VECTOR-VALUED LOGARITHMIC RESIDUES AND THE EXTRACTION OF ELEMENTARY FACTORS

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ABSTRACT. An analysis is presented of the circumstances under which, by the extraction of elementary factors, an analytic Banach algebra valued function can be transformed into one taking invertible values only. Elementary factors are generalizations of the simple scalar expressions $\lambda - \alpha$, that is functions of the type $e - p + (\lambda - \alpha)p$ with p an idempotent. The analysis elucidates old results (such as on Fredholm operator valued functions) and yields new insights which are brought to bear on the study of vector-valued logarithmic residues. Examples illustrate the subject matter and show that new ground is covered. Also a long standing open problem is discussed from a fresh angle.

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1. INTRODUCTION

Let $\mathcal{B}$ be a unital complex Banach algebra. A logarithmic residue in $\mathcal{B}$ is a contour integral of a logarithmic derivative of an analytic $\mathcal{B}$-valued function F. There is a left version and there is a right version of this notion. The left version corresponds to the left logarithmic derivative $F'(\lambda)F(\lambda)^{-1}$, the right version to the right logarithmic derivative $F(\lambda)^{-1}F'(\lambda)$.

The first to consider integrals of this type in a vector valued context, was L. Mittenthal [M]. His goal was to generalize the spectral theory of a single Banach algebra element (i.e., the case where $F(\lambda) = \lambda e - b$ with e the unit element in $\mathcal{B}$ and $b \in \mathcal{B}$). He succeeded in giving sufficient conditions for a logarithmic residue to be an idempotent. The conditions in question, however, are very restrictive. On the other hand they elucidate why the spectral case "works". An unclear point in [M] was clarified in [B]).

Logarithmic residues also appear in the paper [GS1] by I. Gohberg and E. Sigal. The setting there is $\mathcal{B} = \mathcal{L}(X)$, the Banach algebra of all bounded linear operators on a complex Banach space, and F is a Fredholm operator valued function. For such functions Gohberg and Sigal introduced the concept of algebraic (or null) multiplicity. It turns out that the algebraic multiplicity of F with respect to a given contour is equal to the trace of the corresponding (left/right) logarithmic residues (see also [BKL2] and [GGK]). For analytic matrix functions, such a result was obtained in [MS].

Further progress was made in [BES2]–[BES8]. In these papers, logarithmic residues are studied from different angles and perspectives. For an overview of the issues dealt with, see for instance the Introduction of [BES8]. Two of those lie at the heart of the present paper. First, and mainly,

Problem 1 *What kind of elements are logarithmic residues?*

Second, and to a lesser extent,

Problem 2 *If a logarithmic residue vanishes, under what circumstances does it follow that the function in question takes invertible values inside the integration contour?*

These problems are investigated here for functions satisfying certain restrictions: they allow for a certain type of representation modelled after what has been observed in important cases such as analytic Fredholm operator valued functions. These representations are interesting in their own right as they involve the extraction of elementary factors of the form $e - p + (\lambda - \alpha)p$ with $p \in \mathcal{B}$ an idempotent. Such elementary factors generalize the scalar functions $\lambda - \alpha$ familiar from complex function theory.

Apart from the introduction (Section 1) and the list of references, the paper consists of eight sections. Here is a brief description of their contents.

Section 2 is concerned with vector valued logarithmic residues and is mainly meant for easy reference. The material can be skipped, at least in first reading, and later be consulted as far as need arises.

Section 3 deals with elementary functions. As already indicated above, these are functions of the type $e - p + (\lambda - \alpha)p$ where $p \in \mathcal{B}$ is an idempotent. The emphasis is on commutativity properties of elementary functions which are important for the study of plain functions.

Section 4 contains the basic facts about plain functions. Their defining property is that they can be written as the product of an everywhere invertible analytic function and an elementary polynomial. Here an elementary polynomial is a product of elementary functions (so simply a polynomial in the scalar case). In view of the commutativity properties of elementary functions referred to above, plain functions are characterized by the fact that, in the sense of [GKL], they are analytically equivalent to an elementary polynomial.

The notion of a plain function has its roots [GS1], [GS2], [BKL1] and [T]. The idea of extracting factors that are in a sense elementary or irreducible also appears in systems theory. See, for instance, the material on the Callier-Desoer class of transfer functions in [CZ], Chapter 7; cf., also [BGK], Sections 1.3 and 3.2, and [BKZ].

Section 5 continues the analysis of plain functions. Matrix valued analytic functions and, more generally, analytic Fredholm valued functions, are plain on each open set containing only a finite number of points where non-invertible values are assumed. A similar theorem is true for functions having a simply meromorphic resolvent. Underlying these results is the notion of an annihilating idempotent for a zero product in the Banach algebra under consideration. With the help of this concept it is possible to formulate a quite general sufficient condition for a function to be plain.

Section 7 contains three striking examples on the extraction of elementary factors. Two of these are concerned with non-plain functions.

The general sufficient condition for a function to be plain meant in the all but last paragraph should be appreciated in light of examples where it is satisfied. Such examples are discussed in Section 6. One of them covers the situation where the underlying Banach algebra is of the type $\mathcal{L}(H)$, where H is a complex Hilbert space. Special attention is paid to the finite dimensional case.

Section 8 deals with logarithmic residues of plain functions. In earlier publications, several instances have been given where logarithmic residues are sums of idempotents. On the other hand there are also cases in which they not even belong to the closed algebra generated by the idempotents. The plain functions considered here occupy an intermediate position: their logarithmic residues turn out to be linear combinations of (monomials in) idempotents with integer coefficients. This adds to the understanding of Problem 1. The section also contains a contribution to the discussion of Problem 2. Here families of traces on Banach algebras play an essential role. The paper closes with a discussion concerning a long standing open question concerning Problem 2 and zero sums of idempotents.

`logres`

2. Basics on vector valued logarithmic residues

We shall here recall the introductory background material on vector valued logarithmic residues needed later. Also some fundamental facts concerning the relationship with sums of idempotents are reviewed (cf., Problem 1 from the Introduction). One can postpone consulting this section until need arises.

As before, $\mathcal{B}$ is a unital complex Banach algebra. If F is a $\mathcal{B}$-valued function with domain Δ, then F^{-1} stands for the *resolvent* of F. Thus F^{-1} is the function given by $F^{-1}(\lambda) = F(\lambda)^{-1}$, the domain being the *resolvent set* of F, that is the set of all $\lambda \in \Delta$ such that $F(\lambda)$ is invertible. If Δ is an open subset of the complex plane $\mathbb{C}$ and $F : \Delta \to \mathcal{B}$ is analytic, then so is F^{-1} on the resolvent set of F. Further, the *left*, respectively *right*, *logarithmic derivative* of F is the function given by $F'(\lambda)F^{-1}(\lambda)$, respectively $F^{-1}(\lambda)F'(\lambda)$, where F' denotes the derivative of F. These logarithmic derivatives are defined and analytic on the resolvent set of F.

Logarithmic residues are contour integrals of logarithmic derivatives. To make this notion more precise, we shall employ bounded Cauchy domains in $\mathbb{C}$ and their positively oriented boundaries. For a discussion of these notions, see [TL].

Let D be a bounded Cauchy domain in $\mathbb{C}$. The (positively oriented) boundary of D will be denoted by ∂D. We write $\mathcal{A}_\partial(D; \mathcal{B})$ for the set of all $\mathcal{B}$-valued functions F with the following properties: F is defined and analytic on a neighborhood of the closure $\overline{D} = D \cup \partial D$ of D and F takes invertible values on all of ∂D (hence F^{-1} is analytic on a neighborhood of ∂D). For $F \in \mathcal{A}_\partial(D; \mathcal{B})$, one can define the contour integrals

$$LR_{\mathrm{left}}(F; D) = \frac{1}{2\pi i} \int_{\partial D} F'(\lambda)F^{-1}(\lambda)\, d\lambda, \qquad (1) \qquad \texttt{p1}$$

$$LR_{\mathrm{right}}(F; D) = \frac{1}{2\pi i} \int_{\partial D} F^{-1}(\lambda)F'(\lambda)\, d\lambda. \qquad (2) \qquad \texttt{p2}$$

The elements of the form (1) or (2) are called logarithmic residues in $\mathcal{B}$. More specifically, we call $LR_{\mathrm{left}}(F; D)$ the *left* and $LR_{\mathrm{right}}(F; D)$ the *right logarithmic residue of F with respect to D*.

It is convenient to introduce a local version of these concepts too. Given a complex number λ_0, we let $\mathcal{A}(\lambda_0; \mathcal{B})$ be the set of all $\mathcal{B}$-valued functions F with the following properties: F is defined and analytic on an open neighborhood of λ_0 and

F takes invertible values on a deleted neighborhood of λ_0. For $F \in \mathcal{A}(\lambda_0; \mathcal{B})$, one can introduce

$$LR_{\text{left}}(F; \lambda_0) = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} F'(\lambda) F^{-1}(\lambda) d\lambda, \tag{3}$$ p3

$$LR_{\text{right}}(F; \lambda_0) = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} F^{-1}(\lambda) F'(\lambda) d\lambda, \tag{4}$$ p4

where ϱ is a positive number such that both F and F^{-1} are analytic on an open neighborhood of the punctured closed disc with center λ_0 and radius ϱ. The orientation of the integration contour $|\lambda - \lambda_0| = \varrho$ is, of course, taken positively, that is counterclockwise. Note that the right hand sides of (3) and (4) do not depend on the choice of ϱ. In fact, (3) and (4) are equal to the coefficient of $(\lambda - \lambda_0)^{-1}$ in the Laurent expansion at λ_0 of the left and right logarithmic derivative of F at λ_0, respectively. Obviously, $LR_{\text{left}}(F; \lambda_0)$, respectively $LR_{\text{right}}(F; \lambda_0)$, is a left, respectively right, logarithmic residue of F in the sense of the definitions given in the preceding paragraph (take for D the open disc with radius ϱ centered at λ_0). We call $LR_{\text{left}}(F; \lambda_0)$ the *left* and $LR_{\text{right}}(F; \lambda_0)$ the *right logarithmic residue of F at λ_0*.

In certain cases, the study of logarithmic residues with respect to bounded Cauchy domains can be reduced to the study of logarithmic residues with respect to single points. The typical situation is as follows. Let D be a bounded Cauchy domain, let $F \in \mathcal{A}_\partial(D; \mathcal{B})$ and suppose F takes invertible values on D except in a finite number of distinct points $\alpha_1, \ldots, \alpha_n \in D$. Then

$$LR_{\text{left}}(F; D) = \sum_{j=1}^{n} LR_{\text{left}}(F; \alpha_j), \tag{5}$$ p5

$$LR_{\text{right}}(F; D) = \sum_{j=1}^{n} LR_{\text{right}}(F; \alpha_j). \tag{6}$$ p6

This occurs, in particular, when F^{-1} is meromorphic on D with a finite number of poles in D, a state of affairs that we will encounter in what follows.

The particular case when all poles of F^{-1} of are *simple*, i.e., they have order one, is of special interest here. Indeed, the functions that we will investigate turn out to be products of functions of this type. Also, there is an important connection with Problem 1 from the Introduction, as is seen from the following two results taken out of [BES6].

tp3 **Theorem 2.1.** *Let $\lambda_0 \in \mathbb{C}$, let $F \in \mathcal{A}(\lambda_0; \mathcal{B})$, and suppose F^{-1} has a simple pole at λ_0. Write p and q for the left and right logarithmic residue of F at λ_0, i.e.,*

$$p = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} F'(\lambda) F^{-1}(\lambda) d\lambda,$$

$$q = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} F^{-1}(\lambda) F'(\lambda) d\lambda,$$

where ϱ is positive and sufficiently small. Then p and q are non-zero idempotents. Also p and q are similar, i.e., $p = s^{-1} q s$ for some invertible $s \in \mathcal{B}$.

tp1 **Theorem 2.2.** *Let $x \in B$, where $\mathcal{B}$ is a complex Banach algebra, and let D be a bounded Cauchy domain in $\mathbb{C}$. The following statements are equivalent:*

(i) *x is a sum of idempotents in $\mathcal{B}$;*
(ii) *x is the left logarithmic residue with respect to D of a function F in $\mathcal{A}_\partial(D;\mathcal{B})$ such that F^{-1} is meromorphic on D with a finite number of poles each of which is simple;*
(iii) *x is the right logarithmic residue with respect to D of a function F in $\mathcal{A}_\partial(D;\mathcal{B})$ such that F^{-1} is meromorphic on D with a finite number of poles each of which is simple.*

The implications (ii)⇒(i) and (iii)⇒(i) follow from Theorem 2.1 with the help of (5) and (6). The fact that (i) implies both (ii) and (iii) is easy to prove when the number of connected components in D is larger than or equal to the number of terms in the sum of idempotents x (cf., [BES2]). Things are considerably more complicated when this is not the case, especially when D is connected. This situation is covered by the following theorem which is a slight reformulation of the result obtained by one of the authors (T. Ehrhardt) in [E]. A Banach algebra valued function is called *entire* when it is defined and analytic on all of $\mathbb{C}$.

tp2 **Theorem 2.3.** *Let $p_1,\dots,p_n$ be nonzero idempotents in the complex Banach algebra $\mathcal{B}$ and let $\lambda_1,\dots,\lambda_n$ be distinct (but otherwise arbitrary) complex numbers. Then there exists an entire analytic $\mathcal{B}$-valued function F such that F takes invertible values on all of $\mathbb{C}$, except for $\lambda_1,\dots,\lambda_n$, where F^{-1} has simple poles, while in addition,*

$$LR_{\text{left}}(F;\lambda_j) = LR_{\text{right}}(F;\lambda_j) = p_j, \qquad j = 1,\dots,n.$$

For completeness, we mention that the function F constructed in [E] is a product of $3n$ factors, each of them a function of the form $e-p+\varphi(\lambda)p$ where p is one of the given idempotents and φ is an entire scalar function. For a refinement of Theorem 2.3, see Theorem 2.1 in [BES8].

elfcalc 3. Elementary functions: a calculus

As before $\mathcal{B}$ stands for a unital Banach algebra. Its unit element will be denoted by e.

Given an idempotent p in $\mathcal{B}$ and a complex number α, let $E_{p,\alpha}$ be the (entire) function defined by

$$E_{p,\alpha}(\lambda) = e - p + (\lambda-\alpha)p, \qquad \lambda \in \mathbb{C}.$$

Such functions will be called *elementary*. More precisely, we will say that $E_{p,\alpha}$ is an *elementary function based at α*.

If p is the zero element in $\mathcal{B}$, then $E_{p,\alpha}(\lambda)$ is identically equal to e, so multiplication with $E_{p,\alpha}$ amounts to the same as multiplication with the scalar 1. If p is the unit element element in $\mathcal{B}$, then $E_{p,\alpha}(\lambda)$ is equal to $(\lambda-\alpha)e$, so multiplication with $E_{p,\alpha}$ amounts to the same as multiplication with the scalar function $\lambda-\alpha$.

Taking α equal to zero, we get $E_{p,0}$ with $E_{p,0}(\lambda) = e - p + \lambda p$. This function is multiplicative:

$$E_{p,0}(\lambda\mu) = E_{p,0}(\lambda)E_{p,0}(\mu), \qquad \lambda,\mu \in \mathbb{C}.$$

Also $E_{p,0}(1) = e$ and $E_{p,0}\left(\frac{1}{\lambda}\right) = E_{p,0}(\lambda)^{-1}$ whenever $\lambda \neq 0$.

Returning to the general situation, note that

$$E_{p,\alpha}(\lambda) = E_{p,0}(\lambda-\alpha), \qquad \lambda \in \mathbb{C}.$$

and the multiplicativity property for $E_{p,0}$ translates into

$$E_{p,\alpha}(\alpha+\lambda\mu) = E_{p,\alpha}(\alpha+\lambda)E_{p,\alpha}(\alpha+\mu), \qquad \lambda,\mu\in\mathbb{C}.$$

Finally, $E_{p,\alpha}(\lambda)$ is invertible whenever $\lambda\neq\alpha$ and

$$E_{p,\alpha}(\lambda)^{-1} = e-p+\frac{1}{(\lambda-\alpha)}p = E_{p,\alpha}\Big(\alpha+\frac{1}{\lambda-\alpha}\Big), \qquad \lambda\in\mathbb{C};\ \lambda\neq\alpha.$$

In case p is not equal to the zero element in $\mathcal{B}$, the function $E_{p,\alpha}^{-1}$ has a simple pole (that is a pole of order one) at α. For $\alpha=0$ the expression for the inverse reduces to the simple identity

$$E_{p,0}(\lambda)^{-1} = e-p+\lambda^{-1}p = E_{p,0}\big(\lambda^{-1}\big), \qquad \lambda\in\mathbb{C};\ \lambda\neq 0 \tag{7}$$ inver

which will be used in Example 8.1 below.

Elementary functions have certain commutativity properties that will be used later. The first result in this direction is the following proposition which features as Remark 4.1 in [BES6]. For the convenience of the reader we present it with proof.

comelf1 **Proposition 3.1.** *Let Δ be a non-empty open subset of $\mathbb{C}$, let $G:\Delta\to B$ be analytic, let $p\in\mathcal{B}$ be an idempotent and let $\alpha\in\Delta$. Suppose G takes invertible values on all of Δ and put $q=G(\alpha)^{-1}pG(\alpha)$. Then q is an idempotent (similar to p) and there exists an analytic function $H:\Delta\to\mathcal{B}$ such that H takes invertible values on all of Δ and*

$$E_{p,\alpha}(\lambda)G(\lambda) = H(\lambda)E_{q,\alpha}(\lambda), \qquad \lambda\in\Delta. \tag{8}$$ cfe1

Proof. Introduce

$$H(\lambda) = \begin{cases} E_{p,\alpha}(\lambda)G(\lambda)E_{q,\alpha}^{-1}(\lambda), & \lambda\in\Delta,\ \lambda\neq\alpha,\\[2mm] G(\alpha)+(e-p)G'(\alpha)G(\alpha)^{-1}pG(\alpha), & \lambda=\alpha.\end{cases}$$

Then H is analytic on $\Delta\setminus\{\alpha\}$ and takes invertible values there. Also $H(\lambda)\to H(\alpha)$ when $\lambda\to\alpha$, so H is analytic on all of Δ. One verifies easlily that $H(\alpha)$ is invertible with inverse $H(\alpha)^{-1} = G(\alpha)^{-1}-G(\alpha)^{-1}(e-p)G'(\alpha)G(\alpha)^{-1}p$. For values of λ different from α, the identity (8) is obvious from the definition of $H(\lambda)$. For $\lambda=\alpha$ it follows by continuity, but a direct computation using the definition of $H(\alpha)$ works too. □

Proposition 3.1 says that when we have a product with an elementary function $E_{p,\alpha}$ on the left and an everywhere invertible function G on the right, we can bring the elementary function to the right and the everywhere invertible function to the left at the expense of changing p into a similar idempotent q and G into another everywhere invertible function H. Here "everywhere" means: everywhere on the given domain of G, i.e., the open subset Δ of $\mathbb{C}$.

