# WIDE SENSE ONE-DEPENDENT PROCESSES WITH EMBEDDED HARRIS CHAINS AND THEIR APPLICATIONS IN INVENTORY MANAGEMENT

EMŐKE BÁZSA* AND PETER DEN ISEGER†

*Econometric Institute Report EI 2002-44*

Abstract. In this paper we consider stochastic processes with an embedded Harris chain. The embedded Harris chain describes the dependence structure of the stochastic process. That is, all the relevant information of the past is contained in the state of the embedded Harris chain. For these processes we proved a powerful reward theorem. Further, we show how we can control these type of processes and give a formulation similar to semi-Markov decision processes. Finally we discuss a number of applications in inventory management.

## 1. Introduction

Although very general, the analysis of single item inventory system, as it was done in Bázsa and den Iseger [4], can be extended substantially. The first point to tackle would be incorporating nonstationary policies. The second is to consider more general demand processes, or even more general systems. For example, demand could depend on an exogenous factor (cf. [20], [18]) or the dependence could be of endogenous nature (cf. [14], [11]). In fact, such a system leads to a solution method for networks, since there the primal difficulty is to deal with the various inter-dependence of the components of the networks. In the present paper we only give examples of single-item single-echelon systems, with more complicated dependence structures. Bázsa and den Iseger [5]depicts a two echelon decentralized system, which can be solved with the theory developed in this paper, for more general demand processes than before (cf. [3]).However, the power of the applicability of the model really shows in even more complexer networks (e.g. queuing networks), which is the topic of future research.

*Date*: December 10, 2002.

*Erasmus University Rotterdam, Econometric Institute, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands, e-mail: bazsa@few.eur.nl.

†Erasmus University Rotterdam, Econometric Institute, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands, e-mail: iseger@few.eur.nl.

Under the simultaneous influence of the works of Song and Zipkin (cf. [20]) and Glynn and Sigman (cf. [11]), we aimed first to develop a more powerful reward theorem for synchronous processes with an embedded Markov chain. Song and Zipkin (cf. [20]) considered inventory models, in which the demand process is described by a Markov modulated Poisson process, that is, the arrival process of customers is driven by an exogenous state-of-the-world variable: when the world is in state $i$, demand follows a Poisson process with rate $\lambda_i$. Glynn and Sigman (cf. [11]), on the other hand, develop a reward theorem for synchronous processes: a process with identically distributed but dependent cycles.

However, the applicability of synchronous processes in inventory management, for instance, is quite restricted. In order to illustrate this, let us consider a basic inventory control model. Let **IP** denote the inventory position process, **IN** the net inventory process, and **D** the demand process (for the precise definitions see for instance [4]). Now, it is rather easy and not excessively restrictive to make the necessary assumptions and show that **IP** and **D** are synchronous processes. However, the 'synchronousness' property in general does not preserve under measurable functions of more than one variable, hence it is very difficult to prove that the netstock process **IN**, needed for the actual reward theorem, is also synchronous.

Our approach is to consider general stochastic processes for which all the relevant information about the past can be described by a Harris chain. This means that there exists an embedded Harris chain (its points being a subset of the arrival points of the stochastic process). In this setup, the stochastic process inherits some very advantageous properties from the embedded Harris chain: it can be shown that it has a general regenerative property, which is less restrictive than the wide-sense regeneration property (cf. [2]). Now, this general regeneration does preserve under measurable functions of more than one variable. Hence in the above simple inventory example we can make use of the flow conservation law $\mathbf{IN}(t+L) = \mathbf{IP}(t) - \mathbf{D}(t, t+L]$, yielding the crucial (general) regenerative property needed for a powerful reward theorem. Moreover, the average cost expression we deduced is in terms of the simple 'arrival cycles', instead of the regeneration cycles. In general the regeneration cycles for classical regenerative processes do not coincide with the simple arrival cycles, consequently being considerably more difficult to determine. As it turns out, many relevant operations research problems can be modeled with this technique. We show how we can control these type of processes and give a formulation similar to semi-Markov decision

processes. With this technique a nonstationary optimal policy can be obtained, for instance when demand is nonstationary.

This paper is organized in the following way. In section 2 we discuss the concept of synchronous processes, Harris chains and Harris processes. We discuss in section 3 almost surely convergence and a reward theorem for synchronous processes. In section 5 we introduce the main topic of this paper, general stochastic processes, with an embedded Harris chain. We give a powerful reward theorem for these processes. In section 6 we discuss how we can control these type of processes and give a formulation similar to semi-Markov decision processes, while Section 7 provides examples of relevant models from inventory theory, which can be solved with the new technique. The Appendix (sections A up to B.4) provide the background and essential properties needed for the limit theorems.

## 2. Two specific stochastic processes

In order to be able to appreciate the difference between the earlier work and the present approach we pursue to briefly introduce synchronous processes and Harris chains. In the remaining part of this paper we assume that every stochastic process is shift measurable (cf. B.1).

2.1. **Synchronous processes.** Consider now a stochastic process $\mathbf{X}$, defined on $(\Omega, \mathcal{F}, I\!P)$, with Polish state space $(E, \mathcal{E})$, and a Polish path space $(H, \mathcal{H})$ of right continuous maps from $[0, +\infty)$ to $E$ with left limits. Let us define now a so called cemetery state $\Delta$ external to $E$, and $\mathcal{E}$ is now endowed with the one-point compactification topology (cf. [11]).

**Definition 2.1.** [1] *The stochastic process* $\mathbf{X}$ *is said to be a synchronous process with respect to the random sequence* $0 \leq \mathbf{t}_0 < \mathbf{t}_1 < \ldots$, *if* $\{\mathbf{X}_n : n \geq 1\}$ *forms a stationary sequence in* $(H, \mathcal{H})$, *where*

$$\mathbf{X}_n(t) := \begin{cases} \mathbf{X}(\mathbf{t}_{n-1} + t), & \text{if} \quad 0 \leq t < \mathbf{T}_n, \\ \Delta & \text{if} \quad t \geq \mathbf{T}_n. \end{cases} \tag{2.1}$$

Let $I\!P^0$ denote the probability measure under which $\mathbf{X}$ is non-delayed, that is, $I\!P^0\{\mathbf{X} \in B\} = I\!P\{\phi_{\mathbf{t}_0} \circ \mathbf{X} \in B\}$ (or $\mathbf{t}_0 = 0$). We refer to $\mathbf{t}_n$ as the *synch-times* for $\mathbf{X}$.

[1] The definitions and basic properties of (related to) synchronous processes were taken from the article of Glynn and Sigman (cf. [11])

**Definition 2.2.** *The process* $\mathbf{X}$ *is called positive recurrent if* $I\!E(\mathbf{T}_1) < \infty$*, null recurrent otherwise. Moreover,* $\mathbf{X}$ *is called ergodic if it is positive recurrent and the invariant* $\sigma$*-field,* $\mathcal{I}$*, defined by relation (B.1) of* $\{\mathbf{T}_n, \mathbf{X}_n\}$ *is trivial. Furthermore,* $\lambda := 1/I\!E(\mathbf{T}_1)$ *is called the rate of the synch-times, while* $\hat{\lambda} := 1/I\!E(\mathbf{T}_1|\mathcal{I})$ *is called the conditional rate.*

Although synchronous processes start over probabilistically at the synch-times, the future is not necessarily independent of the past, in contrast with regenerative processes. Therefore, the synch-times do not form a renewal process. Due to this dependence, one needs to establish extra conditions in order to be able to construct limit theorems, similar to those known for classical regenerative processes (cf.e.g. [16]).

2.2. **Harris chains and Harris processes.** A discrete or continuous time stochastic process $\mathbf{X}$, with state space $(E, \mathcal{E})$ and general path space $(H, \mathcal{H})$ is a Markov process (cf. [21]) if the future depends on the past only through the present. The Markov process is time–homogeneous if the conditional distribution of $\phi_t\mathbf{X}$ given the value of $\mathbf{X}_t$ does not depend on $t$. For time–homogeneous Markov processes, the transition kernel $P^t$, $t \in [0, \infty)$ is defined by

$$P^t(x, A) := I\!P\{\mathbf{X}_{s+t} \in A | \mathbf{X}_s = x\}, \quad x \in E, A \in \mathcal{E}, s \in [0, \infty). \tag{2.2}$$

Now we define Harris chains (discrete time) and Harris processes (continuous time) consecutively (cf. [21]).

**Definition 2.3. (Regeneration sets)** *A discrete- time Markov process* $\mathbf{X} = (\mathbf{X}_k)_0^\infty$*, with state space* $(E, \mathcal{E})$ *and one-step transition probabilities* $P$*, is a Harris chain if it has a regeneration set, that is if there is a set* $A \in \mathcal{E}$ *such that the hitting time of the set* $A$*,*

$$\tau_A := \inf\{n \in I\!N \cup \{0\} : \mathbf{X}_n \in A\}$$

*is finite with probability one for all initial distributions, and there is an* $l > 0$*, a* $p \in (0, 1]$*, and a probability measure* $\mu$ *on* $(E, \mathcal{E})$ *with*

$$I\!P\{\mathbf{X}_l \in \cdot | \mathbf{X}_0 = x\} = P^l(x, \cdot) \geq p\,\mu(\cdot), \quad x \in A. \tag{2.3}$$

**Remark 2.4.** *If* $E$ *is finite or countable, and* $\mathbf{X}$ *is irreducible and recurrent then* $\mathbf{X}$ *is a Harris chain.*

Now, in order to extend regeneration to continuous time we need the strong Markov property: a shift measurable Markov process $\mathbf{X} = (\mathbf{X}_s)_{s\in[0,\infty)}$ with semigroup of transition probabilities $P^s, 0 \le s < \infty$ is a strong Markov process (cf. [21]) if the Markov property holds at all stopping times $\tau$, that is,

$$\phi_\tau \mathbf{X} \text{ depends on } (\mathbf{X}_s)_{s\in[0,\tau]} \text{ only through } \mathbf{X}_\tau \text{ and}$$
$$I\!P\{\phi_\tau \mathbf{X} \in \cdot | \mathbf{X}_\tau = x\} = I\!P\{\mathbf{X} \in \cdot | \mathbf{X}_0 = x\}, \;\; x \in E.$$

Now the definition of a continuous time Harris process stays the same as for Harris chains, in the sense that it is defined to be a strong Markov process which has a regeneration set with property (2.3). However the definition of a regeneration set for the continuous time case is different (cf. [21]):

**Definition 2.5.** *A set $A \in \mathcal{E}$ is a regeneration set for the strong Markov process $\mathbf{X}$, if the hitting time $\tau_A$ is measurable and finite with probability one for all initial distributions, such that $\mathbf{X}_{\tau_A} \in A$, and if there is an $l > 0$, a $p \in (0, 1]$, and a probability measure $\mu$ on $(E, \mathcal{E})$ such that (2.3) holds.*

Intuitively, (2.3) means, that whenever $\mathbf{X}$ enters $A$ it lag–$l$ regenerates $l$ time units later with probability $p$. Thorisson (cf. [21]) proves that with a so called conditional splitting one can construct an increasing sequence $\mathbf{S} = (\mathbf{S}_n)_0^\infty$ such that $(\mathbf{X}, \mathbf{S})$ is lag–$l$ regenerative, and the distribution of its zero-delayed version (under $I\!P^0$) $(\mathbf{X}^0, \mathbf{S}^0)$ does not depend on the initial distribution of $\mathbf{X}$.

