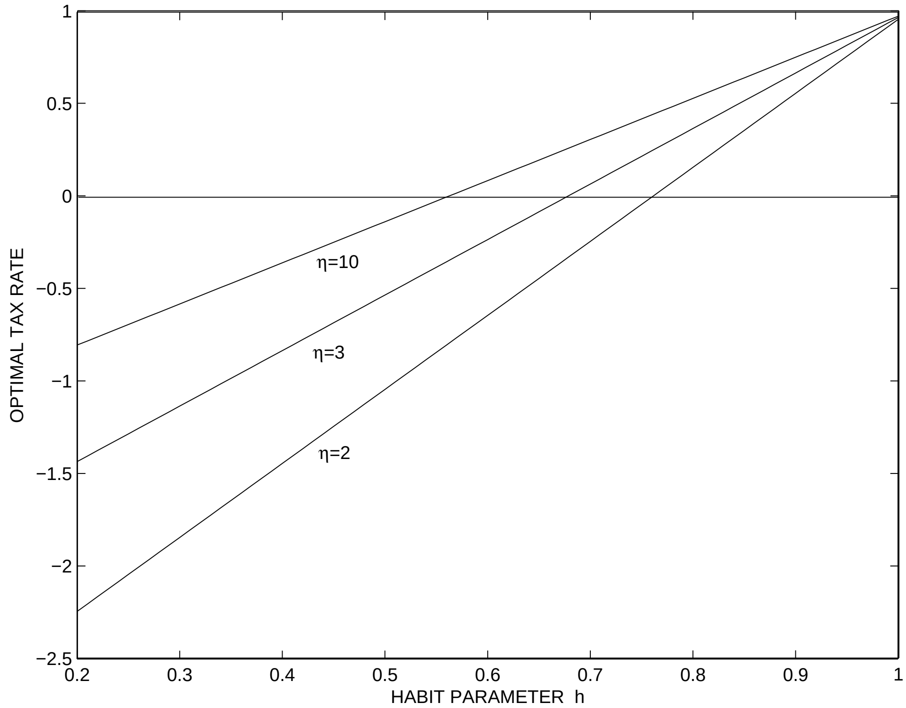

Figure 1: **The Optimal Tax Rate as h and η vary.**

6 Optimal Monetary Stabilization Policy

6.1 Formulating the Policymaker's Loss Function

Much of the optimal monetary policy literature has stayed with the ad hoc loss function (4) which, with a interest rate lower bound constraint, becomes

$$\Omega_0 = E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t \left[(y_t - \hat{y}_t - k)^2 + w_\pi \pi_t^2 + w_r r_t^2 \right] \right] \tag{85}$$

Indeed Clarida *et al.* (1999) provide a stout defence of a hybrid research strategy that combines a loss function based on the stated objectives of central banks with a micro-founded macro-model. A normative assessment of policy rules requires welfare analysis and for this, given our linear-quadratic framework,[20] we require a quadratic approximation of the representative consumer's utility function.

[20]We have emphasized the convenience of the LQ approach to optimal policy. However, recent developments in numerical methods now allow the researcher to go beyond linear approximations of their models and to conduct analysis of both the dynamics and welfare using higher-order (usually second-order) approximations (see, Kim *et al.* (2003) and for an application to simple monetary policy rules, Juillard *et al.* (2006)).

A common procedure for reducing optimal policy to a LQ problem is as follows. Linearize the model about a deterministic steady state as we have already done. Then expand the consumer's utility function as a second-order Taylor series after imposing the economy's resource constraint. In general this procedure is incorrect unless the steady state is not too far from the efficient outcome (see Woodford (2003), chapter 6, Benigno and Woodford (2004), Kim and Kim (2006) and Levine *et al.* (2006)). This we assume and for this case we show in Appendix C that a quadratic single-period loss function that approximates the utility takes the form

$$\begin{aligned} U_t &= w_c(c_t - hc_{t-1})^2 + w_l l_t^2 + w_\pi(\pi_t - \gamma_p \pi_{t-1})^2 + w_{\Delta w}(\Delta w_t - \gamma_w \Delta w_{t-1})^2 \\ &+ w_{lk}(l_t - k_{t-1} - z_t - \frac{1}{1-\alpha} a_t)^2 + w_z(z_t + \psi a_t)^2 - w_{al} a_t l_t - w_i(i_t - i_{t-1})^2 \end{aligned} \quad (86)$$

where positive weights w_c etc are defined in Appendix C. All variables are in log-deviation form about the steady state as in the linearization.[21] The first four terms in (86) give the welfare loss from consumption, employment, price inflation and wage inflation variability respectively. The remaining terms are contributions from arise from the resource constraint in our quadratic approximation procedure.

6.2 Imposing the Interest Rate Zero Lower Bound

In the analysis that follows we adopt a single period loss function of the form

$$L_t = U_t + w_r(r_t - r_t^*)^2 \quad (87)$$

where U_t is given by (86). As explained in section 2.2.1, the policymaker's optimization problem is to choose an unconditional distribution for r_t (i.e., the steady state variance) shifted to the right about a new non-zero steady state inflation rate and a higher nominal interest rate, such that the probability, p, of the interest rate hitting the lower bound is very low. This is implemented by calibrating the weight w_r *for each of our policy rules* so that $z_0(p)\sigma_r < R$ where $z_0(p)$ is the critical value of a standard normally distributed variable Z such that prob $(Z \leq z_0) = p$, $R = \frac{1}{\beta} - 1 + \pi^*$ is the steady state nominal interest rate, σ_r is the unconditional variance and π^* is the new steady state inflation rate. Given σ_r the steady state positive inflation rate that will ensure $r_t \geq 0$ with probability $1 - p$ is

[21] Our quadratic approximation is along the lines of Onatski and Williams (2004) with some differences. Note the expression is not exactly positive definite, but this is of no consequence since positive-definiteness is a sufficient but far from necessary second-order condition for optimality (see Appendix C and Levine *et al.* (2006)).

given by[22]

$$\pi^* = \max[z_0(p)\sigma_r - \left(\frac{1}{\beta} - 1\right) \times 100, 0] \tag{88}$$

In our linear-quadratic framework we can write the intertemporal expected welfare loss at time $t = 0$ as the sum of stochastic and deterministic components, $\Omega_0 = \tilde{\Omega}_0 + \bar{\Omega}_0$. By increasing w_r we can lower σ_r thereby decreasing π^* and reducing the deterministic component, but at the expense of increasing the stochastic component of the welfare loss. By exploiting this trade-off, we then arrive at the optimal policy that, in the vicinity of the steady state, imposes the zero lower bound constraint, $r_t \geq 0$ with probability $1 - p$.

Tables 3a and 3b show the results of this optimization procedure under discretion and commitment respectively using the loss function given by (86), with w_π and other weights functions of fundamental parameters given in Appendix C.[23] We choose $p = 0.025$. Given w_r, denote the expected intertemporal loss (stochastic plus deterministic components) at time $t = 0$ by $\Omega_0(w_r)$. This includes a term penalizing the variance of the interest rate which does not contribute to utility loss as such, but rather represents the interest rate lower bound constraint. Actual utility, found by subtracting the interest rate term, is given by $\Omega_0(0)$. The steady state inflation rate, π^*, that will ensure the lower bound is reached only with probability $p = 0.025$ is computed using (88). Given π^*, we can then evaluate the deterministic component of the welfare loss, $\bar{\Omega}_0$. Since in the new steady state the real interest rate is unchanged, the steady state involving real variables are also unchanged, so from (86) we can write

$$\bar{\Omega}_0(0) = \left[w_\pi(1-\gamma_p)^2 + w_{\Delta w}(1-\gamma_w)^2\right]\pi^{*2} \tag{89}$$

Both the ex-ante optimal and the optimal time-consistent deterministic welfare loss that guide the economy from a zero-inflation steady state to $\pi = \pi^*$ differ from $\bar{\Omega}_0(0)$ (but not by much because the steady state contributions by far outweighs the transitional one). From a *timeless perspective* (see Appendix A.1.2), however, the policymaker will jump immediately to the new steady state justifying the use of (89).

[22]If the inefficiency of the steady-state output is negligible, then $\pi^* \geq 0$ is a credible the new steady state inflation rate. It contrasts with a transitional *deflationary bias* highlighted by Krugman (1998), Eggertsson (2006) and Adam and Billi (2006) which arises under discretion because the central bank cannot credibly lower the expected real interest rate, following a negative demand shock, by a promise to raise the inflation rate in the future. It must therefore rely on lowering the interest rate, hitting the zero lower bound more often. Reduced inflationary expectations, in turn, causes a temporary negative inflation bias. This effect is absent in the approximate approach to imposing the constraint in this paper.

[23]The solution procedures set out in Appendix A actually require a very small weight on the instrument. One can get round this without significantly changing the result by letting inflation be the instrument and then setting the interest rate at a second stage of the optimization to achieve the optimal path for inflation.

Tables 3a and 3b demonstrate the trade-off between reducing the stochastic component of policy at the expense of a higher steady state inflation rate and therefore higher deterministic component of policy. Under discretion in table 3a the optimal combination (i.e., the minimum of $\Omega_0^{TC}(0)$) is achieved at $\pi^* = 0.52$, or at an inflation rate around 2% per year. This pins down the parameter penalizing the variability of the interest rate at $w_r = 4$. The same exercise for optimal policy under commitment sees $\pi^* = 0.26$ with $w_r = 20$, but in the case the loss function is very flat as w_r falls from the value that results in $\pi^* = 0$, so there is little to gain from raising the steady inflation and interest rates.

Figure 2 further demonstrates the results in table 3a. The top-left figure shows the distribution for the nominal interest with zero steady state inflation for the case $w_r^{TC} = 2$ where $\sigma_r = 1.00$. The probability of hitting the zero lower bound is now high, of the order $p = 0.30$. If in the top-right figure, the steady state inflation increases to $\pi^* = 0.95$, thus shifting the distribution by this amount to the right, the probability of $r_t \leq 0$ falls to $p = 0.025$. However this choice of π^* and σ_r is sub-optimal. In the bottom-left figure keeping $p = 0.025$, the total welfare loss falls if we set $\sigma_r = 0.74$ and $\pi^* = 0.68$, values obtained by tightening the variability constraint to $w_r = 3$. Finally in the bottom right figure illustrates the optimal combination of $\sigma_r = 0.62$ and $\pi^* = 0.52$ at $p = 0.025$, obtained at $w_r = 4$ and highlighted in table 3a.

By reporting the expected intertemporal utility loss at time $t = 0$ under both the time-consistent discretionary policy and optimal commitment, $\Omega_0^{TC}(0)$ and $\Omega_0^{OP}(0)$ respectively, we can now assess the stabilization gains from commitment as the interest rate lower bound takes greater effect. We compute these gains as equivalent permanent percentage increases in consumption and inflation, c_e^{gain} and π_e^{gain} respectively. From Appendix C these are given by

$$c_e = \frac{\Omega^{TC}(0) - \Omega^{OP}(0)}{1-h} \times 10^{-2}\,; \qquad \pi_e = \sqrt{\frac{2(\Omega^{TC}(0) - \Omega^{OP}(0))}{w_\pi}} \tag{90}$$

A further useful expression is the *minimum cost of fluctuations*[24] in consumption and inflation equivalent terms obtained under the optimal commitment rule given by

$$c_e^{min} = \frac{\Omega^{OP}(0)}{1-h} \times 10^{-2}\,; \quad \pi_e^{min} = \sqrt{\frac{2\Omega^{OP}(0)}{w_\pi}} \tag{91}$$

Table 3b reports c_e^{min} and table 3c the gains from commitment under three scenarios: the first where the lower bound constraint on the nominal interest rate is ignored ($w_r^{TC} = w_r^{OP} = 0$), the second under optimal combinations of σ_r and π^* highlighted in tables 3a

[24] But it should be noted that our quadratic approximation to the utility function omits terms independent of policy so the cost of fluctuations is under-estimated.