Of course the proposition has an analogue in which one starts with a function H and an idempotent q, and comes up with a function G and an idempotent p such that (8) holds. The details are as follows: $p=H(\alpha)qH(\alpha)^{-1}$, so p is an idempotent similar to q,

$$G(\lambda) = \begin{cases} E_{p,\alpha}^{-1}(\lambda)H(\lambda)E_{q,\alpha}(\lambda), & \lambda\in\Delta,\ \lambda\neq\alpha,\\[2mm] H(\alpha)+H(\alpha)qH(\alpha)^{-1}H'(\alpha)(e-q), & \lambda=\alpha,\end{cases}$$

and $G(\alpha)^{-1} = H(\alpha)^{-1}-qH(\alpha)^{-1}H'(\alpha)(e-q)H(\alpha)^{-1}$.

Now start with the data of Proposition 3.1. With these we produce H and q as indicated in the proof. But then, in turn, we can go on along the lines indicated in the previous paragraph and transform H and q into a new function and a new idempotent. Let us denote these now by G_0 and p_0, respectively. At this point, it is natural to ask for the relationship between G_0 and p_0 on the one hand and the original function G and idempotent p on the other. We look here at the idempotents leaving the functions to the reader. From the definitions, we have $p_0 = H(\alpha)qH(\alpha)^{-1} = H(\alpha)G(\alpha)^{-1}pG(\alpha)H(\alpha)^{-1}$ and a simple computation gives

$$p_0 = p + (e-p)G'(\alpha)G^{-1}(\alpha)p. \tag{9}$$ relipot

At first sight, this might be somewhat surprising because the natural thing to expect is $p_0 = p$. Things become transparent when one notes that $q = G(\alpha)^{-1}pG(\alpha)$ is the right logarithmic residue (cf., Section 2 for the definition) at the point α of the function $E_{p,\alpha}(\lambda)G(\lambda)$ and, similarly, $p_0 = H(\alpha)qH(\alpha)^{-1}$ is the left logarithmic residue of the function $H(\lambda)E_{q,\alpha}(\lambda)$. However, the functions $E_{p,\alpha}(\lambda)G(\lambda)$ and $H(\lambda)E_{q,\alpha}(\lambda)$ coincide. Thus $p_0 = p$ if and only if p (the idempotent that we started with) happens to be the left logarithmic residue at α of the function $E_{p,\alpha}(\lambda)G(\lambda)$. By the way, it can also be seen directly that this left logarithmic residue is equal to the right hand side of (9). Indeed, observe that for the left logarithmic derivative of the function $E_{p,\alpha}(\lambda)G(\lambda)$ we have the expression

$$(\lambda-\alpha)^{-1}p + \big(e - p + (\lambda-\alpha)p\big)G'(\lambda)G^{-1}(\lambda)\big(e - p + (\lambda-\alpha)^{-1}p\big)$$

and determine the coefficient of $(\lambda-\alpha)^{-1}$.

To finish the discussion of Proposition 3.1, consider once more the function $E_{p,\alpha}(\lambda)G(\lambda)$. From what we have seen, we may suspect that certain changes in p and G will not affect this function, and this is indeed true. In fact, the freedom one has (and, in fact, all of it) is to replace p by $p + (e-p)bp$ (again an idempotent) and G by

$$\Big(e + (1+\alpha-\lambda)(e-p)bp\Big)G(\lambda)$$

(again invertible on all of Δ), were $b \in \mathcal{B}$ is arbitrary. One of the possible choices concerns a "distinguished" idempotent, namely the left logarithmic residue p_0 of the function $E_{p,\alpha}(\lambda)G(\lambda)$ at the point α, and the preceding paragraph makes clear that in a sense this is the natural choice (cf., (9); see also [BES6]). Analogous remarks can be made, of course, concerning functions of the alternative type $H(\lambda)E_{q,\alpha}(\lambda)$.

Proposition 3.1 is concerned with elementary functions based at one given point α. In our second commutativity observation, we look at a situation involving two elementary functions based at possibly different points. The result can be obtained quickly from [BES6], Lemma 3.2, but we prefer to give a direct proof providing some details.

comelf2 **Proposition 3.2.** *Let p and q be idempotents in $\mathcal{B}$, and let α and β be points in $\mathbb{C}$. Then there exist idempotents $\hat{p}$ and $\hat{q}$ in $\mathcal{B}$, and an entire function $G : \mathbb{C} \to \mathcal{B}$, such that G takes invertible values on all of $\mathbb{C}$ and*

$$E_{p,\alpha}(\lambda)E_{q,\beta}(\lambda) = E_{\hat{q},\beta}(\lambda)E_{\hat{p},\alpha}(\lambda)G(\lambda), \quad \lambda \in \mathbb{C}. \tag{10}$$ cfe4

In case α and β are different, $\hat{p}$ and $\hat{q}$ can be chosen such that $\hat{p}$ is similar to p and $\hat{q}$ is similar to q.

Combining this with Proposition 3.1, one sees that there is an analogous result where G is the first factor in the right hand side of (10) or G is in the middle between the elementary functions $E_{\hat{q},\beta}$ and $E_{\hat{p},\alpha}$.

Proof. The situation $\alpha = \beta$ is trivial: just take $\hat{q} = p$, $\hat{p} = q$ and let G be identically equal to the unit element in $\mathcal{B}$. So we (may) assume that $\alpha \neq \beta$. In view of the remarks made above, it need not come as a surprise that logarithmic residues play a role in the argument (see also [BES6]).

Let $\hat{q}$ be the left logarithmic residue of the function $E_{p,\alpha}(\lambda)E_{q,\beta}(\lambda)$ at β. Looking at the Laurent expansion of the left logarithmic derivative of this function, and actually computing its coefficient for $(\lambda-\beta)^{-1}$, we get

$$\hat{q} = E_{p,\alpha}(\beta) q E_{p,\alpha}^{-1}(\beta).$$

Note here that the invertibility of $E_{p,\alpha}(\beta)$ is guaranteed by the assumption $\alpha \neq \beta$. Clearly, $\hat{q}$ is an idempotent similar to q. Further, let $\hat{p}$ be the left logarithmic residue of the function $E_{\hat{q},\beta}^{-1}(\lambda)E_{p,\alpha}(\lambda)E_{q,\beta}(\lambda)$ at α. Again using Laurent expansions, we obtain

$$\hat{p} = E_{\hat{q},\beta}^{-1}(\alpha)\Big(p + \frac{1}{\alpha-\beta}(e-p)qp\Big)E_{\hat{q},\beta}(\alpha),$$

which can be rewritten as

$$\begin{aligned}\hat{p} &= E_{\hat{q},\beta}^{-1}(\alpha)\Big(e + \frac{1}{\alpha-\beta}(e-p)qp\Big)p\Big(e + \frac{1}{\beta-\alpha}(e-p)qp\Big)E_{\hat{q},\beta}(\alpha)\\ &= E_{\hat{q},\beta}^{-1}(\alpha)\Big(e + \frac{1}{\beta-\alpha}(e-p)qp\Big)^{-1}p\Big(e + \frac{1}{\beta-\alpha}(e-p)qp\Big)E_{\hat{q},\beta}(\alpha).\end{aligned}$$

Thus $\hat{p}$ is an idempotent similar to p. Define $G : \mathbb{C} \to \mathcal{B}$ by

$$G(\lambda) = \begin{cases} E_{\hat{p},\alpha}^{-1}(\lambda)E_{\hat{q},\beta}^{-1}(\lambda)E_{p,\alpha}(\lambda)E_{q,\beta}(\lambda), & \lambda \neq \alpha, \beta,\\[2mm] E_{\hat{p},\alpha}^{-1}(\beta)E_{p,\alpha}(\beta)\Big(e + \frac{1}{\beta-\alpha}qp(e-q)\Big), & \lambda = \beta,\\[2mm] E_{\hat{q},\beta}^{-1}(\alpha)\Big(e + \frac{1}{\alpha-\beta}(e-p)qp\Big)\Big(e + \frac{1}{\beta-\alpha}p\hat{q}(e-p)\Big)E_{q,\beta}(\alpha), & \lambda = \alpha. \end{cases}$$

Then G is analytic. The (rather tedious) detailed verification of this (by considering the Laurent expansions of G at β and α) is omitted. Evidently G takes invertible values on $\mathbb{C} \setminus \{\alpha, \beta\}$ and it is easily seen that $G(\alpha)$ and $G(\beta)$ are invertible too. We finish the proof by noting that (10) is satisfied. □

With respect to Proposition 3.2, one can raise a "back and forth issue" somewhat analogous to the one discussed above for Proposition 3.1. We refrain from pursuing this point here. Instead we present an interpretation of Propositions 3.1 and 3.2 which is perhaps somewhat loose, but nevertheless elucidating. Write $\mathcal{I}$ for the set of $\mathcal{B}$-valued functions that take invertible values on all of their domain and $\mathcal{E}_\alpha$ for the set of $\mathcal{B}$-valued elementary functions based at α. On and between these sets the multiplication in $\mathcal{B}$ induces an (associative) multiplicative structure. Proposition 3.1 can now be rephrased (with a notation that speaks for itself) as

$$\mathcal{E}_\alpha \times \mathcal{I} = \mathcal{I} \times \mathcal{E}_\alpha.$$

Thus it is fair to say that the sets $\mathcal{I}$ and $\mathcal{E}_\alpha$ commute. Proposition 3.2 can be written as $\mathcal{E}_\alpha \times \mathcal{E}_\beta \subset \mathcal{E}_\beta \times \mathcal{E}_\alpha \times \mathcal{I}$. Since $\mathcal{I} \times \mathcal{I} = \mathcal{I}$, it follows that

$$(\mathcal{E}_\alpha \times \mathcal{I}) \times (\mathcal{E}_\beta \times \mathcal{I}) = (\mathcal{E}_\beta \times \mathcal{I}) \times (\mathcal{E}_\alpha \times \mathcal{I}).$$

So the sets $\mathcal{E}_\alpha \times \mathcal{I}$ or, what comes down to the same, the sets $\mathcal{I} \times \mathcal{E}_\alpha$, are mutually commuting.

We conclude this section with a simple observation showing that the presence of $\mathcal{I}$ is essential here. In other words, in Proposition 3.2 the factor G cannot be missed.

comelf3 **Proposition 3.3.** *Let $p, q, \hat{p}, \hat{q}$ be idempotents in $\mathcal{B}$ and let α, β be complex numbers, $\alpha \neq \beta$. Then*

$$E_{p,\alpha}(\lambda)E_{q,\beta}(\lambda) = E_{\hat{q},\beta}(\lambda)E_{\hat{p},\alpha}(\lambda), \quad \lambda \in \mathbb{C} \tag{11}$$ cfe5

if and only if $p = \hat{p}$, $q = \hat{q}$ and $pq = qp$.

Proof. For the only if part, the argument is as follows. In (11), substitute first $\lambda = 1 + \alpha$ to obtain $q = \hat{q}$, and then $\lambda = 1 + \beta$ to get $p = \hat{p}$. Here the assumption $\alpha \neq \beta$ is used. With $p = \hat{p}$ and $q = \hat{q}$, the expression (11) becomes

$$E_{p,\alpha}(\lambda)E_{q,\beta}(\lambda) = E_{q,\beta}(\lambda)E_{p,\alpha}(\lambda), \quad \lambda \in \mathbb{C}.$$

Comparing the coefficients of λ^2 in the two sides of this identity gives $pq = qp$ as desired. The if part of the proposition is trivial (and true also when $\alpha = \beta$). $\square$

4. Plain functions: preliminaries and first results

plainprel

Products of elementary functions will be called *elementary polynomials*. Thus P is an elementary polynomial if it admits a representation

$$P(\lambda) = \prod_{k=1}^{n} E_{p_k,\alpha_k}(\lambda), \tag{12}$$ elpol1

with $\alpha_1, \ldots, \alpha_n$ points in $\mathbb{C}$ (not necessarily distinct) and $p_1, \ldots, p_n$ idempotents in $\mathcal{B}$. To avoid possible confusion: in products written in the Π-notation and involving possibly non-commuting factors, the order of the factors corresponds to the order of the indices. So in (12), the first factor is $E_{p_1,\alpha_1}(\lambda)$ and the last factor is $E_{p_n,\alpha_n}(\lambda)$:

$$P(\lambda) = E_{p_1,\alpha_1}(\lambda) \cdots E_{p_n,\alpha_n}(\lambda). \tag{13}$$ elpol

To include the case $n = 0$, we adhere to the standard practice of letting an empty product be equal to the unit element in $\mathcal{B}$. In the expressions (12) and (13), one can leave out the factors with $p_j = 0$, thereby obtaining representations involving non-zero idempotents only.

elpres **Proposition 4.1.** *Let P be an elementary polynomial given by* (13) *and with all the idempotents $p_1, \ldots, p_n$ non-zero. Then P takes invertible values on $\mathbb{C}$, except in the points $\alpha_1, \ldots, \alpha_n$ where P takes non-invertible values and the meromorphic resolvent P^{-1} has its poles.*

Poles are always meant to have positive order.

Proof. For λ not one of the points $\alpha_1, \ldots, \alpha_n$, we have that $P(\lambda)$ is invertible while

$$P(\lambda)^{-1} = \prod_{j=1}^{n} \left(e - p_{n+1-j} + \frac{1}{\lambda - \alpha_{n+1-j}} p_{n+1-j} \right).$$

Hence P^{-1} is analytic on $\mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_n\}$ and meromorphic on $\mathbb{C}$. Also, the poles of P^{-1} are clearly among the points $\alpha_1, \ldots, \alpha_n$.

Let α be one of these points. We shall first prove that $P(\alpha)$ is not invertible. Let k be the smallest index such that $\alpha = \alpha_k$. Then $\alpha \neq \alpha_1, \dots, \alpha_{k-1}$ and, therefore, $E_{p_1,\alpha_1}(\alpha), \dots, E_{p_{k-1},\alpha_{k-1}}(\alpha)$ are invertible. But then the product of these elements $\prod_{j=1}^{k-1} E_{p_j,\alpha_j}(\alpha)$ is invertible too. Suppose $P(\alpha)$ is invertible. From

$$P(\alpha) = \Big(\prod_{j=1}^{k-1} E_{p_j,\alpha_j}(\alpha)\Big)\Big(\prod_{j=k}^{n} E_{p_j,\alpha_j}(\alpha)\Big)$$

we may then infer the invertibility of $\prod_{j=k}^{n} E_{p_j,\alpha_j}(\alpha)$. Now $E_{p_k,\alpha_k}(\alpha) = e - p_k$ and so

$$p_k \prod_{j=k}^{n} E_{p_j,\alpha_j}(\alpha) = p_k(e-p_k) \prod_{j=k+1}^{n} E_{p_j,\alpha_j}(\alpha) = 0.$$

Thus it would follow that $p_k = 0$, contrary to assumption. And so $P(\alpha)$ is not invertible indeed.

Next assume that P^{-1} does not have a pole (of positive order) at α, i.e., the principal part of the Laurent expansion of P^{-1} at α vanishes. Then $P_\alpha = \lim_{\lambda\to\alpha} P(\lambda)^{-1}$ exists and a continuity argument gives $P(\alpha)P_\alpha = e = P_\alpha P(\alpha)$. This contradicts the result obtained in the previous paragraph. $\square$

Let Δ be a non-empty open subset of the complex plane $\mathbb{C}$ and let F be a $\mathcal{B}-$valued function which is analytic on Δ. We say that F is *plain* on Δ if there exist an elementary polynomial P and an analytic function $G : \Delta \to B$ such that G takes invertible values on all of Δ and

$$F(\lambda) = P(\lambda)G(\lambda), \quad \lambda \in \Delta,$$

i.e., assuming P is given by (13),

$$F(\lambda) = E_{p_1,\alpha_1}(\lambda) \cdots E_{p_n,\alpha_n}(\lambda)G(\lambda), \quad \lambda \in \Delta. \tag{14}$$ plain1

In the next section, we will encounter interesting classes of plain vector or operator valued functions. For the moment, we just note that the notion of a plain function is clearly modelled after the situation for scalar analytic functions.

Here are some remarks in connection with the representation (14). First, changing (if necessary) G and the idempotents $p_1, \dots, p_n$, we can bring G to the left and position it as the first (instead of the last) factor. This follows from Proposition 3.1. More generally, one can put G somewhere in between, so that the representation is of the type

$$F(\lambda) = \left(\prod_{j=1}^{k-1} E_{p_j,\alpha_j}(\lambda)\right) G(\lambda) \left(\prod_{j=k}^{n} E_{p_j,\alpha_j}(\lambda)\right), \quad \lambda \in \Delta.$$

Here k can be any integer among $1, \dots, n+1$.

Along these lines, we have the following characterization of plain functions. Let Δ be a non-empty open subset of $\mathbb{C}$ and let $F : \Delta \to \mathcal{B}$. Then F is plain on Δ if and only if there exist an elementary polynomial P and analytic functions $G, H : \Delta \to \mathcal{B}$, taking invertible values on all of Δ, such that

$$F(\lambda) = H(\lambda)P(\lambda)G(\lambda), \quad \lambda \in \Delta.$$

Using terminology from [GKL] (see also [GGK]), this can be rephrased as follows:

an.eq **Proposition 4.2.** *Let Δ be a non-empty open subset of $\mathbb{C}$ and let $F : \Delta \to \mathcal{B}$. Then F is plain on Δ if and only if F is analytically equivalent on Δ to an elementary polynomial.*

Next let us consider the location of the points $\alpha_1, \dots, \alpha_n$. An elementary function based at a point outside Δ takes invertible values on all of Δ. Thus, applying Proposition 3.1 once again, one can come to the case where $\alpha_1, \dots, \alpha_n$ belong to Δ. On top of that the factors involving zero idempotents can always be left out. In the resulting situation where $\alpha_1, \dots, \alpha_n \in \Delta$ and $p_1, \dots, p_n$ are non-zero, Proposition 4.1 guarantees that F takes invertible values on Δ, except in the points $\alpha_1, \dots, \alpha_n$ where F takes non-invertible values and the (meromorphic) resolvent F^{-1} of F has poles (of positive order).

We now push this line of reasoning a little further and bring to bear Proposition 3.2 too.

cfeT2 **Proposition 4.3.** *Let Δ be a non-empty open subset of $\mathbb{C}$, let $F : \Delta \to \mathcal{B}$ be plain, and let $\beta_1, \dots, \beta_n$ be the distinct points in Δ where F takes non-invertible values (in any given order). Then there exist an analytic function $Q : \Delta \to \mathcal{B}$ taking invertible values on all of Δ, positive integers $m_1, \dots, m_n$, and non-zero idempotents $q_j^{(i)}$ $(i = 1, \dots, m_j; j = 1, \dots, n)$ in $\mathcal{B}$ such that*

$$F(\lambda) = \left(\prod_{i=1}^{m_1} E_{q_1^{(i)},\beta_1}(\lambda)\right) \dots \left(\prod_{i=1}^{m_n} E_{q_n^{(i)},\beta_n}(\lambda)\right) Q(\lambda), \quad \lambda \in \Delta.$$

The case $n = 0$ corresponds to the situation where F coincides with Q and takes invertible values on all of Δ.