There is also an other, equivalent definition for Harris processes, used often in the literature (see for example Sigman(1992)):

**Definition 2.6. ($\varphi$-recurrence)** *A Markov process $\mathbf{X}$, with Polish state space $(E, \mathcal{E})$, satisfying the strong Markov property, is called Harris recurrent if there exists a non-trivial $\sigma$-finite measure $\varphi$ on $(E, \mathcal{E})$, such that for any $B \in \mathcal{E}$, with $\varphi(B) > 0$, the total time spent by $\mathbf{X}$ in the set $B$ is infinite with probability one for all initial distributions, that is,*

$$I\!P_z\left(\int_0^\infty 1_B \circ \mathbf{X}(t)dt = \infty\right) = 1 \text{ for all } z \text{ starting states.}$$

In the discrete case it can be shown (cf. [15]) that $\varphi$–recurrence for some $\varphi$, and the existence of regeneration sets are equivalent properties. However, in continuous time this relation has

not been proven yet. We conclude this section with some of the properties of Harris processes which are important for our limit theorems.

(i) A Harris chain is aperiodic if the inter-regeneration times are aperiodic; and this holds independently of the choice of the regeneration set and of $l$ and $p$ at (2.3).

(ii) Glynn showed (cf. [10]) that if $\mathbf{X}$ has a stationary distribution , then $\mathbf{X}$ is a Harris process if and only if for all initial distributions and all $A \in \mathcal{H}$,

$$I\!P\{\phi_{\mathbf{U}t}\mathbf{X} \in A\} \to I\!P\{\mathbf{X}^* \in A\}, \text{ as } t \to \infty,$$

where $\mathbf{X}^*$ is a stationary version of $\mathbf{X}$ and $\mathbf{U}$ is uniform on $[0,1]$.

(iii) Sigman (cf. [18]) proves that Harris processes are one-dependent regenerative, possessing a unique invariant $\sigma$–finite measure $\mu$. Conversely, if a Markov process is positive recurrent one-dependent regenerative, then it is a positive recurrent Harris process.

Now we are ready to proceed with the limit results.

## 3. Limit theorems

3.1. **Birkhoff's Ergodic Theorem.** In order to state the already known limit results we make use of Birkhoff's ergodic theorem, which can be found in the book of Billigsley (cf. [7]). The following concepts are strongly related to the one of invariant $\sigma$–fileds, dealt with in subsection B.3. A mapping $T : \Omega \to \Omega$, with $(\Omega, \mathcal{F}, P)$ the underlying probability space, is called a measure-preserving transformation if it is measurable $\mathcal{F}/\mathcal{F}$ and $P(T^{-1}A) = P(A)$ for all $A$ in $\mathcal{F}$ (in the definition of the invariant $\sigma$–field in Section B.3, the measure-preserving transformation considered is the shift–map $\phi_1 = T$ in discrete time). The set $A \in \mathcal{F}$ is invariant under $T$ if $T^{-1}A = A$, and it is nontrivial invariant if $0 < P(A) < 1$. Likewise for stochastic processes, the transformation $T$ is called *ergodic* if there are no nontrivial invariant sets in $\mathcal{F}$. A measurable function is *invariant* if $f(Tw) = f(w)$ for all $w \in \Omega$; $A$ is invariant if and only if $1_A$ is.

**Theorem 3.1.** *(Birkhoff's ergodic theorem) Let $T$ be a measure-preserving transformation on the triplet $(\Omega, \mathcal{F}, I\!P)$, and $g$ a measurable and integrable function. Then*

$$\lim_{n\uparrow\infty} \frac{1}{n} \sum_{k=1}^{n} g(T^{k-1}w) = I\!E(g|\mathcal{I}) \quad a.s., \tag{3.1}$$

*where $\mathcal{I}$ is the invariant $\sigma$-field related to $g$. If $T$ is ergodic then $I\!E(g|\mathcal{I}) = I\!E(g)$ a.s.*

This theorem leads us to the first limit result. Consider a synchronous process $\mathbf{X}$, a measurable cost function $f$, and denote the cost of a cycle, related to the process $\mathbf{X}$, as

$$\mathbf{J}_n = \mathbf{J}_n(f) := \int_{\mathbf{t}_{n-1}}^{\mathbf{t}_n} f(\phi_t \circ \mathbf{X})dt. \tag{3.2}$$

**Corollary 3.2.** *Consider $g := \mathbf{J}_1(f)$, if $I\!E(\mathbf{T}_n|\mathcal{I}) > 0$, and if $\mathbf{X}$ is positive recurrent such that $I\!E(\mathbf{J}_1(|f|)) < \infty$ and $\int_0^{\mathbf{t}_0} f(\phi_s \circ \mathbf{X}ds < \infty$ a.s. then*

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t f(\phi_s \circ \mathbf{X})ds = \frac{I\!E(\mathbf{J}_1|\mathcal{I}_J)}{I\!E(\mathbf{T}_1|\mathcal{I})} \quad a.s. \tag{3.3}$$

*If in addition $\{\mathbf{T}_n\}$ is ergodic, then*

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t f(\phi_s \circ \mathbf{X})ds = \frac{I\!E(\mathbf{J}_1|\mathcal{I}_J)}{I\!E(\mathbf{T}_1)}. \tag{3.4}$$

See also Theorem A1 of Glynn and Sigman (cf. [11]). Using the notation

$$\pi(f) := I\!E\left(\frac{I\!E(\mathbf{J}_1|\mathcal{I}_J)}{I\!E(\mathbf{T}_1|\mathcal{I})}\right), \tag{3.5}$$

$\pi$ defines a measure on $(H, \mathcal{H})$, which is called the *stationary probability measure for* $\mathbf{X}$: under $\pi$, the shift $\phi = (\phi_s)$ is measure preserving on $(H, \mathcal{H})$. In particular, if $\mathbf{X}$ has distribution $\pi$, then $\mathbf{X}$ is time stationary (cf. [11]). Observe that if $\{\mathbf{T}_n\}$ is ergodic then, by Corollary 3.2, $\pi(f) = \lambda I\!E(\mathbf{J}_1)$. The question is now: under what kind of conditions can one obtain convergence in $\mathcal{L}^1$ for the limit (3.3)?

## 4. The results of Glynn and Sigman for synchronous processes

Define now the functional

$$\mu_t(f) := \frac{1}{t}\int_0^t I\!E f(\phi_s \circ \mathbf{X})ds. \tag{4.1}$$

A necessary and sufficient condition for $\mathcal{L}^1$ convergence, $\mu_t(f) \to \pi(f)$, has been given in subsection A.2. The results established there yield that

$$\mu_t(f) \to \pi(f) \iff \{\frac{1}{t}\int_0^t f(\phi_s \circ \mathbf{X})ds : t \geq 0\} \text{ U.I.} \tag{4.2}$$

In particular, the Cesaro averaged distributions converge weakly. Glynn and Sigman (cf. [11]) established additional conditions under which $\mu_t(f)$ converges uniformly (over a class of functions) to $\pi(f)$. Their main result reads as follows:

**Proposition 4.1.** *If* $\mathbf{X}$ *is a positive recurrent synchronous process,* $g \in \mathcal{L}_+^1(\pi)$ *is such that* $(1/t)I\!E \int_0^{t\wedge \mathbf{t}_0} g(\phi_s \circ \mathbf{X})ds \to 0$ *(where* $\mathbf{t}_0$ *stands for the delay) and*

$$\left\{ \frac{1}{t} \int_0^t g(\phi_s \circ \mathbf{X})ds \,:\, t \geq 0 \right\} \text{ is uniformly integrable} \tag{4.3}$$

*under the non-delay distribution, then*

$$\sup_{|f|\leq g} |\mu_t(f) - \pi(f)| \to 0. \tag{4.4}$$

*In particular, if either there exists an* $\varepsilon > 0$ *such that* $I\!P^0\{\mathbf{T}_n > \varepsilon\} = 1$*, or the cycles* $\mathbf{X}_n$ *form a k-dependent process, then (4.3) holds for all* $g \in L_1^+(\pi)$ *which satisfy* $I\!E \int_0^{\mathbf{t}_0} g(\phi_s \circ \mathbf{X})ds < \infty$.

In accordance with relation (4.2), uniform integrability, that is, condition (4.3) is necessary and sufficient; in particular k-dependence implies uniform integrability (cf. Proposition 3.1, [11])).

## 5. General stochastic processes with an embedded Harris chain

The following section depicts a limit theorem for shift measurable stochastic processes $\mathbf{X}$, with state space $(D, \mathcal{D})$. The stochastic process is general in the sense that any kind of dependence structure of the underlying point process $(\mathbf{t}_n)_0^\infty$ (arrivals) is allowed, as long as this dependence structure can be modeled through a Harris chain. This means, that we define an embedded Harris chain at the points $\mathbf{t}_n$, such that $\mathbf{A}_n := \mathbf{A}(\mathbf{t}_n)$ (with state space $(E, \mathcal{E})$ and path space $(H, \mathcal{H})$), such that it 'takes care' of the dependence structure of $\mathbf{X}$, that is, $\mathbf{X}(\mathbf{t}_n)$ is conditionally independent of the past given $\mathbf{A}_n$. Thus, we consider mathematical models, where such an embedded Harris chain can be constructed. As it is illustrated later, a large class of models in operations research can be covered by this construction. Moreover, the limit theorem derived for these types of models has the advantage of using exclusively the simple cycles determined by two consecutive points of the process $\mathbf{X}$, $\mathbf{T}_n = \mathbf{t}_n - \mathbf{t}_{n-1}$, which we call 'simple arrival cycles'.

Consider for instance a marketing problem: if the total demand for a planning horizon is known, then the magnitude of the demand up to the present gives a lot of information about demand in the remaining of the planning horizon. Define therefore the embedded Harris chain $\mathbf{A}$ as the total demand up to the present. Conditioning on $\mathbf{A}_n$ at point $\mathbf{t}_n$ yields the desired

independence. Certainly, we need to have the conditions implying that $\mathbf{A}$ is indeed a Harris chain.