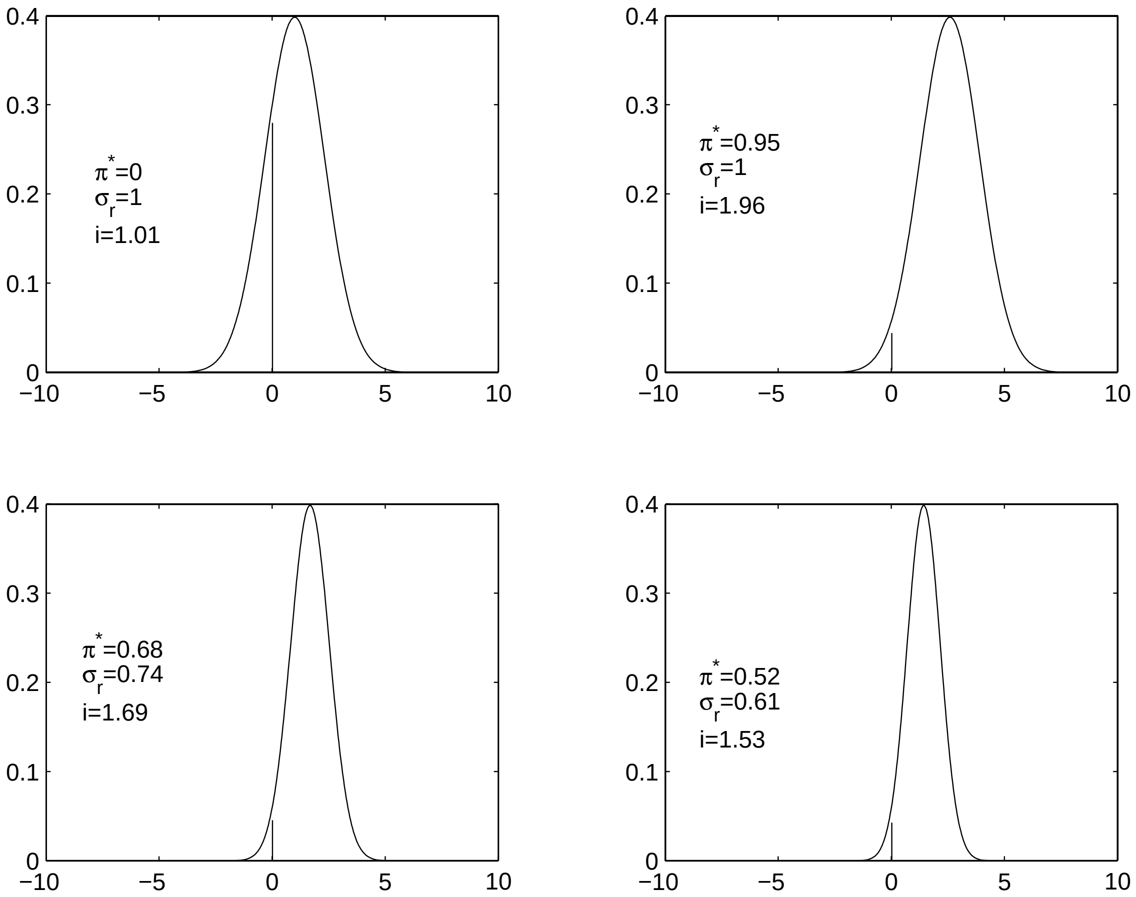

Figure 2: **Imposing the Interest Rate Zero Lower Bound under Discretion.**

and 3b ($w_r^{TC} = 4$, $w_r^{OP} = 20$), and third, under an added constraint that the steady-state inflation rate remains at zero ($w_r^{TC} = 60$, $w_r^{OP} = 45$). A number of interesting points emerge from these tables. First using (91) the minimal cost of consumption fluctuations is given by $\Omega^{OP} = 0.55$ a value much larger than the welfare cost reported by Lucas (1987) which was of the order 0.05%. Taking into account the fact we have omitted fluctuation costs from terms independent of policy, our figures are of the order of those reported in Levin *et al.* (2006) for a similar model but without the nominal interest rate lower bound. The reason why they are much larger in these models is down to the welfare costs of price and wage inflation not included in the Lucas calculations and to the estimated variances of the shocks.[25] Our figure rises when we impose the interest rate lower bound and is

[25]The Lucas calculation is based on $\mathrm{sd}(c_t) = 1.5\%$ which is somewhat lower than the standard deviation we found under optimal commitment of $\mathrm{sd}(c_t) = 2.7\%$. Reworking the Lucas calculation would then give a consumption equivalent loss from fluctuations of 0.16%, still a low figure and far below the loss reported above and by Levin *et al.* (2006).

further increased by the existence of internal habit which reduces the utility increase from a increase in consumption. Thus in our set-up the answer to the question posed by Lucas, "Is there a Case for Stabilization Policy?" is very much in the affirmative.

Second, the most important point from these tables endorses the conclusion reached by Adam and Billi (2006) discussed in the Introduction, namely that the lower bound constraint on the nominal interest rate increases the gains from commitment several fold. In terms of the consumption equivalent for the welfare-based case c_e^{gain} we can see that the stabilization gain from commitment rises until at the optimal combination of σ_r and π^* it reaches $c_e^{gain} = 0.42\%$ and $\pi_e^{gain} = 0.62\%$. They report the gains in terms of a percentage increase in welfare loss as one proceeds from commitment to discretion. Our results indicate an increase of 76% with is remarkably close to the 65% increase reported for the baseline calibration in the paper (but of course for a much simpler New Keynesian model). If we require that there is no long-run inflation under discretion, the commitment gain increases dramatically to $c_e^{gain} = 10.8\%$ and $\pi_e^{gain} = 3.20\%$.

Weight w_r^{TC}	$(\sigma_r^{TC})^2$	$\tilde{\Omega}_0^{TC}(w_r)$	$\tilde{\Omega}_0^{TC}(0)$	π^*	$\bar{\Omega}_0^{TC}(0)$	$\Omega_0^{TC}(0)$
0	103.6	22.2	22.2	18.9	7.08×10^3	7.10×10^3
1	1.74	27.8	26.9	1.58	49.5	76.4
2	1.00	31.4	30.5	0.95	17.9	48.4
3	0.74	34.6	33.4	0.68	9.2	42.6
4	**0.61**	**37.5**	**36.2**	**0.52**	**5.4**	**41.6**
5	0.54	40.3	39.0	0.43	3.7	42.7
10	0.40	55.7	53.7	0.23	1.0	54.7
50	0.31	420	412	0.08	0.1	412
60	0.25	497	489	0	0	489

Table 3a. Core Model: Optimal Discretion.

$\pi^* = \max[z_0(p)\sigma_r^{TC} - (\frac{1}{\beta} - 1) \times 100, 0] = \max[1.96\sigma_r^{TC} - 1.01, 0]$ with $p = 2.5\%$ probability of hitting the zero-lower bound and $\beta = 0.99$.

$\bar{\Omega}_0^{TC}(0) = \frac{1}{2}\left[w_\pi(1-\gamma_p)^2 + w_{\Delta w}(1-\gamma_w)^2\right]\pi^{*2} = 19.81\pi^{*2}$.

$\Omega_0^{TC}(0) = \tilde{\Omega}_0^{TC}(0) + \bar{\Omega}_0^{TC}(0)$.

Weight w_r^{OP}	$(\sigma_r^{OP})^2$	$\tilde{\Omega}_0^{OP}(w_r)$	$\tilde{\Omega}_0^{OP}(0)$	π^*	$\bar{\Omega}_0^{OP}(0)$	$\Omega_0^{OP}(0)$	c_e^{min}
0	28.1	17.1	17.1	9.4	1.75×10^3	1.75×10^3	41.2
10	0.65	23.9	20.8	0.57	6.44	27.2	0.63
20	**0.42**	**26.3**	**22.4**	**0.260**	**1.34**	**23.7**	**0.55**
30	0.33	28.0	23.5	0.116	0.27	23.8	0.55
40	0.27	29.4	24.3	0.01	0.002	24.3	0.56
45	0.25	30.0	24.8	0	0	24.8	0.58
50	0.24	30.5	25.1	0	0	25.1	0.58

Table 3b. Core Model: Optimal Commitment.
π^*, $\Omega^{OP}(0)$ defined as for discretion above.

(w_r^{TC}, w_r^{OP})	$(\sigma_r^{TC}, \sigma_r^{OP})$	$((\pi^*)^{TC}, (\pi^*)^{OP})$	c_e^{gain}	π_e^{gain}
$(0, 0)$	(10.1, 5.3)	$(0, 0)$	0.12	0.44
$(4, 20)$	(0.78, 0.65)	(0.52, 0.26)	0.42	0.62
$(60, 45)$	(0.5, 0.5)	$(0, 0)$	10.8	3.20

Table 3c. Core Model: Stabilization Gains From Commitment:
% Consumption Equivalent (c_e^{gain}) and % Inflation Equivalent (π_e^{gain})
π^*, $\Omega^{OP}(0)$ defined as for discretion above.

In tables 4 and 5 we repeat the same exercise for first, the preferred model variant without any indexation, and then for the low price stickiness variant of the model $\xi_p = 0.75$, as opposed to $\xi_p = 0.91$ or $\xi_p = 0.90$ freely estimated for the core and no indexation variants respectively.

From tables 4a–4c we see that similar results to those of the core model are obtained for the no indexation variant. The welfare gains from commitment are now a little higher at $c_e^{gain} = 0.47\%$ but without the inflation inertia bought about by indexation, this is achieved at a lower optimal steady-state inflation rates under discretion, $\pi^* = 0.18\%$ as opposed to $\pi^* = 0.52\%$ for the core model. Similarly under commitment the optimal steady-state inflation rate is lower in the no indexation model. In the absence of inflation inertia it is now far less costly to impose a zero long-run inflation under discretion and doing so increases the commitment gain by a modest amount.

Weight w_r^{TC}	$(\sigma_r^{TC})^2$	$\tilde{\Omega}_0^{TC}(w_r)$	$\tilde{\Omega}_0^{TC}(0)$	π^*	$\bar{\Omega}_0^{TC}(0)$	$\Omega_0^{TC}(0)$
0	32.8	19.5	19.5	10.2	1.06×10^4	1.06×10^4
1	2.03	25.2	25.2	1.78	323	348
2	1.09	29.8	28.7	1.04	110	139
3	0.76	32.5	31.4	0.70	95.8	127
4	0.59	34.9	33.7	0.50	25.5	59.2
5	0.49	37.0	35.8	0.36	13.2	49.0
6	0.42	39.0	37.7	0.26	6.9	44.6
7	**0.37**	**40.9**	**39.6**	**0.18**	**3.30**	**42.9**
10	0.28	46.1	44.7	0.03	0.09	44.8
12	0.25	49.5	48.0	0	0	48.0

Table 4a. No Indexation Model ($\gamma_p = \gamma_w = 0$): Optimal Discretion.
$\pi^* = \max[z_0(p)\sigma_r^{TC} - (\frac{1}{\beta} - 1) \times 100, 0] = \max[1.96\sigma_r^{TC} - 1.01, 0]$ with $p = 2.5\%$ probability of hitting the zero-lower bound and $\beta = 0.99$.
$\bar{\Omega}_0^{TC}(0) = \frac{1}{2}\left[w_\pi(1-\gamma_p)^2 + w_{\Delta w}(1-\gamma_w)^2\right]\pi^{*2} = 101.8\pi^{*2}$. $\Omega_0^{TC}(0) = \tilde{\Omega}_0^{TC}(0) + \bar{\Omega}_0^{TC}(0)$.

Weight w_r^{OP}	$(\sigma_r^{OP})^2$	$\tilde{\Omega}_0^{OP}(w_r)$	$\tilde{\Omega}_0^{OP}(0)$	π^*	$\bar{\Omega}_0^{OP}(0)$	$\Omega_0^{OP}(0)$	c_e^{min}
0	14.6	14.7	14.7	6.48	4.27×10^3	4.29×10^3	99.8
10	0.67	22.3	19.1	0.59	35.4	54.5	1.27
20	0.41	24.8	20.9	0.25	6.36	27.3	0.63
25	0.35	25.7	21.5	0.15	2.29	23.8	0.55
30	**0.31**	**26.4**	**22.0**	**0.08**	**0.65**	**22.7**	**0.53**
40	0.26	27.8	23.0	0	0	23.0	0.53
42	0.25	28.0	23.1	0	0	23.1	0.54

Table 4b. No Indexation Model: Optimal Commitment.
π^*, $\Omega^{OP}(0)$ defined as above.

(w_r^{TC}, w_r^{OP})	$(\sigma_r^{TC}, \sigma_r^{OP})$	$((\pi^*)^{TC}, (\pi^*)^{OP})$	c_e^{gain}	π_e^{gain}
$(0,0)$	(5.7, 3.8)	$(0,0)$	0.11	0.32
$(7,30)$	(0.61, 0.56)	(0.18, 0.08)	0.47	0.66
$(12,42)$	(0.5, 0.5)	$(0,0)$	0.58	0.66

Table 4c. No Indexation: Stabilization Gains From Commitment
π^*, $\Omega^{OP}(0)$ defined as above.