Here are two simple non-trivial example of a non-plain function. For another one (with spectacular additional features), see Example 7.2 below.

nonplain **Example 4.4.** Let $n \geq 2$ and write $\mathcal{T}_n$ for the (commutative) Banach subalgebra of $\mathbb{C}^{n\times n}$ consisting of all upper triangular $n \times n$ Toeplitz matrices. Let $F : \mathbb{C} \to \mathcal{T}_n$ be given by $F(\lambda) = \lambda I_n - J_n$, where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ upper triangular nilpotent Jordan block. Then F is entire and takes invertible values on all of $\mathbb{C}$, except in the origin where

$$F^{-1}(\lambda) = \sum_{k=1}^{n} \frac{1}{\lambda^k} J_n^{k-1}$$

has a pole of order n. The function F is not plain on any open subset of $\mathbb{C}$ containing the origin. Suppose it is. Then F admits a representation of the form

$$F(\lambda) = \big(I_n - P_1 + \lambda P_1\big) \dots \big(I_n - P_k + \lambda P_k\big) G(\lambda) \tag{15}$$ tpl

with $P_1, \dots, P_k$ non-zero idempotents in $\mathcal{T}_n$, the function G analytic on an open neighborhood of the origin, and $G(0)$ invertible. However, the only non-zero idempotent in $\mathcal{T}_n$ is the $n \times n$ identity matrix. So in the present situation, the expression (15) amounts to $F(\lambda) = \lambda^k G(\lambda)$, and F has a zero at the origin of order k. Since $F(0) = -J_n \neq 0$ (because $n \geq 2$), it follows that $k = 0$. Hence $F(0) = G(0)$ has to be invertible, which $F(0) = -J_n$ obviously is not. $\square$

The next example involves a Banach algebra that has been examined in [BES7] and [BES8]. The space of all bounded linear operators on a Banach space X will be denoted by $\mathcal{L}(X)$.

nonplain.comp

Example 4.5. Let X be an infinite-dimensional separable Hilbert space (for instance ℓ_2), and let I be the identity operator on X. Consider the Banach subalgebra $\mathcal{L}_{\mathcal{C}}(X)$ of $\mathcal{L}(X)$ given by $\mathcal{L}_{\mathcal{C}}(X) = \{\alpha I + C \mid \alpha \in \mathbb{C},\ C \in \mathcal{C}(X)\}$, where $\mathcal{C}(X)$ is the closed ideal in $\mathcal{L}(X)$ consisting of all compact (bounded) linear operators on X. Define $F : \mathbb{C} \to \mathcal{L}_{\mathcal{C}}(X)$ by $F(\lambda) = \lambda I - A$, where A is a compact operators on X with $A^2 = 0$ and having range and null space of infinite dimension and codimension, respectively. Concrete examples of such an operator A are easy to construct when $X = \ell_2$. The function F is entire and takes invertible values on all of $\mathbb{C}$, except in the origin where F^{-1} has pole of order 2. Indeed,

$$F^{-1}(\lambda) = \frac{1}{\lambda^2}A + \frac{1}{\lambda}I_X,$$

where I_X the identity operator on X. We shall now prove that F is not plain on any open subset of $\mathbb{C}$ containing the origin. Suppose it is. Then F admits a representation of the form

$$F(\lambda) = \big(I_X - P_1 + \lambda P_1\big) \dots \big(I_X - P_k + \lambda P_k\big) G(\lambda),$$

with $P_1, \dots, P_k$ non-zero idempotents in $\mathcal{L}_{\mathcal{C}}(X)$, the function G analytic on an open neighborhood of the origin, and $G(0)$ invertible. It is sufficient to show that $P_1, \dots, P_k$ then necessarily are of finite rank. Indeed, because this would imply that $A = -F(0) = -\big(I_X - P_1\big) \dots \big(I_X - P_1\big) G(0)$ is a Fredholm operator, which obviously it is not.

The function $\big(I_X - P_1 + \lambda^{-1}P_1\big)F(\lambda) = \big(I_X - P_1 + \lambda P_1\big)^{-1}F(\lambda)$ is analytic at the origin (or, if one prefers, it has a removable singularity there). Hence $P_1 A = 0$. In other words, $\operatorname{Im} A$ is contained in $\operatorname{Ker} P_1$, and we may conclude that $\operatorname{Ker} P_1$ has infinite dimension. From [BES7], Proposition 2.1 we know that the idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ are the projections on X for which either the range or the null space has finite dimension, and it follows that P_1 has finite rank. We proceed by (finite) induction. So assume $P_1, \dots, P_{l-1}$ are of finite rank, $2 \le l \le k$. We need to show that P_l is then of finite rank too. Let L be the function defined by the expression

$$L(\lambda) = \left(I_X - P_l + \frac{1}{\lambda}P_l\right) \dots \left(I_X - P_1 + \frac{1}{\lambda}P_1\right) F(\lambda).$$

Then L is analytic at the origin. Also introduce

$$L_0(\lambda) = I_X - \left(I_X - P_{l-1} + \frac{1}{\lambda}P_{l-1}\right) \dots \left(I_X - P_1 + \frac{1}{\lambda}P_1\right),$$

and note that L_0 is a polynomial in λ^{-1} with finite rank coefficients (including the constant term). Hence the coefficients in the Laurent expansion of $L_0(\lambda)F(\lambda) = L_0(\lambda)(\lambda I_X - A)$ at the origin have finite rank too. For $\lambda \neq 0$, we have

$$\left(I_X - P_l + \frac{1}{\lambda}P_l\right) F(\lambda) = L(\lambda) + \left(I_X - P_l + \frac{1}{\lambda}P_l\right) L_0(\lambda)F(\lambda).$$

The coefficient of λ^{-1} in the left hand side of this expression is $-P_l A$, that in the right hand side clearly has finite rank. Thus $P_l A$ is of finite rank again. From this we see that the codimension of $\operatorname{Im} A \cap \operatorname{Ker} P_l$ in $\operatorname{Im} A$ is finite. As $\operatorname{Im} A$ is infinite dimensional, it follows that so is $\operatorname{Im} A \cap \operatorname{Ker} P_l$. But then, a fortiori, the dimension of $\operatorname{Ker} P_l$ is infinite. Using [BES7], Proposition 2.1 once more, we may now conclude that P_l is of finite rank, as desired. $\square$

We could have introduced a somewhat more general notion of a plain function by requiring that a representation of the type (14) should hold, maybe not on all of Δ, but on every bounded open set whose closure (in $\mathbb{C}$) is contained in Δ. Such a definition would accommodate situations where the number of points where F takes non-invertible values is infinite without accumulation point in Δ. Since we are mainly interested in logarithmic residues and the Cauchy domains involved are bounded (cf., Section 2), there is no need to consider such a generalization here.

zdivA

5. Plain functions and zero divisors

In this section, we will exhibit some relevant classes of plain functions. Let us start with a heuristic remark.

Let Δ be a non-empty open subset of the complex plane $\mathbb{C}$ and let F be an analytic $\mathcal{B}-$valued function. Then F is plain on Δ if (and only if) it can be transformed into an analytic function taking invertible values on all of Δ by multiplying from the left (or from the right) with resolvents of elementary functions, i.e., with functions of the type

$$E_{p,\alpha}^{-1}(\lambda) = e - p + \frac{1}{\lambda - \alpha} p \tag{16}$$

zdiv.1A

with $p \in \mathcal{B}$ an idempotent. In such a transformation process the analytic function F, whose resolvent F^{-1} (generally) does have poles, changes into a function, analytic still, but with a resolvent which is analytic on all of Δ too. Thus the effect of the multiplication is pole order reduction while keeping analyticity intact. It is instructive to consider this phenomenon in some detail. For simplicity we will take $\alpha = 0$, so that the right hand side of (16) becomes $e - p + \lambda^{-1}p$.

Let the $\mathcal{B}$-valued function F be analytic on a neighborhood U of the origin, taking invertible values on the deleted neighborhood $U \setminus \{0\}$, but with $F(0)$ not invertible. Suppose the resolvent F^{-1} has a pole of order m at the origin, so the Laurent expansion of F^{-1} at the origin looks like

$$F^{-1}(\lambda) = \sum_{j=-m}^{\infty} \lambda^j B_j$$

with $B_{-m} \neq 0$ and, as $F(0)$ is not invertible, $m \geq 1$. Now let p be an idempotent in $\mathcal{B}$ such that the function $\widehat{F}(\lambda) = \big(e - p + \lambda^{-1}p\big)F(\lambda)$ is analytic at the origin (so on all of U) while, moreover, there is pole order reduction. The latter in the sense that the Laurent expansion of $\widehat{F}^{-1}$ at the origin has the form

$$\widehat{F}^{-1}(\lambda) = F^{-1}(\lambda)\big(e - p + \lambda p\big) = \sum_{j=-m+1}^{\infty} \lambda^j \widehat{B}_j$$

with $\widehat{B}_{-m+1}$ automatically non-zero, otherwise B_{-m} would be zero too which is obvious from the identity $F^{-1}(\lambda) = \widehat{F}^{-1}(\lambda)\big(e - p + \lambda^{-1}p\big)$. The analyticity of $\widehat{F}$ at the origin amounts to the identity $pF(0) = 0$, the pole order reduction to $B_{-m}(e-p) = 0$. Equivalently: $F(0) = (e-p)F(0)$ and $B_{-m} = B_{-m}p$. These identities are in line with the fact that $B_{-m}F(0) = 0$ (obtained by multiplying the Laurent expansion for F^{-1} with the Taylor expansion for F, both of course taken at the origin). Indeed, writing $B_{-m}F(0) = 0$ as $B_{-m}F(0) = B_{-m}p(e-p)F(0) = 0$, we see that "behind" $B_{-m}F(0) = 0$, there is the particular zero product $p(e-p) = 0$.

We now take the converse route starting from $B_{-m}F(0) = 0$. Suppose this zero product comes about in the way indicated above, so by the existence of an idempotent $p \in \mathcal{B}$ such that $F(0) = (e-p)F(0)$ and $B_{-m} = B_{-m}p$. The first identity, rewritten as $pF(0) = 0$, makes it possible to extract an elementary factor from F in the sense that F can be written as

$$F(\lambda) = \big(e - p + \lambda p\big)\widehat{F}(\lambda), \quad \lambda \in U,$$

with $\widehat{F}$ analytic, not only on the deleted neighborhood $U \setminus \{0\}$ of the origin, but also at the origin itself. Just put $\widehat{F}(\lambda) = \big(e - p + \lambda^{-1}p\big)F(\lambda)$ and observe that $pF(0) = 0$ implies the analyticity of $\widehat{F}$ at the origin. The second identity $B_{-m} = B_{-m}p$, restated as $B_{-m}(e-p) = 0$, guarantees that there is pole reduction in the sense that either $\widehat{F}^{-1}$ has a pole at the origin of order $m-1$ (namely when $m \geq 2$), or (when $m = 1$) the principal part of the Laurent expansion of $\widehat{F}^{-1}$ at the origin vanishes altogether, in which case $\widehat{F}(0)$ is invertible.

Similar things arise in considering the extraction of elementary factors at the right. The basic identity $F(0)B_{-m} = 0$ playing a role there comes about now as $F(0)(e-p)pB_{-m} = 0$ and so corresponds to the special zero product $(e-p)p = 0$. In this sense the roles of the idempotent and its complementary idempotent are interchanged.

These considerations suggest the following definition. An idempotent $p \in \mathcal{B}$ is called *annihilating* for the (ordered) pair a, b of elements in $\mathcal{B}$ if

$$pa = b(e-p) = 0,$$

or, what amounts to the same,

$$a = (e-p)a, \quad b = bp.$$

A necessary condition for such an idempotent to exist is that $ba = 0$. Indeed, given the above identities, we have $ba = bp(e-p)a = 0$. Note that e is an annihilating idempotent for the pair $0, b$ and 0 is one for the pair $a, 0$. Thus the definition is only relevant for the situation where we have zero divisors. In this connection we also observe that if q is an idempotent $\mathcal{B}$, then q is annihilating for the pair $e-q, q$.

As a first application of these ideas, we consider functions possessing a simply meromorphic resolvent. A $\mathcal{B}$-valued function is said to be *simply meromorphic* on an open subset Δ of $\mathbb{C}$ if it is meromorphic on Δ with only simple poles (i.e., poles of order one). The following result improves on [BES6], Lemma 3.2. We present it with a fair indication of the proof.

simplyA **Theorem 5.1.** *Let Δ be a non-empty open subset of $\mathbb{C}$ and let $F : \Delta \to \mathcal{B}$ be analytic. If F^{-1} is simply meromorphic on Δ with a finite number of poles, then F is plain. In fact, the following more detailed result holds true. Suppose F takes invertible values on Δ, except in a finite number of distinct points $\alpha_1, \ldots, \alpha_n$ where F^{-1} has simple poles. Then there exist an analytic function $G : \Delta \to \mathcal{B}$ taking invertible values on all of Δ and non-zero idempotents $p_1, \ldots, p_n$ in $\mathcal{B}$ such that*

$$F(\lambda) = E_{p_1,\alpha_1}(\lambda) \ldots E_{p_n,\alpha_n}(\lambda)G(\lambda), \quad \lambda \in \Delta.$$

In such a representation, the idempotents $p_1, \ldots, p_n$ are necessarily similar to the left logarithmic residues $LR_{\mathrm{left}}(F;\alpha_1), \ldots, LR_{\mathrm{left}}(F;\alpha_n)$, respectively.

This is the left version of the theorem. There is also a right version in which the elementary factors are on the right hand side of G. We leave the details to the reader. Note also that we can now characterize the plain functions as those that are the product of analytic functions possessing a simply meromorphic resolvent.

Proof. In the case $n = 0$ there is nothing to prove. So assume n is positive.

Write $p_1 = LR_{\text{left}}(F; \alpha_1)$ and

$$F^{-1}(\lambda) = \frac{1}{\lambda - \alpha_1} F_{-1} + F_0 + (\lambda - \alpha_1) F_1 + \dots .$$

Then $p_1 = F'(\alpha_1) F_{-1}$. Now $F(\alpha_1) F_{-1} = 0$ and $F_{-1} F'(\alpha_1) + F_0 F(\alpha_1) = e$. Hence

$$p_1^2 = F'(\alpha_1)\Big(e - F_0 F(\alpha_1)\Big) F_{-1} = F'(\alpha_1) F_{-1} = p_1,$$

so p_1 is an idempotent. In an analogous fashion we have

$$(e - p_1) F(\alpha_1) = F(\alpha_1), \quad F_{-1} p_1 = F_{-1}.$$

Thus p_1 is an annihilating idempotent for the pair $F(\alpha_1), F_{-1}$. Put

$$\widehat{F}(\lambda) = \Big(e - p_1 + (\lambda - \alpha_1)^{-1} p_1\Big) F(\lambda)$$

Then $\widehat{F}$ is analytic at α_1 and we have pole reduction there, which in the present case of pole order one means that $\widehat{F}(\alpha_1)$ is invertible. Proceeding by induction, one obtains a representation as in the theorem.

The only thing left to verify is that in such a representation the idempotents $p_1, \dots, p_n$ are necessarily similar to the left logarithmic residues

$$LR_{\text{left}}(F; \alpha_1), \dots, LR_{\text{left}}(F; \alpha_n),$$

respectively. For this the argument is as follows. Let j be one of the integers $1, \dots, n$, put $\alpha = \alpha_j$, $p = p_j$ and write

$$F(\lambda) = H(\lambda) E_{p,\alpha}(\lambda) G(\lambda), \quad \lambda \in \Delta,$$

with $H(\alpha)$, $G(\alpha)$ invertible. Suppressing the variable λ where convenient, one has

$$\begin{aligned} F' F^{-1} &= H' H^{-1} + H E'_{p,\alpha} E^{-1}_{p,\alpha} H^{-1} + H E_{p,\alpha} G' G^{-1} E^{-1}_{p,\alpha} H^{-1} \\ &= H' H^{-1} + \frac{1}{\lambda - \alpha} H p H^{-1} + \\ &\quad + H\Big(e - p + (\lambda - \alpha) p\Big) G' G^{-1} \Big(e - p + \frac{1}{\lambda - \alpha} p\Big) H^{-1}. \end{aligned}$$

As $LR_{\text{left}}(F; \alpha)$ is the coefficient of $(\lambda - \alpha)^{-1}$ in this expression, we get

$$\begin{aligned} LR_{\text{left}}(F; \alpha) &= H(\alpha) p H^{-1}(\alpha) + H(\alpha)(e - p) g p H^{-1}(\alpha) \\ &= H(\alpha)\Big(p + (e - p) g p\Big) H^{-1}(\alpha) \end{aligned}$$

with $g = G'(\alpha) G^{-1}(\alpha)$. Along with p, the element $p + (e - p) g p$ is an idempotent. Also, $e + (e - p) g p$ is invertible with inverse $e - (e - p) g p$ and

$$\Big(p + (e - p) g p\Big)\Big(e + (e - p) g p\Big) = \Big(e + (e - p) g p\Big) p.$$

So, putting $s = H(\alpha)\Big(e + (e - p) g p\Big)$, we have $p = s^{-1} LR_{\text{left}}(F; \alpha) s$. □

Elaborating on the first part of the proof, note that p_1 can be chosen to be $LR_{\text{left}}(F;\alpha_1)$. More generally, for $p_1,\ldots,p_n$ we can take the idempotents inductively defined as follows:

$$\begin{aligned} p_1 &= LR_{\text{left}}(F;\alpha_1),\\ p_2 &= LR_{\text{left}}(E^{-1}_{p_1,\alpha_1}F;\alpha_2),\\ p_3 &= LR_{\text{left}}(E^{-1}_{p_2,\alpha_2}E^{-1}_{p_1,\alpha_1}F;\alpha_3),\\ &\ \ \vdots\\ p_n &= LR_{\text{left}}(E^{-1}_{p_{n-1},\alpha_{n-1}}\ldots E^{-1}_{p_1,\alpha_1}F;\alpha_n). \end{aligned}$$

An argument analogous to that presented in the second part of the proof yields that this can be rewritten as

$$\begin{aligned} p_1 &= q_1,\\ p_2 &= E_{p_1,\alpha_1}(\alpha_2)^{-1}q_2E_{p_1,\alpha_1}(\alpha_2),\\ p_3 &= E_{p_2,\alpha_2}(\alpha_3)^{-1}E_{p_1,\alpha_1}(\alpha_3)^{-1}q_3E_{p_1,\alpha_1}(\alpha_3)E_{p_2,\alpha_2}(\alpha_3),\\ &\ \ \vdots\\ p_n &= E_{p_{n-1},\alpha_{n-1}}(\alpha_n)^{-1}\ldots E_{p_1,\alpha_1}(\alpha_n)^{-1}q_nE_{p_1,\alpha_1}(\alpha_n)\ldots E_{p_{n-1},\alpha_{n-1}}(\alpha_n), \end{aligned}$$

where $q_1,\ldots,q_n$ are the left logarithmic residues of F at $\alpha_1,\ldots,\alpha_n$, respectively. Note that in this way, the idempotents $p_1,\ldots,p_n$ are fully (albeit implicitly) expressed in terms of $q_1,\ldots,q_n$ and $\alpha_1,\ldots,\alpha_n$.