Sigman (cf. [18]) introduces the notion of marked point processes governed by a Harris chain, however his motivation and consequently his model is different. The examples he considers essentially consist of marked point processes with an underlying renewal sequence, while his limit theorem is given in terms of regeneration cycles, which in general are different from the simple arrival cycles (hence more difficult to determine).

5.0.1. *Independence realized with* $\mathbf{A}$. Let us now give the formal conditions which yield the desired properties of the Harris chain $\mathbf{A}$, which realize the independence in our model. Throughout this paper we assume the following:

**Assumption 5.1.** *The Harris chain* $\mathbf{A}_n = \mathbf{A}(\mathbf{t}_n)$ *satisfies the following conditions:*

$$\mathbf{T}_k \text{ depends on } \{\mathbf{T}_j, j < k\} \text{ only through } \mathbf{A}_{k-1}, \tag{5.1}$$

$$\text{and} \quad \mathbf{A}_k \text{ depends on } \{\mathbf{T}_i, i \leq j < k\} \text{ only through } \mathbf{A}_j,$$

$$\mathbf{J}_k \text{ depends on } \{\mathbf{T}_j, j < k\} \text{ only through } \mathbf{A}_{k-1}. \tag{5.2}$$

*and, for all* $k \in I\!N$,

$$I\!E(\mathbf{T}_k | \mathbf{A}_{k-1} = s) = I\!E(\mathbf{T}_1 | \mathbf{A}_0 = s), \text{ for almost every } s \in E \tag{5.3}$$

$$I\!E(\mathbf{J}_n | \mathbf{A}_{n-1} = s) = I\!E(\mathbf{J}_1 | \mathbf{A}_0 = s) \text{ for almost every } s \in E. \tag{5.4}$$

Conditions (5.1) and (5.2) provide precisely the desired independence: $\mathbf{A}$ *contains all the relevant information about the past*. Hence, conditioning on $\mathbf{A}$ yields independence of the cycles. Furthermore, a sort of 'time- homogeneousness' property is assumed for the conditional distributions of the cycles, given the realizations of $\mathbf{A}$.

5.1. **A limit result with an embedding technique.** The aim of this section is to prove that the limit theorem 4.1 also holds for the general stochastic process $\mathbf{X}$ with the embedded Harris chain $\mathbf{A}$, defined by relations (5.1) – (5.4). Knowing that every Harris chain forms a one-dependent process, we will show that this implies together with relations (5.1) and (5.2) that the process $\mathbf{X}$ (hence also $\mathbf{J}$, cf. Observation B.4) forms a wide-sense one-dependent process, which suffices to satisfy all the conditions of Proposition 4.1, thus we obtain the desired limit results.

**Remark 5.2.** *In 1992 Glynn and Sigman (cf.* [11]*) established limit theorems for Harris chains, however, although the limit theorems in the present paper are related to Harris chains they are more general: they are established for general stochastic processes, of which dependence structure can be modeled through Harris chains satisfying Assumption 5.1.*

In Section 2.2, property (iii) established that for every Harris chain there exists a set of points $\{\mathbf{N}\}$, such that $(\mathbf{A}, \mathbf{N})$ is one-dependent regenerative. Define now

$$\mathbf{S}_k := \mathbf{t}_{\mathbf{N}_k}, \tag{5.5}$$

the arrival moments (in continuous time!) of the process $\mathbf{X}$ corresponding to the regeneration times of $\mathbf{A}$ in discrete time. Furthermore, by the definition of the process $\mathbf{A}$ (cf. relations (5.1) and (5.2)) we have that if $\mathbf{A}_{k-1}$ is independent of $\mathbf{A}_j$, then $\mathbf{A}_{k-1}$ is independent of $\mathbf{T}_j$ $j = 0, \ldots, k-2$, which in turn implies that $(\mathbf{T}_k, \mathbf{J}_k)$ (or equivalently, $(\mathbf{T}_k, \phi_{\mathbf{t}_{k-1}}\mathbf{X})$) is independent of $\mathbf{T}_j$, $j = 0, \ldots, k-2$. Letting $\mathbf{S}$ denote the sequence $\{\mathbf{S}_k : k \in I\!N \cup \{0\}\}$, with $\mathbf{S}_k$ defined by relation (5.5), we can summarize the findings in the following theorem:

**Theorem 5.3.** $(\mathbf{X}, \mathbf{S})$*, or equivalently* $(\mathbf{J}, \mathbf{S})$*, is a wide-sense one-dependent process.*

Hence, applying Proposition 3.1 of Glynn and Sigman (cf. [11]) yields that, if the expected cost of the delay cycle is finite then in the long run expected average cost expression we can ignore this term, considering only the process which started in the first regeneration point, $\mathbf{S}_0$ (see Appendix C for the proof). Moreover, if $I\!E\mathbf{S}_1 > 0$ and the expected cost of this first cycle is also finite, then the necessary and sufficient uniform integrability condition (4.3) is satisfied (see Proposition 4.1 and the remark afterwards), hence the long run average cost of the system is given by

$$I\!E\bar{\mathbf{J}}_1 \,/\, I\!E\mathbf{S}_1, \tag{5.6}$$

where

$$\bar{\mathbf{J}}_1 := \sum_{k=1}^{\mathbf{N}_1} \mathbf{J}_k.$$

**Remark 5.4.** *Although the statement of Proposition 3.1 of Glynn and Sigman (cf.* [11]*) requires one-dependence, its proof only uses the conditions of wide-sense one-dependence, which makes it possible for us to apply it for our case.*

Consider the trivial identity

$$I\!E\left(\sum_{k=1}^{\mathbf{N}_1}\mathbf{J}_k\right)=I\!E\left(\sum_{k=1}^{\mathbf{N}_1}I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right)+I\!E\left(\sum_{k=1}^{\mathbf{N}_1}(\mathbf{J}_k-I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}))\right).$$

By the construction, $\mathbf{N}_1$ is a stopping time with respect to $\mathbf{A}_k$, that is, $\{\mathbf{N}_1\le k\}\subseteq\sigma(\mathbf{A}_j: j\le k)$; on the other hand, $\mathbf{J}_k-I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})$ is independent of $\sigma(\mathbf{A}_j: j\le k-1)$ (cf. [23]), hence $\mathbf{N}_1$ and $\{\mathbf{J}_k-I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\}$ are independent for every $k$. This yields that

$$I\!E\left(\sum_{k=1}^{\mathbf{N}_1}(\mathbf{J}_k-I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}))\right)=I\!E\left(\sum_{k=1}^{\mathbf{N}_1}I\!E(\mathbf{J}_k-I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}))\right)=0.$$

This means that we only need to deal with

$$I\!E\left(\sum_{k=1}^{\mathbf{N}_1}I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right). \tag{5.7}$$

Define now the functions $Y_n$ and $Z_n$, defined on $E$ as

$$Y_n(s):=I\!E(\mathbf{J}_n|\mathbf{A}_{n-1}=s)\quad\text{and}\quad Z_n(s):=I\!E(\mathbf{T}_n|\mathbf{A}_{n-1}=s) \tag{5.8}$$

(functions of the realizations of $\mathbf{A}_{n-1}$) for all $n\in I\!N$. Assume that

$$\int_E Y_1(u)\pi_\infty(du)<\infty,\quad\text{and}\quad\int_E Z_1(u)\pi_\infty(du)>0, \tag{5.9}$$

where $\pi_\infty$ is the limiting distribution in the Cesaro sense (cf. Bázsa and den Iseger [4]) of the Harris chain $\mathbf{A}$, given by

$$\pi_\infty:=\lim_{n\uparrow\infty}\frac{1}{n}\sum_{k=1}^{n}\pi_k. \tag{5.10}$$

**Theorem 5.5.** *Under the conditions of Assumption 5.1 and (5.9) the expected first cycle is finite, that is, $I\!E\bar{\mathbf{J}}_1<\infty$ and $I\!E\mathbf{S}_1>0$. Moreover, the expected long run average cost expression (5.6) can be expressed in terms of the arrival cycles:*

$$\frac{I\!E\bar{\mathbf{J}}_1}{I\!E\mathbf{S}_1}=\frac{\int_E I\!E(\mathbf{J}_1|\mathbf{A}_0=s)\pi_\infty(ds)}{\int_E I\!E(\mathbf{T}_1|\mathbf{A}_0=s)\pi_\infty(ds)}, \tag{5.11}$$

*for almost all initial states of $\mathbf{A}$, or equivalently, of the process $\mathbf{X}$, w.r.t. $\pi_\infty$.*

**Proof:** As mentioned before, the proof of the finiteness of the expected first cycle is given in Appendix C. In order to prove that the long run average cost is indeed given by

expression (5.11), observe that having obtained expression (5.7), we have

$$\frac{1}{I\!E\mathbf{N}_1} I\!E\bar{\mathbf{J}}_1 = \frac{1}{I\!E\mathbf{N}_1} I\!E\left(\sum_{k=1}^{\mathbf{N}_1} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right).$$

Conditioning on $\mathbf{A}_{k-1}$ with distribution $\pi_{k-1}$, the previous expression is equal to

$$\int_E \frac{1}{I\!E\mathbf{N}_1} I\!E\left(\sum_{k=1}^{\mathbf{N}_1} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}=u)\right)\pi_{k-1}(du).$$

Using again condition (5.4) this can be evaluated as

$$\int_E Y_1(s)\left(\frac{1}{I\!E\mathbf{N}_1}\sum_{k=1}^{I\!E\mathbf{N}_1}\pi_{k-1}(ds)\right).$$

Since $\mathbf{N}_1$ constitutes a regeneration point for the Harris chain $\mathbf{A}$ it yields that

$$\frac{1}{I\!E\mathbf{N}_1}\sum_{k=1}^{I\!E\mathbf{N}_1}\pi_{k-1} = \pi_\infty,$$

where $\pi_\infty$ was defined by relation (5.10). In conclusion,

$$I\!E\bar{\mathbf{J}}_1 = I\!E\mathbf{N}_1\int_E Y_1(s)\pi_\infty(ds) < \infty \tag{5.12}$$

by condition (5.9). Note that $I\!E\mathbf{N}_1 < \infty$ since $\mathbf{A}$ is positive recurrent. Similarly as for $\mathbf{J}_k$ we can use the same argumentation for $\mathbf{T}_k$, obtaining

$$I\!E\mathbf{S}_1 = I\!E\mathbf{N}_1\int_E Z_1(s)\pi_\infty(ds) > 0, \tag{5.13}$$

by condition (5.9). This immediately yields expression (5.11) for the expected long run average cost, completing the proof.

