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A Note on Log Concave Survivor Functions in Auctions

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Abstract:

In a standard English auction in which bidders' valuations are independently drawn from a common distribution, a standard regularity condition is that the survivor function of the distribution be log concave. In an auction where the seller sets a fixed price, the equivalent condition requires log concavity of a survivor function derived from the primitive distribution. In this note we show that log concavity of the primitive survivor function implies log concavity in the derived functions. This result is of interest when studying on-line auctions that combined aspects of fixed-price and English auctions.

Keywords: English Auction; Log Concavity; Survivor Function
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A Note on Log-Concave Survivor Functions in Auctions

1. Introduction:

Consider a simple English or second-price private-value auction in which a seller with valuation wishes to sell a single unit of a good in an auction with n bidders, each of whom has a valuation drawn from the commonly known distribution, F, with support drawn from some interval on the positive real line. The seller's choice problem is to set the reservation price, R, to maximise his expected revenue from the auction.

A sufficient condition for the revenue function in this problem to be strictly quasi-concave (and hence for the first-order condition to yield a unique maximum), is that the survivor function, 1-$F(x)$, be log concave, independent of the number of bidders.

Now consider instead the same environment in which the seller will sell using a simple fixed-price auction, again with n bidders, in which the seller posts a set price, p_f, and the good is sold if at least one of the n bidders has a valuation in excess of p_f. As we will show below, a sufficient condition for the revenue function of this problem to be strictly quasi-concave is that $1-\left(F(p_f)\right)^n$ be log concave. Clearly this condition does depend on the number of bidders.

Many on-line auctions combine features of both a standard English auction, and a fixed-price auction. For instance, *E-Bay* and many other on-line auctions allow sellers to offer a "buy now" option, in which bidders have the option of purchasing the good at a fixed price announced at the start of the auction as long as the reserve price has not yet been met. In the New Zealand on-line auction, *Trade Me,* in addition to allowing buy-now prices, sellers can, in the event that the auction fails to attract a bid that meets reserve, make a fixed-price offer to all buyers who registered as "watchers" during the unsuccessful English-auction phase.

Given this combination of auction styles, it would be useful to have a general set of conditions for there to be a unique revenue-maximising value for the decision variable for the seller in either type of auction, for any number of bidders.

It turns out that the sufficient condition for characterising the profit maximum in the English auction—that the survivor function be log concave—is also sufficient for the fixed-price auction to have a unique maximum for any number of bidders. The purpose of this paper is to prove this result.

Note that $\left(1-\left(F(x)\right)^n\right)$ is a concave transformation of 1-F(x)) for $n \geq 2$ and $F(x) \in (0,1),$ so it is intuitive that log concavity of the latter would imply log concavity of the former. It is not true in general, however, that for some log concave function, *f*, and concave function, *g*, the composite function *g*(*f*) is log concave, and so the result still needs to be established formally.

In the following section, we briefly summarise the literature to establish why the two log concavity assumptions generate unique maxima for the two types of auction, respectively. Section 3 then proves the main result.

2. The Role of Log Concavity:

Let *f* be the associated probability density function of *F*, and let x_0 be the valuation of the seller for the good if it does not sell. The objective functions and the resulting first-order and second-order conditions

In the case of an English auction, it is well-known (see, for instance, Krishna, 2002, Chapter 2.5), that the first-order condition for the profit-maximising reserve price is

$$1-(R-x_0)\lambda(R)=0, \qquad (1)$$

where $\lambda(x)$ is the hazard rate,

$$\lambda(x)=\frac{f(x)}{1-F(x)}.$$

A sufficient condition for Equation (1) to describe a unique maximum is that $\lambda(x)$ be increasing in *x*. As shown in Bagnoli and Bergstrom (2005), this condition is equivalent to the requirement that the survivor function, 1-*F*(*x*), be log concave.

