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A note on a stochastic location problem

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In this note we give a short and easy proof of the equivalence of Hakimi's one-median problem and the k-server-facility-loss median problem as discussed by Chiu and Larson in *Computer and Operation Research*. The proof makes only use of a stochastic monotonicity result for birth and death processes and the insensitivity of the $M/G/k/k$ loss model.

stochastic location; Hakimi median; stochastic monotonicity

1. Model formulation and results

In [1] Chiu and Larson consider the so-called k-server-single-facility-loss median model (k-SFLM). In this model customers at fixed locations generate calls according to a Poisson process with rate $\lambda > 0$. If upon arrival of a customer's call at the service facility all its k identical servers are busy this customer is lost at cost $Q > 0$. Moreover, if upon arrival some of the servers are idle one of them is assigned to this customer and travels to the location of that customer at a fixed cost rate. The objective is now to determine among a set $\mathscr{F} \subseteq \mathbb{R}^2$ of feasible locations that location for the service facility which will minimize the average cost of the system. In order to analyze this model define

- $\mathscr{N}_t(\boldsymbol{x}) :=$ number of customers being served at time t by one of the k servers if the facility is located at $\boldsymbol{x} \in \mathscr{F}$.

As argued in [1] the queueing process underlying the k-SFLM location problem is a $M/G/k/k$ loss model. For this queueing process it is well-known (cf. [2]) that for fixed $\boldsymbol{x} \in \mathscr{F}$ the random variable $\mathscr{N}_t(\boldsymbol{x})$ converges in distribution to the random variable $\mathscr{N}(\boldsymbol{x})$ representing the number of customers being served in the steady state if $\boldsymbol{x} \in \mathscr{F}$ denotes the location. Moreover, the distribution of $\mathscr{N}(\boldsymbol{x})$ depends only on the arrival rate and the first moment $m(\boldsymbol{x})$ of the service time distribution (Erlang's Loss Formula). If the facility is located at $\boldsymbol{x} \in \mathscr{F}$, the cost function $z(\boldsymbol{x})$ takes the following form (cf. [1]).

$$z(\boldsymbol{x}) = P_k(\boldsymbol{x})Q + (1 - P_k(\boldsymbol{x}))m(\boldsymbol{x}) \tag{1}$$

with

- $P_k(\boldsymbol{x}) :=$ Pr{customer finds upon arrival in steady state all k servers busy} = Pr{customer arriving in steady state is lost},
- $m(\boldsymbol{x}) :=$ expected total travel time of server to arbitrary customer when the facility is located at $\boldsymbol{x}$, and
- $Q :=$ cost per lost customer, $Q \geq 0$.

The main result proved in [1] using lengthy calculations states that the cost function $z(\boldsymbol{x})$ is increasing in $m(\boldsymbol{x})$. This implies that the k-SFLM location problem is solved by determining the location $\boldsymbol{x} \in \mathscr{F}$ which minimizes the expected

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total travel time $m(x)$. Hence, in the special case where $\mathscr{F}$ denotes some network $\mathscr{N}$ this reduces to finding the so-called Hakimi median (cf. [3]) at one of the nodes of $\mathscr{N}$. The above result can be verified easily without any calculations by using a well-known stochastic monotonicity result for birth and death processes. Before proving this we need the following observations. By Little's formula (cf. [5]) the quantity $(1 - P_k(x))m(x)$ equals $(1/\lambda)L(x)$ where $L(x)$ is the expected number of customers in the system and λ is the arrival rate of the Poisson process. Moreover, by the PASTA property, i.e. Poisson Arrivals See Time Averages (cf. [5]), we obtain

$$P_k(x) = \Pr\{\mathscr{N}(x) = k\}$$

and hence by (1)

$$z(x) = Q \Pr\{\mathscr{N}(x) = k\} + \frac{1}{\lambda}L(x). \tag{2}$$

Using (2) one can now prove the following result.

Lemma 1.1. *Let $x_1, x_2 \in \mathscr{F}$. Then $m(x_1) \le m(x_2)$ implies $z(x_1) \le z(x_2)$.*

Proof. Since the expected number $L(x)$ of customers in the steady state equals $\sum_{j=1}^{k} \Pr\{\mathscr{N}(x) \ge j\}$, it is sufficient by (2) to show that $\mathscr{N}(x_1)$ is stochastically smaller than $\mathscr{N}(x_2)$($\mathscr{N}(x_1) \overset{\text{d}}{\le} \mathscr{N}(x_2)$), i.e. $\Pr\{\mathscr{N}(x_1) \ge j\} \le \Pr\{\mathscr{N}(x_2) \ge j\}$ for every $0 \le j \le k$. In order to prove this we observe that for the $M/G/k/k$ loss model corresponding to a facility location in x (cf. [2])

$$\Pr\{\mathscr{N}(x) = j\} = \frac{\rho(x)^j/j!}{\sum_{i=0}^{k} \rho(x)^i/i!}$$

for $0 \le j \le k$ with $\rho(x) = \lambda m(x)$. This holds in particular for the Markovian loss model with arrival rate λ and service rate $1/m(x)$ and so we are finished by showing that $\mathscr{N}^{(1)} \overset{\text{d}}{\le} \mathscr{N}^{(2)}$ with $\mathscr{N}^{(i)}$ the number of customers in the steady state in a Markovian $M/M/k/k$ loss system i with arrival rate λ and service rate $1/m(x_i)$. If $\mathscr{N}_t^{(i)}$ denotes the number of customers at time t in the same Markovian loss system i then the stochastic processes $\{\mathscr{N}_t^{(i)}, t \ge 0\}$, $i = 1, 2$, are birth and death processes on the finite state space $\{0, 1, \dots, k\}$ with nonzero transition rates $q_{j,j+1} = \lambda$, $0 \le j \le k-1$ and $q_{j,j-1} = j/m(x_i)$ for system i, $j = 1, \dots, k$. By assumption we know that $m(x_1) \le m(x_2)$ and hence by a well-known monotonicity result for birth and death processes (cf. Prop. 4.2.10 of [4]) it follows that for every $t > 0$ $\mathscr{N}_t^{(1)} \overset{\text{d}}{\le} \mathscr{N}_t^{(2)}$. This implies $\mathscr{N}^{(1)} \overset{\text{d}}{\le} \mathscr{N}^{(2)}$ and so the result is proved. □

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