Now consider the low price stickiness variant. The inflation costs of a given rate of inflation are now much lower and, in the absence of commitment, the incentive to raise or lower inflation following shocks correspondingly higher. The lower interest rate bound acts as a greater constraint for optimal discretion and as a consequence, as tables 5a–5c show, the gains from commitment rise considerably to $c_e^{gain} = 2.35\%$ and $\pi_e^{gain} = 4.39$.

Weight w_r^{TC}	$(\sigma_r^{TC})^2$	$\tilde{\Omega}_0^{TC}(w_r)$	$\tilde{\Omega}_0^{TC}(0)$	π^*	$\bar{\Omega}_0^{TC}(0)$	$\Omega_0^{TC}(0)$
0	116	26.1	26.1	20.1	2.8×10^3	2.9×10^3
0.5	6.9	35.2	33.4	4.14	120	153
0.6	6.3	37.1	35.2	3.91	107	142
0.7	6.0	39.2	37.1	3.78	100	137
0.8	5.7	41.4	39.1	3.68	95	134
0.9	**5.6**	**43.9**	**41.4**	**3.61**	**92**	**133**
1.00	5.5	46.5	43.8	3.59	90.3	134
27	0.25	375	371	0	0	375

Table 5a. Low Price Stickiness Model ($\xi_p = 0.75$): Optimal Discretion.
$\pi^* = \max[z_0(p)\sigma_r^{TC} - (\frac{1}{\beta} - 1) \times 100, 0] = \max[1.96\sigma_r^{TC} - 1.01, 0]$ with $p = 2.5\%$ probability of hitting the zero-lower bound and $\beta = 0.99$.
$\bar{\Omega}_0^{TC}(0) = \frac{1}{2}\left[w_\pi(1-\gamma_p)^2 + w_{\Delta w}(1-\gamma_w)^2\right]\pi^{*2} = 7.02\pi^{*2}$. $\Omega_0^{TC}(0) = \tilde{\Omega}_0^{TC}(0) + \bar{\Omega}_0^{TC}(0)$.

Weight w_r^{OP}	$(\sigma_r^{OP})^2$	$\tilde{\Omega}_0^{OP}(w_r)$	$\tilde{\Omega}_0^{OP}(0)$	π^*	$\bar{\Omega}_0^{OP}(0)$	$\Omega_0^{OP}(0)$	c_e^{min}
0	36	19.8	19.8	10.8	819	839	19.5
10	1.1	31.4	26.3	1.05	7.74	34.0	0.79
15	0.81	33.6	28.1	0.75	3.99	32.1	0.74
20	**0.62**	**35.2**	**29.6**	**0.53**	**1.97**	**31.6**	**0.73**
30	0.41	37.4	31.9	0.25	0.44	32.3	0.75
45	0.25	39.5	34.5	0	0	34.5	0.80

Table 5b. $\xi_p = 0.75$ Model: Optimal Commitment.

(w_r^{TC}, w_r^{OP})	$(\sigma_r^{TC}, \sigma_r^{OP})$	$((\pi^*)^{TC}, (\pi^*)^{OP})$	c_e^{gain}	π_e^{gain}
$(0, 0)$	(116, 36)	$(0, 0)$	0.15	1.10
$(0.9, 25)$	(5.56, 0.62)	$(3.61, 0.53)$	2.35	4.39
$(27, 45)$	(0.25, 0.25)	$(0, 0)$	7.83	8.02

Table 5c. $\xi_p = 0.75$ Model: Stabilization Gains From Commitment.

The overall conclusion that emerges from these results for three variants of our model is that the stabilization gains from commitment are significantly greater than those previously reported in the literature. For the empirically supported model variants, the core model and the alternative with no indexation we find these gains to be a $0.4-0.5\%$ equivalent permanent increase in consumption corresponding to a $0.6-0.7\%$ permanent increase in quarterly inflation. The latter, for instance, compares with a range of $0.04-0.4\%$ found in the comprehensive study of Dennis and Söderström (2006) across several models.[26] Moreover in our variant with a more plausible degree of price stickiness, gains of over 2% consumption equivalent are found.

6.3 Stabilization Gains with Simple Rules

We now turn to results for simple commitment rules of the general form:

$$r_t \;=\; \rho r_{t-1} + \Theta_\pi E_t \pi_{t+j} + \Theta_y (y_t - \hat{y}_t) + \Theta_{\Delta w} \Delta w_t + \Theta_{wr} w r_t \tag{92}$$

where $\rho \in [0,1]$, $\Theta_\pi, \Theta_y, \Theta_{\Delta w}, \Theta_{wr} > 0, j \geq 0$. Putting $\Theta_{\Delta w} = \Theta_{wr} = 0$ gives the standard Taylor rule where the interest rate only to current price inflation and the output gap, $\Theta_{\Delta w} = \Theta_{wr} = \Theta_y = 0$ gives a price inflation rule, $\Theta_\pi = \Theta_{wr} = \Theta_y = 0$ gives a wage inflation rule and $j = \Theta_{\Delta w} = \Theta_y = 0$ gives a current price inflation and real wage rule.

Results for these rules are summarized in table 6 for the core model. Since the welfare gains from increasing the steady state inflation rate and widening the interest rate distribution consistent with $p = 0.025$ is very small, we confine ourselves to $\pi^* = 0$. There are two notable results that emerge. First, we assess the effect of using an arbitrary rather than an optimized simple commitment rule by examining the outcome when a minimal rule $i_i = 1.001\pi_t$ that just produces saddle-path stability. This is the worst case and we see that the costs are substantial: $c_e^{gain} = 7.03\%$. Interestingly, this outcome is still better than that under discretion if the same constraint on the variance of the interest rate as for optimal commitment is imposed. Second, simple price inflation or wage inflation rules perform reasonably well in that they achieve over 80% of the commitment gains achieved by the optimal rule when $\pi^* = 0$ is imposed. The simple rule that closely mimics optimal commitment for the welfare-based case is the inflation and real wage rule.[27] From table 5

[26] We have adjusted their reported annual inflation rate equivalents. Note that they examine models without explicit micro-foundations and therefore employ an ad hoc loss function.

[27] This finding that simple rules should respond to labour-market conditions is in broad agreement with the result in Levin *et al.* (2006). However their study, which did not incorporate a nominal interest rate zero lower bound, found that the wage inflation rule performed a lot better than the price inflation and closely mimicked optimal commitment.

almost all the gains from commitment are achieved by this rule though simplicity per se still leaves a small cost of $c_e^{gain} = 0.02\%$ and $\pi_e^{gain} = 0.15\% \approx 0.60\%$ on an annual basis.

Tables 7 and 8 again repeats the same exercise for the variants of the model with no indexation and with $\xi_p = 0.75$ imposed. A similar story emerges: by far the best simple interest rate rule is one that responds to current inflation and the real wage with a higher, but still quite small, costs of simplicity of $c_e^{gain} = 0.02 - 0.07\%$. For the model without indexation, the wage inflation rule performs a lot better than the price inflation rule and, in that respect, reproduces the finding of Levin *et al.* (2006). It is of interest to note that, in these alternative variants, the costs of the minimal rules are far less.

Rule	ρ	Θ_π	$\Theta_{\Delta w}$	Θ_{wr}	w_r	$\Omega_0(w_r)$	$\Omega_0(0)$	c_e^{gain}	π_e^{gain}	σ_r^2	π^*
Minimal	0	1.001	0	0	0	327	327	7.03	2.57	0.46	0
π_t	0.37	1.42	0	0	45	82.4	77.0	1.21	1.07	0.24	0
Δw_t	0.95	0	0.68	0	25	80.3	77.2	1.22	1.08	0.25	0
wr_t, π_t	0.96	0.15	0	0.20	55	32.9	25.8	0.02	0.15	0.26	0
OP	n.a.	n.a.	n.a.	n.a.	45	30.0	24.8	0	0	0.25	0

Table 6. Comparison of Optimal Commitment Rules. Core model.

Rule	ρ	Θ_π	$\Theta_{\Delta w}$	Θ_{wr}	w_r	$\Omega(w_r)$	$\Omega_0(0)$	c_e^{gain}	π_e^{gain}	σ_r^2	π^*
Minimal	0	1.001	0	0	0	397	8.70	2.86	4.50	0.13	0
π_t	0.43	2.55	0	0	23	77.2	74.3	1.19	1.06	0.25	0
Δw_t	1.00	0	0.71	0	42	59.4	54.2	0.73	0.82	0.25	0
wr_t, π_t	0.93	0.37	0	0.23	43	29.4	24.0	0.02	0.14	0.25	0
OP	n.a.	n.a.	n.a.	n.a.	42	28.0	23.1	0	0	0.25	0

Table 7. Comparison of Optimal Commitment Rules: No Indexation Model ($\gamma_p = \gamma_w = 0$).

Rule	ρ	Θ_π	$\Theta_{\Delta w}$	Θ_{wr}	w_r	$\Omega(w_r)$	$\Omega_0(0)$	c_e^{gain}	π_e^{gain}	σ_r^2	π^*
Minimal	0	1.001	0	0	0	90.4	90.4	1.30	3.27	2.87	0
π_t	0.67	0.39	0	0	80	85.4	75.4	0.95	2.80	0.25	0
Δw_t	0.72	0	0.52	0	16	86.3	84.3	1.16	3.09	0.25	0
wr_t, π_t	1.00	0.06	0	0.08	62	45.4	37.3	0.07	0.73	0.26	0
OP	n.a.	n.a.	n.a.	n.a.	45	39.5	34.5	0	0	0.25	0

Table 8. Comparison of Optimal Commitment Rules: Low Price Stickiness Model ($\xi_p = 0.75$).

6.4 Impulse Responses Under Commitment and Discretion

Figures 5-12 compare the responses under the optimal commitment, discretion and the optimized simple inflation/real wage rule following an unanticipated government spending shock ($g_0 = 1$) and an unanticipated productivity shock ($a_0 = 1$).

In order to interpret these graphs it is useful to consider the four sources of the time-inconsistency problem in our model; namely, from forward-looking pricing, consumption, investment and wage setting. Following a shock which diverts the economy from its steady state, given expectations of inflation, the opportunist policy-maker can increase or decrease output by reducing or increasing the interest rate which increases or decreases inflation. Consider the case where the economy is below the its steady state level of output. A reduction in the interest rate then causes consumption demand rise. Firms locked into price contracts respond to an increase in demand by increasing output and increasing the price according to their indexing rule. Those who can re-optimize increase only increase their price. Given inflationary expectations, a reduction in the interest rate sees Tobin's Q rise, and with it investment and capital stock. This increases output on the supply side. Given inflationary expectations an inflationary impulse results in a fall in the real wage and an increase in labour supply, adding further to the supply side boost to output. All these changes are for *given* inflationary expectations and illustrates the incentive to inflate when the output gap increases. In an non-commitment equilibrium however the incentive is anticipated and the result is higher inflation compared with the commitment case. This contrast between the commitment and discretionary cases is seen clearly in the figures. Finally, comparing the optimal commitment and the simple inflation-real wage rules, we see how the latter closely mimics the former.

6.5 Sustaining Commitment as an Equilibrium

We now examine numerically the no-deviation condition for commitment to be a perfect Bayesian equilibrium. We confine ourselves to reporting results for the core SW variant and for the form of the condition given by (29) which assume an instantaneous loss of reputation following deviation. Experiment revealed this to give very similar results to those using (27), and this in turn implied that the condition relevant for our simple inflation/real wage, (32), was satisfied.

Figure 4 plots a histogram from $10,000$ draws of the sector $[z_t^T p_{2,t}^T]^T$ in the vicinity of the steady state of the economy under the optimal commitment rule. The probability of the weak government deviating from the optimal rule, q_t, is then the proportion of these draws for which (29) does not hold; i.e., $\Phi = \text{tr}((1-\beta)(N_{11}+S)(Z_t+\beta\Sigma)+\text{tr}((1-\beta)N_{22}p_{2t}p_{2t}^T) <$ 0. For our model and sample of 10,000 draws we see that in fact $q_t = 0$ so that optimal commitment for a weak government turns out to be a perfect Bayesian equilibrium.