Now consider the fixed-price auction. The revenue for the seller as a function of the fixed price is

$$\pi(p_f) = \left(1 - \left(F(p_f)\right)^n\right)(p_f - x_0).$$

A sufficient condition for this function to be strictly quasi-concave is that the quantity function, $1 - \left(F(p_f)\right)^n$, be log concave (see the "applications to monopoly theory" in Bergstrom and Bagnoli (2005)).

These two sufficient conditions are well known, but the derivations are dispersed throughout the literature. For completeness, therefore, we derive both in the Appendix to this note.

3. The Main Result:

Let *F* be a distribution defined over some variable *x*. Let there be *n* independent draws from this distribution. The distribution function for the maximum of the *n* draws is then $\left(F(x)\right)^n$. The survivor function for a single draw is (1-*F*(*x*)) and the survivor function for the *n*-draw maximum is $\left(1 - \left(F(x)\right)^n\right)$.

Theorem:

If the survivor function (1-*F*(*x*)) is log concave, then the survivor function for the *n*-draw maximum, $\left(1 - \left(F(x)\right)^n\right)$, will also be log concave for all $n \geq 2$.

Proof:

If $\left(1 - F(x)\right)$ is log concave, then by definition,

$$\frac{d^2}{dx^2}\left(\ln\left(1 - F(x)\right)\right) < 0$$

$$\Rightarrow \qquad f'(x) > \frac{-\left(f(x)\right)^2}{\left(1 - F(x)\right)}.$$

Define

$$z = f'(x) + \frac{\left(f(x)\right)^2}{\left(1 - F(x)\right)} > 0$$

so that

$$f'(x) = \frac{-\left(f(x)\right)^2}{\left(1 - F(x)\right)} + z. \qquad (2)$$

We want to show that

$$\frac{d^2}{dx^2}\left(\ln\left(1 - \left(F(x)\right)^n\right)\right) < 0.$$

Differentiating twice gives

$$\frac{d^2}{dx^2}\left(\ln\left(1 - \left(F(x)\right)^n\right)\right) = \frac{nF(x)^{n-2}\left(\left(1 - \left(F(x)\right)^n\right)\left(-(n-1)\left(f(x)\right)^2 - F(x)f'(x)\right) - n\left(F(x)\right)^n\left(f(x)\right)^2\right)}{\left(1 - \left(F(x)\right)^n\right)^2}$$

Substituting in Equation (2) gives

$$\begin{aligned}\frac{d^2}{dx^2}\left(\ln\left(1 - \left(F(x)\right)^n\right)\right) &= \frac{nF(x)^{n-2}\left(f(x)\right)^2\left(\left(1 - \left(F(x)\right)^n\right)\left(-(n-1)(1 - F(x)) + F(x)\right) - n\left(F(x)\right)^n(1 - F(x))\right)}{\left(1 - \left(F(x)\right)^n\right)^2(1 - F(x))} \\ &\quad - \frac{nF(x)^{n-1}}{\left(1 - \left(F(x)\right)^n\right)}z \\ &= \frac{nF(x)^{n-2}\left(f(x)\right)^2}{\left(1 - \left(F(x)\right)^n\right)^2(1 - F(x))}H(x) - \frac{nF(x)^{n-1}}{\left(1 - \left(F(x)\right)^n\right)}z,\end{aligned}$$

where

$$H(x) = \left(\left(1 - \left(F(x)\right)^n\right) - n(1 - F(x)\right).$$

Since $z > 0,$ we only need to show that $H(x) < 0 \ \forall n \geq 2.$ Note that

$\lim_{x\to\infty} H(x) = 0,$ and

$H'(x) = nf(x)\left(1 - \left(F(x)\right)^{n-1}\right) > 0,$

which establishes the proposition.