**Remark 5.6.** (1) *If $\mathbf{A}_0$ is distributed with the invariant distribution, Theorem 5.5 remains true (see Proposition 4.3 of* [11]*).*

(2) *The Harris chain $\mathbf{A}$ from Section 5 can consist of two chains: $\mathbf{A}^1$ satisfying conditions (5.1) and (5.3) and $(\mathbf{A}^1, \mathbf{A}^2)$ satisfying conditions (5.2), (5.4). Certainly, $(\mathbf{A}^1, \mathbf{A}^2)$ needs to be a Harris chain as well.*

Although at first sight it might look difficult to prove that $\mathbf{A} := (\mathbf{A}^1, \mathbf{A}^2)$ (or more generally the collection $\mathbf{A} := (\mathbf{A}^1, \ldots, \mathbf{A}_n)$) is a Harris chain, as it is stated in the previous remark, the argument is rather simple: it the consequence of result (ii) of Section 2.2. Hence, if we can prove that $\mathbf{A}$ is convergent in Cesaro total variation (that is, the time stationary version of $\mathbf{A}$

exists) it implies that it is a Harris chain. In case of inventory models, as discussed in Bázsa and den Iseger [4], it is often the case that the joint time- average distribution of $(\mathbf{IP}, \mathbf{N})$ exists. This is the result we will use later to show for several models that the embedded Harris chain exists.

## 6. An MDP formulation

Section 5, namely the construction of the Harris chain $\mathbf{A}$, suggest that one can influence the system through this Harris chain, hence dealing with models with control policies. More specifically, one wants to control the transitions of the process from one state to another. This section is aimed as a guideline for solving models of the type of the previous sections with generalized Markov decision theory, that is, Markov decision processes with continuous state space. Since the theory of Markov decision processes is vast, this section is only meant to formulate the problem, and not to give precise conditions under which an optimal solution exists, and there is also no exact solution procedure provided. Besides, the problem formulated below is very general, one needs to solve the optimization algorithms for each specified problem.

Knowing that every Harris chain possesses a unique invariant $\sigma$–finite measure $\pi_\infty^R$ (cf. property $(iii)$ of Section 2.2) defined by relation (5.10), the problem is formulated with the normalization

$$\int_E \pi_\infty^R(ds) = 1. \tag{6.1}$$

We proved that the average cost of the system is given by

$$g(R) = \frac{\int_E I\!E(\mathbf{J}_1|\mathbf{A}_0 = s)\pi_\infty^R(ds)}{\int_E I\!E(\mathbf{T}_1|\mathbf{A}_0 = s)\pi_\infty^R(ds)} + \int_E \int_{\mathcal{E}} K^R(s,y)P^R(s,dy)\pi_\infty^R(ds), \tag{6.2}$$

where $P$ is the transition kernel associated with the Harris chain $A$ (defined by relation (2.2)), with $P^R\,\pi_\infty^R = \pi_\infty^R$. There is a nonnegative cost $K \geq 0$ associated with the control of the system, which also depends on the transitions, hence on the policy $R$. Take for instance an inventory system with a positive order policy (that is, no disposal allowed). Then $K^R(x,y) = K1_{\{x<y\}}$. As usual, a policy $R^*$ is optimal if $g(R^*) \leq g(R)$, for all stationary policies $R$. We ought to remark here that we indeed are looking for nonstationary control policies, however this nonstationarity will be realized through the transitions of the embedded Harris chain $\mathbf{A}$:

the control policy depends on the state of $\mathbf{A}$. The optimization problem is hence of the form:

$$\begin{cases} \min_R \left\{ \frac{\int_E I\!E(\mathbf{J}_1|\mathbf{A}_0=s)\pi_\infty^R(ds)}{\int_E I\!E(\mathbf{T}_1|\mathbf{A}_0=s)\pi_\infty^R(ds)} + \int_E \int_{\mathcal{E}} K^R(s,y)P^R(s,dy)\pi_\infty^R(ds) \right\}, \\ P^R\,\pi_\infty^R = \pi_\infty^R. \end{cases} \tag{6.3}$$

In the case when $E$ is finite or countable, the problem reduces to a classical semi Markov decision problem. Methods for solving semi Markov decision problems can be found in any standard textbook (see for example [22], [17]); one can use policy iteration or value iteration (in the latter case, with a data transformation technique the problem is reduced to a simple Markov decision problem). In the continuous state space case, in the inventory applications the state space can almost always be assumed compact, but certainly Borel. For standard solution techniques the reader is referred to De Leve et al (cf. [8]).

Note that Federgruen and Zipkin (cf. [9]) solve a similar continuous state space Markov decision problem for the optimization of a stationary $(s, S)$ policy (only depending on the inventory position process).

## 7. Models 'under control': Examples

7.1. **Demand forecasting with time series.** One of the forecasting techniques which accounts for seasonal and trend factors in the demand process is the autoregressive moving average modeling (ARMA). It is believed that these discrete- time models are the most likely to be found in the real world (cf. [12]). The ARMA process, unlike the first- order autoregressive model or exponential smoothing model, is not a Markov process, since $\mathbf{D}_{k+1}$ depends not just on $\mathbf{D}_k$ but also on values at certain earlier times, say $\mathbf{D}_{k-1}$ and $\mathbf{D}_{k-2}$ (cf. [25]). It also depends on scalar noise factors over earlier time periods, say $\varepsilon_k$ and $\varepsilon_{k-1}$ ($\varepsilon_k$ are independent over all $k$, with 0 mean for all $k$). Thus the dynamics of the system is given by

$$\mathbf{D}_k = \sum_{j=1}^{m} \alpha_j \mathbf{D}_{k-j} + \sum_{j=0}^{m} \beta_j \varepsilon_{k-j}.$$

For the sake of simplicity, in this example we will only consider the following dynamics, characteristic for seasonal demand processes:

$$\mathbf{D}_k = \alpha_1 \mathbf{D}_{k-1} + \alpha_2 \mathbf{D}_{k-2} + \varepsilon_k + \beta \varepsilon_{k-1},$$

The stability condition (as in [25]) is $|\alpha_i| < 1$, $i = 1, 2$. Observe that the above dynamics is time-homogeneous since $\alpha_i$, $i = 1, 2$ and $\beta$ are constants. Defining now $\mathbf{A}_{k-1} := (\mathbf{D}_{k-1}, \mathbf{D}_{k-2})$

(with $\mathbf{D}_{-1} := 0$) it is a Harris chain: the Markov property is satisfied by the construction of the demand process, while the stability condition leads to stationarity.

Now the joint process $(\mathbf{A}, \mathbf{IP})$ realizes the desired independence (5.2) and the homogeneousness like property (5.4), that is for all $n$

$$I\!E\left(f(\mathbf{IP}_n - \sum_{k=n+1}^{n+L-1} \mathbf{D}_k)|(\mathbf{A}_n, \mathbf{IP}_n) = (s,u)\right)$$

are the same for almost every $(s,u)$ (with $s = (s_1, s_2)$), follows from stationarity. One alternative in order to show that $(\mathbf{A}, \mathbf{IP})$ is also a Harris chain, is to follow again statement $(ii)$ of Section 2.2 and the theory of Bázsa and den Iseger [4] to prove that its distribution converges to its time stationary version in Cesaro total variation. However, following this method, we only know for sure that the time average distribution exists if we assume that the control policy is stationary, that is, $\mathbf{IP}$ has a pointwise limiting distribution. Since the time- average distribution of $\mathbf{A}$ exists, the time average distribution of $(\mathbf{A}, \mathbf{IP})$ also exists (cf. Bázsa and den Iseger [4]). Hence, the long run expected average cost is given by the expression

$$I\!E_{(\mathbf{A},\mathbf{IP})^c_\infty}\left(I\!E\left(f(\mathbf{IP}_0 - \sum_{k=1}^{L-1} \mathbf{D}_k|(\mathbf{A}, \mathbf{IP})_0)\right)\right).$$

However, the second alternative, that is, using Definition 2.3 yields the possibility to construct nonstationary policies. Although a proof, following Definition 2.3 can be somewhat more tedious, the result is worth the trouble. To illustrate this, in the next section we consider a nonhomogeneous compound Poisson process together with a nonstationary $(s(\lambda), S(\lambda))$ policy.

7.2. **Inventory systems with Harris–modulated demand.** Let us assume that demand is a Harris–modulated time–nonhomogeneous stochastic process, that is, the rate of the process, $\lambda(t)$ is described by a Harris chain $\mathbf{A}_n := \lambda(\mathbf{t}_n)$ $n \in I\!N \cup \{0\}$. Furthermore we assume that the individual demand $\mathbf{Y}_n$ is i.i.d., independent of $\mathbf{A}$ and the associated cumulative distribution function $F_Y$ is spread-out, such that the density function $g$ in Definition A.2 has positive support on all sets of positive measure. Assume further that $\mathbf{A}$ realizes conditions (5.1) and (5.3), meaning that conditioning on $\mathbf{A}$ realizes the independence among the cycles, while if the rate is constant then the cycles become identically distributed. The applied decision rule is a so called Harris–modulated $(s,S)$ policy, defined in every decision moment

$\mathbf{t}_n, n \in I\!N \cup \{0\}$ as

$$\Gamma(\mathbf{A}_n) = \begin{cases} S(\mathbf{A}_n) & \text{if} \quad \widetilde{\mathbf{IP}}_n \leq s(\mathbf{A}_n), \\ \mathbf{IP}(\mathbf{t}_n) & \text{if} \quad \widetilde{\mathbf{IP}}_n > s(\mathbf{A}_n), \end{cases} \tag{7.1}$$

where $\widetilde{\mathbf{IP}}_n := \mathbf{IP}_{n-1} - \mathbf{Y}_n$. The action space is assumed compact, such that $\alpha := \sup_a\{s(\mathbf{A}) : \mathbf{A} = a\} < \beta := \inf_a\{S(\mathbf{A}) : \mathbf{A} = a\}$. In this setting $\mathbf{IP}$ depends on $\mathbf{A}$ only through the decision parameters $s(\mathbf{A})$ and $S(\mathbf{A})$. We aim to show now that $(\mathbf{A}, \mathbf{IP})$ forms a Harris chain, which realizes the desired independence conditions (5.2) and (5.4). That is, we want to show that there exist a regeneration set $B \times C$, associated with the joint process $(\mathbf{A}, \mathbf{IP})$, such that $\varphi(B, C) > 0$ implies $\varphi$-recurrence (see Definition 2.6 and the observation thereafter):

$$I\!P\left\{\sum_{n=1}^{\infty} (1_{B\times C} \circ (\mathbf{A}_n, \mathbf{IP}_n)) = \infty\right\} = 1. \tag{7.2}$$

The reason for using this method in order to prove that $(\mathbf{A}, \mathbf{IP})$ is a Harris chain, instead of using $(ii)$ of Section 2.2, is that in this case neither $\mathbf{A}$ nor $\mathbf{IP}$ has a pointwise limiting distribution, thus we would need additional information for proving that the joint limiting distribution in the Cesaro sense of $\mathbf{A}$ and $\mathbf{IP}$ exists.