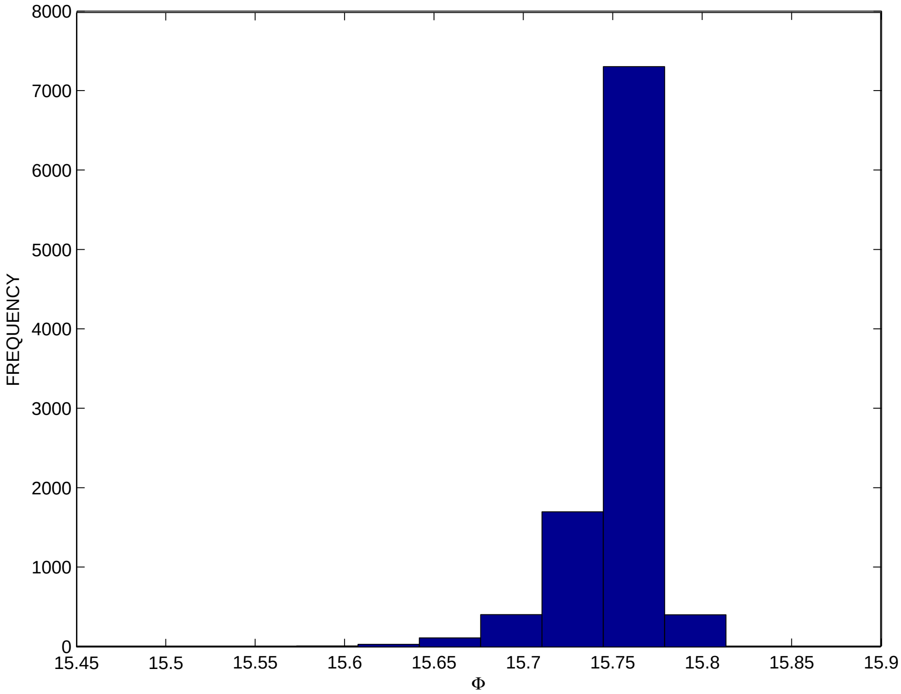

Figure 3: **The No-Deviation Condition:** $\Phi = \mathbf{tr}((1-\beta)(N_{11}+S)(Z_t+\beta\Sigma)+\mathbf{tr}((1-\beta)N_{22}p_{2t}p_{2t}^T)$**.** $\beta = 0.99$

As discussed in section 3 the no-deviation condition compares the temporary stabilization gains from reneging ('temptation') with the long-run stabilization loss from losing reputation (the 'penalty'). The latter depends crucially on the policymaker's rate of discount β. In all our welfare-based results we have set $\beta = 0.99$ on a quarterly basis for both the policymaker and theeprivate sector. But suppose that the policymaker was more myopic than the private sector. For all $\beta \geq 0.75$ we find that $q_t = 0$. In figure 4 we set $\beta = 0.5$ which could be appropriate for a non-independent central bank in which optimal monetary policy depends on the probability of the survival (re-election) of government was very low, in fact 0.1250 per year. We find that there is now a very small probability of a a break-down in the no-deviation condition, namely $q_t = 0.002$. Thus our result that commitment can be sustained as a PBE is very robust to variations in the policymaker's discount factor for all conceivable institutional arrangements in the euro-area.

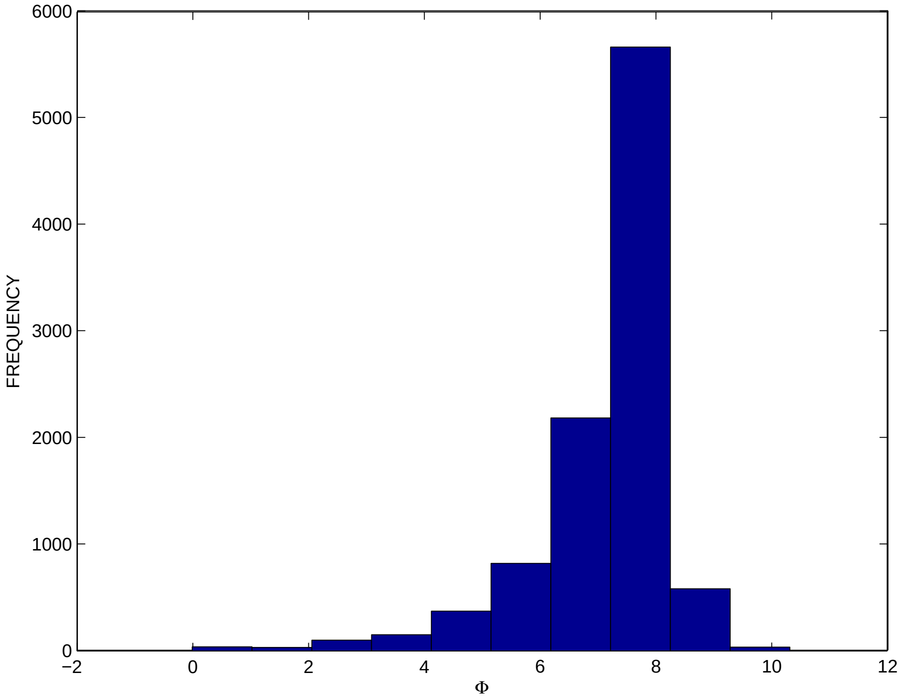

Figure 4: **The No-Deviation Condition:** $\Phi = \mathbf{tr}((1-\beta)(N_{11}+S)(Z_t+\beta\Sigma) + \mathbf{tr}((1-\beta)N_{22}p_{2t}p_{2t}^T)$. $\beta = 0.5$

7 Conclusions

This paper has examined the credibility problem in an empirical DSGE model with four sources of time-inconsistency: from forward-looking pricing, consumption, investment and wage setting. In the absence of commitment, following a shock which diverts the economy from its steady state and given expectations of inflation, the opportunist policy-maker can increase or decrease output by reducing or increasing the interest rate which increases or decreases inflation. This results in a higher variability of inflation and the nominal interest rate under discretion. The latter means that the interest rate zero lower bound constraint is tighter under discretion and its presence increases the stabilization gains from commitment. The constraint can be relaxed by increasing the steady state inflation rate, but this comes at a cost of an increase in the deterministic component of the welfare loss.

The main findings of this paper can be summarized as follows:

1. Our welfare-based loss function uses the 'small distortions' quadratic approximation to the consumer's utility which is accurate if the steady state is close to the social optimum. In assessing this condition we highlight a neglected aspect of typical New Keynesian models: external habit in consumption tends to make labour supply and

the natural rate of output *too high* compared with the social optimum. If the habit effect is sufficiently high and labour market and product market distortions are not too big then, with a sufficiently small tax wedge, the natural rate can actually be *above* the social optimum. This would then render the long-run 'inflationary bias' negative.

2. Whilst the validity of an inflationary bias arising from the pursuit of an ambitious output target above its natural rate has been criticized (notably in Blinder (1998)), our analysis suggests a rather different form of bias arising from the interest rate zero lower bound. We find that the optimal steady state inflation rate necessary to avoid the lower bound is far lower under commitment than under discretion, so there is a new sense in which there is a long-run inflationary bias which is really an integral part of the stabilization bias.

3. In terms of an equivalent permanent increase in consumption, c_e^{gain} for the welfare-based loss function and a permanent decrease in inflation π_e^{gain}, the stabilization gains from commitment rise considerably if the lower bound effect is taken into account. Using empirical estimates from the core model and the preferred variant without indexation, we find an average consumption and inflation-equivalent gains of $c_e^{gain} = 0.4 - 0.5\%$ and $\pi_e^{gain} = 0.6 - 0.7\%$ respectively, the latter on a quarterly basis. For the variant of the model with lower price stickiness, these rise considerably to $c_e^{gain} = 2.35\%$ and $\pi_e^{gain} = 4.39\%$

4. Given these large gains from commitment, the incentive for central banks to avoid a loss of reputation for commitment is substantial. Consequently, unless the policymaker is implausibly myopic, a commitment rule can be sustained as a perfect Bayesian equilibrium in which deviation from commitment hardly ever happens despite the possibility of large exogenous shocks.

5. Simple interest rate rules should respond to labour-market conditions as well as inflation. The optimal commitment rule can be closely approximated in terms of its good stabilization properties by an interest rate rule that responds positively to current inflation and to the current real wage.

There are a number of possible directions for future research. First, the robustness of our finding that gains from commitment may be far higher than previously thought needs to investigated further across an number of other DSGE models, including the SW model fitted to US data and small open economy models such as Adolfson *et al.* (2004). Second, a more accurate quadratic approximation of the household utility can be obtained from the 'large distortions' procedure of Benigno and Woodford (2004).[28] Finally, using estimates for posterior model probabilities and, for each model variant, estimates of the posterior densities of the parameters, a consistently Bayesian approach to both the estimation and the design of *robust* interest rate rules can be employed as in Batini *et al.* (2006).

A Details of Policy Rules

First consider the purely deterministic problem. In general policy involving several (for example monetary and fiscal) instruments starts with a model in state-space form:

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{x}^e_{t+1,t} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + B\mathsf{w}_t \tag{A.1}$$

where z_t is an $(n-m)\times 1$ vector of predetermined variables including non-stationary processed, z_0 is given, w_t is a vector of policy variables, x_t is an $m \times 1$ vector of non-predetermined variables and $\mathsf{x}^e_{t+1,t}$ denotes rational (model consistent) expectations of x_{t+1} formed at time t. Then $\mathsf{x}^e_{t+1,t} = \mathsf{x}_{t+1}$ and letting $\mathsf{y}_t^T = \left[\mathsf{z}_t^T \; \mathsf{x}_t^T\right]$ (A.1) becomes

$$\mathsf{y}_{t+1} = A\mathsf{y}_t + B\mathsf{w}_t \tag{A.2}$$

Define target variables s_t by

$$\mathsf{s}_t = M\mathsf{y}_t + H\mathsf{w}_t \tag{A.3}$$

and the policy-maker's loss function at time t by

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty} \beta^t [\mathsf{s}_{t+i}^T Q_1 \mathsf{s}_{t+i} + \mathsf{w}_{t+i}^T Q_2 \mathsf{w}_{t+i}] \tag{A.4}$$

which we can rewrite as

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty} \beta^t [\mathsf{y}_{t+i}^T Q \mathsf{y}_{t+i} + 2\mathsf{y}_{t+i}^T U \mathsf{w}_{t+i} + \mathsf{w}_{t+i}^T R \mathsf{w}_{t+i}] \tag{A.5}$$

where $Q = M^T Q_1 M$, $U = M^T Q_1 H$, $R = Q_2 + H^T Q_1 H$, Q_1 and Q_2 are symmetric and non-negative definite, R is required to be positive definite and $\beta \in (0,1)$ is discount factor. The procedures for evaluating the three policy rules are outlined in the rest of this appendix (or Currie and Levine (1993) for a more detailed treatment).

A.1 The Optimal Policy with Commitment

Consider the policy-maker's *ex-ante* optimal policy at $t = 0$. This is found by minimizing Ω_0 given by (A.5) subject to (A.2) and (A.3) and given z_0. We proceed by defining the Hamiltonian

$$\mathcal{H}_t(y_t, y_{t+1}, \mu_{t+1}) = \frac{1}{2}\beta^t(\mathsf{y}_t^T Q \mathsf{y}_t + 2\mathsf{y}_t^T U \mathsf{w}_t + \mathsf{w}_t^T R \mathsf{w}_t) + \mu_{t+1}(A\mathsf{y}_t + B\mathsf{w}_t - \mathsf{y}_{t+1}) \tag{A.6}$$

[28] See Levine *et al.* (2006) for a method to computationally implement this procedure.

where μ_t is a row vector of costate variables. By standard Lagrange multiplier theory we minimize

$$\mathcal{L}_0(y_0, y_1, \ldots, w_0, w_1, \ldots, \mu_1, \mu_2, \ldots) = \sum_{t=0}^{\infty} \mathcal{H}_t \tag{A.7}$$

with respect to the arguments of L_0 (except z_0 which is given). Then at the optimum, $\mathcal{L}_0 = \Omega_0$.

Redefining a new costate column vector $\mathsf{p}_t = \beta^{-t}\mu_t^T$, the first-order conditions lead to

$$\mathsf{w}_t = -R^{-1}(\beta B^T \mathsf{p}_{t+1} + U^T \mathsf{y}_t) \tag{A.8}$$

$$\beta A^T \mathsf{p}_{t+1} - \mathsf{p}_t = -(Q\mathsf{y}_t + U\mathsf{w}_t) \tag{A.9}$$

Substituting (A.8) into (A.2)) we arrive at the following system under control

$$\begin{bmatrix} I & \beta BR^{-1}B^T \\ 0 & \beta(A^T - UR^{-1}B^T) \end{bmatrix} \begin{bmatrix} \mathsf{y}_{t+1} \\ \mathsf{p}_{t+1} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}U^T & 0 \\ -(Q - UR^{-1}U^T & I \end{bmatrix} \begin{bmatrix} \mathsf{y}_t \\ \mathsf{p}_t \end{bmatrix} \tag{A.10}$$

To complete the solution we require $2n$ boundary conditions for (A.10). Specifying z_0 gives us $n-m$ of these conditions. The remaining condition is the 'transversality condition'

$$\lim_{t\to\infty} \mu_t^T = \lim_{t\to\infty} \beta^t \mathsf{p}_t = 0 \tag{A.11}$$

and the initial condition

$$\mathsf{p}_{20} = 0 \tag{A.12}$$

where $\mathsf{p}_t^T = \left[\mathsf{p}_{1t}^T \, \mathsf{p}_{2t}^T\right]$ is partitioned so that p_{1t} is of dimension $(n-m) \times 1$. Equation (A.3), (A.8), (A.10) together with the $2n$ boundary conditions constitute the system under optimal control.

