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1998

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Abstract

Abstract. This paper treats stochastic games. A nonzero-sum average payoff stochastic games with arbitrary state spaces and the stopping games are considered. Such models of games very well fit in some studies in economic theory and operations research. A correlation of strategies of the players, involving "public signals", is allowed in the nonzero-sum average payoff stochastic games. The main result is an extension of the correlated equilibrium theorem proved recently by Nowak and Raghavan for dynamic games with discounting to the average payoff stochastic games. The stopping games are special model of stochastic games. The version of Dynkin's game related to observation of Markov process with random priority assignment mechanism of states is presented in the paper. The zero-sum and nonzero-sum games are considered. The paper also provides a brief overview of the theory of nonzero-sum stochastic games and stopping games which are very far from being complete.

AMS 1980 *Subject Classification.* Primary:90D15, Secondary: 93C30.
Key Words: Average payoff stochastic games, correlated stationary equilibria, nonzero-sum games, stopping time, stopping games.

1 Stochastic Markov Games

The theory of nonzero-sum stochastic games with the average payoffs per unit time for the players started with the papers by Rogers [1] and Sobel [2]. They considered finite state spaces only and assumed that the transition probability matrices induced by any stationary strategies of the players are unichain. Till now only special classes of nonzero-sum average payoff stochastic games are shown to possess Nash equilibria (or ϵ-equilibria). Parthasarathy and Raghavan [3] considered games in which one player is able to control transition probabilities and proved the existence of stationary equilibria in such a case. Non-stationary ϵ-equilibria were shown to exist in games with state independent transitions by Thuijsman [4] and in games with absorbing states by Vrieze and Thuijsman [5].

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Parthasarathy [6] first considered nonzero-sum stochastic games with countable state spaces and proved that every discounted stochastic game always has a stationary Nash equilibrium solution. Federgruen [7] extended the works of Rogers and Sobel to average payoff nonzero-sum stochastic games with countably many states, satisfying a natural uniform geometric ergodicity condition. Federgruen's result [7] was strengthened by Borkar and Ghosh [8]. In [9] the overtaking optimality criterion in the class of stationary strategies of the players is considered for undiscounted stochastic games, satisfying a strong ergodicity condition.

In many applications of stochastic games, especially in economic theory, it is desirable to assume that the state spaces are not discrete; see for example Duffie et al. [10], Dutta [11], Karatzas et al. [12], or Majumdar and Sundaram [13]. The mentioned papers deal with dynamic programming or discounted stochastic games only. There are also some papers devoted to nonzero-sum average payoff stochastic games with uncountable state spaces. Dutta and Sundaram [14] studied a class of dynamic economic games. They proved the existence of stationary Nash equilibria in a class of games satisfying a number of specific conditions and a convergence condition imposed on discounted Nash equilibria as the discount factor tends to one. Ghosh and Bagchi [15] studied games under some separability assumptions and a recurrence condition which is stronger than uniform geometric ergodicity.

Our main objective in this section is to describe the idea of correlated equilibrium notion and report a correlated equilibrium theorem proved for discounted stochastic games by Nowak and Raghavan [16]. We will also report an extension of this result to undiscounted stochastic games obtained by Nowak [17].

To describe the model, we need the following definition. Let X be a metric space, (S, Σ) a measurable space. A set-valued map or a correspondence F from S into a family of subsets of X is said to be *lower measurable* if for any open subset G of X the set $\{s \in S : F(s) \cap G \neq \emptyset\}$ belongs to Σ. For a broad discussion of lower measurable correspondences with some applications to control and optimization theory consult Castaing and Valadier [18] or Himmelberg [19].

An *N-person nonzero-sum stochastic game* is defined by the following objects:

$$((S, \Sigma), X_k, A_k, r_k, q)$$

with the interpretation that

(i) (S, Σ) is a measurable space, where S is the *set of states* for the game, and Σ is a countably generated σ-algebra of subsets of S.

(ii) X_k is a non-empty compact metric *space of actions* for player k. We put $X = X_1 \times X_2 \times \cdots \times X_N$.

(iii) A_k's are lower measurable correspondences from S into non-empty compact subsets of X_k. For each $s \in S$, $A_k(s)$ represents the *set of actions available* to player k in state s. We put

$$A(s) = A_1(s) \times A_2(s) \times \cdots \times A_N(s), \quad s \in S.$$

(iv) $r_k : S \times X \to R$ is a bounded product measurable *payoff function* for player k. It is assumed that $r_k(s, \cdot)$ is continuous on X, for every $s \in S$.