The following example is concerned with annihilating idempotents in $\mathcal{L}(X)$, the Banach algebra of all bounded linear operators on a complex Banach space X. If $S\in\mathcal{L}(X)$, the expressions $\operatorname{Ker} S$ and $\operatorname{Im} S$ stand for the null space and image of S, respectively. The closure in X of $\operatorname{Im} S$ is denoted by $\overline{\operatorname{Im}}\, S$.

zdiv.ex1A **Example 5.2.** Let $S,T\in\mathcal{L}(X)$ where X is a complex Banach space. An idempotent $P\in\mathcal{L}(X)$, i.e., a bounded projection on X, is annihilating for the pair S,T if and only if

$$\operatorname{Im} S \subseteq \operatorname{Ker} P \subseteq \operatorname{Ker} T.$$

We conclude that there exists an annihilating idempotent for the pair S,T if and only if $\operatorname{Im} S \subseteq Z \subseteq \operatorname{Ker} T$ for some complemented subspace Z of X. Thus a sufficient condition for the pair S,T to have an annihilating idempotent is that $TS=0$ (which amounts to $\operatorname{Im} S\subseteq \operatorname{Ker} T$) and either $\overline{\operatorname{Im}}\, S$ or $\operatorname{Ker} T$ is complemented (take $Z=\overline{\operatorname{Im}}\, S\subseteq\operatorname{Ker} T$ or $Z=\operatorname{Ker} T\supseteq \operatorname{Im} S$). Hence, in the situation when X is isomorphic to a Hilbert space, the pair $S,T\in\mathcal{L}(X)$ has an annihilating idempotent if and only if $TS=0$. When X actually is a Hilbert space, the annihilating idempotent can be chosen to be an orthogonal projection on X. $\square$

We will now consider analytic Fredholm operator valued operator functions. For background material (including relevant references), see [GGK], Ch. XI. Here we briefly review a few results directly pertaining to the topic of the present paper.

Let X be a complex Banach space. A bounded linear operator T on X is called a Fredholm operator if its null space $\operatorname{Ker} T$ is finite dimensional and its range $\operatorname{Im} T$ has finite codimension in X (and is therefore closed). For what follows it is important to recall that $T\in\mathcal{L}(X)$ is Fredholm if and only if T is invertible modulo the finite rank operators on X. Thus T is Fredholm if and only if there exist $L,R\in\mathcal{L}(X)$

such that both $LT - I$ and $TR - I$ have finite rank. Here I is the identity operator on X.

Let Δ be a non-empty open subset of $\mathbb{C}$ and let $F : \Delta \to \mathcal{L}(X)$ be analytic and Fredholm operator valued. Assume the resolvent set of F is non-empty, i.e., $F(\lambda)$ is invertible for at least one $\lambda \in \Delta$, and Δ is connected. Then the set

$$\Xi = \{\lambda \in \Delta \,|\, F(\lambda) \text{ is not invertible}\}$$

is at most countable and has no accumulation point in Δ. The points of Ξ are poles of the resolvent F^{-1} of F (of positive order). Further the coefficients of the principal part of the Laurent expansion of F^{-1} at such a pole are finite rank operators on X and the constant term is a Fredholm operator (of index zero). Finally, if D is a bounded set whose closure (in $\mathbb{C}$) is contained in Δ, then the intersection of Ξ and D is a finite set (cf., the last paragraph of Section 4).

The following result is a slight refinement of [BES5], Proposition 3.1. Relevant references that were already given there are [T] and [GS2]. We will not repeat the proof here, but there is also no need to consult [BES5] on this point: the essentials can be found in the argument given for Theorem 5.5 below.

FredholmA

Theorem 5.3. *Let Δ be a non-empty open subset of $\mathbb{C}$ and let $F : \Delta \to \mathcal{L}(X)$ be analytic and Fredholm operator valued. If the number of points in Δ where F assumes non-invertible values is finite, then F is plain on Δ. In fact, the following more detailed result holds true. Let $\alpha_1, \ldots, \alpha_n$ be the distinct points in Δ where F takes non-invertible values (in any given order) and, for $j = 1, \ldots, n$, let m_j be the (positive) order of α_j as a pole of F^{-1}. Then there exist an analytic function $G : \Delta \to \mathcal{B}$ taking invertible values on all of Δ and non-zero finite rank projections $P_j^{(i)}$ $(i = 1, \ldots, m_j; j = 1, \ldots, n)$ on X such that*

$$F(\lambda) = \left(\prod_{i=1}^{m_1} E_{P_1^{(i)},\alpha_1}(\lambda)\right) \cdots \left(\prod_{i=1}^{m_n} E_{P_n^{(i)},\alpha_n}(\lambda)\right) G(\lambda), \quad \lambda \in \Delta.$$

Here we have the "everywhere" invertible factor G as the last. There is an analogous result where it is the first. Allowing for such factors both in the first and the last position, we enter the situation where F is analytically equivalent to the elementary polynomial in the middle (see Proposition 4.2). It is a remarkable fact that in the Fredholm case the middle term can be chosen to be of diagonal type involving mutually disjoint (commuting) projections, as indicated in [GGK], Ch. XI and [GS2]. For the matrix case (where X is finite dimensional), things come down to the Smith canonical form (see, for instance, [G], Ch. VI or [LT], Ch.7).

Returning to the general situation, let $\mathcal{J}$ be a (possibly non-closed two-sided) ideal in $\mathcal{B}$. An element y from $\mathcal{B}$ is said to be *$\mathcal{J}-$invertible* if there exist x and z in $\mathcal{B}$ such that both $xy - e$ and $yz - e$ belong to $\mathcal{J}$. Here, without loss of generality, x and z may be taken to be the same. Note that each invertible element of $\mathcal{B}$ is $\mathcal{J}-$invertible too. Also, the product of two $\mathcal{J}-$invertible elements is $\mathcal{J}-$invertible again. Finally, if u and y are elements in $\mathcal{B}$, at least one of the products uy and yu belongs to $\mathcal{J}$ and y is $\mathcal{J}-$invertible, then $u \in \mathcal{J}$. Here is the (routine) argument for this. Suppose $yu \in \mathcal{J}$. Since y is $\mathcal{J}-$invertible, there exists $x \in \mathcal{B}$ such that $xy - e \in \mathcal{J}$. But then $xyu - u = (xy - e)u \in \mathcal{J}$ too. Since $yu \in \mathcal{J}$ by assumption, $xyu \in \mathcal{J}$ as well. Hence $u = xyu - (xyu - u) \in \mathcal{J}$. The case when $uy \in \mathcal{J}$ can be treated similarly.

The following proposition will be used in the proof of the next theorem. The choice for the origin as the point under consideration is made for simplicity of notation. Any other point in the complex plane will do of course.

idealmeromorphy **Proposition 5.4.** *Let $\mathcal{J}$ be an ideal in $\mathcal{B}$ and let F be a $\mathcal{B}-$valued function, defined and analytic on a neighborhood of the origin. Suppose the resolvent F^{-1} of F has a pole at the origin. Then $F(0)$ is $\mathcal{J}-$invertible if and only if the coefficients of the Laurent expansion of F^{-1} at the origin belong to $\mathcal{J}$.*

In the case when $\mathcal{J}$ is closed, the argument is practically trivial (and an essential singularity at the origin does just as well as a pole): just pass through the quotient algebra $\mathcal{B}/\mathcal{J}$.

Proof. Suppose $F(0)$ is $\mathcal{J}-$invertible and write

$$F(\lambda) = \sum_{k=0}^{\infty} \lambda^k F_k, \qquad F^{-1}(\lambda) = \sum_{k=-m}^{\infty} \lambda^k G_k$$

with m a positive integer. Then $G_{-m}F_0 = 0$. Since $F_0 = F(0)$ is $\mathcal{J}-$invertible, we may conclude that $G_{-m} \in \mathcal{J}$. Assume now that $G_{-m}, \dots, G_{-n}$ are all in $\mathcal{J}$, where n is one of the integers $2, \dots, m$. Clearly

$$G_{-n+1}F_0 + G_{-n}F_1 + \cdots + G_{-m}F_{-n+m+1} = 0.$$

But then $G_{-n+1}F_0 = -(G_{-n}F_1 + \cdots + G_{-m}F_{-n+m+1}) \in \mathcal{J}$ and again we obtain $G_{-n+1} =\in \mathcal{J}$, as desired.

By (finite) induction this proves the only if part of the proposition. The if part follows immediately from the identities

$$G_0F_0 + G_{-1}F_1 + \cdots + G_{-m}F_{-m} = e,$$
$$F_0G_0 + F_1G_{-1} + \cdots + F_{-m}G_{-m} = e,$$

combined with the assumption $G_{-1}, \dots, G_{-m} \in \mathcal{J}$. □

Let $\mathcal{J}$ be an ideal in $\mathcal{B}$ (two-sided and possibly non-closed, as always in this paper) and let $\mathcal{P}$ be a (non-empty) family of idempotents in $\mathcal{B}$. We say that $\mathcal{P}$ is *$\mathcal{J}-$annihilating for the commuting zero divisors in* $\mathcal{B}$ if for each (ordered) pair a, b of elements in $\mathcal{B}$, with $\mathcal{J}-$invertible a and $ba = ab = 0$ (hence $b \in \mathcal{J}$), there exist idempotents p and q in $\mathcal{P}$ such that

$$pa = b(e-p) = 0, \quad (e-q)b = aq = 0,$$

i.e., p is annihilating for the pair a, b and $e-q$ is annihilating for the pair b, a. Combining the $\mathcal{J}-$invertibility of a with the identities $pa = aq = 0$, we see that p and q necessarily belong to the ideal $\mathcal{J}$. Thus, if the family $\mathcal{P}$ is $\mathcal{J}-$annihilating for the commuting zero divisors in $\mathcal{B}$, then so is $\mathcal{P} \cap \mathcal{J}$. In other words, a $\mathcal{J}-$annihilating family for the commuting zero divisors in $\mathcal{B}$ can always be taken to be a subset of $\mathcal{J}$. The examples of annihilating families that we will give in the next section reflect this fact.

ann **Theorem 5.5.** *Let $\mathcal{J}$ be an ideal in $\mathcal{B}$, let $\mathcal{P}$ be a family of idempotents in $\mathcal{B}$, and assume $\mathcal{P}$ is $\mathcal{J}-$annihilating for the commuting zero divisors in $\mathcal{B}$. Let Δ be a non-empty open subset of $\mathbb{C}$, let $F : \Delta \to \mathcal{B}$ be analytic and suppose F takes invertible values on Δ except for a finite number of points where F^{-1} has a pole. Suppose, in addition, that the (non-invertible) values of F are $\mathcal{J}-$invertible. Then F is plain.*

In fact, the following more detailed result holds true. Let $\alpha_1, \dots, \alpha_n$ *be the distinct points in* Δ *where* F *has non-invertible (but* $\mathcal{J}-$*invertible) values (in any given order) and, for* $j = 1, \dots, n$*, let* m_j *be the (positive) order of* α_j *as a pole of* F^{-1}*. Then there exist analytic functions* $G, H : \Delta \to \mathcal{B}$*, taking invertible values on all of* Δ*, and non-zero idempotents* $p_j^{(i)}$*,* $q_j^{(i)} \in \mathcal{P} \cap \mathcal{J}$*,* $i = 1, \dots, m_j$*,* $j = 1, \dots, n$*, such that*

$$F(\lambda) = \left(\prod_{i=1}^{m_1} E_{p_1^{(i)},\alpha_1}(\lambda)\right) \dots \left(\prod_{i=1}^{m_n} E_{p_n^{(i)},\alpha_n}(\lambda)\right) G(\lambda), \quad \lambda \in \Delta$$

and

$$F(\lambda) = H(\lambda) \left(\prod_{i=1}^{m_1} E_{q_1^{(i)},\alpha_1}(\lambda)\right) \dots \left(\prod_{i=1}^{m_n} E_{q_n^{(i)},\alpha_n}(\lambda)\right), \quad \lambda \in \Delta.$$

Proof. Put $M = m_1 + \cdots + m_n$. We may assume that M is positive, otherwise there is nothing to prove.

Consider the situation at the point α_1. Write

$$F^{-1}(\lambda) = \sum_{j=-m}^{\infty} (\lambda - \alpha_1)^j F_j \tag{17}$$

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for the Laurent expansion of F^{-1} there. Here $m = m_1$, the order of α_1 as a pole of F^{-1}. Clearly $F_{-m}F(\alpha_1) = F(\alpha_1)F_{-m} = 0$ and, by assumption, $F(\alpha_1)$ is $\mathcal{J}-$invertible. But then there exist idempotents $p, q \in \mathcal{P}$ such that p is annihilating for the pair $F(\alpha_1)$, F_{-m} and $e - q$ is annihilating for the pair F_{-m}, $F(\alpha_1)$. This means that the following identities hold

$$pF(\alpha_1) = F_{-m}(e - p) = 0, \qquad (e - q)F_{-m} = F(\alpha_1)q = 0.$$

Note that p and q are non-zero as the same is true for F_{-m}. Introduce

$$\widehat{F}(\lambda) = \Big(e - p + (\lambda - \alpha_1)^{-1}p\Big)F(\lambda),$$

$$\widetilde{F}(\lambda) = F(\lambda)\Big(e - q + (\lambda - \alpha_1)^{-1}q\Big),$$

Then $\widehat{F}$ is analytic at α_1 and we have pole reduction there in the sense that either $\widehat{F}^{-1}$ has a pole at α_1 of order $m - 1$ (when $m \geq 2$), or the principal part of the Laurent expansion of $\widehat{F}^{-1}$ at α_1 vanishes altogether (when $m = 1$), in which case $\widehat{F}(\alpha_1)$ is invertible. In fact, at α_1 the function $\widehat{F}^{-1}$ has the expansion

$$\widehat{F}^{-1}(\lambda) = F^{-1}(\lambda)\Big(e - p + (\lambda - \alpha_1)p\Big) = \sum_{j=-m+1}^{\infty} (\lambda - \alpha_1)^j \Big(F_j(e - p) + F_{j-1}p\Big).$$

Now $F_{-m}, \dots, F_{-1}$ belong to $\mathcal{J}$ and the latter is an ideal in $\mathcal{B}$. Hence the coefficients of the principal part of the the Laurent expansion of $\widehat{F}^{-1}$ at α_1 belong to $\mathcal{J}$ which, by Proposition 5.4, amounts to the same as saying that $\widehat{F}(\alpha_1)$ is $\mathcal{J}-$invertible.

The function $\widehat{F}$ takes invertible values on Δ except in the points $\alpha_2, \dots, \alpha_n$ and possibly α_1. As we saw $\widehat{F}(\alpha_1)$ is $\mathcal{J}-$invertible (possibly even invertible). For $k = 2, \dots, n$, the function value $\widehat{F}(\alpha_k)$ is the product of a $\mathcal{J}-$invertible element, namely $F(\alpha_k)$, and an invertible one. Hence $\widehat{F}(\alpha_k)$ is $\mathcal{J}-$invertible itself. Finally, $\widehat{F}^{-1}$ has poles at $\alpha_2, \dots, \alpha_n$ of order $m_2, \dots, m_n$, respectively, and a pole at α_1 of order $m_1 - 1$ $(= m - 1)$. Summarizing, $\widehat{F}$ is a function of the same type as F but with the sum of the pole orders reduced with one from M to $M-1$. The same conclusion holds for $\widetilde{F}$. But then the desired result follows by induction. $\square$

It is possible to refine Theorem 5.5 in the sense that it can be split into two results, one dealing with left ideals (and extraction of elementary factors from the right), the other with right ideals (and extraction of elementary factors from the left). The refinement is straightforward and will not be pursued here (but see Example 7.4 below).

Theorem 5.5 should be considered in combination with examples of annihilating families of idempotents. They will be given in Section 6. Here we just emphasize that Theorem 5.3 is a particular case of Theorem 5.5 (see Example 6.1 below); Theorem 5.1, however, does not correspond to such a specialization of Theorem 5.5. Its proof, however, has a significant overlap with that of Theorem 5.5.

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6. Annihilating families of idempotents

As was already remarked, Theorem 5.5 should be appreciated in combination with examples of annihilating families of idempotents. The first example is the one that lies "behind" Theorem 5.3.

ann0A **Example 6.1.** Let $\mathcal{B} = \mathcal{L}(X)$ where X is a complex Banach space and let $\mathcal{J}$ be the ideal in $\mathcal{L}(X)$ consisting of all finite rank operators. Then the family of finite rank projections on X is $\mathcal{J}-$annihilating for the commuting zero divisors in $\mathcal{L}(X)$. This is clear by looking at Example 5.2. Recall that $\mathcal{J}-$invertibility here amounts to Fredholmness. $\square$

Closely related to this example is the following one, involving a Banach algebra a special instance of which featured already in Example 4.5.

ann0A.comp **Example 6.2.** Let X be an infinite-dimensional complex Banach space, and let I be the identity operator on X. Consider the Banach algebra

$$\mathcal{L}_{\mathcal{C}}(X) = \{\alpha I + C \mid \alpha \in \mathbb{C},\ C \in \mathcal{C}(X)\},$$

where $\mathcal{C}(X)$ is the closed ideal in $\mathcal{L}(X)$ consisting of all compact (bounded) linear operators on X. First let $\mathcal{J}$ be any (possibly non-closed) ideal contained in $\mathcal{C}(X)$. Then $\mathcal{J}-$invertibility implies $\mathcal{C}(X)-$invertibility, hence it comes down to Fredholmness. Thus, as in the previous example, the family of finite rank projections on X is $\mathcal{J}-$annihilating for the commuting zero divisors in $\mathcal{L}_{\mathcal{C}}(X)$. Next let $\mathcal{J}$ be all of $\mathcal{L}_{\mathcal{C}}(X)$, so that $\mathcal{J}-$ invertiblity is an empty requirement. Then the above conclusion cannot be drawn. To illustrate this, let A be a a compact operators on X with $A^2 = 0$ and having range and null space of infinite dimension and codimension, respectively. Concrete examples are easy to construct when X is a separable Hilbert space. Suppose P is an annihilating idempotent for the pair A, A. So $PA = A(I-P) = 0$. Then the dimension and codimension of the null space of P are both infinite. However, we know from [BES7], Proposition 2.1 that the idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ are the projections on X for which either the range or the null space has finite dimension, and a contradiction is immediate. Thus, when $\mathcal{J} = \mathcal{L}_{\mathcal{C}}(X)$, no $\mathcal{J}-$annihilating family of idempotents for the commuting zero divisors in $\mathcal{L}_{\mathcal{C}}(X)$ exists. $\square$

Here are some more examples related to Theorem 5.5.

ann1A **Example 6.3.** Consider the (commutative) Banach algebra $L^\infty(X,\mu)$ where (X,μ) is a measure space. Fix a measurable subset X_0 of X and let $\mathcal{J}$ be the set of all $f \in L^\infty(X,\mu)$ such that f vanishes a.e. on X_0. Then $\mathcal{J}$ is an ideal in $L^\infty(X,\mu)$. Let $\mathcal{P}$ be the subset of $L^\infty(X,\mu)$ determined by the characteristic functions vanishing a.e. on X_0. Then $\mathcal{P}$ is a $\mathcal{J}-$annihilating family of idempotents for the (commuting) zero divisors in $L^\infty(X,\mu)$. As each non-zero idempotent in $L^\infty(X,\mu)$ has norm one, this family is norm bounded (cf., Example 6.4). Note that $f \in L^\infty(X,\mu)$ is $\mathcal{J}-$invertible if and only if the absolute value of f is essentially bounded away from zero on X_0. When $X_0 = X$, one has the uninteresting case $\mathcal{J} = \{0\}$; when X_0 has measure zero, $\mathcal{J}$ is the full Banach algebra $L^\infty(X,\mu)$. $\square$

Next we indicate how to build new examples from given ones.

ann3Aideal **Example 6.4.** Let $\mathcal{F} = \{\mathcal{B}_\omega\}_{\omega\in\Omega}$ be a (non-empty) family of unital Banach algebras and let $\mathbf{B}(\Omega,\mathcal{F})$ consist of all functions $f : \Omega \to \bigcup_{\omega\in\Omega} \mathcal{B}_\omega$ such that $f(\omega) \in \mathcal{B}_\omega$ for each ω in Ω while, in addition,

$$\|f\|_{\mathcal{F}} = \sup_{\omega\in\Omega} \|f(\omega)\|_\omega < \infty.$$

Here $\|.\|_\omega$ stands for the norm in $\mathcal{B}_\omega$. With $\|.\|_{\mathcal{F}}$ as norm and, of course, the usual point-wise operations, $\mathbf{B}(\Omega,\mathcal{F})$ is a unital Banach algebra.