Solving the system under control leads to the following rule

$$\mathsf{w}_t = -F \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \equiv D \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} = -F \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_{2t} \end{bmatrix} \tag{A.13}$$

where

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{p}_{2t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ S_{21} & S_{22} \end{bmatrix} G \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \equiv H \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \tag{A.14}$$

$$N = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{A.15}$$

$$\mathsf{x}_t = -\begin{bmatrix} N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \tag{A.16}$$

where $F = -(R + B^T SB)^{-1}(B^T SA + U^T), G = A - BF$ and

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{A.17}$$

partitioned so that S_{11} is $(n-m) \times (n-m)$ and S_{22} is $m \times m$ is the solution to the steady-state Ricatti equation

$$S = Q - UF - F^T U^T + F^T RF + \beta(A - BF)^T S(A - BF) \tag{A.18}$$

The cost-to-go for the optimal policy (OP) at time t is

$$\Omega_t^{OP} = -\frac{1}{2}(\mathrm{tr}(N_{11}Z_t) + \mathrm{tr}(N_{22}\mathsf{p}_{2t}\mathsf{p}_{2t}^T)) \tag{A.19}$$

where $Z_t = \mathsf{z}_t\mathsf{z}_t^T$. To achieve optimality the policy-maker sets $\mathsf{p}_{20} = 0$ at time $t = 0$. At time $t > 0$ there exists a gain from reneging by resetting $\mathsf{p}_{2t} = 0$. It can be shown that $N_{11} < 0$ and $N_{22} < 0$.[29], so the incentive to renege exists at all points along the trajectory of the optimal policy. This is the time-inconsistency problem.

A.1.1 Implementation

The rule may also be expressed in two other forms: First as

$$\mathsf{w}_t = D_1\mathsf{z}_t + D_2 H_{21} \sum_{\tau=1}^{t} (H_{22})^{\tau-1} \mathsf{z}_{t-\tau} \tag{A.20}$$

where $D = [D_1\ D_2]$ is partitioned conformably with z_t and p_{2t}. The rule then consists of a feedback on the lagged predetermined variables with geometrically declining weights with lags extending back to time $t = 0$, the time of the formulation and announcement of the policy.

The final way of expressing the rule is express the process for w_t in terms of the target variables only, s_t, in the loss function. This in particular eliminates feedback from the exogenous processes in the vector z_t. Since the rule does not require knowledge of these processes to design, Woodford (2003) refers to this as "robust" in describing it as the *Robust Optimal Explicit* rule.

A.1.2 Optimal Policy from a Timeless Perspective

Noting from (A.16) that long the optimal policy we have $\mathsf{x}_t = -N_{21}\mathsf{z}_t - N_{22}\mathsf{p}_{2t}$, the optimal policy "from a timeless perspective" proposed by Woodford (2003) replaces the initial condition for optimality $p_{20} = 0$ with

$$J\mathsf{x}_0 = -N_{21}\mathsf{z}_0 - N_{22}\mathsf{p}_{20} \tag{A.21}$$

[29] See Currie and Levine (1993), chapter 5.

where J is some $1 \times m$ matrix. Typically in New Keynesian models the particular choice of condition is $\pi_0 = 0$ thus avoiding any once-and-for-all initial surprise inflation. This initial condition applies only at $t = 0$ and only affects the deterministic component of policy and not the stochastic, stabilization component.

A.2 The Dynamic Programming Discretionary Policy

The evaluate the discretionary (time-consistent) policy we rewrite the cost-to-go Ω_t given by (A.5) as

$$\Omega_t = \frac{1}{2}[\mathsf{y}_t^T Q \mathsf{y}_t + 2\mathsf{y}_t^T U \mathsf{w}_t + \mathsf{w}_t^T R \mathsf{w}_t + \beta\Omega_{t+1}] \tag{A.22}$$

The dynamic programming solution then seeks a stationary solution of the form $w_t = -Fz_t$ in which Ω_t is minimized at time t subject to (1) in the knowledge that a similar procedure will be used to minimize Ω_{t+1} at time $t+1$.

Suppose that the policy-maker at time t expects a private-sector response from $t+1$ onwards, determined by subsequent re-optimization, of the form

$$\mathsf{x}_{t+\tau} = -N_{t+1}\mathsf{z}_{t+\tau}, \ \tau \geq 1 \tag{A.23}$$

The loss at time t for the *ex ante* optimal policy was from (A.19) found to be a quadratic function of x_t and p_{2t}. We have seen that the inclusion of p_{2t} was the source of the time inconsistency in that case. We therefore seek a lower-order controller

$$\mathsf{w}_t = -F\mathsf{z}_t \tag{A.24}$$

with the cost-to-go quadratic in z_t only. We then write $\Omega_{t+1} = \frac{1}{2}\mathsf{z}_{t+1}^T S_{t+1}\mathsf{z}_{t+1}$ in (A.22). This leads to the following iterative process for F_t

$$\mathsf{w}_t = -F_t\mathsf{z}_t \tag{A.25}$$

where

$$\begin{aligned}
F_t &= (\overline{R}_t + \lambda\overline{B}_t^T S_{t+1}\overline{B}_t)^{-1}(\overline{U}_t^T + \beta\overline{B}_t^T S_{t+1}\overline{A}_t) \\
\overline{R}_t &= R + K_t^T Q_{22} K_t + U^{2T} K_t + K_t^T U^2 \\
K_t &= -(A_{22} + N_{t+1}A_{12})^{-1}(N_{t+1}B^1 + B^2) \\
\overline{B}_t &= B^1 + A_{12}K_t \\
\overline{U}_t &= U^1 + Q_{12}K_t + J_t^T U^2 + J_t^T Q_{22} J_t \\
\overline{J}_t &= -(A_{22} + N_{t+1}A_{12})^{-1}(N_{t+1}A_{11} + A_{12})
\end{aligned}$$

$$\begin{aligned}
\overline{A}_t &= A_{11} + A_{12}J_t \\
S_t &= \overline{Q}_t - \overline{U}_t F_t - F_t^T \overline{U}^T + \overline{F}_t^T \overline{R}_t F_t + \beta(\overline{A}_t - \overline{B}_t F_t)^T S_{t+1}(\overline{A}_t - \overline{B}_t \overline{F}_t) \\
\overline{Q}_t &= Q_{11} + J_t^T Q_{21} + Q_{12}J_t + J_t^T Q_{22} J_t \\
N_t &= -J_t + K_t F_t
\end{aligned}$$

where $B = \begin{bmatrix} B^1 \\ B^2 \end{bmatrix}$, $U = \begin{bmatrix} U^1 \\ U^2 \end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, and Q similarly are partitioned conformably with the predetermined and non-predetermined components of the state vector.

The sequence above describes an iterative process for F_t, N_t, and S_t starting with some initial values for N_t and S_t. If the process converges to stationary values, F, N and S say, then the time-consistent feedback rule is $w_t = -Fz_t$ with loss at time t given by

$$\Omega_t^{TC} = \frac{1}{2}\mathsf{z}_t^T S \mathsf{z}_t = \frac{1}{2}\text{tr}(SZ_t) \tag{A.26}$$

A.3 Optimized Simple Rules

We now consider simple sub-optimal rules of the form

$$\mathsf{w}_t = D\mathsf{y}_t = D\begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{A.27}$$

where D is constrained to be sparse in some specified way. Rule (A.27) can be quite general. By augmenting the state vector in an appropriate way it can represent a PID (proportional-integral-derivative)controller.

Substituting (A.27) into (A.5) gives

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty}\beta^t \mathsf{y}_{t+i}^T P_{t+i}\mathsf{y}_{t+i} \tag{A.28}$$

where $P = Q + UD + D^TU^T + D^TRD$. The system under control (A.1), with w_t given by (A.27), has a rational expectations solution with $\mathsf{x}_t = -N\mathsf{z}_t$ where $N = N(D)$. Hence

$$\mathsf{y}_t^T P\mathsf{y}_t = \mathsf{z}_t^T T\mathsf{z}_t \tag{A.29}$$

where $T = P_{11} - N^TP_{21} - P_{12}N + N^TP_{22}N$, P is partitioned as for S in (A.17) onwards and

$$\mathsf{z}_{t+1} = (G_{11} - G_{12}N)\mathsf{z}_t \tag{A.30}$$

where $G = A + BD$ is partitioned as for P. Solving (A.30) we have

$$\mathsf{z}_t = (G_{11} - G_{12}N)^t\mathsf{z}_0 \tag{A.31}$$

Hence from (A.32), (A.29) and (A.31) we may write at time t

$$\Omega_t^{SIM} = \frac{1}{2}z_t^T V z_t = \frac{1}{2}\text{tr}(VZ_t) \tag{A.32}$$

where $Z_t = z_t z_t^T$ and V satisfies the *Lyapunov* equation

$$V = T + H^TVH \tag{A.33}$$

where $H = G_{11} - G_{12}N$. At time $t = 0$ the optimized simple rule is then found by minimizing Ω_0 given by (A.32) with respect to the non-zero elements of D given z_0 using a standard numerical technique. An important feature of the result is that unlike the previous solution the optimal value of D, D^* say, is not independent of z_0. That is to say

$$D^* = D^*(z_0)$$

A.4 The Stochastic Case

Consider the stochastic generalization of (A.1)

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{x}^e_{t+1,t} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + B\mathsf{w}_t + \begin{bmatrix} \mathsf{u}_t \\ 0 \end{bmatrix} \tag{A.34}$$

where u_t is an $n \times 1$ vector of white noise disturbances independently distributed with $\text{cov}(\mathsf{u}_t) = \Sigma$. Then, it can be shown that certainty equivalence applies to all the policy rules apart from the simple rules (see Currie and Levine (1993)). The expected loss at time t is as before with quadratic terms of the form $\mathsf{z}_t^T X \mathsf{z}_t = \text{tr}(X\mathsf{z}_t, Z_t^T)$ replaced with

$$E_t\left(\text{tr}\left[X\left(z_t z_t^T + \sum_{i=1}^{\infty}\beta^t \mathsf{u}_{t+i}\mathsf{u}_{t+i}^T\right)\right]\right) = \text{tr}\left[X\left(z_t^T z_t + \frac{\lambda}{1-\lambda}\Sigma\right)\right] \tag{A.35}$$

where E_t is the expectations operator with expectations formed at time t.

Thus for the optimal policy with commitment (A.19) becomes in the stochastic case

$$\Omega_t^{OP} = -\frac{1}{2}\text{tr}\left(N_{11}\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right) + N_{22}\mathsf{p}_{2t}\mathsf{p}_{2t}^T\right) \tag{A.36}$$

For the time-consistent policy (A.26) becomes

$$\Omega_t^{TC} = -\frac{1}{2}\text{tr}\left(S\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right)\right) \tag{A.37}$$

and for the simple rule, generalizing (A.32)

$$\Omega_t^{SIM} = -\frac{1}{2}\text{tr}\left(V\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right)\right) \tag{A.38}$$

The optimized simple rule is found at time $t = 0$ by minimizing Ω_0^{SIM} given by (A.38). Now we find that

$$D^* = D^*\left(\mathsf{z}_0\mathsf{z}_0^T + \frac{\beta}{1-\beta}\Sigma\right) \tag{A.39}$$

or, in other words, the optimized rule depends both on the initial displacement z_0 and on the covariance matrix of disturbances Σ.