(v) q is a product measurable transition probability from $S \times X$ to S, called the *law of motion* among states. If s is a state at some stage of the game and the players select an $x \in A(s)$, then $q(\cdot \mid s, x)$ is the probability distribution of the next state of the game. We assume that the transition probability q has a density function, say z, with respect to a fixed probability measure μ on (S, Σ), satisfying the following L_1 continuity condition:
For any sequence of joint action tuples $x^n \to x^0$,

$$\int_S \mid z(s,t,x^n) - z(s,t,x^0) \mid \mu(dt) \to 0 \quad \text{as } n \to \infty.$$

The L_1 continuity above is satisfied via Scheffe's theorem when $z(s,t,\cdot)$ is continuous on X. It implies the norm continuity of the transition probability $q(\cdot \mid s, x)$ with respect to $x \in X$.

The game is played in discrete time with past history as common knowledge for all the players. An *individual strategy* for a player is a map which associates with each given history a probability distribution on the set of available to him actions. A *stationary strategy* for player k is a map which associates with each state $s \in S$ a probability distribution on the set $A_k(s)$ of actions available to him at s, independent of the history that lead to the state s. A stationary strategy for player k can thus be identified with a measurable transition probability f from S to X_k such that $f(A_k(s) \mid s) = 1$, for every $s \in S$.

Let $H = S \times X \times S \times \cdots$ be the space of all infinite histories of the game, endowed with the product σ-algebra. For any profile of strategies $\pi = (\pi_1, \ldots, \pi_N)$ of the players and every initial state $s_1 = s \in S$, a probability measure P_s^π and a stochastic process $\{\sigma_n, \alpha_n\}$ are defined on H in a canonical way, where the random variables σ_n and α_n describe the state and the actions chosen by the players, respectively, on the n-th stage of the game (cf. Chapter 7 in Bertsekas and Shreve [20]). Thus, for each profile of strategies $\pi = (\pi_1, \ldots, \pi_N)$, any finite horizon T, and every initial state $s \in S$, the *expected T-stage payoff* to player k is

$$\Phi_k^T(\pi)(s) = E_s^\pi(\sum_{n=1}^{T} r_k(\sigma_n, \alpha_n)).$$

Here E_s^π means the expectation operator with respect to the probability measure P_s^π. If β is a fixed real number in $(0, 1)$, called the *discount factor*, then we can also consider the *β-discounted expected payoff* to player k defined as

$$\Phi_k^\beta(\pi)(s) = E_s^\pi\left(\sum_{n=1}^{\infty}\beta^{n-1}r_k(\sigma_n,\alpha_n)\right).$$

The *average payoff per unit time* for player k is defined as

$$\Phi_k(\pi)(s) = \limsup_T \frac{1}{T}\Phi_k^T(\pi)(s).$$

Let $\pi^* = (\pi_1^*,\ldots,\pi_N^*)$ be a fixed profile of strategies of the players. For any strategy π_k of player k, we write (π_{-k}^*,π_k) to denote the strategy profile obtained from π^* by replacing π_k^* with π_k.

A strategy profile $\pi^* = (\pi_1^*,\ldots,\pi_N^*)$ is called a *Nash equilibrium* for the average payoff stochastic game if no unilateral deviations from it are profitable, that is, for each $s \in S$,

$$\Phi_k(\pi^*)(s) \geq \Phi_k(\pi_{-k}^*,\pi_k)(s),$$

for every player k and any his strategy π_k. Of course, Nash equilibria are analogously defined for the β-discounted stochastic games.

It is still an *open problem* whether the β-discounted stochastic games with uncountable state space have stationary equilibrium solutions. A positive answer to this problem is known only for some special classes of games, where the transition probabilities satisfy certain additional separability assumptions (cf. Himmelberg et al., [21]), or some other specific conditions (cf. Majumdar and Sundaram [13], Dutta and Sundaram [14], Karatzas et al., [12]). Whitt [22] and Nowak [23] proved the existence of stationary ϵ-equilibrium strategies in discounted stochastic games using some (different) approximations by games with countably many states. The assumptions on the model in Nowak [23] are as in (i) - (v) above plus some extra integrability condition on the transition probability density. Whitt [22] assumed that the state spaces are separable metric and imposed some uniform continuity conditions on the payoffs and transition probabilities. Breton and L'Ecuyer [24] extended Whitt's result to games with a weaker form of discounting. Mertens and Parthasarathy [25] proved the existence of non-stationary Nash equilibria for discounted stochastic games with arbitrary state spaces. Finally, Nowak and Raghavan [16] obtained stationary equilibrium solutions in the class of correlated strategies of the players with symmetric information or "public signals" (see Theorem 1 below). A related result is reported in Duffie et al. [10]. They used some stronger assumptions about the primitive data of the game, but showed that there exists a stationary correlated equilibrium which induces an ergodic process. Nonstationary correlated equilibria in a class of dynamic games with weakly continuous transition probabilities were studied by Harris [26]. As already mentioned, Dutta and Sundaram [14] proved an existence theorem for stationary Nash equilibria in some undiscounted dynamic economic games.