Since $\mathbf{A}$ is a Harris chain, there exists a regeneration set $B$, together with $l_A > 0$, $\mathbf{A} > 0$ and a probability measure $\mu_A$ satisfying (2.3), such that the hitting time $\tau_A$ of $B$ is finite with probability one. Consider the sequence $\{\tau_A^k : k \in I\!N\}$ generated by the hitting times of the set $B$, such that

$$\tau_A^k := \inf\{m \in I\!N : \mathbf{A}_m \in B \text{ and } m > \tau_A^{k-1}\}, \quad k \geq 2, \quad \tau_A^1 := \tau_A.$$

Since the sequence $\{\tau_A^k : k \in I\!N\}$ is a subset of the arrival times $\{\mathbf{t}_n : n \in I\!N\}$, define $\widehat{\mathbf{IP}}_k := \mathbf{IP}_{\tau_A^k}$. Since $\mathbf{Y}$ has a spread-out distribution, it means (see Definition A.2) that there exists an $n_0 \in I\!N$ (finite) such that $F_Y^{n_0*}$ has an absolutely continuous component with a density $g$. Consequently, supposing that an average order cycle $\mathbf{T}$ is completed in $k$ arrivals, we have that $I\!P\{\mathbf{T} = k\} = (F_Y^{k*} - F^{(k+1)*})(S(\mathbf{A}_0) - s(\mathbf{A}_k)) > 0$ if $k \geq n_0$. Considering $C := (\alpha, \beta)$ yields that there exist $m, k \geq n_0$ such that

$$I\!P\{\widehat{\mathbf{IP}}_k \in C\} \geq \inf_a I\!P\{S(a) - \mathbf{D}_m \in C\} I\!P\{\text{replenishment order placed at time } \mathbf{t}_{k-m}\} > 0,$$

since $\mathbf{D}_m = F_Y^{m*}$ has an absolutely continuous component for $m > n_0$, yielding $F_Y^{m*}(x) > 0$ for all $x > 0$. We can conclude thus that there exist a $p > 0$ such that $I\!P\{\widehat{\mathbf{IP}}_k \in C\} > p$.

Combining this with relation (7.2) yields that

$$I\!P\left\{\sum_{n=1}^{\infty}(1_{B\times C}\circ(\mathbf{A}_n,\mathbf{IP}_n))=\infty\right\}=I\!P\left\{\sum_{k=n_0}^{\infty}\left(1_C\circ\widehat{\mathbf{IP}}_k\right)=\infty\right\}=1.$$

Finally we can conclude that $(\mathbf{A},\mathbf{IP})$ is a Harris chain satisfying conditions (5.1)–(5.4), yielding that the long run average cost of the system is given by

$$g_R=\frac{1}{I\!E_{\mathbf{A}^c_\infty}(I\!E(\mathbf{T}_1|\mathbf{A}_0))}I\!E_{(\mathbf{A},\mathbf{IP})^c_\infty}\left(I\!E\left(\int_0^{\mathbf{t}_1}f(\mathbf{IP}_0-\mathbf{D}_L(t)|(\mathbf{A},\mathbf{IP})_0)\right)\right),$$

where $I\!E_{(\mathbf{A},\mathbf{IP})^c_\infty}$ denotes the expectation w.r.t. the limiting distribution in the Cesaro sense of the distribution of $(\lambda(\mathbf{t}_n),\mathbf{IP}_n)$. Solving the optimization problem (6.3) for this $g_R$ yields a dynamic (nonstationary) optimal policy $(s(\lambda),S(\lambda))$.

7.2.1. *Markov modulated arrivals.* Markov modulated demand processes have been considered in several articles in the literature, however their approach is quite different from ours. The closest to our approach is that of Sigman (cf. [18]), however his motivation and consequently his results are different. As mentioned earlier, he proves that every Harris process is a one-dependent process, but the limit result he deduces is in terms of the one-dependent regeneration cycles.

Song and Zipkin (cf. [20]) consider an exogenous 'world-driven' Poisson demand: when the world is in state $i$, demand follows a Poisson distribution with rate $\lambda_i$. Their model is a discrete-state dynamic program with two state variables, the world and the inventory position.

Lovejoy (cf. [14]) considers demand processes as functions of a Markovian information process. This information process may depend on the past of the demand process as well as on an exogenous variable. Further it concentrates on the efficacy of myopic policies.

7.3. **Inventory with returns.** Consider a general single item inventory model which allows returns, with the following characteristics. Demand up to time $t$, $\mathbf{D}(t)$, is a compound renewal process with arrival process $\{\mathbf{t}_n : n\in I\!N\cup\{0\}\}$ and individual demands $\{\mathbf{Y}_n, n\in I\!N\cup\{0\}\}$. We assume that control actions, denoted by $\Gamma$, are only permitted at purchase arrivals, that is, at times $\{\mathbf{t}_n : n\in I\!N\}$; furthermore, $\Gamma$ is a stationary policy, depending only on the inventory position process. There is also a so called returns process $\mathbf{R}$, a nonhomogeneous compound Poisson process with fixed batch return sizes $\mu$. Denote the total amount of items on the

market at time $t$ as $\mathbf{A}(t)$ (that is, demand minus returns) and let us assume that the returns rate is $a$, given $\mathbf{A} = a$. Thus, fixing a sample paths of $\mathbf{A}$, $\mathbf{R}$ is just a compound Poisson process with a variable rate. Assuming that every item is returned is not very restrictive, since we can use the following correction: introduce the binary variable $(\mathbf{y}/\mathbf{n})$ (yes or no), for deciding whether the returned item is suitable for remanufacturing or not. Hence, the returns process has the form $\sum_{k=1}^{\mathbf{N}_r(t)} (\mathbf{y}/\mathbf{n})\mu$ (we assume here that non-suitable items can be instantaneously disposed of). Since we only need to concentrate on the times $\mathbf{t}_n$ of purchase arrivals, it is useful to introduce the notations $\mathbf{IP}_n := \mathbf{IP}(\mathbf{t}_n^+)$ and $\mathbf{A}_n := \mathbf{A}(\mathbf{t}_n^+)$. By the definition of the model it follows now that

$$\begin{aligned} \mathbf{IP}_{n+1} &= \Gamma(\mathbf{IP}_n) - \mathbf{Y}_{n+1} + \mathbf{R}(\mathbf{t}_n, \mathbf{t}_{n+1}], \\ \mathbf{A}_{n+1} &= \mathbf{A}_n + \mathbf{Y}_{n+1} - \mathbf{R}(\mathbf{t}_n, \mathbf{t}_{n+1}]. \end{aligned} \tag{7.3}$$

This means that the $\{\mathbf{IP}_n\}$ and $\{\mathbf{A}_n\}$ are Markov processes. A common assumption for inventory models (see for instance Zipkin [24]) to assume that the embedded Markov chain $\mathbf{IP}_n$ is positive recurrent – since our model assumes continuous state space we assume $\varphi$–recurrence (see Section 2.2), hence $\mathbf{IP}_n$ is a Harris chain. Having assumed that the control policy is stationary, the pointwise limiting distribution of $\mathbf{IP}_n$ exists. This implies that $(\mathbf{A}, \mathbf{IP})$ converges to its stationary version in Cesaro total variation, hence it is a Harris chain (statement $(ii)$, Section 2.2). Furthermore, the equivalent of the classical flow-conservation law remains valid:

$$\mathbf{IN}(t+L) = \mathbf{IP}(t) - (\mathbf{D}(t, t+L] - \mathbf{R}(t, t+L]). \tag{7.4}$$

Since $(\mathbf{A}, \mathbf{IP})$ is a Harris chain, we aim to show that it satisfies the conditions of Assumption 5.1. Since $\{\mathbf{t}_n\}$ is a renewal process, conditions (5.1) and (5.3) follow. Since $\mathbf{R}$ is a compound Poisson process an $\mathbf{D}$ is a compound renewal process, conditions (5.2) and (5.4) are satisfied too. Hence, applying Theorem 5.5, it follows that the long run average cost of the system is given by the expression

$$\frac{1}{I\!E\mathbf{T}_1} I\!E_{(\mathbf{A},\mathbf{IP})^{\mathrm{c}}_{\infty}} \left( \int_0^{\mathbf{t}_1} f(\mathbf{IP}_0 - \mathbf{D}_L(t) + \mathbf{R}_L(t))dt \middle| (\mathbf{A}, \mathbf{IP})_0) \right),$$

where $I\!E_{(\mathbf{A},\mathbf{IP})^{\mathrm{c}}_{\infty}}$ stands for the expectation with respect to the joint time average distribution of the Harris chain $(\mathbf{A}, \mathbf{IP})$.

The so called multi–echelon models prove to be even more difficult to solve exactly than the single echelon models considered so far. Such a model is a two- level decentralized distribution system, consisting of one depot and several retailers. This application is worked out in Bázsa and den Iseger [5]. There are many more OR models, not only inventory models, which can be included under the framework of general stochastic processes with an embedded Harris chain. In the following subsection we give such an example.

7.4. **Generalized Semi – Markov Processes (GSMP's).** A Generalized Semi – Markov process has less restrictive assumptions than a continuous time Markov Chain. The following intuitive summarizing definition is taken from the lecture notes of Haas (2000). That is, a GSMP $\{\mathbf{X}(t) : t \geq 0\}$ makes stochastic state transitions when one or more events associated with the occupied state occur: events associated with a state compete to trigger the next state transition; each event has its own distribution for determining the next state; new events can be scheduled at each state transition; for each new event a *clock* is set with a reading that indicates the time until the next transition (when the clock runs down to 0 the event occurs); old events don't trigger state transitions but are associated with the next state, their clocks continue to run down; cancelled events don't trigger a state transition and are not associated with the next state, their clock readings are discarded; clocks can run down at state dependent speed. The mathematical definition of a GSMP has the following elements: the set of states $S$ is countable, there is a finite set of events $E := \{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$, and $E(s)$ is the set of events scheduled to occur in state $s \in S$. Furthermore, $p(s'; s, E^*)$ is the probability that the new state is $s'$ given that the events in $E^*$ simultaneously occur in $s$. If $E^* = \{\mathbf{e}^*\}$ for some $\mathbf{e}^* \in E(s)$, then write $p(s'; s, \mathbf{e}^*)$. $r(s, \mathbf{e})$ denotes the nonnegative finite speed at which clock for $\mathbf{e}$ runs down in state $s$, and $F(\cdot; s', \mathbf{e}', s, E^*)$ the distribution function used to set the clock for the new event $e$, when the simultaneous occurrence of the events in $E^*$ triggers a state transition from $s$ to $s'$. $\mu$ is the initial distribution function for the clock and state readings. It is assumed however, that $\mu$ is such that the initial state $s$ is chosen according to a distribution $\nu$, and for each event $\mathbf{e} \in E(s)$ the clock is set independently according to $F_0(\cdot; \mathbf{e}, s)$. Hence, formally, $\mathbf{X} = \{(S_n, \mathbf{C}_n) : n \geq 0\}$, where $S_n$ is the state after the $n$th transition and $\mathbf{C}_n = (\mathbf{C}_{n,1}, \ldots, \mathbf{C}_{n,M})$ is a clock-reading vector after the $n$th transition. The solution method for such a GSMP model is generally simulation.