For $\omega \in \Omega$, let $\mathcal{J}_\omega$ be an ideal in $\mathcal{B}_\omega$. Write $\mathbf{J}$ for the collection of all functions $j : \Omega \to \bigcup_{\omega\in\Omega} \mathcal{J}_\omega$ such that $j(\omega) \in \mathcal{J}_\omega$ for each ω in Ω. Then, clearly, $\mathbf{J}$ is an ideal in $\mathbf{B}(\Omega,\mathcal{F})$.

Suppose now that for each ω in Ω, we have a family $\mathcal{P}_\omega$ of idempotents in $\mathcal{B}_\omega$, each of them norm bounded, and even with a uniform upper bound in the sense that

$$\sup_{\omega\in\Omega} \sup_{p\in\mathcal{P}_\omega} \|p\|_\omega < \infty$$

(cf., Example 6.3 above and Example 6.8 below). Introduce $\mathbf{P}$ as the collection of all functions $p : \Omega \to \bigcup_{\omega\in\Omega} \mathcal{P}_\omega$ such that $p(\omega) \in \mathcal{P}_\omega$ for each ω in Ω. Then, clearly, $\mathbf{P}$ is a (norm bounded) family of idempotents in $\mathbf{B}(\Omega,\mathcal{F})$. If for all $\omega \in \Omega$, the family $\mathcal{P}_\omega$ is $\mathcal{J}_\omega-$annihilating for the commuting zero divisors in $\mathcal{B}_\omega$, then $\mathbf{P}$ is $\mathbf{J}-$annihilating for the commuting zero divisors in $\mathbf{B}(\Omega,\mathcal{F})$. The straightforward argument is left to the reader. $\square$

annJ1 **Example 6.5.** Consider the situation of the previous example but without the (uniform) boundedness condition on the families of idempotents $\mathcal{P}_\omega$. Write $\mathbf{J}_0$ for the subset of $\mathbf{J}$ consisting of all functions $j \in \mathbf{J}$ such that $j(\omega) = 0$ for all but a finite number of points in Ω. Similarly, let $\mathbf{P}_0$ be the subset of $\mathbf{P}$ consisting of all functions $p \in \mathbf{P}$ such that $p(\omega) = 0$ for all but a finite number of points in Ω. Then $\mathbf{J}_0$ is an ideal in $\mathbf{B}(\Omega,\mathcal{F})$. Also $\mathbf{P}_0$ is a family of idempotents in $\mathbf{B}(\Omega,\mathcal{F})$. This is true (due to the finiteness condition) in spite of the fact that, as we dropped the (uniform) boundedness requirement on the families $\mathcal{P}_\omega$, $\mathbf{P}$ need not be subset of $\mathbf{B}(\Omega,\mathcal{F})$ any more. Suppose now that for all $\omega \in \Omega$, the family $\mathcal{P}_\omega$ is $\mathcal{J}_\omega-$annihilating for the commuting zero divisors in $\mathcal{B}_\omega$. Then $\mathbf{P}_0$ is $\mathbf{J}_0-$annihilating for the commuting zero divisors in $\mathbf{B}(\Omega,\mathcal{F})$. The argument is as follows. Assume we have $a, b \in \mathbf{B}(\Omega,\mathcal{F})$ with $ba = ab = 0$ and $b \in \mathbf{J}_0$. Then

$$b(\omega)a(\omega) = a(\omega)b(\omega) = 0, \quad b(\omega) \in \mathcal{J}_\omega, \qquad \omega \in \Omega.$$

Hence there exist $p(\omega), q(\omega) \in \mathcal{P}_\omega$ such that

$$p(\omega)a(\omega) = b(\omega)\big(e - p(\omega)\big) = 0, \quad \omega \in \Omega,$$
$$\big(e - q(\omega)\big)b(\omega) = a(\omega)q(\omega) = 0, \quad \omega \in \Omega,$$

where by slight (and customary) abuse of notation e stands for the unit element in $\mathcal{B}_\omega$. The functions $p, q : \Omega \to \bigcup_{\omega\in\Omega} \mathcal{B}_\omega$ arising in this way will generally not belong to $\mathbf{P}_0$ and, due to possible unboundedness, not even to $\mathbf{B}(\Omega, \mathcal{F})$. However, we can make them belong to $\mathbf{P}_0$ by taking $p(\omega) = q(\omega) = 0$ whenever $b(\omega) = 0$ which happens outside a finite subset of Ω as $b \in \mathbf{J}_0$. □

Specifying the data in the previous example in an certain way, one gets the following rather concrete example.

annJ1A **Example 6.6.** Consider the sequence space $\ell_\infty^{n\times n}$ of all bounded sequences of complex $n\times n$ matrices, and let $\mathcal{J}$ be the set of all such sequences having finite support. Then $\mathcal{J}$ is an ideal in $\ell_\infty^{n\times n}$. Further, let $\mathcal{P}$ be the subset of $\mathcal{J}$ consisting of all $P \in \mathcal{J}$ such that the matrix $P(\omega)$ is idempotent for each $\omega \in \Omega$. Clearly, $\mathcal{P}$ is a family of idempotents in $\ell_\infty^{n\times n}$. We claim that it is $\mathcal{J}-$annihilating for the commuting zero divisors in $\ell_\infty^{n\times n}$. The argument is as follows.

Assume we have $A, B \in \ell_\infty^{n\times n}$ with $\mathcal{J}-$ invertible A and $BA = AB = 0$ (hence $B \in \mathcal{J}$). Then

$$B_k A_k = A_k B_k = 0, \qquad k = 1, 2, 3, \dots .$$

From Example 6.1 (with $X = \mathbb{C}^n$), we see that there exist idempotent $n\times n$ matrices P_k and Q_k such that

$$P_k A_k = B_k\big(I_n - P_k\big) = 0, \qquad k = 1, 2, 3, \dots,$$
$$\big(I_n - Q_k\big)B_k = A_k Q_k = 0, \qquad k = 1, 2, 3, \dots,$$

where I_n stands for the $n \times n$ identity matrix. Due to possible unboundedness, the sequences P and Q arising in this way will generally not belong to $\ell_\infty^{n\times n}$. However, we can make them belong to $\ell_\infty^{n\times n}$ by taking $P_k = Q_k = 0$ whenever $B_k = 0$ which happens for all but a finite number of positive integers k. Doing this, we even get $P, Q \in \mathcal{P}$, and we finish the argument by observing that P and $I - Q$ are annihilating idempotents for, respectively, the pairs A, B and B, A. Here I denotes the identity element in $\ell_\infty^{n\times n}$ (i.e., the sequence each term of which is equal to I_n).

What does $\mathcal{J}-$invertibility mean in this case? Let $Y \in \ell_\infty^{n\times n}$ be $\mathcal{J}-$invertible. Then there exists $X \in \ell_\infty^{n\times n}$ such that both $XY - I$ and $YX - I$ belong to $\mathcal{J}$. But then, for all but a finite number of k,

$$X_k Y_k = Y_k X_k = I_n.$$

Thus, if $Y \in \ell_\infty^{n\times n}$ is $\mathcal{J}-$invertible, the matrix Y_k is invertible for all but a finite number of k and

$$\sup\{\|Y_k^{-1}\| \,|\, Y_k \text{ is invertible}, k = 1, 2, 3, \dots\} < \infty.$$

As is easily verified, the converse also holds. □

In Example 5.2 we already touched on the situation $\mathcal{B} = \mathcal{L}(H)$ where H is a complex Hilbert space. Here we will look at this situation more closely. We shall do this under the assumption that the ideal $\mathcal{J}$ coincides with the full Banach algebra (and so $\mathcal{J}-$invertibility becomes an empty requirement). In such cases we will drop the reference to the ideal altogether and simply write annihilating instead of $\mathcal{B}$-annihilating. So a (non-empty) family $\mathcal{P}$ of idempotents in $\mathcal{B}$ is called *annihilating for the commuting non-zero divisors in* $\mathcal{B}$ if for each (ordered) pair a, b of elements in $\mathcal{B}$ with $ba = ab = 0$, there exist $p,\, q \in \mathcal{P}$ such that $pa = b(e-p) = 0$ and $(e-q)b = aq = 0$.

hilbert **Proposition 6.7.** *Let H be a complex Hilbert space and let $\mathcal{P}$ be a family of idempotents in $\mathcal{L}(H)$. Then $\mathcal{P}$ is annihilating for the commuting zero divisors in $\mathcal{L}(H)$ if and only if both*

$$\{\operatorname{Ker} P \mid P \in \mathcal{P}\} \ \text{ and } \ \{\operatorname{Im} P \mid P \in \mathcal{P}\} \tag{18}$$ famH

coincide with the collection of all closed subspaces of H.

The two sets in (18) coincide provided that $\mathcal{P}$ is closed under the operation of taking the complementary projection.

Proof. Take $A,\, B \in \mathcal{L}(H)$, and assume $BA = AB = 0$. This means that $\operatorname{Im} A \subseteq \operatorname{Ker} B$ and $\operatorname{Im} B \subseteq \operatorname{Ker} A$ or, alternatively, $\operatorname{Im} A \subseteq \operatorname{Ker} B$ and $\overline{\operatorname{Im}}\, B \subseteq \operatorname{Ker} A$. Assuming (18), it is possible to choose $P, Q \in \mathcal{P}$ such that $\operatorname{Ker} P = \operatorname{Ker} B$ and $\operatorname{Im} Q = \overline{\operatorname{Im}}\, B$. But then $\operatorname{Im} A \subseteq \operatorname{Ker} B = \operatorname{Ker} P = \operatorname{Im}(I-P)$ and $\operatorname{Ker}(I-Q) = \operatorname{Im} Q = \overline{\operatorname{Im}}\, B \subseteq \operatorname{Ker} A$, where I is the identity operator on H. Hence $PA = B(I-P) = 0$ and $(I-Q)B = AQ = 0$, and we have proved the if part of the proposition.

For the only if part, we argue as follows. Suppose $\mathcal{P}$ is annihilating for the commuting zero divisors in $\mathcal{L}(H)$, and let N be an arbitrary closed subspace of H. As H is a Hilbert space, N is complemented and so there is an idempotent R in $\mathcal{L}(H)$ such that $\operatorname{Ker} R = N$. Now $R(I-R) = (I-R)R = 0$ and, by assumption, we have $P(I-R) = R(I-P) = 0$ for some $P \in \mathcal{P}$. But then $\operatorname{Ker} P = \operatorname{Ker} R = N$. Interchanging the roles of R and $I-R$, we also find an idempotent $Q \in \mathcal{P}$ for which $\operatorname{Im} Q = \operatorname{Ker} R = N$, and the proof is complete. $\square$

From Proposition 6.7 it is evident that there is an abundance of annihilating families of idempotents in $\mathcal{L}(H)$. Here is a specific one.

ann2A **Example 6.8.** Consider $\mathcal{L}(H)$ where H is a complex Hilbert space. The family of orthogonal projections on H is annihilating for the commuting zero divisors in $\mathcal{L}(H)$. Since each non-zero orthogonal projection on H has norm one, this family is norm bounded (cf., Example 6.4). $\square$

In the remainder of this section, we give additional details for the finite dimensional case. So we consider $\mathbb{C}^{n\times n}$, the Banach algebra of all complex $n \times n$ matrices. The orthogonal projections mentioned in Example 6.8 correspond here to the selfadjoint idempotent matrices. So these constitute an annihilating family of idempotents for the commuting zero divisors in $\mathbb{C}^{n\times n}$. As will become clear from the subsequent analysis, there are other structural elements (besides selfadjointness) that can be taken into account too. We begin with the following general matrix result.

uppA **Proposition 6.9.** *Let M be an $n \times n$ matrix. Then there exist invertible $n \times n$ matrices F and G such that FM and MG are upper triangular idempotents.*

Taking transposes, one sees that upper triangular may be replaced by lower triangular.

Proof. Via row operations, bring M into reduced row echelon form. Then permute the rows in the echelon form in such a way that the pivots enter the diagonal. The resulting matrix P is clearly upper triangular and easily seen to be idempotent (see [L], the discussion on Algorithm III or [SW], Corollary of Lemma 1; cf., also [S], the material on special solutions). Obviously P is of the form $P = FM$ with an invertible matrix F.

This proves the existence of F. Next we turn to G. Let E be the $n \times n$ reversed identity matrix (having ones on the antidiagonal and zeros everywhere else). Applying the result of the previous paragraph to $M^\top E$, we find an invertible matrix H such that $HM^\top E$ is an upper triangular idempotent. But then $EHM^\top = E\big(HM^\top E\big)E$ is a lower triangular idempotent and, taking transposes, we see that $MH^\top E$ is an upper triangular idempotent again. Thus $G = H^\top E$ has the desired properties. □

ann4A **Example 6.10.** Combining Proposition 6.7 and Proposition 6.9, one sees that the family of all upper triangular idempotents in $\mathbb{C}^{n\times n}$ is annihilating for the commuting zero divisors in $\mathbb{C}^{n\times n}$. Of course this remains true when upper triangular is replaced by lower triangular. □

We end this section with an example showing that there are situations where, formally, Theorem 5.5 applies but where, materially, the result is empty.

ann5A **Example 6.11.** The Banach algebra will be that of the upper triangular $n \times n$ complex matrices, denoted by $\mathbb{C}^{n\times n}_{\rm upp}$. From the preceding example it is obvious that the family of all idempotents in $\mathbb{C}^{n\times n}_{\rm upp}$ is annihilating for the commuting zero divisors in $\mathbb{C}^{n\times n}_{\rm upp}$. This means that it is $\mathcal{J}-$annihilating for every ideal $\mathcal{J}$ in $\mathbb{C}^{n\times n}_{\rm upp}$. Thus Theorem 5.5 applies, regardless of how one selects the ideal $\mathcal{J}$. There is, however, a special choice of $\mathcal{J}$ for which the result becomes empty. Take $\mathcal{J}$ to be the radical in $\mathbb{C}^{n\times n}_{\rm upp}$, that is the collection of all $n\times n$ matrices that are strictly upper triangular. Then ordinary invertibility in $\mathbb{C}^{n\times n}_{\rm upp}$ and $\mathcal{J}-$invertibility come down to the same. So, with this choice of $\mathcal{J}$, the hypothesis in Theorem 5.5 concerning the function F means that F takes invertible values only, a state of affairs rendering the situation completely trivial. Note that with this choice of $\mathcal{J}$, the singleton $\{0\}$ is a $\mathcal{J}-$annihilating family of idempotents. Indeed, if $A, B \in \mathbb{C}^{n\times n}_{\rm upp}$ and A is $\mathcal{J}-$invertible (hence invertible), then $BA = AB = 0$ implies $B = 0$. □

ex.extr.el.fact 7. Examples on extraction of elementary factors

We shall now present some (counter)examples that shed additional light on the issues discussed in the previous sections.

In the situations of Theorems 5.3, 5.1 and 5.5, the number of elementary factors extracted in order to arrive at a function which is everywhere invertible is equal to the sum of the relevant pole orders. This is what happens when one can make use of the step by step method suggested in the opening paragraphs of this section. In general, however, the situation is different: in the following example we are considering a function which is the product of three elementary factors (based at the origin), whose inverse has a pole of order two at the origin, but which nonetheless cannot be written in the form discussed above involving only two elementary factors.

extr1A **Example 7.1.** Let $\mathbb{C}_0^{6\times 6}$ be the (inverse closed) Banach subalgebra of $\mathbb{C}^{6\times 6}$ consisting of the matrices of the type

$$\begin{pmatrix} u & 0 & 0 & 0 & 0 & 0 \\ x & v & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 \\ 0 & 0 & y & w & 0 & 0 \\ 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & z & w \end{pmatrix}$$

with u, v, w, x, y, z in $\mathbb{C}$. For $\lambda \in \mathbb{C}$, introduce

$$F(\lambda) = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ \lambda - 1 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda - 1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda - 1 & \lambda \end{pmatrix},$$

and note that $F = E_1 E_2 E_3$ with

$$E_1(\lambda) = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ \lambda - 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$E_2(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda - 1 & 1 \end{pmatrix},$$

$$E_3(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

As is easily seen E_1, E_2 and E_3 are elementary functions based at the origin and with corresponding idempotents

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

members of $\mathbb{C}_0^{6\times 6}$ indeed. So F is an elementary polynomial taking invertible values on $\mathbb{C}\setminus\{0\}$. The order of the origin as a pole of F^{-1} is two. Nevertheless F cannot be written in the form $F = E^{(1)}E^{(2)}G$ with $E^{(1)}, E^{(2)}$ elementary functions based at the origin and with G an entire function taking invertible values on all of $\mathbb{C}$. Assume, contrarily, that it can. Then we come to a contradiction by arguing as follows.

Suppose the idempotents in $\mathbb{C}_0^{6\times 6}$ corresponding to $E^{(1)}$ and $E^{(2)}$ are $Q^{(1)}$ and $Q^{(2)}$, respectively. So

$$E^{(j)}(\lambda) = I_6 - Q^{(j)} + \lambda Q^{(j)}, \quad \lambda \in \mathbb{C};\ j = 1, 2,$$

where I_6 is the 6×6 identity matrix. Now introduce the block forms

$$Q^{(j)} = \begin{pmatrix} Q_1^{(j)} & 0 & 0 \\ 0 & Q_2^{(j)} & 0 \\ 0 & 0 & Q_3^{(j)} \end{pmatrix} \tag{19}$$ $\boxed{\texttt{Q}}$

with $Q_1^{(j)}, Q_2^{(j)}, Q_3^{(j)}$ lower triangular matrices of order 2 and, analogously,

$$G(\lambda) = \begin{pmatrix} G_1(\lambda) & 0 & 0 \\ 0 & G_2(\lambda) & 0 \\ 0 & 0 & G_3(\lambda) \end{pmatrix}.$$

Then, for $k = 1, 2, 3$ and $\lambda \in \mathbb{C}$, the lower triangular 2×2 matrix $G_k(\lambda)$ is invertible and

$$\begin{pmatrix} \lambda & 0 \\ \lambda - 1 & \lambda \end{pmatrix} = \Big(I - Q_k^{(1)} + \lambda Q_k^{(1)}\Big)\Big(I - Q_k^{(2)} + \lambda Q_k^{(2)}\Big) G_k(\lambda)$$

where I is the 2×2 identity matrix. The left hand side assumes a non-zero value at the origin and its inverse has a pole of order two there. It follows that the idempotents $Q_k^{(j)}$ cannot be equal to 0 or I. Thus $Q_k^{(j)}$ is a non-trivial idempotent lower triangular 2×2 matrix and must therefore have one of the following two forms

$$\begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ * & 1 \end{pmatrix}.$$

But with these two possibilities for $Q_k^{(j)}$, the right hand side of (19) can never be a member of $\mathbb{C}_0^{6\times 6}$, and we have the desired contradiction. $\square$

Confronting the above situation with Theorem 5.5, we conclude that the Banach algebra $\mathbb{C}_0^{6\times 6}$ lacks a sufficient supply of annihilating idempotents. In fact, if A is the coefficient of λ^{-2} in the Laurent expansion of F^{-1} at the origin and $B = F(0)$, then $AB = BA = 0$ but the pair A, B does not have annihilating idempotent in $\mathbb{C}_0^{6\times 6}$. The verification of the details is left to the reader.