1.1 Correlated equilibria

In this subsection we extend the sets of strategies available to the players in the sense that we allow them to correlate their choices in a natural way described below. The resulting solution is a kind of extensive-form correlated equilibrium (cf. Forges [27]).

Suppose that $\{\xi_n : n \geq 1\}$ is a sequence of so-called *signals*, drawn independently from $[0,1]$ according to the uniform distribution. Suppose that at the beginning of each period n of the game the players are informed not only of the outcome of the preceding period and the current state s_n, but also of ξ_n. Then the information available to them is a vector $h^n = (s_1, \xi_1, x_1, \ldots, s_{n-1}, \xi_{n-1}, x_{n-1}, s_n, \xi_n)$, where $s_i \in S$, $x_i \in A(s_i)$, $i = 1, \ldots, n-1$. We denote the set of such vectors by H^n.

An *extended strategy* for player k is a sequence $\pi_k = (\pi_k^1, \pi_k^2, \ldots)$, where every π_k^n is a (product) measurable transition probability from H^n to X_k such that $\pi_k^n(A_k(s_n) \mid h^n) = 1$ for any history $h^n \in H^n$. (Here s_n is the last state in h^n.) An *extended stationary strategy* for player k is a strategy $\pi_k = (\pi_k^1, \pi_k^2, \ldots)$ such that each π_k^n depends on the current state s_n and the last signal ξ_n only. In other words, a strategy π_k of player k is called stationary if there exists a transition probability f from $S \times [0,1]$ to X_k such that for every period n of the game and each history $h^n \in H^n$, we have $\pi_k^n(\cdot \mid h^n) = f(\cdot \mid s_n, \xi_n)$. Assuming that the players use extended strategies we actually assume that they play a stochastic game in the sense of Section 1, but with the extended state space $S \times [0,1]$. The law of motion, say $\bar{q}$, in the extended state space model is obviously the product of the original law of motion q and the uniform distribution η on $[0,1]$. More precisely, for any $s \in S$, $\xi \in [0,1]$, $a \in A(s)$, any set $C \in \Sigma$ and any Borel set $D \subseteq [0,1]$, $\bar{q}(C \times D \mid s, \xi, a) = q(C \mid s, a)\eta(D)$.

For any profile of extended strategies $\pi = (\pi_1, \ldots, \pi_N)$ of the players, the undiscounted [β-discounted] payoff to player k is a function of the initial state s_1 and the first signal ξ_1 and is denoted by $E_k(\pi)(s_1, \xi_1)$ [$E_k^\beta(\pi)(s_1, \xi_1)$].

We say that $f^* = (f_1^*, \ldots, f_N^*)$ is a *Nash equilibrium in the average payoff stochastic game in the class of extended strategies* if for each initial state $s_1 \in S$,

$$\int_0^1 \Phi_k(f^*)(s_1, \xi_1)\eta(d\xi_1) \geq \int_0^1 \Phi_k(f_{-k}^*, \pi_k)(s_1, \xi_1)\eta(d\xi_1), \tag{1}$$

for every player k and any his extended strategy π_k.

A Nash equilibrium in extended strategies is also called a *correlated equilibrium with public signals.* The reason is that after the outcome of any period of the game, the players can coordinate their next choices by exploiting the next (known to all of them, i.e.,public) signal and using some coordination mechanism telling which (pure or mixed) action is to be played by each of them. In many applications, we are particularly interested in stationary equilibria. In such a case the coordination mechanism can be represented by a family of $N+1$ measurable functions $\lambda^1, \ldots, \lambda^{N+1} : S \to [0,1]$ such that $\sum_{i=1}^{N+1} \lambda^i(s) = 1$