B Dynamic Representation as Difference Equations

The linearizations in the main text, especially that for the real wage equation, requires us to express the price and wage-setting first order conditions as stochastic non-linear difference equations.[30] To do this first define

$$\Pi_t \equiv \frac{P_t}{P_{t-1}} = 1 + \pi_t \tag{B.1}$$

$$\Phi_t \equiv P_t^0/P_t \tag{B.2}$$

$$\tilde{\Pi}_t \equiv \frac{\Pi_t}{\Pi_{t-1}^{\gamma}} \tag{B.3}$$

and use $D_{t+k} = \beta^k \frac{MU_{t+k}^C}{P_{t+k}}$ where $MU_t^C = C_t^{-\sigma} H_{C,t}^{1-\sigma}$ is the marginal utility of consumption. Then we can write the first order condition for optimal price-setting, (51) as

$$\Phi_t \Xi = \Lambda_t \tag{B.4}$$

where new variables Ξ_t and Λ_t are defined by

$$\Xi_t - \xi\beta E_t[\tilde{\Pi}_{t+1}^{\zeta-1}\Xi_{t+1}] = Y_t MU_t^C \tag{B.5}$$

$$\Lambda_t - \xi\beta E_t[\tilde{\Pi}_{t+1}^{\zeta}\Lambda_{t+1}] = \frac{U_{L,t}\frac{W_t}{P_t}L_t MU_t^C}{(1-1/\zeta)(1-1/\eta)(1-T_t)} \tag{B.6}$$

(B.7)

From our definitions (B.2) and (B.3), (52) can now be written as

$$1 = \xi_p \tilde{\Pi}_t^{\zeta-1} + (1-\xi_p)\Phi_t^{1-\zeta} \tag{B.8}$$

Five equations (B.2) to (B.8) in Π_t, Φ_t, $\tilde{\Pi}_t$, Ξ_t and Ω_t now provide the dynamics of optimal setting in a convenient form

Similarly we can carry out the same exercise for wage setting. We can now use $\beta^k \Lambda_{t+k}(r) = D_{t+k}$, obtained from (38), and $\Lambda_t(r) = \frac{MU_t^C(r)}{P_t}$, and from (37) we have

$$L_{t+k}(r) = L_{t+k}\left(\frac{W_t^0(r)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_w}}{W_{t+k}}\right)^{-\eta} \tag{B.9}$$

to write (54) as

$$\left(\frac{W_t^0}{P_t}\right)^{1+\eta\phi} E_t \sum_{k=0}^{\infty}(\xi_w\beta)^k(1-T_{t+k})L_{t+k}MU_{t+k}^C\left(\frac{W_{t+k}}{P_{t+k}}\right)^{\eta}\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_w(1-\eta)}\left(\frac{P_t}{P_{t+k}}\right)^{1-\eta}$$
$$= \frac{\eta}{(\eta-1)}E_t\sum_{k=0}^{\infty}(\xi_w\beta)^k L_{t+k}^{1+\phi}\left(\frac{W_{t+k}}{P_{t+k}}\right)^{\eta(1+\phi)}\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{-\gamma_w\eta(1+\phi)}\left(\frac{P_{t+k}}{P_t}\right)^{\eta(1+\phi)} \tag{B.10}$$

[30] This is also necessary if one wants to set up and solve numerically in standard software the non-linear DSGE model.

We can now see that for labour supply habit in ratio form[31] we can proceed as for the price dynamics. The following difference equations corresponding to (B.4) to (B.8) now apply:

$$\left(\frac{W_t^0}{P_t}\right)^{1+\eta\phi}\Upsilon_t = \Gamma_t \tag{B.11}$$

$$\Upsilon_t - \xi_w\beta E_t[\tilde{\Pi}_{t+1}^{\eta-1}\Upsilon_{t+1}] = \left(\frac{W_t}{P_t}\right)^{\eta}(1-T_t)L_t MU_t^C \tag{B.12}$$

$$\Gamma_t - \xi_w\beta E_t[\tilde{\Pi}_{t+1}^{\eta(1+\phi)}\Gamma_{t+1}] = \left(\frac{W_t}{P_t}\right)^{\eta(1+\phi)}\frac{U_{L,t}L_t^{1+\phi}}{(1-1/\eta)} \tag{B.13}$$

$$\left(\frac{W_{t+1}}{P_{t+1}}\right)^{1-\eta} = \xi_w\left(\frac{W_t}{P_t}\right)^{1-\eta}\tilde{\Pi}_{t+1}^{\eta-1} + (1-\xi_w)\left(\frac{W_{t+1}^0}{P_{t+1}}\right)^{1-\eta} \tag{B.14}$$

C Welfare Quadratic Approximation for the Case of An Approximately Efficient Steady State

We denote by $L_t(r)$ the total labour supplied by household r and denote by $L_t(f)$ the index of differentiated labour employed by firm f. Defining $L_t(f,r)$ as the labour supplied to firm f by household r, we have

$$L_t(r) = \int L_t(r,f)df \qquad L_t(f)^{(\eta-1)/\eta} = \int L_t(r,f)^{(\eta-1)/\eta}dr \tag{C.1}$$

To clarify the exposition we first consider the case without capital.

C.1 Labour The Only Factor

Ignoring the welfare implications of monetary frictions, the utility of household r is given by

$$E_0\sum_{t=0}^{\infty}\beta^t\left[\frac{(C_t(r)-hC_{t-1})^{1-\sigma}}{1-\sigma} - \kappa\frac{L_t(r)^{1+\phi}}{1+\phi}\right] \tag{C.2}$$

Since we assume complete risk-sharing within each bloc, we may regard each consumer as being identical with every other. From the point of view of leisure, to obtain the social welfare function, we need to sum over all workers. Before doing this, we obtain the expected value of $L_t(r)^{1+\phi}$. We note that

$$\begin{aligned} L_t(r) &= \int L_t(r,f)df = \left(\frac{W_t(r)}{W_t}\right)^{-\eta}\int L_t(f)df \\ &= \left(\frac{W_t(r)}{W_t}\right)^{-\eta}\int\frac{Y_t(f)}{A_t}df = \left(\frac{W_t(r)}{W_t}\right)^{-\eta}\frac{Y_t}{A_t}\int\left(\frac{P_t(f)}{P_t}\right)^{-\zeta}df \end{aligned} \tag{C.3}$$

[31] This is reason we choose the ratio form over the difference form.

Assuming that $lnW_t(r) \sim N(\mu_t^W, D_t^W)$ and $lnP_t(r) \sim N(\mu_t^P, D_t^P)$, from subsection C.4 we have that

$$\int \left(\frac{P_t(f)}{P_t}\right)^{-\zeta} df \simeq 1 + \frac{1}{2}\zeta D_t^P \qquad \int \left(\frac{W_t(r)}{W_t}\right)^{-\eta} dr \simeq 1 + \frac{1}{2}\eta D_t^W \tag{C.4}$$

and in addition

$$\int \left(\frac{W_t(r)}{W_t}\right)^{-\eta(1+\phi)} dr \simeq 1 + \frac{1}{2}\eta(1+\phi)(1+\eta\phi)D_t^W \tag{C.5}$$

It follows from this that summing over all r, we obtain the social welfare loss approximately as

$$E_0 \sum_{t=0}^{\infty} \beta^t \Big[\frac{(C_t - hC_{t-1})^{1-\sigma}}{1-\sigma} - \kappa \frac{(Y_t/A_t)^{1+\phi}}{1+\phi} \Big(1 + \frac{1}{2}(1+\phi)(\zeta D_t^P + \eta(1+\eta\phi)D_t^W) \Big) \Big] \tag{C.6}$$

where

$$D_t^P = \xi_p D_{t-1}^P + \frac{\xi_p}{1-\xi_p}(\pi_t - \gamma_p \pi_{t-1})^2 \tag{C.7}$$

and

$$D_t^W = \xi_w D_{t-1}^W + \frac{\xi_w}{1-\xi_w}(\Delta w_t - \gamma_w \Delta w_{t-1})^2 = \xi_w D_{t-1}^W + \frac{\xi_w}{1-\xi_w}(\Delta wr_t + \pi_t - \gamma_w(\Delta wr_{t-1} + \pi_{t-1}))^2 \tag{C.8}$$

where for convenience we have written the *log* of the real wage relative to domestic producer prices $wr_t = w_t - p_t$, and π_t is the inflation rate for domestic producer prices.

C.2 Labour, Capital and Fixed costs F

With capital and fixed costs of production, the previous analysis changes to

$$Y_t(f) = A_t Z_t^\alpha K_{t-1}^\alpha L_t(f)^{1-\alpha} - F \qquad K_{t-1}(f) = \frac{1-\alpha}{\alpha}\frac{W_t L_t(f)}{P_t R_{K,t}} \qquad R_{K,t} = \Psi'(Z_t) \tag{C.9}$$

Hence we can write output per industry as

$$Y_t(f) = \left(\frac{\alpha}{1-\alpha}\right)^\alpha L_t(f) \left(\frac{W_t}{P_t}\right)^\alpha A_t Z_t^\alpha R_{K,t}^{-\alpha} - F \tag{C.10}$$

and labour supply of type r as

$$\begin{aligned} L_t(r) &= \int L_t(r,f)df = \left(\frac{W_t(r)}{W_t}\right)^{-\eta} \int L_t(f)df \\ &= \left(\frac{W_t(r)}{W_t}\right)^{-\eta} \int \frac{Y_t(f)+F}{A_t Z_t^\alpha} \left(\frac{1-\alpha}{\alpha}\right)^\alpha \left(\frac{P_t}{W_t}\right)^\alpha R_{K,t}^\alpha df \\ &= \left(\frac{W_t(r)}{W_t}\right)^{-\eta} \left(\frac{1-\alpha}{\alpha}\right)^\alpha \left(\frac{P_t}{W_t}\right)^\alpha \frac{R_{K,t}^\alpha}{A_t Z_t^\alpha}(F + Y_t \int \left(\frac{P_t(f)}{P_t}\right)^{-\zeta} df) \end{aligned} \tag{C.11}$$

and after defining $\frac{1}{B_t} = \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} \frac{R_{K,t}^{\alpha}}{A_t Z_t^{\alpha}}$ we deduce from this that

$$L_t = \int L_t(r)dr = \left(\frac{P_t}{W_t}\right)^{\alpha} \left(\frac{Y_t}{B_t}(1 + \frac{1}{2}(\eta D_t^W + \zeta D_t^P)) + \frac{F}{B_t}(1 + \frac{1}{2}\eta D_t^W)\right) \quad \text{(C.12)}$$

We also infer that when we sum over all individuals, we obtain

$$\int L_t(r)^{1+\phi}dr = \left(\frac{P_t}{W_t}\right)^{\alpha(1+\phi)} \left(\frac{1}{B_t}\right)^{1+\phi} \left(F + Y_t(1 + \frac{1}{2}\zeta D_t^P)\right)^{1+\phi} (1 + \frac{1}{2}\eta(1+\eta\phi)(1+\phi)D_t^W)$$

$$\cong \left(\frac{P_t}{W_t}\right)^{\alpha(1+\phi)} \left(\frac{F+Y_t}{B_t}\right)^{1+\phi} \left(1 + \frac{Y_t}{F+Y_t}\frac{1}{2}\zeta(1+\phi)D_t^P\right)(1 + \frac{1}{2}\eta(1+\eta\phi)(1+\phi)D_t^W) \quad \text{(C.13)}$$

Note that

$$\left(\frac{P_t}{W_t}\right)^{\alpha} \left(\frac{F+Y_t}{B_t}\right) = \left(\frac{P_t K_{t-1} R_{K,t}}{W_t L_t}\right)^{\alpha} L_t \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} = L_t \quad \text{(C.14)}$$

which is obtained by substituting for B_t and Y_t from (C.9), and then using the second minimum cost condition in the same equation. it follows that the utility function can be approximately written as

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t - hC_{t-1})^{1-\sigma}}{1-\sigma} - \kappa\frac{L_t^{1+\phi}}{1+\phi}\left(1 + \right.\right.$$
$$\left.\left.\frac{1}{2}(1+\phi)(\frac{Y_t}{F+Y_t}\zeta D_t^P + \eta(1+\eta\phi)D_t^W)\right)\right] \quad \text{(C.15)}$$

We shall expand this about the efficient steady state level described earlier, using the resource constraints in Z_t, L_t, K_{t-1}. We use proportional deviations for all variables, denoted by the corresponding lower-case letters, so that for example $c_t = \frac{C_t - C}{C}$.

Result: The first order terms in this expansion are zero.