The number of factors in the elementary polynomial in Example 7.1 is three. Working with higher order matrices, one can also describe a situation where the number of factors is equal to a given arbitrary integer n larger than three while the elementary polynomial in question cannot be reduced to an everywhere invertible function via extraction of less than n elementary factors. In fact one can do this by allowing for matrices of order $n(n-1)$ built up of the same 2×2 constituents as used above (cf., Example 7.2 below where this idea is even stretched further).

Example 7.1 was cast in the language of 6×6 matrices and for the extension suggested in the previous paragraph matrices of order $n(n-1)$ were indicated. We could have worked instead with 3–tuples, more generally with $\frac{1}{2}n(n-1)$–tuples, of lower triangular 2×2 matrices. Next let us work with matrix tuples of infinite length (so basically sequences) in order to exhibit a striking phenomenon that can occur in the context of extracting elementary factors.

extr2A **Example 7.2.** Let Ω be the set of all ordered pairs (k, j) of positive integers j and k satisfying $k > j$. With the familiar lexicographic ordering

$$(2,1),\ (3,1),\ (3,2),\ (4,1),\ (4,2),\ (4,3),\ (5,1),\ (5,2),\ \dots$$

this set can be viewed as a simple sequence. Write $\mathbb{C}^{2\times 2}_{\mathrm{lower}}$ for the Banach algebra of lower triangular 2×2 matrices and let $\mathbf{B}_0$ be the Banach subalgebra of $\mathbf{B}(\Omega, \mathbb{C}^{2\times 2}_{\mathrm{lower}})$ consisting of all (norm bounded) functions $M : \Omega \to \mathbb{C}^{2\times 2}_{\mathrm{lower}}$ satisfying the following extra condition: there exists a (bounded) sequence $\mathrm{Seq}_M = (a_1, a_2, a_3, \dots)$ of complex numbers (depending on M) such that $M(k, j)$ has the form

$$M(k,j) = \begin{pmatrix} a_j & 0 \\ * & a_k \end{pmatrix}$$

for all $(k, j) \in \Omega$. For $\lambda \in \mathbb{C}$ and $(k, j) \in \Omega$, put

$$F(\lambda)(k,j) = \begin{pmatrix} \lambda & 0 \\ \lambda - 1 & \lambda \end{pmatrix}.$$

Then $F(\lambda) \in \mathbf{B}_0$, the corresponding sequence $\mathrm{Seq}_{F(\lambda)}$ consisting of nothing else than λ's. The resulting function $F : \mathbb{C} \to \mathbf{B}_0$ is entire (actually F is a linear pencil with coefficients in $\mathbf{B}_0$), takes invertible values on $\mathbb{C} \setminus \{0\}$, and its resolvent F^{-1} has a pole of order two at the origin. We shall analyze the situation on the point of the extraction of elementary factors based at 0. All the time n is allowed to take the values $1, 2, 3, \dots$.

For $\lambda \in \mathbb{C}$ and $(k, j) \in \Omega$, let $E_n(\lambda)(k, j)$ be given by

$$E_n(\lambda)(k,j) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & j,k \neq n, \\ \begin{pmatrix} \lambda & 0 \\ \lambda - 1 & 1 \end{pmatrix}, & j = n, \\ \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, & k = n. \end{cases}$$

Observe that $E_n(\lambda) \in \mathbf{B}_0$, the determining sequence $\mathrm{Seq}_{E_n(\lambda)}$ having λ at the n-th position and ones everywhere else. The function $E_n : \mathbb{C} \to \mathbf{B}_0$ defined this way is elementary, the corresponding idempotent P_n being given by

$$P_n(k,j) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & j,k \neq n, \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, & j = n, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & k = n, \end{cases}$$

where $(k, j) \in \Omega$. Also introduce

$$G_n(\lambda)(k,j) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & k \leq n, \\ \begin{pmatrix} \lambda & 0 \\ \lambda - 1 & \lambda \end{pmatrix}, & k > j > n, \\ \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, & k > n \geq j. \end{cases}$$

As is easily seen $G_n(\lambda) \in \mathbf{B}_0$, the sequence $\mathrm{Seq}_{G_n(\lambda)}$ starting with n ones and continuing with λ's. The function $G_n : \mathbb{C} \to \mathbf{B}_0$ thus introduced is entire (again it is a pencil) , with invertible values on $\mathbb{C} \setminus \{0\}$, and its resolvent G_n^{-1} has a pole of order two at the origin. So on these counts G_n has the same properties as F.

Inspection shows that for all $\lambda \in \mathbb{C}$

$$F(\lambda) = E_1(\lambda) \cdots E_n(\lambda) G_n(\lambda). \tag{20}$$ fact1A

So it is possible to extract an arbitrary number of non-trivial elementary factors from F without losing the analyticity. On the other hand, the resolvents F^{-1} and G_n^{-1} both have a pole at the origin of order two, ie., (20) does not involve pole reduction. In fact, the function F is not plain. To see this, we assume to the contrary that it is, and argue as follows. Write F in the form

$$F(\lambda) = E^{(1)}(\lambda) \dots E^{(n)}(\lambda) G(\lambda) \tag{21}$$ fact2A

with $E^{(1)}, \dots, E^{(n)}$ elementary functions based at the origin and with G an entire function taking invertible values on all of $\mathbb{C}$. For $m = 1, \dots, n$, let

$$E^{(m)}(\lambda) = I - P^{(m)} + \lambda P^{(m)}$$

where I is the unit element and $P^{(m)}$ is an idempotent in $\mathbf{B}_0$. Specifying $\mathrm{Seq}_{P^{(m)}}$ as $(p_1^{(m)}, p_2^{(m)}, p_3^{(m)}, \ldots)$, we have that $P^{(m)}(k,j)$ is of the form

$$P^{(m)}(k,j) = \begin{pmatrix} p_j^{(m)} & 0 \\ * & p_k^{(m)} \end{pmatrix}. \tag{22}$$

seqA

Here, as always, $(k,j) \in \Omega$. As $P^{(m)}$ is an idempotent in $\mathbf{B}_0$, the lower triangular 2×2 matrices in (22) are idempotents too. In particular the sequence $(p_1^{(m)}, p_2^{(m)}, p_3^{(m)}, \ldots)$ consists of no other numbers than zeros and ones.

For $\lambda \in \mathbb{C}$, let $\mathrm{Seq}_{G_n(\lambda)} = (g_1(\lambda),\, g_2(\lambda),\, g_3(\lambda),\, \ldots)$, so

$$G(\lambda)(k,j) = \begin{pmatrix} g_j(\lambda) & 0 \\ * & g_k(\lambda) \end{pmatrix}.$$

The functions $g_1, g_2, g_3, \ldots$ are entire and do not vanish on $\mathbb{C}$. In view of (21) and the definition of F we have that for $(k,j) \in \Omega$, the matrix

$$\begin{pmatrix} g_j(\lambda) \prod_{m=1}^n \left(1 - p_j^{(m)} + \lambda p_j^{(m)}\right) & 0 \\ * & g_k(\lambda) \prod_{m=1}^n \left(1 - p_k^{(m)} + \lambda p_k^{(m)}\right) \end{pmatrix}$$

is equal to $\begin{pmatrix} \lambda & 0 \\ \lambda - 1 & \lambda \end{pmatrix}$ and, consequently,

$$g_j(\lambda) \prod_{m=1}^n \left(1 - p_j^{(m)} + \lambda p_j^{(m)}\right) = \lambda, \quad \lambda \in \mathbb{C};\ j = 1, 2, 3, \ldots .$$

As observed above, $p_j^{(m)}$ is either zero or 1. In the first case the term $1 - p_j^{(m)} + \lambda p_j^{(m)}$ is 1, in the second it is λ. Hence, given j, there is an m_j among the integers $1, \ldots, n$ with $p_j^{(m_j)} = 1$. The integers $m_1, m_2, m_3, \ldots$ are all from $\{1, \ldots, n\}$ and so it is impossible that they are all different. Suppose $m_s = m_t$ for some s and t with $t > s$. Then $(t,s) \in \Omega$ and, with $l = m_s = m_t$, the matrix $P^{(l)}(t,s)$ has the form

$$P^{(l)}(t,s) = \begin{pmatrix} p_s^{(l)} & 0 \\ * & p_t^{(l)} \end{pmatrix} = \begin{pmatrix} p_s^{(m_s)} & 0 \\ * & p_t^{(m_t)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

Since this matrix is an idempotent, it follows that $P^{(l)}(t,s)$ is the 2×2 identity matrix. But then

$$E^{(l)}(\lambda)(t,s) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

and we see from (21) that $F(0)(t,s)$ is the 2×2 zero matrix. This, however, contradicts the definition of F which says that

$$F(0)(t,s) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Elaborating on the above, we note that if on a deleted neighborhood of the origin, one has a factorization of the type (21) with $E^{(1)}, \ldots, E^{(n)}$ elementary functions based at 0 and G analytic there, then G^{-1} necessarily has a pole at the origin of order two. Here is the reasoning. If the principal part of G^{-1} vanishes, then the usual continuity argument (cf., the last paragraph of the proof of Proposition 4.1) gives that $G(0)$ is invertible and F would be plain, which as we have seen it is not. Hence G^{-1} has a (genuine) pole at the origin, clearly of order not exceeding two (the order of 0 as a pole of F^{-1}). However, the order of 0 as a pole of G^{-1} cannot be one. If it is, Theorem 5.1 guarantees that G is plain. But then again we would arrive at the conclusion that F is plain, contrary to the facts.

Summarizing, on the one hand one can extract as many elementary non-trivial factors from F as desired without losing the analyticity, while on the other hand, no matter how this is done, pole reduction does not come to pass. $\square$

The Banach algebra in Example 7.2 is infinite dimensional. Is a similar example possible in a finite dimensional context, more precisely can one make do in a matrix setting? The answer is negative. Suppose F is an $n \times n$ matrix valued function, defined and analytic on a neighborhood U of the origin, and taking invertible values on $U \setminus \{0\}$. Assume, in addition, that F can be written as

$$F(\lambda) = (I_n - P_1 + \lambda P_1) \dots (I_n - P_k + \lambda P_k) G(\lambda),$$

where $P_1, \dots, P_k$ are non-zero idempotents and G is analytic on U. Observe that G takes invertible values on $U \setminus \{0\}$. Also note that for an idempotent $n \times n$ matrix P, one has

$$\det(I_n - P + \lambda P) = \lambda^{r(P)},$$

with $r(P) = \operatorname{rank} P = \operatorname{trace} P$. This can be seen by diagonalizing P via a similarity transformation. It follows that

$$\det F(\lambda) = \lambda^{r(P_1)+\dots+r(P_k)} \det G(\lambda), \qquad \lambda \in U.$$

Now $\det F(\lambda)$ and $\det G(\lambda)$ do not vanish on $U \setminus \{0\}$. Hence the orders of the origin as zeros af the analytic functions $\det F(\lambda)$ and $\det G(\lambda)$ are finite, say $m_F(0)$ and $m_G(0)$, respectively. But then

$$m_F(0) = r(P_1) + \dots + r(P_k) + m_G(0),$$

and we see that $k \le r(P_1) + \dots + r(P_k) = m_F(0) - m_G(0) \le m_F(0)$.

In both the Examples 7.1 and 7.2, the resolvent F^{-1} of the function F has a pole at the origin of order two. It is possible to refine the examples in such a way that this pole order is a prescribed number m larger than two. We refrain from presenting this refinement here.

A question that comes to mind in connection with the heuristic observations from the beginning of Section 5, and in view of the left versus right symmetry for the representation of plain functions exhibited in Section 4, is the following. Let the $\mathcal{B}$-valued function F be analytic on a neighborhood of the origin and assume the resolvent F^{-1} has a pole there. Suppose it is possible to extract an elementary factor (based at the origin) from the left in the sense that the analyticity is kept intact and that there occurs pole order reduction for the resolvent. Does it follow that a non-trivial elementary factor can be extracted from the right such that analyticity is retained? The following example shows that the answer is negative.

lvr1 **Example 7.3.** Let $\mathbb{C}^{3\times 3}_*$ be the (inverse closed) Banach subalgebra of $\mathbb{C}^{3\times 3}$ consisting of the matrices of the type

$$\begin{pmatrix} u & x & y \\ 0 & u & z \\ 0 & 0 & v \end{pmatrix}$$

with u, v, x, y, z in $\mathbb{C}$. For $\lambda \in \mathbb{C}$, let $F(\lambda)$ be the 3×3 upper triangular Jordan block with eigenvalue λ, so

$$F(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Then $F : \mathbb{C} \to \mathbb{C}_*^{3\times 3}$ is analytic and F^{-1} has a pole at the origin of order 3. Writing F in the form

$$F(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

we have extracted an elementary factor from the left (keeping the analyticity intact and with pole order reduction (to order 2) for the resolvent, as stipulated above). We shall now prove that it is impossible to extract a non-trivial elementary factor from the right such that analyticity is kept intact. Suppose, contrarily, that we can, so for some non-zero idempotent $P \in \mathbb{C}_*^{3\times 3}$ we have a representation

$$F(\lambda) = \widetilde{F}(\lambda)\big(I - P + \lambda P\big)$$

such that $\widetilde{F}$ is analytic at the origin. Here I is the unit element in $\mathbb{C}_*^{3\times 3}$, i.e., the 3×3 identity matrix. Taking $\lambda = 0$, we get $F(0) = \widetilde{F}(0)(I - P)$ and hence $F(0)P = 0$. But $F(0)$ is the 3×3 upper triangular nilpotent Jordan block. Hence P has the form

$$\begin{pmatrix} u & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $u, x, y \in \mathbb{C}$. As P belongs to $\mathbb{C}_*^{3\times 3}$, the first and second element on the diagonal have to be equal. Thus $u = 0$, so $P = P^2 = 0$, and we have reached a contradiction.

Without going into the details, we mention that the above argument can easily be transformed to the level of annihilating idempotents. □

Elaborating on this example we note that one cannot do with a pole order for the resolvent F^{-1} less than 3. Indeed, suppose F^{-1} has a pole at the origin of order 2 and assume that we can extract an elementary factor from the left (in the sense given to this above). The resolvent $\widetilde{F}^{-1}$ of the resulting function $\widetilde{F}$ then has a simple pole at the origin and this is a special situation in that then it is always possible to extract an elementary factor (see Theorem 5.1). Doing this, one arrives at a function which takes invertible values on a neighborhood of the origin and it would follow that F is plain on this neighborhood. But then it is possible to extract elementary factors from the right too.

The asymmetry with regard to left and right extraction of elementary factors is further illustrated by the next example (cf., the remark made just after the proof of Theorem 5.5).

lvr2 **Example 7.4.** Let $\mathbb{C}_{\mathrm{upp}}^{2\times 2}$ be the Banach algebra of 2×2 upper triangular complex matrices, and let $\mathcal{J}$ be the ideal in $\mathbb{C}_{\mathrm{upp}}^{2\times 2}$ consisting of all matrices of the form

$$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

with $x, y \in \mathbb{C}$. Note that a matrix

$$\begin{pmatrix} a_1 & a_0 \\ 0 & a_2 \end{pmatrix}$$

in $\mathbb{C}^{2\times 2}_{\text{upp}}$ is $\mathcal{J}-$invertible if and only if a_2 is non-zero.

Now consider an analytic $\mathbb{C}^{2\times 2}_{\text{upp}}$-valued function F, defined on an open neighborhood of the origin, and write

$$F(\lambda) = \begin{pmatrix} f_1(\lambda) & f_0(\lambda) \\ 0 & f_2(\lambda) \end{pmatrix}.$$

Suppose $F(0)$ is $\mathcal{J}-$invertible or, what amounts to the origin of (positive) order m. This means that the scalar function f_1 has a zero at the origin of order m, and so we can write

$$F(\lambda) = \begin{pmatrix} \lambda^m g(\lambda) & f_0(\lambda) \\ 0 & f_2(\lambda) \end{pmatrix}.$$

with $g(0)$ and $f_2(0)$ both non-zero.

Let $\mathcal{P}_{\text{right}}$ be the singleton set having the upper triangular idempotent

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

as its only member, and let $\mathcal{P}_{\text{left}}$ be the (infinite) family of upper triangular idempotents consisting of the matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

with $x \in \mathbb{C}$. Clearly F can be written as

$$F(\lambda) = \begin{pmatrix} g(\lambda) & f_0(\lambda) \\ 0 & f_2(\lambda) \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}^m$$

with the first factor invertible at the origin and the second the m-th power of the elementary function

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

involving the single element of $\mathcal{P}_{\text{right}}$.

So far about extraction from the right. Now what about extraction from the left? This is also possible, after m steps resulting in function which is invertible at the origin. However, here one cannot do with one single idempotent, but one has to use the idempotents from the (infinite) family $\mathcal{P}_{\text{left}}$. We describe the first step, leaving the rest of the (straightforward induction) argument to the reader. Put

$$x = -\frac{f_0(0)}{f_2(0)},$$

and introduce

$$h(\lambda) = \begin{cases} \frac{1}{\lambda}\big(f_0(\lambda) + x f_2(\lambda)\big) - x f_2(\lambda), & \lambda \neq 0, \\ f_0'(0) + x f_2'(0) - x f_2(0), & \lambda = 0. \end{cases}$$

Then $h(\lambda) \to h(0)$ when $\lambda \to 0$, so h is analytic. One easily verifies the identity

$$F(\lambda) = \begin{pmatrix} \lambda & \lambda x - x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{m-1} g(\lambda) & h(\lambda) \\ 0 & f_2(\lambda) \end{pmatrix}.$$

The resolvent of the second factor has a pole at the origin of order $m-1$. The first factor can be written as

$$\begin{pmatrix} 0 & -x \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

which shows that it is an elementary function involving an idempotent from the family $\mathcal{P}_{\text{left}}$. $\square$

logrpl

8. Logarithmic residues of plain functions

Theorem 8.3 below is concerned with Problem 1 from the introduction. It implies that logarithmic residues of plain functions are always in the (possibly non-closed) subalgebra of $\mathcal{B}$ generated by the idempotents in $\mathcal{B}$.