□

4. Discussion:

This result in this note provides a simple sufficient condition to guarantee a unique solution for both English (and hence 2nd-price) auctions and fixed-price auctions for any number of bidders. Interestingly, in the case of 1st-price or Dutch auctions, a sufficient condition for there to be a unique equilibrium is that the distribution function, *F*, be log concave. (See Lebrun, 2006). As shown in Bagnoli and Bergstrom (2005), a sufficient condition for both the distribution function, *F*, and the associated survivor function, 1-*F*, to be log concave is that the underlying density function, *f*, be log concave. We are not aware of any actual auction that combines aspects of 2nd-price and fixed-price auctions, but should such a combination become empirically interesting, the result in this note can be easily applied in that case.

5. References:

Bagnoli, M., Bergstrom, T., 2005. Log Concavity and its Applications. Econ. Theory 26, 445-469.

Krishna, V., 2002. Auction Theory. Academic Press, London, San Diego.

Lebrun, B., 2006. Uniqueness of the Equilibrium in First-Price Auctions. Games and Econ. Behav. 55, 131-151.

Appendix

Sufficient Conditions for an English Auction:

Let $F_n(\upsilon)$ be the distribution function for the maximum of *n* draws from a common distribution *F*, and let $F_{n-1}(x|\upsilon)$ be the distribution function for the maximum of *n*-1 draws from the same distribution, conditional on the information that all draws are no greater than υ. $F_{n-1}(x|\upsilon)$ then is the distribution for the second-highest of *n* draws, conditional on the information that the maximum draw is υ. We then have

$$F_n(x) = \left(F(x)\right)^n, \tag{A1}$$

$$F_{n-1}(x|\upsilon) = \frac{\left(F(x)\right)^{n-1}}{F(\upsilon)^{n-1}}. \tag{A2}$$

Let $E[p\,|\,\upsilon]$ be the expected price paid to the seller in an English auction, conditional on the highest valuation being $\upsilon > R,$ where *R* is the seller-set reservation price. Let the support of *F* be [*a*,*b*]. We then have

$$E[p\,|\,\upsilon] = R\int_a^R dF_{n-1}(x|\upsilon) + \int_R^\upsilon x dF_{n-1}(x|\upsilon).$$

Integrating by parts, this becomes

$$E[p\,|\,\upsilon] = \upsilon - \int_R^\upsilon F_{n-1}(x|\upsilon) d\upsilon. \tag{A3}$$

In an English auction, the unconditional expected price is the integral of the conditional expected price weighted over the relevant range of υ:

$$E[p] = \int_R^b E[p\,|\,\upsilon] dF_n(\upsilon). \tag{A4}$$

Putting (A2) and (A3) into (A4) yields

$$\begin{aligned} E[p] &= \int_R^b \left(\upsilon - \int_R^\nu \frac{\left(F(x)\right)^{n-1}}{\left(F(\upsilon)\right)^{n-1}} dx \right) dF_n(\upsilon) \\ &= \int_R^b \upsilon dF_n(\upsilon) - \int_R^b \left(\int_R^\nu \frac{\left(F(x)\right)^{n-1}}{\left(F(\upsilon)\right)^{n-1}} dx \right) dF_n(\upsilon). \end{aligned}$$

Note from (A1) that

$$\frac{dF_n(\upsilon)}{d\nu} = n\left(F(\upsilon)\right)^{n-1} f(\upsilon),$$

so we can write

$$E[p] = \int_R^b \upsilon dF_n(\upsilon) - \int_R^b n\left(\int_R^\upsilon F(x)^{n-1}dx\right) f(\upsilon)d\upsilon.$$

Integrating by parts produces

$$\begin{aligned} E[p] &= b - R\left(F(R)\right)^n - \int_R^b \left(F(\upsilon)\right)^n d\upsilon - n\int_R^b \left(F(\upsilon)\right)^{n-1} d\upsilon + n\int_R^b \left(F(\upsilon)\right)^n d\upsilon \\ &= b - R\left(F(R)\right)^n + (n-1)\int_R^b \left(F(\upsilon)\right)^n d\upsilon - n\int_R^b \left(F(\upsilon)\right)^{n-1} d\upsilon. \end{aligned}$$