Proof: The latter are given by

$$dU = E_0 \sum_{t=0}^{\infty} \beta^t \left[C^{1-\sigma}(1-h)^{-\sigma}(c_t - hc_{t-1}) - \kappa N^{1+\phi} l_t\right]$$
$$= E_0 \sum_{t=0}^{\infty} \beta^t \left[C^{1-\sigma}(1-h)^{-\sigma}(1-\beta h)c_t - \kappa N^{1+\phi} l_t\right] \quad \text{(C.16)}$$

But aggregate consumption is given by

$$C_t = A_t Z_t^{\alpha} L_t^{1-\alpha} K_{t-1}^{\alpha} - F - G_t - I_t(1 - S(I_t/I_{t-1})) - \Psi(Z_t)K_{t-1} \quad \text{(C.17)}$$

which is a consequence of the fact that $K_{t-1}(f)/L_t(f)$ is the same for all firms. From this we can calculate c_t, so recalling that $\Psi(1) = 0$, and ignoring for the moment the second-order deviations in c_t, the first-order deviations of the resource constraints in utility (C.16) become

$$dU = E_0 \sum_{t=0}^{\infty} \beta^t \left[C^{-\sigma}(1-h)^{-\sigma}(1-\beta h)\left(\alpha(Y+F)z_t + (1-\alpha)(Y+F)l_t\right.\right.$$
$$\left.\left. + (\alpha(Y+F) - \frac{1}{\beta} + 1 - \delta)Kk_{t-1} - \Psi'(1)Kz_t\right) - \kappa N^{1+\phi} l_t\right] \quad \text{(C.18)}$$

The terms in $z_t,\ l_t,\ k_{t-1}$ are all zero as a consequence of first-order conditions (77) to (80).

It follows that the second-order terms in the Taylor-series approximation of the welfare loss is given by the sum of two expressions: the second-order terms in $c_t,\ l_t$ from (C.15), and the second-order terms in $z_t,\ l_t,\ k_{t-1}$ from the expansion of c_t in (C.16). The former expression is given by

$$
\begin{aligned}
&-\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t[C^{1-\sigma}(1-h)^{-1-\sigma}\sigma(c_t-hc_{t-1})^2\\
&+\kappa L^{1+\phi}\left(\phi l_t^2+\frac{Y}{F+Y}\frac{\zeta\xi_p}{(1-\beta\xi_p)(1-\xi_p)}(\pi_t-\gamma_p\pi_{t-1})^2\right.\\
+&\left.\frac{\eta(1+\eta\phi)\xi_w}{(1-\beta\xi_w)(1-\xi_w)}(\Delta wr_t+\pi_t-\gamma_w(\Delta wr_{t-1}+\pi_{t-1}))^2\right)]
\end{aligned}
\tag{C.19}
$$

where all the steady state values C, L, Y correspond to their *efficient* values. Note that we can replace $\kappa L^{1+\phi}$ in this expression by $C^{1-\sigma}(1-h)^{-\sigma}(1-1/\eta)(1-\alpha)/c_y$ where $c_y=\frac{C}{Y+F}$, which holds for the zero-inflation decentralized equilibrium. Thus (C.19) may be rewritten as

$$
\begin{aligned}
&-\tfrac{C^{1-\sigma}(1-h)^{-\sigma}}{2}E_0\textstyle\sum_{t=0}^{\infty}\beta^t[(1-h)^{-1}\sigma(c_t-hc_{t-1})^2\\
&+\tfrac{(1-1/\eta)(1-\alpha)}{c_y}\left(\phi l_t^2+\tfrac{Y}{F+Y}\tfrac{\zeta\xi_p}{(1-\beta\xi_p)(1-\xi_p)}(\pi_t-\gamma_p\pi_{t-1})^2\right.\\
&\left.+\tfrac{\eta(1+\eta\phi)\xi_w}{(1-\beta\xi_w)(1-\xi_w)}(\Delta wr_t+\pi_t-\gamma_w(\Delta wr_{t-1}+\pi_{t-1}))^2\right)]
\end{aligned}
\tag{C.20}
$$

Finally the second-order terms which arise from the resource constraint (C.17) are given by

$$
\begin{aligned}
&-\tfrac{1-\beta h}{2}C^{-\sigma}(1-h)^{-\sigma}E_0\textstyle\sum_{t=0}^{\infty}\beta^t\Big[(Y+F)\Big(\alpha(1-\alpha)(l_t-k_{t-1})^2-2[(1-\alpha)l_t+\alpha k_{t-1}](a_t+\alpha z_t)\\
&+\alpha(1-\alpha)z_t^2\Big)+K(\Psi''(1)z_t^2+2\Psi'(1)z_tk_{t-1})-S''(1)(i_t-i_{t-1})^2\Big]
\end{aligned}
$$

Using the definition $\psi=\Psi'(1)/\Psi''(1)$, and the deterministic equilibrium conditions $\Psi'(1)=R_K$ and $\alpha(Y+F)=R_KK$, this may be rewritten as

$$
\begin{aligned}
&-\tfrac{1-\beta h}{2c_y}C^{1-\sigma}(1-h)^{-\sigma}E_0\textstyle\sum_{t=0}^{\infty}\beta^t\Big[\alpha(1-\alpha)(l_t-k_{t-1}-z_t-\tfrac{1}{1-\alpha}a_t)^2\\
&+\tfrac{\alpha}{\psi}\Big(z_t+\psi a_t\Big)^2-2a_tl_t-\tfrac{\delta K}{Y+F}S''(1)(i_t-i_{t-1})^2\Big]
\end{aligned}
\tag{C.21}
$$

To summarise, the quadratic form of the welfare is given by the sum of (C.20) and (C.21).

A number of points are worthy of note:

1. Ignoring exogenous terms independent of policy, the quadratic approximation to the utility is negative definite only if $S''(1) < 0$. In our assumption and estimate for this parameter is positive, so the utility is not completely positive definite.

2. There are of course no second order contributions from first order changes in L_t.

3. When there is no capital stock, habit, wage-stickiness and government spending we end up with the loss function in Woodford (2003):

$$\Omega_0 = E_0 \left[\frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left[(y_t - \hat{y}_t)^2 + w_\pi (\pi_t - \gamma_p \pi_{t-1})^2 \right] \right] \tag{C.22}$$

   where $\hat{y}_t = \frac{1+\phi}{\sigma+\phi} a_t$ is potential output achieved when prices are flexible and

$$w_\pi = \frac{\zeta \xi}{(1-\xi)(1-\beta\xi)(\sigma+\phi)} \tag{C.23}$$

4. To work out the welfare in terms of a consumption equivalent percentage increase, expanding $U(C) = \frac{C^{1-\sigma}(1-h)^{1-\sigma}}{1-\sigma}$ as a Taylor series, a 1% permanent increase in consumption of 1 per cent yields a first-order welfare increase $(1-h)^{1-\sigma} C^{-\sigma} \Delta C = (1-h_c) C^{1-\sigma} (1-h)^{-\sigma} \times 0.01$. Since standard deviations are expressed in terms of percentages, the welfare loss terms which are proportional to the covariance matrix (and pre-multiplied by 1/2) are of order 10^{-4}. Letting X be these losses reported in the paper. Then $c_e = \frac{X}{(1-h)} \times 0.01$ as given in (90). The expressions in (91) are derived using only the quadratic terms.

C.3 Derivation of (C.4) and (C.5)

It is convenient though not essential to assume a normal distribution with $lnW_t(r) \sim N(\mu, \sigma^2)$. By definition,

$$W_t^{1-\eta} = \int W_t(r)^{1-\eta} dr = exp((1-\eta)\mu + (1-\eta)^2 \frac{1}{2} \sigma^2) \tag{C.24}$$

Hence

$$W_t = exp(\mu + (1-\eta) \frac{1}{2} \sigma^2) \tag{C.25}$$

Thus it follows that

$$\int W_t(r)^{-\eta} di = exp(-\eta\mu + \eta^2 \frac{1}{2} \sigma^2) \qquad W_t^{-\eta} = exp(-\eta\mu - \eta(1-\eta) \frac{1}{2} \sigma^2) \tag{C.26}$$

from which we obtain (C.4). Similarly

$$\int W_t(r)^{-\eta(1+\phi)} dr = exp(-\eta(1+\phi)\mu + \eta^2 (1+\phi)^2 \frac{1}{2} \sigma^2) \tag{C.27}$$

$$W_t^{-\eta(1+\phi)} = exp(-\eta(1+\phi)\mu - \eta(1+\phi)(1-\eta) \frac{1}{2} \sigma^2) \tag{C.28}$$

and hence (C.5).

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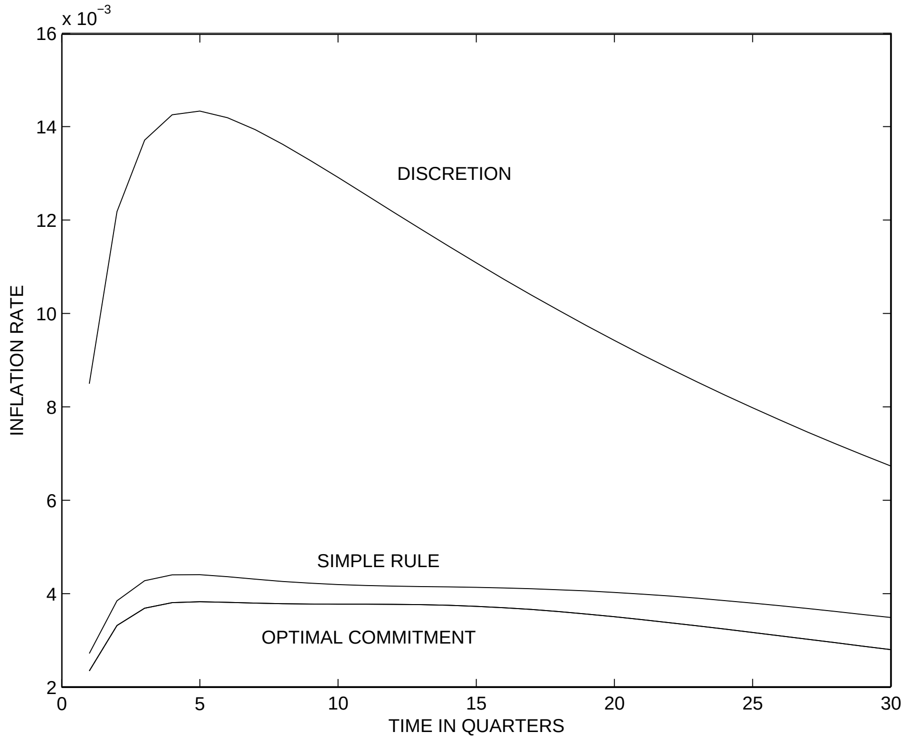