To give the theorem its proper background, we first present two examples. The first, taken from [BES6] and repeated here as a service to the reader, features a function having a logarithmic residue not belonging to the closure of the subalgebra mentioned above. The second example is motivated by Proposition 4.3 and deals with the special situation where we have a "pure" elementary polynomial, i.e., a product of elementary functions based at a single point (taken to be the origin for simplicity).

lrpfE2 **Example 8.1.** Let $\mathbb{C}_0^{3\times 3}$ be the the Banach algebra of all 3×3 matrices $(a_{ij})_{i,j=1}^3$ such that $a_{ij}=0$ $(i,j=1,2,3;\ i>j)$ and $a_{kk}=a_{11}$ $(k=1,2,3)$. In other words, $\mathbb{C}_0^{3\times 3}$ is the Banach subalgebra of $\mathbb{C}^{3\times 3}$ consisting of all upper triangular 3×3 matrices with constant diagonal. Introduce $F:\mathbb{C}\to\mathbb{C}_0^{3\times 3}$ by

$$F(\lambda) = \begin{pmatrix} \lambda & \lambda^2 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Then F is entire and F takes invertible values on all of $\mathbb{C}$, except in the origin. The left logarithmic residue of F at the origin is

$$LR_{\text{left}}(F;0) = \frac{1}{2\pi i}\int_{|\lambda|=1} F'(\lambda)F^{-1}(\lambda)d\lambda = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now the only idempotents in $\mathbb{C}_0^{3\times 3}$ are the unit element and the zero element in $\mathbb{C}_0^{3\times 3}$. So the algebra generated by the idempotents in $\mathbb{C}_0^{3\times 3}$ consists of the scalar multiples of the 3×3 identity matrix. It is now clear that $LR_{\text{left}}(F;0)$ does not belong to (the closure of) this algebra. $\square$

Returning to the case where there are no restrictions on the underlying Banach algebra $\mathcal{B}$, we now compute the logarithmic residues of an elementary polynomial involving only factors based at the origin.

lrpfE1 **Example 8.2.** Let $p_1,\dots,p_n$ be idempotents in the Banach algebra $\mathcal{B}$, and introduce $P=E_{p_1,0}\dots E_{p_n,0}$, i.e.,

$$P(\lambda) = \prod_{j=1}^{n}\big(e-p_j+\lambda p_j\big), \qquad \lambda\in\mathbb{C}. \tag{23}$$ pp

The aim is to compute $LR_{\text{left}}(P;0)$ and $LR_{\text{right}}(P;0)$.

For $m=0,\dots,n-1$, let P_m by given by $P_m=E_{p_1,0}\dots E_{p_m,0}$, where as usual the empty product (in the case $m=0$) is read as e, the unit element in $\mathcal{B}$. Recall that

$LR_{\text{left}}(P;0)$ is the coefficient of λ^{-1} in the Laurent expansion of the left logarithmic derivative of P at the origin given by

$$P'(\lambda)P^{-1}(\lambda) = \sum_{m=1}^{n} P_{m-1}(\lambda)\left(\lambda^{-1}p_m\right)P_{m-1}(\lambda)^{-1}, \qquad \lambda \neq 0. \tag{24}$$

lrpf03

It follows that $LR_{\text{left}}(P;0)$ is the constant term in the Laurent expansion at the origin of the function

$$\sum_{m=1}^{n} P_{m-1}(\lambda)p_m P_{m-1}(\lambda)^{-1}. \tag{25}$$

lrpf03A

From this we immediately see that $LR_{\text{left}}(P;0)$ is a linear combination of monomials in idempotents in $\mathcal{B}$, with non-negative integers as coefficients and the idempotents coming from $p_1,\dots,p_n$ and the complementary idempotents $e-p_1,\dots,e-p_n$. The same conclusion holds for the right logarithmic residue $LR_{\text{right}}(P;0)$. It is, however, possible to make things more explicit.

Observe that

$$\begin{aligned} P_m(\lambda) &= \prod_{j=1}^{m}\Big(e+(\lambda-1)p_j\Big) \\ &= \sum_{k=0}^{m}\sum_{1\le j_1<j_2<\cdots<j_k\le m}(\lambda-1)^k p_{j_1}p_{j_2}\dots p_{j_k} \\ &= \sum_{k=0}^{m}\sum_{1\le j_1<j_2<\cdots<j_k\le m}\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}\lambda^l p_{j_1}p_{j_2}\dots p_{j_k}, \end{aligned}$$

which, after rearranging terms, gives

$$P_m(\lambda) = \sum_{l=0}^{m}\lambda^l\left(\sum_{k=l}^{m}(-1)^{k-l}\binom{k}{l}\sum_{1\le j_1<j_2<\cdots<j_k\le m}p_{j_1}p_{j_2}\dots p_{j_k}\right).$$

Taking into account that

$$P_m(\lambda)^{-1} = \prod_{j=1}^{m}E_{p_{m+1-j},0}(\lambda^{-1}) = \prod_{j=1}^{m}\Big(e+\big(\lambda^{-1}-1\big)p_{m+1-j}\Big),$$

one also has

$$P_m(\lambda)^{-1} = \sum_{l=0}^{m}\lambda^{-l}\left(\sum_{k=l}^{m}(-1)^{k-l}\binom{k}{l}\sum_{1\le i_1<i_2<\cdots<i_k\le m}p_{i_k}\dots p_{i_2}p_{i_1}\right),$$

where, of course, $\lambda\neq 0$. As was already observed, $LR_{\text{left}}(P;0)$ is the constant term in the Laurent expansion at the origin of (25) and a straightforward computation gives that it is equal to

$$\sum_{m=1}^{n}\sum_{s,t=0}^{m-1}(-1)^{s+t}\sum_{l=0}^{\min\{s,t\}}\binom{s}{l}\binom{t}{l}\sum_{\substack{1\le i_1<i_2<\cdots<i_s\le m-1\\ 1\le j_1<j_2<\cdots<j_t\le m-1}}p_{j_1}p_{j_2}\dots p_{j_t}p_m p_{i_s}\dots p_{i_2}p_{i_1}.$$

Simplifying with the help of

$$\sum_{l=0}^{\min\{s,t\}}\binom{s}{l}\binom{t}{l} = \binom{s+t}{s},$$

an identity which can be quickly verified by comparing the coefficients of x^t in the left and right hand sides of

$$(1+x)^{s+t} = (1+x)^s(1+x^{-1})^t x^t,$$

we arrive at

$$\sum_{m=1}^{n} \sum_{s,t=0}^{m-1} (-1)^{s+t} \binom{s+t}{t} \sum_{\substack{1\le i_1<i_2<\cdots<i_s\le m-1 \\ 1\le j_1<j_2<\cdots<j_t\le m-1}} p_{j_1}p_{j_2}\dots p_{j_t}p_m p_{i_s}\dots p_{i_2}p_{i_1} \tag{26}$$

`left`

for $LR_{\mathrm{left}}(P;0)$.

For the right logarithmic residue $LR_{\mathrm{right}}(P;0)$ there is the analogous expression

$$\sum_{m=1}^{n} \sum_{s,t=0}^{n-m} (-1)^{s+t} \binom{s+t}{t} \sum_{\substack{n\ge i_1>i_2>\cdots>i_s\ge m+1 \\ n\ge j_1>j_2>\cdots>j_t\ge m+1}} p_{j_1}p_{j_2}\dots p_{j_t}p_m p_{i_s}\dots p_{i_2}p_{i_1}. \tag{27}$$

`right`

Note that it can be obtained from (26) by reversing the order of the idempotents. Here is the precise formulation that makes this completely transparent: if $\widetilde{P} = E_{p_n,0}\dots E_{p_1,0}$, i.e.,

$$\widetilde{P}(\lambda) = \prod_{j=1}^{n} \big(e - p_{n+1-j} + \lambda p_{n+1-j}\big), \qquad \lambda \in \mathbb{C},$$

then

$$LR_{\mathrm{left}}(\widetilde{P};0) = LR_{\mathrm{right}}(P;0). \tag{28}$$

`rev`

This can be seen as follows. For $m = 0,\dots,n-1$, write $\widetilde{P}_m = E_{p_n,0}\dots E_{p_{n+1-m},0}$ and $Q_m = E_{p_{n+1-m},0}\dots E_{p_n,0}$. Then $LR_{\mathrm{left}}(\widetilde{P};0)$ and $LR_{\mathrm{right}}(P;0)$ are the constant terms in the Laurent expansions of

$$\widetilde{L}(\lambda) = \lambda\widetilde{P}'(\lambda)\widetilde{P}(\lambda)^{-1} = \sum_{m=1}^{n} \widetilde{P}_{m-1}(\lambda)p_{n+1-m}\widetilde{P}_{m-1}(\lambda)^{-1} \tag{29}$$

`llpf03A`

and

$$R(\lambda) = \lambda P(\lambda)^{-1}P'(\lambda) = \sum_{m=1}^{n} Q_{m-1}(\lambda)^{-1}p_{n+1-m}Q_{m-1}(\lambda) \tag{30}$$

`rrpf03A`

respectively. For $\lambda \neq 0$ and $k = 0,\dots,n-1$, we have

$$\begin{aligned}\widetilde{P}_k(\lambda^{-1}) &= E_{p_n,0}(\lambda^{-1})\dots E_{p_{n+1-k},0}(\lambda^{-1}) \\ &= E_{p_n,0}(\lambda)^{-1}\dots E_{p_{n+1-k},0}(\lambda)^{-1} = \ Q_k(\lambda)^{-1}.\end{aligned}$$

Here we used the simple observation (7) from Section 3. With (29) and (30) it follows that

$$\widetilde{L}\big(\lambda^{-1}\big) = R(\lambda), \qquad \lambda \neq 0.$$

Hence the constant terms in (29) and (30) are the same, and with this the desired identity (28) has been established.

The expressions (26) and (27) are what we will call *integer combinations of the idempotents* $p_1,\dots,p_n$. That is, they are linear combinations of monomials in these idempotents with integer coefficients. The number of different monomials involved is $\frac{1}{3}\,(4^n-1)$ and so it grows fast when n becomes larger. However, in the case where the idempotents $p_1,\dots,p_n$ commute, things can be enormously simplified. Indeed, in that situation all terms except the given idempotents themselves (corresponding to the values $m = 1, \dots, n$; $s = t = 0$ of the summation indices) cancel each other, so that

$$LR_{\mathrm{left}}(P; 0) = LR_{\mathrm{right}}(P; 0) = \sum_{k=1}^{n} p_k.$$

This can be seen by analyzing the coefficients of (26) and (27), but also (and more quickly) from the fact that in the commutative case

$$P'(\lambda)P^{-1}(\lambda) = P^{-1}(\lambda)P'(\lambda) = \sum_{m=1}^{n} \frac{1}{\lambda} p_m, \qquad \lambda \neq 0.$$

To put things in perspective, it is proved in [BES3] that logarithmic residues of commutative functions are always sums of idempotent. It is also elucidating to recall that [BES3], Example 2.4 exhibits a situation where the left logarithmic residue

$$p_1 + p_2 - p_1 p_2 - p_2 p_1 + 2p_1 p_2 p_1$$

of an elementary polynomial P of the form $P = E_{p_1,0} E_{p_2,0}$ is not a sum of idempotents. The underlying Banach algebra in the example is almost commutative in the sense that it is a polynomial identity algebra (cf., [AL]). Similar things can of course be done for the right logarithmic residue

$$p_2 + p_1 - p_2 p_1 - p_1 p_2 + 2p_2 p_1 p_2.$$

obtained from the above formula by interchanging p_1 and p_2. □

Next we turn to logarithmic residues of general plain functions. The difference with Example 8.2 lies in the fact that now we are faced with the disturbing influence of an everywhere invertible factor. Nevertheless some of the basic features of the example remain.

lrpfT2 **Theorem 8.3.** *Let Δ be a bounded Cauchy domain in $\mathbb{C}$, let $F \in \mathcal{A}_\partial(\Delta; \mathcal{B})$ and suppose F is plain on Δ. Then the (left and right) logarithmic residues of F with respect to Δ belong to the* (*possibly non-closed*) *subalgebra of $\mathcal{B}$ generated by the idempotents. In fact, the logarithmic residues of F with respect to Δ are integer combinations of the idempotents of $\mathcal{B}$.*

From the theorem it is clear that Example 8.1 provides another example, besides Example 4.4, of a non-plain function. The advantage of Example 4.4 is that the Banach algebra there is commutative. The one in Example 8.1, being a subalgebra of $\mathbb{C}^{3\times 3}$, is a polynomial identity algebra, but it is not commutative.

Proof. We begin with some preliminary material. Let $p \in \mathcal{B}$ be an idempotent, let $b \in \mathcal{B}$ be arbitrary, and consider the element $p + pb(e-p)$. Clearly $\big(p + pb(e-p)\big)^2 = p + pb(e-p)$, so $p + pb(e-p)$ is again an idempotent. Also $s = e + pb(e-p)$ is invertible with inverse $s^{-1} = e - pb(e-p)$ and $s\big(p + pb(e-p)\big) = ps$. So $p + pb(e-p)$ is similar to p. An analogous observation holds for $p + (e-p)bp$.

Let $q_1, \dots, q_m \in \mathcal{B}$ be idempotents and let $a_1, b_1, \dots, a_m, b_m$ be arbitrary elements in $\mathcal{B}$. For $k = 1, \dots, m$, introduce $L_k = L^{q_1,\dots,q_k}_{a_1,b_1,\dots,a_k,b_k}$ (inductively) as follows:

$$\begin{aligned}
L_1 &= q_1 + q_1a_1(e-q_1) + (e-q_1)b_1q_1,\\
L_2 &= q_2 + q_2L_1q_2 + (e-q_2)L_1(e-q_2) + q_2a_2(e-q_2) + (e-q_2)b_2q_2,\\
&\vdots\\
L_m &= q_m + q_mL_{m-1}q_m + (e-q_m)L_{m-1}(e-q_m) + q_ma_m(e-q_m) + (e-q_m)b_mq_m.
\end{aligned}$$

Note that $L_k = L^{q_1,\dots,q_k}_{a_1,b_1,\dots,a_k,b_k}$ is an integer combination of $3k$ idempotents each of which is equal or similar to one of the idempotents $q_1,\dots,q_k$ (cf., the observations of the first paragraph).

Since the function F is plain on Δ, it has only a finite number of points there where it takes a non-invertible value. This enables us to reduce the situation to the local case (cf., the identities (5) and (6) in Section 2). So we consider the situation at a single point in Δ, for simplicity of notation assumed to be the origin.

From Sections 3 and 4, we know that there exist idempotents $q_1,\dots,q_m$ in $\mathcal{B}$, a neighborhood Δ of the origin and an analytic function $G:\Delta\to\mathcal{B}$ such that G has invertible values on Δ and

$$F(\lambda) = G(\lambda)E_{q_1,0}(\lambda)\dots E_{q_m,0}(\lambda), \quad \lambda\in\Delta.$$

When $m=0$, the functions F and G coincide on Δ and, the logarithmic residues being zero, there is nothing to prove. So we assume that m is positive. Restricting ourselves to the right version of the logarithmic residue, it is sufficient to prove that there exist $a_1,b_1,\dots,a_m,b_m\in\mathcal{B}$ such that

$$LR_{\text{right}}(F;0) = L^{q_1,\dots,q_m}_{a_1,b_1,\dots,a_m,b_m}.$$

The argument goes by induction.

For $m=1$, the situation is simple. Indeed, for $\lambda\in\Delta$ we have

$$\begin{aligned}
F^{-1}(\lambda)F'(\lambda) &= \lambda^{-1}q_1 + \big(e-q_1+\lambda^{-1}q_1\big)G^{-1}(\lambda)G'(\lambda)\big(e-q_1+\lambda q_1\big)\\
&= \lambda^{-1}q_1 + q_1G^{-1}(\lambda)G'(\lambda)q_1 + (e-q_1)G^{-1}(\lambda)G'(\lambda)(e-q_1) +\\
&\quad +\lambda^{-1}q_1G^{-1}(\lambda)G'(\lambda)(e-q_1) + \lambda(e-q_1)G^{-1}(\lambda)G'(\lambda)q_1,
\end{aligned}$$

and, computing the coefficient of λ^{-1}, it follows that

$$LR_{\text{right}}(F;0) = q_1 + q_1G^{-1}(0)G'(0)(e-q_1).$$

So we can take $a_1 = G^{-1}(0)G'(0)$ and $b_1 = 0$.

Next assume that m is at least 2 and write $F(\lambda) = \widetilde{F}(\lambda)E_{q_m,0}$ with

$$\widetilde{F}(\lambda) = G(\lambda)E_{q_1,0}(\lambda)\dots E_{q_{m-1},0}(\lambda), \quad \lambda\in D.$$

Then, for $\lambda\in\Delta$,

$$\begin{aligned}
F^{-1}(\lambda)F'(\lambda) &= \lambda^{-1}q_m + \big(e-q_m+\lambda^{-1}q_m\big)\widetilde{F}^{-1}(\lambda)\widetilde{F}'(\lambda)\big(e-q_m+\lambda q_m\big)\\
&= \lambda^{-1}q_m + q_m\widetilde{F}^{-1}(\lambda)\widetilde{F}'(\lambda)q_m + (e-q_m)\widetilde{F}'(\lambda)\widetilde{F}^{-1}(\lambda)(e-q_m) +\\
&\quad +q_m\Big(\lambda^{-1}\widetilde{F}^{-1}(\lambda)\widetilde{F}'(\lambda)\Big)(e-q_m) + (e-q_m)\Big(\lambda\widetilde{F}^{-1}(\lambda)\widetilde{F}'(\lambda)\Big)q_m,
\end{aligned}$$

and from this one infers that $LR_{\text{right}}(F;0)$ is equal to

$$q_m + q_mLR_{\text{right}}(\widetilde{F};0)q_m + (e-q_m)LR_{\text{right}}(\widetilde{F};0)(e-q_m) + q_ma(e-q_m) + (e-q_m)bq_m,$$

where, for ϱ a sufficiently small positive number,

$$a = \frac{1}{2\pi i}\int_{|\lambda|=\varrho} \lambda^{-1}\widetilde{F}^{-1}(\lambda)\widetilde{F}'(\lambda)d\lambda, \qquad b = \frac{1}{2\pi i}\int_{|\lambda|=\varrho} \lambda\widetilde{F}^{-1}(\lambda)\widetilde{F}'(\lambda)d\lambda.$$

By induction hypothesis, we may assume that

$$LR_{\mathrm{right}}(\widetilde{F};0) = L^{q_1,\dots,q_{m-1}}_{a_1,b_1,\dots,a_{m-1},b_{m-1}}.$$

So with $a_m = a$, $b_m = b$ and $L_{m-1} = L^{q_1,\dots,q_{m-1}}_{a_1,b_1,\dots,a_{m-1},b_{m-1}}$, the above expression for $LR_{\mathrm{right}}(F;0)$ becomes

$$q_m + q_m L_{m-1} q_m + (e-q_m)L_{m-1}(e-q_m) + q_m a_m (e-q_m) + (e-q_m)b_m q_m$$

and this is just $L^{q_1,\dots,q_m}_{a_1,b_1,\dots,a_m,b_m}$. □

We now turn to material connected with Problem 2 from the introduction. Since the example in [BES2] of a non-trivial situation where the logarithmic residue vanishes involves a plain function, some extra structure is needed. In line with [GS1] and [BKL2], we find this in the notion of a *trace*, here meant to be a (possibly non-continuous) linear function $\tau : \mathcal{B} \to \mathbb{C}$ with the following additional commutativity property: $\tau(ab) = \tau(ba)$ for all a and b in $\mathcal{B}$. There are important examples where non-trivial traces do exist. One such an example is concerned with the so called rotation C^*-algebras considered in [Bo]. The tracial state, as it is called there, is even defined on the full algebra itself ($\mathcal{J} = \mathcal{B}$). Another instance is provided by the polynomial identity Banach algebras in the sense of [AL] and [K]. We shall come back to this in the somewhat more general framework of Theorem 8.7 below. First, however, we make the connection with logarithmic residues of plain functions.

lrpfT3 **Theorem 8.4.** *Let Δ be a bounded Cauchy domain in $\mathbb{C}$, let $F \in \mathcal{A}_\partial(\Delta;\mathcal{B})$ be plain on Δ and suppose F is represented in the form*

$$F(\lambda) = H(\lambda)E_{p_1,\alpha_1}(\lambda)\cdots E_{p_n,\alpha_n}(\lambda), \quad \lambda \in \Delta, \tag{31}$$ Aa

with $\alpha_1,\dots,\alpha_n$ points in Δ (not necessarily distinct), $p_1,\dots,p_n$ idempotents in $\mathcal{B}$ and $H : \Delta \to \mathcal{B}$ an analytic function taking invertible values on Δ. Assume τ is a trace on $\mathcal{B}$. Then

$$\tau\big(LR_{\mathrm{left}}(F;D)\big) = \tau\big(LR_{\mathrm{right}}(F;D)\big) = \sum_{j=1}^{n}\tau\left(p_j\right). \tag{32}$$ A

As a consequence, the sum of traces appearing in (32) is independent of the (non-unique) representation (31). There is a version of the theorem with the function H appearing in (31) as the last factor instead of the first.