A simple example for such a GSMP is the GI/G/1 queue. Assuming that the interarrival distribution $F_a$ and the service time distribution $F_s$ are continuous, such that an arrival and a service completion never occur simultaneously, we can model the queue as a GSMP in the following manner. Let $\mathbf{X}(t)$ the number of jobs in service or waiting in queue at time $t$. Then $S = \{0, 1, 2, \ldots\}$, $E = \{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_1$ is the event arrival, and $\mathbf{e}_2$ is the event completion of service. $E(s) = \{\mathbf{e}_1\}$ is $s = 0$ and $E(s) = \{\mathbf{e}_1, \mathbf{e}_2\}$ if $s > 0$. Furthermore $p(s+1; s, \mathbf{e}_1) = 1$ and $p(s-1; s, \mathbf{e}_2) = 1$. $F(x; s', \mathbf{e}', s, e) = F_a(x)$ if $e' = \mathbf{e}_1$ and $F_s(x)$ if $\mathbf{e}' = \mathbf{e}_2$, while $r(s, e) = 1$ for all $s$ and $e$. Finally, $\nu(1) = 1$, $F_0(\cdot; \mathbf{e}_1, s) = F_a(\cdot)$ and $F_0(\cdot; \mathbf{e}_2, s) = F_s(\cdot)$.

With the modeling technique of general stochastic process with an embedded Harris chain one can also model GSMP's in the following way. Extend the state space with the states $S_n$, and the Harris chain with the clock-reading vectors $\mathbf{A} \times \{\mathbf{C}_1, \mathbf{C}_2, \ldots\}$. We need to assume here that the clocks are set independently, which yields that this is then indeed a Harris chain. Suppose that $\mathbf{e}' = \mathbf{e}_n$. Then $\mathbf{T}_k = \mathbf{C}_n(k)$, the $n$th clock at time $\mathbf{t}_k$. Then $\mathbf{C}_j(k+1) = \mathbf{C}_j(k) - \mathbf{T}_k$, $j \neq n$, and $\mathbf{C}_n(k+1)$ is distributed with $F(\cdot; s', \mathbf{e}', s, \mathbf{e}_n)$. The transition probabilities for the Harris chain are given by $I\!P\{\mathbf{A}_{k+1}, \mathbf{A}_k; \mathbf{e}' = \mathbf{e}_n\}$. The event $\mathbf{e}' = \mathbf{e}_n$ is known at time $\mathbf{t}_k$, $\mathbf{e}' = \mathbf{e}_n$ if and only if $\mathbf{C}_n = \min\{\mathbf{C}_1, \mathbf{C}_2, \ldots\}$. In fact, given the clocks at time $\mathbf{t}_k$, $\mathbf{T}_k$ is deterministic, i.e. $\mathbf{T}_k = \min\{\mathbf{C}_1(k), \mathbf{C}_2(k), \ldots\}$.

## 8. Discussion

The essential difference between semi-Markov processes and the 'Harris-modulated stochastic processes' described in Section 5 is, that while in the case of semi-Markov processes the jump can depend on the state of the process when the jump occurs, for Harris–modulated stochastic processes the allowed dependence can be more complex. For instance, the expected interarrival time can depend on the next state (where the process jumps to) instead of depending solely on the state where the process has been before the jump. This means that the class of models which can be analyzed is considerably extended. Moreover, the established limit theorem (cf. Section 5.1, that is, the average cost expression (5.11), is given in terms of the simple arrival cycles. Within this framework, the analysis of many important models becomes straightforward (only identifying the embedded Harris chain), which otherwise would cost a laborious work of many pages; for other models, the analysis has only been performed under simplifying assumptions.

It is very interesting to observe how things fall into their places. In Bázsa and den Iseger [4] performance measures, among others an average cost expression was deduced under the assumption that the limiting distribution in the Cesaro sense of the joint process $(\mathbf{IP}, \mathbf{N})$ exists. The condition of our limit theorem 5.5 and the average cost expression (5.11) was the existence of a Harris chain satisfying the initial assumptions 5.1. Now, do these results really relate to each other? Statement $(ii)$ of Section 2.2 yields the answer: in order to prove that the Markov process constructed according to the conditions of Assumption 5.1 is a Harris chain, one can use the necessary and sufficient condition that 'the process is converging in Cesaro total variation to its stationary version'. This means exactly that if and only if the limiting distribution in the Cesaro sense exists. Although it might not be necessary, we like to stress that this is a necessary and sufficient condition, that is, Theorem 5.5 holds if and only if the Cesaro sense limiting distribution exists. Furthermore, for the case of inventory models, the Harris chain realizing the independence condition is always the joint process of $\mathbf{IP}$ and an embedded chain $\mathbf{A}$ of the arrival process of customers. Since the chain $\mathbf{A}$ is driving the arrival process, it can be intuitively seen as a better candidate for $\mathbf{N}$, inheriting many of its properties. Besides, when $\mathbf{N}$ has independent increments, $\mathbf{N}$ itself can play the role of the embedded Harris chain, leading thus to the condition (cf. Bázsa and den Iseger [4]) of the existence of the Cesaro limiting distribution of $(\mathbf{IP}, \mathbf{N})$.

A sufficient condition for the existence of a Cesaro limiting distribution of the joint process $(\mathbf{IP}, \mathbf{N})$ is that either of them possesses a pointwise limiting distribution while the other a Cesaro limiting distribution (cf. Bázsa and den Iseger [4]). However, this is NOT a necessary condition, therefore we also make use of alternative definitions and properties of Harris chains. For instance, when considering nonhomogeneous demand, we want to consider a non-stationary control policy. This implies that neither $\mathbf{A}$ not $\mathbf{IP}$ will possess a pointwise limiting distribution, however it can be shown that for the joint Markov process $(\mathbf{A}, \mathbf{IP})$ a regeneration set exists, implying that $(\mathbf{A}, \mathbf{IP})$ is a Harris chain. Although this implies that the joint process $(\mathbf{A}, \mathbf{IP})$ possesses a Cesaro limiting distribution, it would be difficult to prove its existence without further information (specification of the problem).

## Appendix A. Measure theoretic background

Let us start with a measure theoretic review of terminology, which we will use throughout the following sections. The definitions are taken from the book of Thorisson (cf. [21]). A random element in a measurable space $(E, \mathcal{E})$, defined on a probability space $(\Omega, \mathcal{F}, I\!P)$, is a measurable mapping $\mathbf{Y}$ from $(\Omega, \mathcal{F}, I\!P)$ to $(E, \mathcal{E})$, that is,

$$\{\mathbf{Y} \in A\} \in \mathcal{F}, \quad \text{for all} \quad A \in \mathcal{E},$$

where

$$\{\mathbf{Y} \in A\} := \{\omega \in \Omega : \mathbf{Y}(\omega) \in A\} =: \mathbf{Y}^{-1}A.$$

**Definition A.1.** *It is also said that* $\mathbf{Y}$ *is supported by* $(\Omega, \mathcal{F}, I\!P)$*, and* $\mathbf{Y}$ *is an* $\mathcal{F}/\mathcal{E}$ *measurable mapping from* $\Omega$ *to* $E$*.*

The distribution of a random element $\mathbf{Y}$ (under $I\!P$) is the probability measure on $(E, \mathcal{E})$ induced by $I\!P\mathbf{Y}^{-1}$. Since $I\!P\{\mathbf{Y} \in A\} = I\!P\mathbf{Y}^{-1}A$ for all $A \in \mathcal{E}$, we use the notation $I\!P\{\mathbf{Y} \in \cdot\}$. A random element $\mathbf{Y}$ is *canonical* if $\mathbf{Y}$ is the identity mapping, that is, if

$$(\Omega, \mathcal{F}) = (E, \mathcal{E}) \ \text{ and } \ \mathbf{Y}(\omega) = \omega, \ \omega \in \Omega. \quad \text{Then } \ I\!P\{\mathbf{Y} \in \cdot\} = I\!P.$$

A random element $\hat{\mathbf{Y}}$ in $(E, \mathcal{E})$, defined on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{I\!P})$ is a *copy* or *representation* of $\mathbf{Y}$ if

$$\hat{I\!P}\{\hat{\mathbf{Y}} \in \cdot\} = I\!P\{\mathbf{Y} \in \cdot\}, \ \text{ that is, } \ \hat{\mathbf{Y}} \stackrel{d}{=} \mathbf{Y}.$$

A random element $\mathbf{Y}$ *always has a canonical representation*, the canonical random element on $(E, \mathcal{E}, I\!P\{\mathbf{Y} \in \cdot\})$.

A *random variable* $\mathbf{Y}$ is a random element in $(I\!R, \mathcal{B})$, where $I\!R$ is the set of real numbers and $\mathcal{B}$ denotes its Borel subsets (i.e. $\mathcal{B}$ is the $\sigma$-algebra generated by the open sets). The following definition is from the book of Thorisson (cf. [21]).

**Definition A.2.** *The random variable* $\mathbf{Y}_1$ *is spread out if there exists a finite* $n \in I\!N$ *and a function* $g \in \mathcal{B}^+$ *such that* $\int_{I\!R} g(x)dx > 0$ *and, with* $\mathbf{Y}_2, \ldots, \mathbf{Y}_n$ *i.i.d. copies of* $\mathbf{Y}_1$*,*

$$I\!P\{\mathbf{Y}_1 + \ldots + \mathbf{Y}_n \in B\} = F_Y^{n*}(B) \geq \int_B g(x)dx, \qquad B \in \mathcal{B}.$$

This means (cf. [1]), that $F_Y$ is spread out if $F_Y^{n*}$ (for some $n \in I\!N$) has an absolutely continuous component $G$ ($0 \neq G \leq F_Y$) which is absolutely continuous, i.e. $G$ has a density $g$ with respect to the Lebesgue measure.