Figure 5: Price Inflation Rate Following a 1% Government Spending Shock

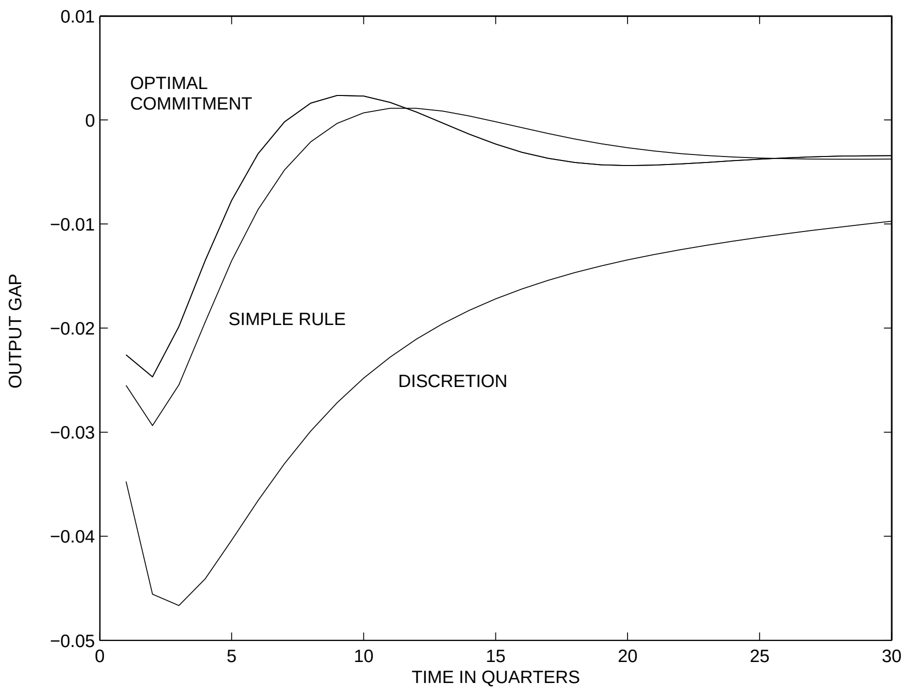

Figure 6: Output Gap Following a 1% Government Spending Shock

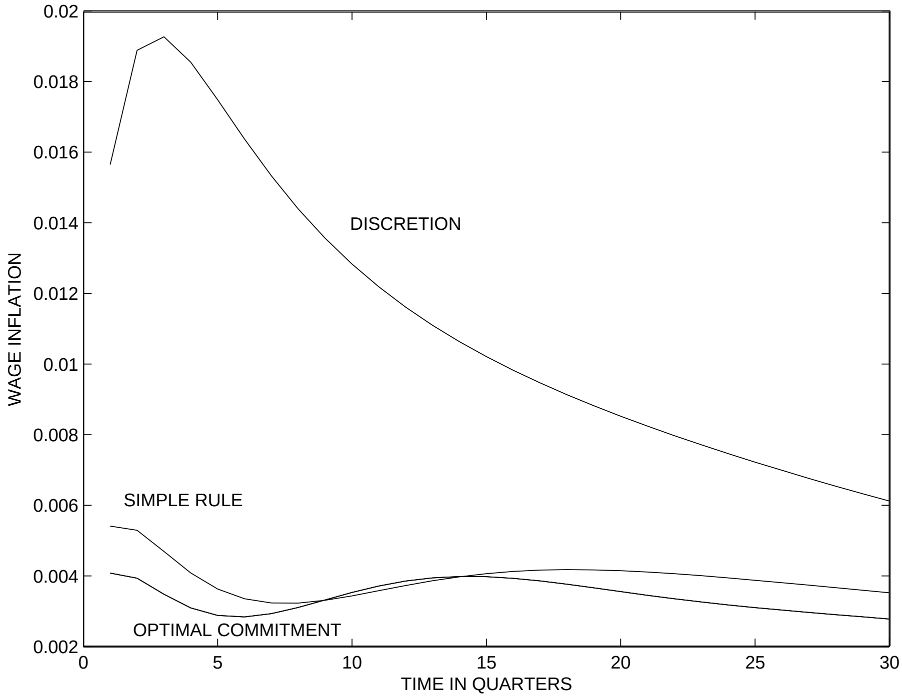

Figure 7: Wage Inflation Rate Following a 1% Government Spending Shock

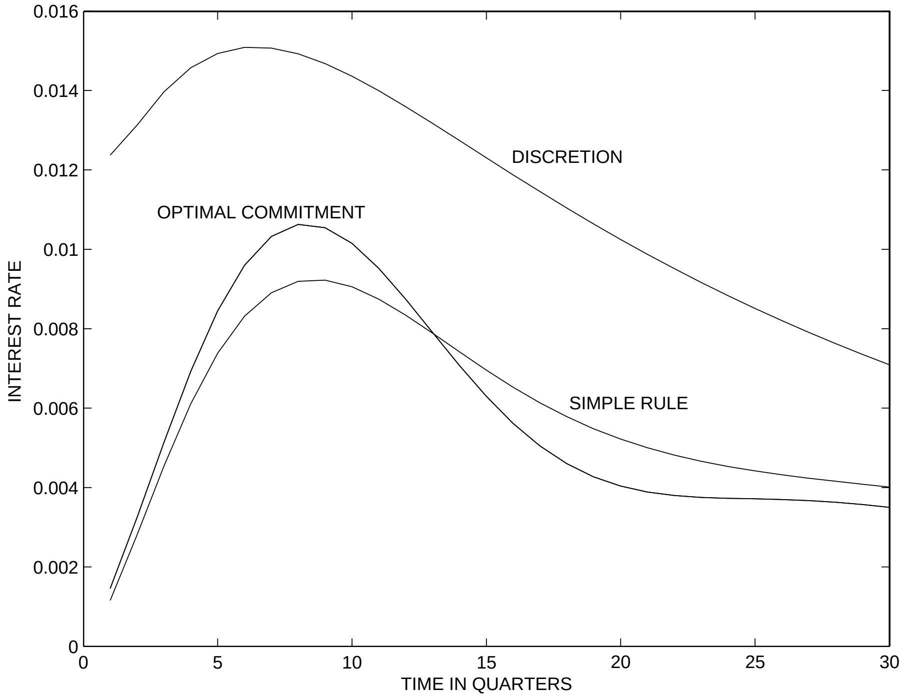

Figure 8: Interest Rate Following a 1% Government Spending Shock

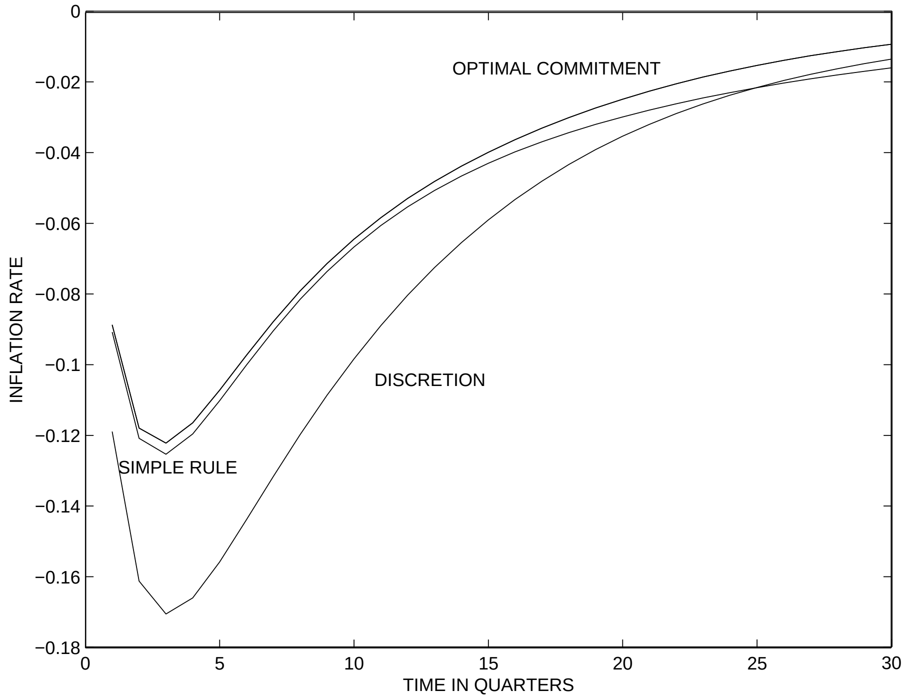

Figure 9: Price Inflation Rate Following a 1% Technology Shock

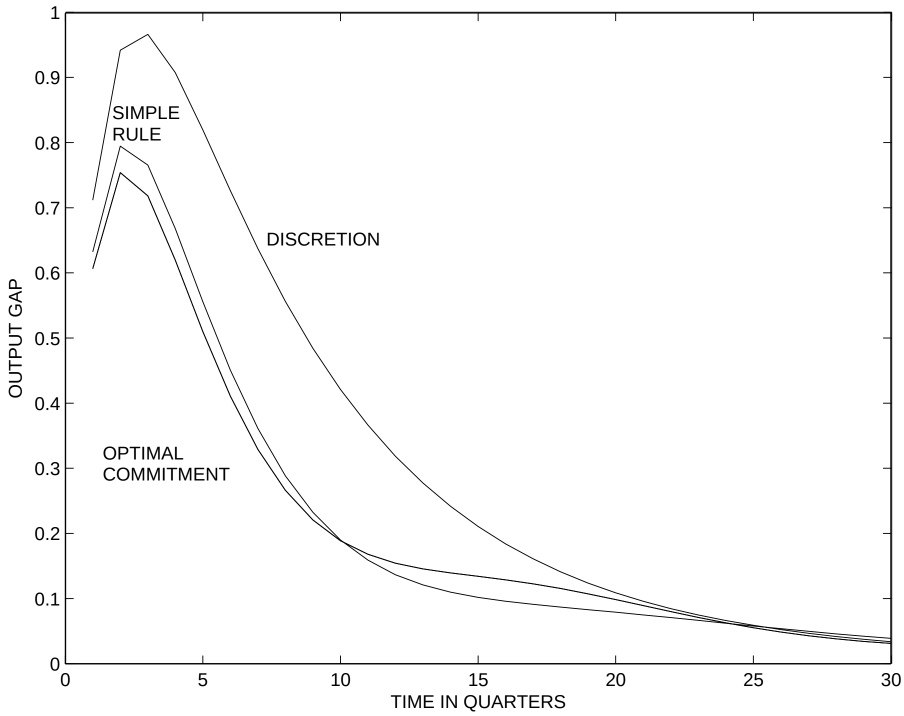

Figure 10: Output Gap Following a 1% Technology Shock

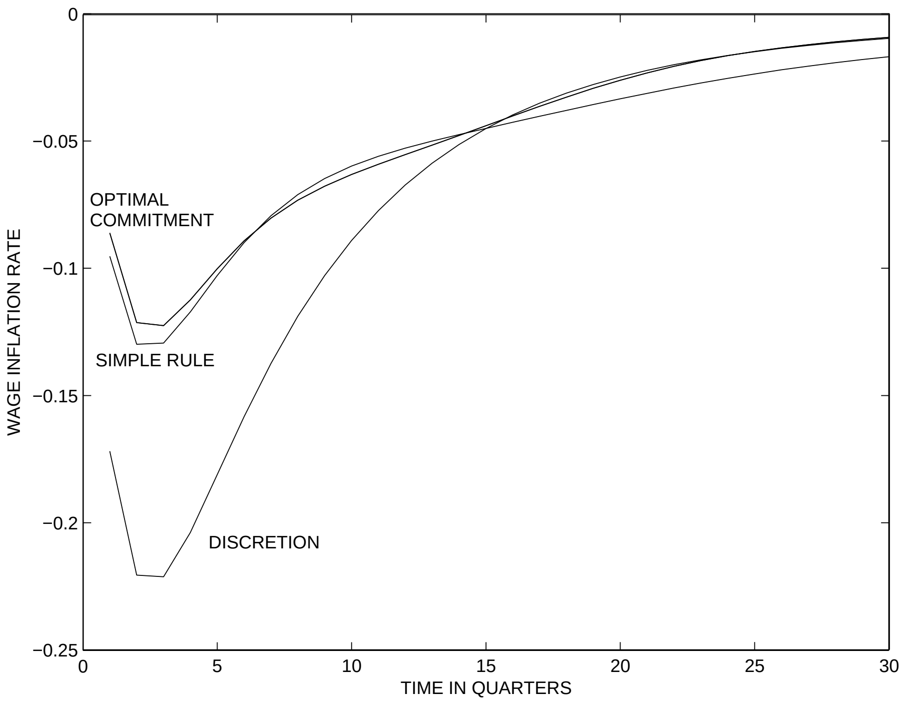

Figure 11: Wage Inflation Rate Following a 1% Technology Shock

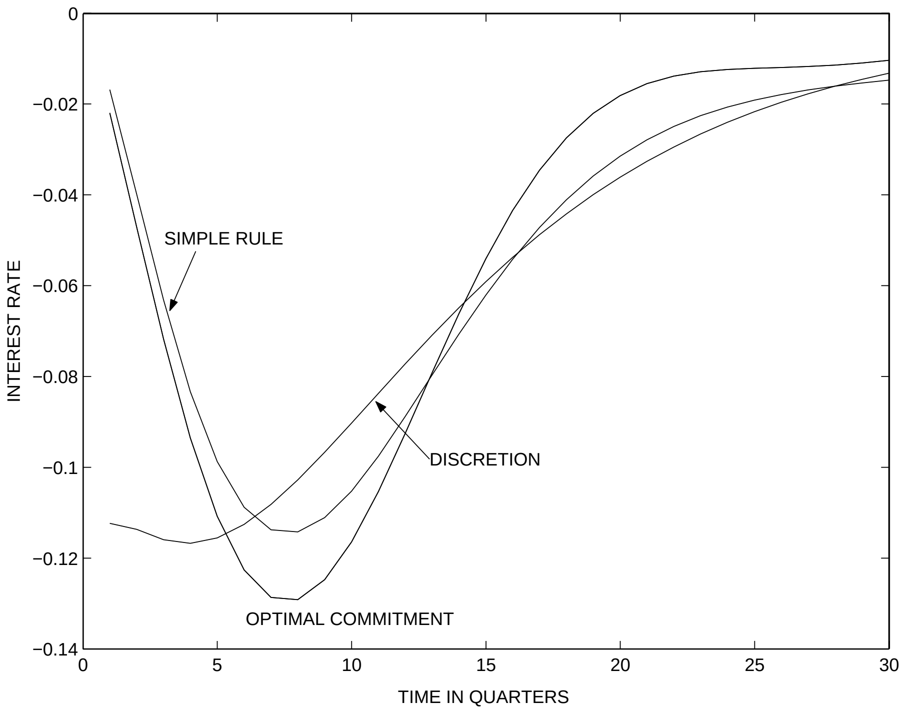

Figure 12: Interest Rate Following a 1% Technology Shock

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