Total surplus to the seller, π, consists of the price received from the sale added to the value received from the good if it does not sell, x_0. The probability that the good does not sell is the probability that all buyers have a valuation less than *R*, $(F(R))^n$, so that expected surplus is

$$E[\pi] = b - R\left(F(R)\right)^n + (n-1)\int_R^b \left(F(\upsilon)\right)^n d\upsilon - n\int_R^b \left(F(\upsilon)\right)^{n-1} d\upsilon + x_0(F(R))^n.$$

The seller sets *R* to maximize $E[\pi]$. We have

$$\begin{aligned} \frac{dE[p]}{dR} &= -\left(F(R)\right)^n - nR\left(F(R)\right)^{n-1} f(R) - (n-1)\left(F(R)\right)^n + n\left(F(R)\right)^{n-1} + x_0 n(F(R))^{n-1} f(R) \\ &= n\left(F(R)\right)^{n-1} H(R). \end{aligned}$$

where

$$H(R) = (x_0 - R) f(R) + (1 - F(R)).$$

The first-order condition for profit maximisation is then,

$$H(R) = 0,$$

giving,

$$R^* - x_0 = \frac{(1 - F(R^*))}{f(R^*)}, \tag{A5}$$

and the sufficient condition for this condition to describe a unique maximum is that $E[p]$ be strictly quasi-concave, for which it is sufficient that $H(R)$ be decreasing in R at the optimum. That is, we need

$$\frac{dH(R)}{dR} = -2f(R) + (x_0 - R)f'(R) < 0,$$

at R=R*. Substituting in the solution for R* from the first-order condition, (A5), we get

$$\frac{dH(R^*)}{dR} = -2f(R) + \frac{(1-F(R))f'(R)}{f(R)} < 0,$$

for which it is sufficient that (1-F(R)) be log concave.

Sufficient Conditions for a Fixed-Price Auction:

Consider a general monopoly profit-maximisation problem of the form

$$\underset{p}{Max}\, \pi = (p-c)q(p) + \phi, \tag{A6}$$

where $q'(p) < 0$ and there is no sign restriction on the lump-sum effect, ϕ.

The first-order condition for this problem is

$$\frac{d\pi}{dp} = q(p) + (p-c)q'(p) = 0$$

$$\Rightarrow \quad p^* - c = -\frac{q(p^*)}{q'(p^*)}. \tag{A7}$$

For solutions to (A7) to uniquely characterise a global maximum, we need $d^2\pi / dp^2 < 0$ at p=p*. We have

$$\frac{d^2\pi}{dp^2} = 2q'(p) + (p-c)q''(p).$$

Substituting in the optimal value of p-c from (A6), we get

$$\frac{d^2\pi}{dp^2} = 2q'(p^*) - \frac{q(p^*)q''(p^*)}{q'(p)}.$$

If $q'(p) < 0,$ then the required condition for a maximum is that

$$q(p^*)q''(p^*) - 2(q'(p^*))^2 < 0.$$

Log concavity of $q(p)$ is equivalent to the condition that

$$q(p)q''(p) - (q'(p))^2 < 0, \forall p,$$

and so log concavity is a sufficient condition for quasi-concavity of the profit function.

Now consider the fixed-price auction. In this case, the expected surplus for the seller is

$$\begin{aligned} \pi(p_f) &= p_f \left(1 - \left(F(p_f)\right)^n\right) + x_0 \left(F(p_f)\right)^n \\ &= (p_f - x_0)\left(1 - \left(F(p_f)\right)^n\right) + x_0. \end{aligned}$$

This has the same functional form as (A6) where the quantity function is

$$\left(1 - \left(F(p_f)\right)^n\right).$$

As we have just shown, a sufficient condition for strict quasi-convexity of the expected surplus function is that this quantity function be log concave.