**Observation A.3.** *Since the derivative of the convolution function $G*F_Y$ is given by $g*F_Y$, it follows that for every integer $m \geq n$, $F_Y^{m*}$ has an absolute continuous component, $G*F_Y^{(m-n)*}$.*

A.1. **Modes of convergence.** Let $\{\mathbf{Y}_n : n \in I\!N\}$ be a sequence of random variables, and $\mathbf{Y}$ a rn=andom variable, all defined on $(\Omega, \mathcal{F}, I\!P)$. Let us summarize the types of convergence used in the following sections (cf. [23]):

- **almost surely:** $\mathbf{Y}_n \to \mathbf{Y}$ a.s. if $I\!P\{\mathbf{Y}_n \to \mathbf{Y}\} = 1$ as $n \to \infty$;
- **in probability:** $\mathbf{Y}_n \xrightarrow{d} \mathbf{Y}$ if for every $\varepsilon > 0$ $I\!P\{|\mathbf{Y}_n - \mathbf{Y}| > \varepsilon\} \to 0$ as $n \to \infty$;
- $\mathcal{L}^p$ **convergence:** if $\mathbf{Y}_n, \mathbf{Y} \in \mathcal{L}^p, n \in I\!N$ and $I\!E(|\mathbf{Y}_n - \mathbf{Y}|^p) \to 0$ as $n \to \infty$;
- **total variation:** $\mathbf{Y}_n \xrightarrow{\text{tv}} \mathbf{Y}$ if $\sup_{A \in \mathcal{E}} |I\!P\{\mathbf{Y}_n \in A\} - I\!P\{\mathbf{Y} \in A\}| \to 0$ as $n \to \infty$;

Almost sure convergence and $\mathcal{L}^p$–convergence (for $p \geq 1$) imply convergence in probability.

A.2. **Uniform integrability.** As we will show later, a necessary and sufficient condition for our limit theorem is uniform integrability. This follows directly from the fact that uniform integrability is a necessary and sufficient condition for $\mathcal{L}^1$–convergence (cf. [23]).

**Definition A.4.** *A family of random variables $\mathbf{Y}_n$, $n \in I\!N \cup \{0\}$ (or, $\mathbf{Y}_s$, $s \in [0, \infty)$) is uniformly integrable if*

$$\sup_{n \geq 0} I\!E\left(\mathbf{Y}_n 1_{\{\mathbf{Y}_n > x\}}\right) \to 0, \qquad x \to \infty.$$

The following result can be found in the book of Williams (cf. [23]).

**Theorem A.5.** *If $(\mathbf{Y}_n)$ is a sequence in $\mathcal{L}^1$ and $\mathbf{Y} \in \mathcal{L}^1$, then $I\!E(|\mathbf{Y}_n - \mathbf{Y}|) \longrightarrow 0$ if and only if the following two conditions are satisfied:*

$$(i) \;\; \mathbf{Y}_n \longrightarrow \mathbf{Y} \text{ in probability,}$$

$$(ii) \text{ the sequence } \{\mathbf{Y}_n\} \text{ is U.I.}$$

Scheffé's Lemma (cf. [23]) proves to be very useful, since the result which is more often needed than $I\!E(|\mathbf{Y}_n - \mathbf{Y}|) \longrightarrow 0$ is $I\!E(\mathbf{Y}_n) \longrightarrow I\!E(\mathbf{Y})$.

**Lemma A.6.** *Suppose that* $\mathbf{Y}_n, \mathbf{Y} \in \mathcal{L}^1$ *such that* $\mathbf{Y}_n \to \mathbf{Y}$ *a.e. Then*

$$I\!E(|\mathbf{Y}_n - \mathbf{Y}|) \to 0 \text{ if and only if } I\!E(\mathbf{Y}_n) \to I\!E(\mathbf{Y}).$$

Since almost surely convergence implies convergence in probability (see the previous subsection), and having $\mathbf{Y}_n, \mathbf{Y} \in \mathcal{L}^1$ with $\mathbf{Y}_n \to \mathbf{Y}$ a.s., it follows that $I\!E(|\mathbf{Y}_n - \mathbf{Y}|) \to 0$ if and only if $I\!E(\mathbf{Y}_n) \to I\!E(\mathbf{Y})$ if and only if the sequence $(\mathbf{Y}_n)$ is U.I.

A.3. $\sigma$–**finite measure.** For the sake of completeness we introduce the notion of a $\sigma$–finite measure, which, although will not be used directly, will be present in some of the statements. A measure $\mu$ on a field $\mathcal{F}$ in $\Omega$ is $\sigma$–finite (cf. [6]) if $\Omega = A_1 \bigcup A_2 \bigcup \dots$ for some finite or countable sequence of $\mathcal{F}$–sets, satisfying $\mu(A_k) < \infty$. A $\sigma$–finite measure can be finite or infinite; a finite measure is by definition $\sigma$–finite. An important result is that if $\mu$ is a $\sigma$–finite measure on the field $\mathcal{F}$, then $\mathcal{F}$ cannot contain an uncountable, disjoint collection of sets of positive $\mu$–measure.

## Appendix B. General stochastic processes

A stochastic process with index set $I\!I$ and state space $(E, \mathcal{E})$ is a family $\mathbf{X} := (\mathbf{X}_s)_{s \in I\!I}$, where the $\mathbf{X}_s$ are random elements defined on a common probability space $(\Omega, \mathcal{F}, I\!P)$ and all taking values in $(E, \mathcal{E})$. Now, rather then regarding $\mathbf{X}$ as a family of random elements in $(E, \mathcal{E})$, we can equivalently regard $\mathbf{X}$ as a random mapping (cf. [21]), that is, a *single random element* in $(E^{I\!I}, \mathcal{E}^{I\!I})$, defined by

$$\mathbf{X}(\omega) = \{\mathbf{X}_s(\omega) : s \in I\!I\}, \ \ \omega \in \Omega.$$

The *paths* of $\mathbf{X}$ are the realizations $\mathbf{X}(\omega), \ \omega \in \Omega$ of the random mapping $\mathbf{X}$. Most of the time there are restrictions put on the path, for our case, that they are right continuous with left limits. More generally, one can say that they lie in a subset $H$ of $E^{I\!I}$ (cf. [21]). In this case it is more natural to say that $\mathbf{X}$ is a random element in $(H, \mathcal{H})$, instead of $(E, \mathcal{E})$, where $\mathcal{H}$ is the $\sigma$-algebra on $H$, generated by the projection mapping taking $x \in H$ to $x_t \in E$, for all $t \in I\!I$. $\mathcal{H}$ is also called the trace of $H$ on $\mathcal{E}^{I\!I}$ (cf. [21]), because

$$\mathcal{H} = \mathcal{E}^{I\!I} \bigcap H := \{A \bigcap H : A \in \mathcal{E}^{I\!I}\}.$$

In conclusion, $\mathbf{X}_t$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$ if and only if $\mathbf{X}$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(H, \mathcal{H})$, and $(H, \mathcal{H})$ is called the *path space* of $\mathbf{X}$.

B.1. **Shift measurability.** Observing a continuous time stochastic process at a random time means the following. Let $\mathbf{T}$ be a random time in $[0, +\infty)$; by $\mathbf{X_T}$ we mean the $E$ valued mapping defined on $\Omega$ as $\mathbf{X_T}(\omega) := \mathbf{X}_{\mathbf{T}(\omega)}(\omega)$, for all $\omega \in \Omega$. This mapping need not be $\mathcal{F}/\mathcal{E}$ measurable! (cf. [21], see also Definition A.1). To take care of this measurability problem, one needs to impose a *canonical joint measurability* condition (cf. [21]):

**Definition B.1.** *The process* $\mathbf{X}$ *is canonically jointly measurable if the mapping taking* $(x, t) \in H \times [0, \infty)$ *to* $x_t \in E$ *is* $\mathcal{H} \otimes \mathcal{B}[0, \infty)/\mathcal{E}$ *measurable.*

This condition suffices for drawing the conclusion that if $\mathbf{X}'$ is also a stochastic process such that $(\mathbf{X}, \mathbf{T}) \stackrel{d}{=} (\mathbf{X}', \mathbf{T})$, then $\mathbf{X_T} \stackrel{d}{=} \mathbf{X}'_{\mathbf{T}}$. However, rather than observing a stochastic process *at* a random time, we need most of the time to observe the whole process *from* that time onwards. Canonical joint measurability is insufficient for this purpose, hence the definition is extended in the following manner (cf. [21]):

**Definition B.2.** *The path set* $H$ *of a continuous time stochastic process* $\mathbf{X}$ *is internally shift–invariant if*

$$\{\phi_t x : x \in H\} = H, \text{ for all } t \in [0, \infty),$$

*where* $\phi_t x = (x_{t+s})_{s \in [0,\infty)}$*, for all* $x \in H$*. The process* $\mathbf{X}$ *is said to be* **shift-measurable** *if its path set* $H$ *is internally shift-invariant and if the mapping taking* $(x, t) \in H \times [0, \infty)$ *to* $\phi_t x \in H$ *is* $\mathcal{H} \otimes \mathcal{B}[0, \infty)/\mathcal{H}$ *measurable.*

Finally, a stochastic process with internally shift-invariant path space is shift measurable if and only if it is canonically jointly measurable. The standard cases, where the paths are right continuous with left limits and the state space $(E, \mathcal{E})$ is Polish[2], are all covered by shift measurability, in fact, completeness of $E$ is not even necessary (cf. [21]).

[2]A measurable space $(E, \mathcal{E})$ is Polish if $E$ is a complete separable metric space and $\mathcal{E}$ is generated by the open sets.

B.2. **Back to convergence.** For the shift measurable stochastic processes $\mathbf{X}$ and $\mathbf{X}'$, plain total variation convergence is similar to that for random elements:

$$\|I\!P\{\phi_t\mathbf{X} \in \cdot\} - I\!P\{\phi_t\mathbf{X}' \in \cdot\}\| \to 0, \text{ as } t \to \infty.$$

If $\mathbf{U}$ is a uniform random variable on $[0,1]$, then the Cesaro (or time-average) total variation convergence is defined as

$$\|I\!P\{\phi_{\mathbf{U}t}\mathbf{X} \in \cdot\} - I\!P\{\phi_{\mathbf{U}t}\mathbf{X}' \in \cdot\}\| \to 0, \text{ as } t \to \infty.$$

B.3. **Invariant $\sigma$–field.** Ergodicity for stochastic processes is defined through the so called *invariant field* or invariant $\sigma$-algebra. The invariant field (cf. [21]) consists of path sets in $\mathcal{H}$ that do not depend on where the origin is placed; it is formally defined as:

$$\mathcal{I} = \{A \in \mathcal{H} \,:\, \phi_t^{-1}A = A,\ 0 \le t < \infty\}. \tag{B.1}$$

This is a $\sigma$-algebra because if $A$ is the union of the sets $A_k$, satisfying $\phi_t^{-1}A_k = A_k$, then $\phi_t^{-1}A = A$; $\mathcal{I}$ is also closed under complementation and it contains $H$.