Proof. We first focus on the local situation at a single point. Take $\alpha \in \{\alpha_1,\dots,\alpha_n\}$ and let $j_1,\dots,j_m$ be the different integers j among $1,\dots,n$ such that $\alpha_j = \alpha$. By Proposition 3.2 we can write F in the form

$$F(\lambda) = G(\lambda)E_{q_1,\alpha}(\lambda)\dots E_{q_m,\alpha}(\lambda), \quad \lambda \in D,$$

where $q_1,\dots,q_m$ are idempotents in $\mathcal{B}$, similar to $p_{j_1},\dots,p_{j_m}$ respectively, and G takes invertible values on an open neighborhood of α. From the proof of Theorem 8.3 we know that for appropriately chosen $a_1,b_1,\dots,a_m,b_m \in \mathcal{B}$

$$LR_{\mathrm{right}}(F;\alpha) = L^{q_1,\dots,q_m}_{a_1,b_1,\dots,a_m,b_m}. \tag{33}$$ B

Here (and below) we employ the special notation introduced in the proof of Theorem 8.3. Now consider the elements $L^{q_1,\dots,q_k}_{a_1,b_1,\dots,a_k,b_k}$, $k=1,\dots,m$. Via a simple induction argument, based on the identities

$$\tau\big(pb(e-p)\big)=\tau\big((e-p)pb\big)=\tau(0)=0,$$
$$\tau\big((e-p)bp\big)=\tau\big(p(e-p)b\big)=\tau(0)=0,$$
$$\tau\big(pbp+(e-p)b(e-p)\big)=\tau\big(pb+(e-p)b\big)=\tau(eb)=\tau(b),$$

holding for arbitrary $b\in\mathcal{B}$ provided $p\in\mathcal{B}$ is an idempotent, one proves that

$$\tau\left(L^{q_1,\dots,q_k}_{a_1,b_1,\dots,a_k,b_k}\right)=\sum_{i=1}^{k}\tau\left(q_i\right),\qquad k=1,\dots,m.$$

Since q_i is similar to p_{j_i}, we have $\tau\left(q_i\right)=\tau\left(p_{j_i}\right)$, and we get

$$\tau\left(L^{q_1,\dots,q_k}_{a_1,b_1,\dots,a_k,b_k}\right)=\sum_{i=1}^{k}\tau\left(p_{j_i}\right),\qquad k=1,\dots,m.$$

Combining this identity (for $k=m$) with (33), one obtains

$$\tau\big(LR_{\mathrm{right}}(F;\alpha)\big)=\tau\left(L^{q_1,\dots,q_m}_{a_1,b_1,\dots,a_m,b_m}\right)=\sum_{i=1}^{m}\tau\left(p_{j_i}\right)=\sum_{\substack{j=1\\ \alpha_j=\alpha}}^{n}\tau\left(p_j\right).$$

So far for the local situation. To make the step to the global level, note that F takes invertible values on $\Delta\setminus\{\alpha_1,\dots,\alpha_n\}$. Hence

$$LR_{\mathrm{right}}(F;\Delta)=\sum_{\alpha\in\{\alpha_1,\dots,\alpha_n\}}LR_{\mathrm{right}}(F;\alpha).$$

But then

$$\tau\big(LR_{\mathrm{right}}(F;\Delta)\big)=\sum_{\alpha\in\{\alpha_1,\dots,\alpha_n\}}\tau\big(LR_{\mathrm{right}}(F;\alpha)\big)=\sum_{\alpha\in\{\alpha_1,\dots,\alpha_n\}}\left(\sum_{\substack{j=1\\ \alpha_j=\alpha}}^{n}\tau\left(p_i\right)\right),$$

and the second identity in (32) follows.

For the first identity in (32) one can argue along the lines indicated in [GS1]) or [BKL2]). It is also possible to take the following route. From Proposition 3.1 we know

$$F(\lambda)=E_{\widehat{p}_1,\alpha_1}(\lambda)\dots E_{\widehat{p}_n,\alpha_n}(\lambda)\widehat{H}(\lambda),\quad \lambda\in\Delta,$$

where $\widehat{H}$ takes invertible values on all of Δ and with $\widehat{p}_1,\dots,\widehat{p}_n$ similar to $p_1,\dots,p_n$, respectively. An argument analogous to the one given above yields

$$\tau\big(LR_{\mathrm{left}}(F;\Delta)\big)=\sum_{j=1}^{n}\tau\left(\widehat{p}_j\right).$$

To finish the proof, note that $\tau\left(\widehat{p}_j\right)=\tau\left(p_j\right),\ j=1,\dots,n.$ $\square$

From the material presented in Section 3 it is clear that the pointwise product of plain functions (on the same domain) is again plain. In connection with this, we now show that traces of logarithmic residues of plain functions satisfy a logarithmic property.

lrpfC1 **Corollary 8.5.** *Let Δ be a bounded Cauchy domain in $\mathbb{C}$ and suppose τ is a trace on $\mathcal{B}$. If F_1 and F_2 in $\mathcal{A}_\partial(\Delta;\mathcal{B})$ are plain on Δ, then the product function F_1F_2 is again plain on Δ and*

$$\tau\big(LR_{\mathrm{left}}(F_1F_2;\Delta)\big) = \tau\big(LR_{\mathrm{left}}(F_1;\Delta)\big) + \tau\big(LR_{\mathrm{left}}(F_2;\Delta)\big).$$

The identity remains true when the left logarithmic residue is replaced by the right logarithmic residue.

Proof. Write F_j in the same form as the function F in Theorem 8.4. Thus, suppressing the variable, $F_j = H_j E_{p_1^{(j)},\alpha_1^{(j)}} \dots E_{p_{n_j}^{(j)},\alpha_{n_j}^{(j)}}$, so that

$$\tau\big(LR_{\mathrm{left}}(F_j;\Delta)\big) = \sum_{i=1}^{n_j} \tau\left(p_i^{(j)}\right), \qquad j=1,2. \tag{34}$$ Ab

For the product F_1F_2, we have

$$F_1F_2 = H_1 E_{p_1^{(1)},\alpha_1^{(1)}} \dots E_{p_{n_1}^{(1)},\alpha_{n_1}^{(1)}} H_2 E_{p_1^{(2)},\alpha_1^{(2)}} \dots E_{p_{n_2}^{(2)},\alpha_{n_2}^{(2)}}.$$

By (possibly) repeated application of Proposition 3.1 this can be transformed into

$$F_1F_2 = H E_{\widetilde{p}_1^{(1)},\alpha_1^{(1)}} \dots E_{\widetilde{p}_{n_1}^{(1)},\alpha_{n_1}^{(1)}} E_{p_1^{(2)},\alpha_1^{(2)}} \dots E_{p_{n_2}^{(2)},\alpha_{n_2}^{(2)}},$$

where H takes invertible values on all of Δ and with $\widetilde{p}_i^{(j)}$ similar to $p_i^{(j)}$ (hence having the same value for τ) for all relevant values of i and j. It follows that

$$\tau\big(LR_{\mathrm{left}}(F_1F_2;\Delta)\big) = \sum_{i=1}^{n_1} \tau\left(\widetilde{p}_i^{(1)}\right) + \sum_{i=1}^{n_2} \tau\left(p_i^{(2)}\right) = \sum_{i=1}^{n_1} \tau\left(p_i^{(1)}\right) + \sum_{i=1}^{n_2} \tau\left(p_i^{(2)}\right),$$

and in combination with (34) the desired result follows. □

Let $\mathcal{T}$ be non-empty family of traces on $\mathcal{B}$. We will say that $\mathcal{T}$ is *resolving* if the only sums of idempotents in $\mathcal{B}$ that are annihilated by all the traces from $\mathcal{T}$ are the trivial ones. More precisely, $\mathcal{T}$ is resolving if (and only if) the following holds: the situation where $p_1,\dots,p_n$ are idempotents in $\mathcal{B}$ and $\tau(p_1+\dots+p_n)=0$ for every trace τ in $\mathcal{T}$ can only occur when $p_j=0,\ j=1,\dots,n$. If $\mathcal{B}$ has a resolving family of traces, then $\mathcal{B}$ has only zero sums of idempotents (cf., [BES1]-[BES8]).

We now have the following partial answer to the Problem (B) from the introduction, specialized to plain functions as indicated after Example 8.2.

lrpfC2 **Corollary 8.6.** *Let Δ be a bounded Cauchy domain in $\mathbb{C}$ and let $F \in \mathcal{A}_\partial(\Delta;\mathcal{B})$ be plain on Δ. Assume $\mathcal{B}$ has a resolving family of traces, $\mathcal{T}$ say. Then the following statements are equivalent:*

(a) *F takes invertible values on all of Δ;*
(b) *$LR_{\mathrm{left}}(F;\Delta)=0$;*
(c) *$\tau\big(LR_{\mathrm{left}}(F;\Delta)\big)=0$ whenever $\tau\in\mathcal{T}$.*

For definiteness, the result is stated in terms of the left version of the logarithmic residue but of course it is also valid for the right variant.

Proof. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are trivial. So we concentrate on (c) $\Rightarrow$ (a). Write F as in (31) from Theorem 8.4 so that (32) holds with $p_1,\dots,p_n$

idempotents in $\mathcal{B}$. Suppose (c) is satisfied. Then, for $\tau \in \mathcal{T}$,

$$\tau\left(\sum_{j=1}^{n} p_j\right) = \sum_{j=1}^{n} \tau\left(p_j\right) = \tau\big(LR_{\text{left}}(F;\Delta)\big) = 0.$$

By assumption, the family $\mathcal{T}$ of traces is resolving. Hence $p_j = 0$, $j = 1, \ldots, n$, and we see from (31) that F coincides with the function G having invertible values on all of Δ. $\square$

A sufficient condition for $\mathcal{T}$ to be resolving is that $\tau(p) \geq 0$ for all idempotents $p \in \mathcal{B}$ and each $\tau \in \mathcal{T}$, while $\tau(p) = 0$ for every $\tau \in \mathcal{T}$ (if and) only if $p = 0$. This is used in the proof of the next theorem for which we now prepare with some definitions (cf., [K]; see also [BES1] and [BES2]).

By a *matrix representation* of $\mathcal{B}$ we mean a (possibly non-continuous) unital homomorphism from $\mathcal{B}$ into a matrix algebra $\mathbb{C}^{n\times n}$. So μ is a matrix representation of $\mathcal{B}$ if there exists a positive integer n_μ such that $\mu : \mathcal{B} \to \mathbb{C}^{n_\mu \times n_\mu}$ is a linear and multiplicative function mapping the unit element e in $\mathcal{B}$ into $\mu(e) = I_n$. Let $\mathcal{M}$ be a (non-empty) family of matrix representations $\mu : \mathcal{B} \to \mathbb{C}^{n_\mu \times n_\mu}$ of $\mathcal{B}$. We say that $\mathcal{M}$ is a *sufficient family of matrix representations* for $\mathcal{B}$ if an element $a \in \mathcal{B}$ is invertible in $\mathcal{B}$ if (and only if) $\mu(a)$ is invertible for each $\mu \in \mathcal{M}$. We emphasize that we do not require the homomorphisms μ to be continuous. Also we do not require that the collection of integers n_μ with μ from $\mathcal{M}$ be bounded.

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Theorem 8.7. *Each unital Banach algebra possessing a sufficient family of matrix representations has a resolving family of continuous traces.*

In particular, each polynomial identity Banach algebra has a resolving family of continuous traces (cf., [AL] and [K]).

Proof. Let $\mathcal{B}$ be a unital Banach algebra, and assume $\mathcal{M}$ is a sufficient family of matrix representations for $\mathcal{B}$. The case when the matrix representations in $\mathcal{B}$ are continuous is easy. Indeed, we then proceed as follows. Take $\mu \in \mathcal{M}$. Then $\mu : \mathcal{B} \to \mathbb{C}^{n\times n}$ for some $n = n_\mu$ depending on μ, μ is continuous, linear and multiplicative, while in addition $\mu(e) = I_n$. Now, for $b \in \mathcal{B}$, let $\tau_\mu(b)$ be the trace of the $n \times n$ matrix $\mu(b)$. Then, obviously, τ_μ is a continuous trace on $\mathcal{B}$. Introducing $\mathcal{T} = \{\tau_\mu \mid \mu \in \mathcal{M}\}$, we obtain a non-empty family of continuous traces on $\mathcal{B}$. Let p be an idempotent in $\mathcal{B}$. If $\mu \in \mathcal{M}$, then $\mu(p)^2 = \mu(p^2) = \mu(p)$. Thus $\mu(p)$ is an idempotent matrix and so its trace is equal to its rank. In particular $\tau_\mu(p) \geq 0$. Now assume $\tau_\mu(p) = 0$ whenever $\mu \in \mathcal{M}$. Each μ in $\mathcal{M}$ sends p into an idempotent matrix $\mu(p)$ with zero trace, that is into the zero matrix of appropriate size. But then $\mu(e - p) = \mu(e) - \mu(p) = \mu(e)$ is an identity matrix, hence invertible. Since the family $\mathcal{M}$ is sufficient, it follows that $e - p$ is invertible. From $p(e - p) = 0$, it is now clear that $p = 0$, as desired.

It remains to consider the general case where the matrix representations in $\mathcal{M}$ are allowed to be non-continuous. This situation is handled by showing that $\mathcal{M}$ can be changed into a sufficient family of continuous matrix representations for $\mathcal{B}$. The argument for this rests heavily on what is known as Burnside's theorem holding that every proper algebra of matrices (over an algebraically closed field) has a non-trivial invariant subspace (cf., [LR]). Here is a sketch of the reasoning. Using Burnside's Theorem one can see that $\mathcal{M}$ can be transformed into a sufficient family of surjective matrix representations for $\mathcal{B}$. However, the null space of a surjective matrix representation of $\mathcal{B}$ is a maximal ideal in $\mathcal{B}$, hence closed. But, having a closed null space, the matrix representation must be continuous. $\square$

Recall from [K] that the class of Banach algebras possessing a sufficient family of matrix representations includes all polynomial identity Banach algebras (cf., [AL]). Here is an example of a Banach algebra not possessing a sufficient family of matrix representations, but nevertheless having a resolving family of traces.

nKr.res **Example 8.8.** Let $\mathcal{A}$ be the unital Banach algebra of bounded sequences $\boldsymbol{A} = (A_1, A_2, A_3, \ldots)$ with A_n an $n \times n$ matrix, $n = 1, 2, 3, \ldots$. Here boundedness is with respect to the usual matrix norm for the entries $A_n \in \mathbb{C}^{n \times n}$. The algebraic operations in $\mathcal{A}$ are defined coordinate-wise, and the norm on $\mathcal{A}$ is given by

$$\|\boldsymbol{A}\| = \sup\{\|A_n\| \mid n = 1, 2, 3, \ldots\}.$$

For $n = 1, 2, 3 \ldots$, let $\tau_n(\boldsymbol{A})$ be the trace of the $n \times n$ matrix A_n. Then τ_n is a trace on $\mathcal{A}$. If $\boldsymbol{P}$ is an idempotent in $\mathcal{A}$, then obviously $\tau_n(\boldsymbol{P}) \geq 0$. Also $\boldsymbol{P} = 0$ whenever $\tau_n(\boldsymbol{P})$ vanishes for every n. Hence $\mathcal{T} = \{\tau_n \mid n = 1, 2, 3 \ldots\}$ is a resolving family of traces for $\mathcal{A}$. The Banach algebra $\mathcal{A}$ is not a polynomial identity algebra. Suppose it is. Then there exists a polynomial Ψ, in k non-commuting variables say, such that Ψ annihilates $\mathcal{A}$. Now clearly Ψ also annihilates $\mathbb{C}^{n \times n}$. But then the number k cannot be smaller than $2n$ (see Theorem 20.2 in [K]) and the references given there). As this holds for all n, we come to a contradiction. Actually, $\mathcal{A}$ does not even have a sufficient family of matrix representations. The argument for this is quite involved, and will be presented elsewhere. $\square$

The material concerning resolving families of traces has been inspired by [BES2], Theorems 3.1 and 4.1. The first of these result deals with Fredholm operator valued functions (cf. [BES5] and Example 6.1 above); the second with the situation of Theorem 8.7. Thus Corollary 8.6 applies to a wider class of functions than those from [BES2], Theorem 3.1 and also to a less restricted class of Banach algebras than the ones featuring in [BES2], Theorem 4.1. On the other hand, the functions in [BES2], Theorems 4.1 are not required to be plain as is the case in Corollary 8.6.

There is a connection here with a question concerning Problem (B) from the introduction, a question which has been open from the start of the series of publications [BES1]–[BES8]. Problem (B) can be summarized as follows: under what circumstances, does Corollary 8.6(b) imply Corollary 8.6(a). Banach algebras for which this implication holds have the property that they possess only trivial zero sums of idempotents (see [BES2], Theorem 5.1). The issue is now: is the converse true? So, if we have a Banach algebra possessing only trivial zero sums of idempotents, does it follow that Corollary 8.6(b) $\Rightarrow$ Corollary 8.6(a)? The negative answer that we expect could come about by constructing an example showing that the assumption that F is plain in Corollary 8.6 is essential for the implication Corollary 8.6(b) $\Rightarrow$ Corollary 8.6(a). Indeed, such an example would exhibit a situation where Corollary 8.6(b) does not imply Corollary 8.6(a) while the underlying Banach algebra (having a resolving family of traces) has only trivial zero sums of idempotents. Although at first sight Example 8.8 might look like a suitable candidate, we have strong reasons that it fails on this count.

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