B.4. **Regeneration.** Let us introduce some properties for general stochastic processes, which generalize the classical regenerative property introduced. The first concept we introduce is called wide-sense regeneration, first defined by Smith in 1955 (cf. [19]), using the term 'equilibrium process'. At that time the term and the property remained unnoticed, and later it was rediscovered independently by Asmussen and Thorisson. Lag–$l$ regeneration is somewhat more restrictive than wide-sense regeneration (cf. [2]), one-dependent regeneration is noted in the dissertation of Glynn in 1982, and can be found in the article of Sigman (cf. [18]). To our best knowledge wide-sense k-dependence is not known in the literature.

A shift measurable stochastic process $\mathbf{X}$ is *wide-sense regenerative* with regeneration times $\mathbf{S} = (\mathbf{S}_n)_0^\infty$ (cf. [21], Chapter 10, Section 4) if

$$\phi_{\mathbf{S}_n}(\mathbf{X}, \mathbf{S}) \stackrel{d}{=} (\mathbf{X}^0, \mathbf{S}^0), \quad n \in I\!N \cup \{0\}, \tag{B.2}$$

where $(\mathbf{X}^0, \mathbf{S}^0)$ is the zero-delayed version of $(\mathbf{X}, \mathbf{S})$, and

$$\phi_{\mathbf{S}_n}(\mathbf{X}, \mathbf{S}) \text{ is independent of } (\mathbf{S}_0, \ldots, \mathbf{S}_n), \quad n \in I\!N \cup \{0\}. \tag{B.3}$$

Furthermore, the process $\mathbf{X}$ is called *k-dependent* (cf. [11]) with $k \in I\!N$ and $\mathbf{S}$, if for each $n \in I\!N$ $(\mathbf{D}, \mathbf{C}_1, \ldots, \mathbf{C}_n)$ and $(\mathbf{C}_{n+k+1}, \ldots)$ are independent, where $\mathbf{D} := (\mathbf{X}_s)_{s\in[0,\mathbf{S}_0)}$ is the

delay, and $\mathbf{C}_n := (\mathbf{X}_{\mathbf{S}_{n-1}+s})_{s\in[0,\mathbf{L}_n)}$, with $\mathbf{L}_n = \mathbf{S}_n - \mathbf{S}_{n-1}$, $n \in I\!N$. As a combination of the two notions, wide-sense regenerative and $k$-dependent processes, we define a third regeneration related notion, which generalizes the wide-sense regeneration, as follows:

**Definition B.3.** *The process* $\mathbf{X}$ *is said to be wide-sense* $k$*-dependent with* $k \in I\!N$ *and the sequence* $\mathbf{S}$ *if* $(\mathbf{X}, \mathbf{S})$ *satisfies (B.2) and*

$$\phi_{\mathbf{S}_n}(\mathbf{X}, \mathbf{S}) \text{ is independent of } (\mathbf{S}_0, \ldots, \mathbf{S}_{n-k}), \quad n \in I\!N. \tag{B.4}$$

**Observation B.4.** *A very important difference between the wide-sense regenerative and classical regenerative properties is the following (cf.* [21]*): If* $(\mathbf{X}, \mathbf{S})$ *is classical regenerative the path process* $(\phi_s\mathbf{X})_{s\in[0,\infty)}$ *with state space* $(H, \mathcal{H})$ *is in general not classical regenerative (unless* $\mathbf{X}$ *is a non-random constant), but it is wide-sense regenerative with regeneration times* $\mathbf{S}$*. The wide-sense regenerative, as well the wide-sense* $k$*-dependent regenerative properties also preserve under measurable functions* $f$*, defined on* $(H, \mathcal{H})$ *into some measurable space.*

Thorisson (cf. [21]) also notes here that although the Markovian property does not preserve under measurable functions, *for any general stochastic process* $\mathbf{X}$ *the path process* $(\phi_s\mathbf{X})_{s\in[0,\infty)}$ *is always a Markov process!*

## Appendix C. The cost of the delay cycle

Our proof for the average cost expression, in Section 5.1 uses Proposition 3.1 of Glynn and Sigman(cf. [11]). In order to make the use of this proposition 'legal', we need to prove that the expected cost of the delay cycle is finite, that is, $I\!E\int_0^{\mathbf{S}_0} g(\phi_s \circ \mathbf{X})ds < \infty$. By the findings of Section 5.1 (relation (5.7)) this is equivalent with proving that

$$I\!E\left(\sum_{k=1}^{\mathbf{N}_0} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right) < \infty. \tag{C.1}$$

The idea of the proof is exactly as that of Proposition 4.2 of [11]. Let

$$r(z) := I\!E_z \sum_{k=1}^{\mathbf{N}_0} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}),$$

and $\varepsilon := \{z : r(z) < \infty\}$, and we need to show (cf.Proposition 4.2, [11]) that $\pi_\infty(\varepsilon) = 1$. Denoting with $I\!E^0$ the expectation associated with the probability measure under which $\mathbf{X}$ is

non-delayed (see Section 2.1),

$$\text{(C.2)} \qquad \begin{aligned} \pi_\infty(\varepsilon) &= \frac{1}{I\!E\mathbf{N}_1} I\!E^0\left(\sum_{k=1}^{\mathbf{N}_1} 1_{\{r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}))<\infty\}}\right) \\ &= \frac{1}{I\!E\mathbf{N}_1}\sum_{k=1}^{\infty} I\!P^0\{r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}))<\infty, \mathbf{N}_1 > k\}. \end{aligned}$$

On the other hand, using the notation $z_k := I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})$, we have

$$\begin{aligned} I\!E^0(r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})), \mathbf{N}_1 > k) &= I\!E^0\left(I\!E^0_{z_k}\left(\sum_{k=1}^{\mathbf{N}_1} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})\right), \mathbf{N}_1 > k\right) \\ &= I\!E^0\left(I\!E^0_{z_k}\left(\sum_{k=1}^{\infty} I\!E(\mathbf{J}_k|\mathbf{A}_{k-1}), \mathbf{N}_1 > k\right)\right) \\ &= I\!E^0\left(I\!E^0\left(\sum_{n=k}^{\infty} I\!E(\mathbf{J}_n|\mathbf{A}_{n-1}), \mathbf{N}_1 > k\right)\right) \\ &= I\!E^0\left(\sum_{n=k}^{\infty} I\!E(\mathbf{J}_n|\mathbf{A}_{n-1})\right) \\ &\le I\!E^0\left(\sum_{n=1}^{\infty} I\!E(\mathbf{J}_n|\mathbf{A}_{n-1})\right) < \infty, \end{aligned}$$

which means that $I\!P^0\{r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})) < \infty, \mathbf{N}_1 > k\} = 1$. In the very first equality of the above evaluation we use the fact that under the non-delay expectation $I\!E^0$, we can consider $r(z)$ as the expected cost of the first regeneration cycle, started in an arbitrary state, $z_k$. Having thus

$$\sum_{k=1}^{\infty} I\!P^0\{r(I\!E(\mathbf{J}_k|\mathbf{A}_{k-1})) < \infty, \mathbf{N}_1 > k\} = \sum_{k=1}^{\infty} I\!P^0\{\mathbf{N}_1 > k\} = I\!E\mathbf{N}_1,$$

it follows immediately by relation (C.2) that $\pi_\infty(\varepsilon) = 1$. This proves thus that the expected cost of the delay-cycle is finite.

## References


1. S. Asmussen, *Applied probability and queues*, John Wiley & Sons, New York, 1987.
2. S. Asmussen and H. Thorisson, *A markov chain approach to periodic queues*, Journal of Applied Probability **24** (1987), 215–225.
3. S. Axsäter, *Simple evaluation of echelon stock* $(r, q)$ *policies for two-level inventory systems*, IIE Transactions **29** (1997), 661–669.

4. E.M. Bázsa and P.W. den Iseger, *Single item inventory models: A time and event averages approach*, Discussion Paper Series 101/4, Tinbergen Institute, Rotterdam, 2001, Submitted to Operations Research.
5. ———, *A two level decentralized distribution system with compound renewal demand*, Econometric Institute Report EI2002-45, Econometric institute, Rotterdam, 2002.
6. P. Billingsley, *Probability and measure*, second ed., Probability and Mathematical Statistics, John Wiley & Sons, New York, 1986.
7. ———, *Probability and measure*, third ed., Probability and Mathematical Statistics, John Wiley & Sons, New York, 1995.
8. G De Leve, A. Federgruen, and H.C. Tijms, *A general markov decision method i: Model and techniques*, Advances in Applied Probability **9** (1977), no. 2, 296–315.
9. A. Federgruen and P.H. Zipkin, *Computing optimal* $(s, S)$ *policies in inventory models with continuous demands*, Advances in Applied Probability **17** (1985), 424–442.
10. P. Glynn, *Some topics in regenerative steady-state simulation*, Acta Applicandae Mathematicae **34** (1994), 225–236.
11. P. Glynn and K. Sigman, *Uniform cesaro limit theorems for synchronous processes with applications to queues*, Stochastic Processes and their Applications **40** (1992), 29–43.
12. C.W.J. Granger, *Forcasting in business and economics*, second ed., Academic Press, London, 1989.
13. P.J. Haas, *Generalized semi – markov processes*, Lecture Notes on Simulation #3 (Stanford University), MS & E 223, 2000.
14. W.S. Lovejoy, *Stopped myopic policies in some inventory models with generalized demand processes*, Management Science **38** (1992), no. 5, 688–707.
15. S. Orey, *Limit theorems for markov chain transition probabilities*, Van Nostrand, London, 1971.
16. S.M. Ross, *Applied probability with optimization applications*, Holden–Day, San Francisco, 1970.
17. ———, *Introduction to stochastic dynamic programming*, Monographs and Textbooks, Academic Press, San Diego, California, 1983.
18. K. Sigman, *One-dependent regenerative processes and queues in continuous time*, Mathematics of Operations Research **15** (1990), no. 1, 175–189.
19. W.A. Smith, *Regenerative stochastic processes*, Proc. Roy. Soc. London Ser. **A 232** (1955), 6–31.
20. J. Song and P.H. Zipkin, *Inventory control in a fluctuating demand environment*, Operations Research **41** (1993), no. 2, 351–370.
21. H. Thorisson, *Coupling, stationarity, and regeneration*, Probability and its Application, Springer - Verlag, New York, 2000.
22. H.C. Tijms, *Stochastic models: An algorithmic approach*, John Wiley & Sons, New York, 1994.
23. D. Williams, *Probability with martingales*, $3^{\text{rd}}$ ed., Cambridge University Press, Cambridge, 1991.
24. P.H. Zipkin, *Stochastic leadtimes in continuous-time inventory models*, Naval Research Logistics Quarterly **33** (1986), 763–774.

25. ———, *Foundations of inventory management*, Operations and Decision Sciences, McGraw-Hill, Higher Education Div., Boston, 2000.