for every $s \in S$. (We remind that N is the number of players. The number $N+1$ appears in our definition, because Caratheodory's theorem is applied in the proofs of the main results in [16] and [17].) A stationary Nash equilibrium in the class of extended strategies can be constructed then by using a family of $N+1$ stationary strategies $f_k^1, \dots, f_k^{N+1}$, given for each player k, and the following coordination rule. If the game is at a state s on the n-th stage and a random number ξ_n is selected, then each player k is suggested to use $f_k^m(\cdot \mid s)$, where m is the least index for which $\sum_{i=1}^m \lambda^i(s) \geq \xi_n$. The λ^i's and f_k^i's fixed above induce an extended stationary strategy f_k^* for each player k as follows

$$f_k^*(\cdot \mid s, \xi) = f_k^1(\cdot \mid s) \qquad \text{if} \quad \xi \leq \lambda^1(s),\ s \in S,$$

and (2)

$$f_k^*(\cdot \mid s, \xi) = f_k^m(\cdot \mid s) \qquad \text{if} \quad \textstyle\sum_{i=1}^{m-1} \lambda^i(s) < \xi \leq \sum_{i=1}^m \lambda^i(s),$$

for $s \in S$, $2 \leq m \leq N+1$. Because the signals are independent and uniformly distributed in $[0,1]$, it follows that at any period of the game and for any current state s, the number $\lambda^i(s)$ can be interpreted as the probability that player k is suggested to use $f_k^i(\cdot \mid s)$ as his mixed action. Now it is quite obvious that a strategy profile $(f_1^*, \dots, f_N^*)$ obtained by the above construction is a stationary Nash equilibrium in the class of extended strategies of the players in a game iff no player k can unilaterally improve upon his expected payoff by changing any of his strategies f_k^i, $i = 1, \dots, N+1$.

The following result was proved by Nowak and Raghavan [16] by a fixed point argument.

Theorem 1 *Every nonzero-sum discounted stochastic game satisfying (i) - (v) has a stationary correlated equilibrium with public signals.*

To report an extension of this result to undiscounted stochastic games obtained in Nowak [17], we need some additional assumptions on the transition probability q. For any stationary strategy profile f and $n \geq 1$, let $q^n(\cdot \mid s, f)$ denote the n-step transition probability determined by q and f. The following condition is used in the theory of Markov decision processes (cf. Tweedie [28], Hernández-Lerma et al. [29, 30] and their references):

C1 (*Uniform Geometric Ergodicity*): There exist scalars $\alpha \in (0,1)$ and $\gamma > 0$ for which the following holds: For any profile f of stationary strategies of the players, there exists a probability measure p_f on S such that

$$\|q^n(\cdot \mid s, f) - p_f(\cdot)\|_\nu \leq \gamma \alpha^n \qquad \text{for each } n \geq 1.$$

Here $\|\cdot\|_\nu$ denotes the total variation norm in the space of finite signed measures on S.

It is well-known that **C1** follows from the following assumption (cf. Theorem 6.15 and Remark 6.1 in Nummelin [31] or page 185 in Neveu [32]):

M (*Minorization Property*): There exists a positive integer p, a constant $\vartheta > 0$, and a probability measure δ on S, such that

$$q^p(D \mid s, f) \geq \vartheta\delta(D),$$

for every stationary strategy profile $f, s \in S$, and for each measurable subset D of S.

Condition **M** was used in stochastic dynamic programming (one person stochastic game) by Kurano [33] who proved only the existence of "p-periodic" optimal strategies in his model. It is satisfied and easy to verify in some inventory models (cf. Yamada [34]) and some control of water resources problems (cf. Yakovitz [35]).

Condition **C1** has often been used (even in some stronger versions) in control theory of Markov chains (cf. Georgin [36], Hernández-Lerma et al. [29, 30], and the references therein). We mention here some conditions which are known to be equivalent to **C1**. By F we denote the set of all stationary strategy N-tuples of the players.

C2 (*Uniform Ergodicity*): For each $f \in F$, there exists a probability measure p_f on S such that, as $n \to \infty$,

$$\|q^n(\cdot \mid s, f) - p_f(\cdot)\|_\nu \to 0, \quad \text{uniformly in } s \in S \text{ and } f \in F.$$

C3: There exist a positive integer r and a positive number $\delta < 1$ such that

$$\|q^r(\cdot \mid s, f) - q^r(\cdot \mid t, f)\|_\nu \leq 2\delta, \quad \text{for all } s, t \in S \text{ and } f \in F.$$

Obviously **C1** implies **C2** and **C3** follows immediadely from **C2** and the triangle inequality for the norm $\|\cdot\|_\nu$. Finally, **C3** implies **C1** by Ueno's lemma [37]. For details consult pages 275-276 in [36].

Another equivalent version of **C1**, called the *simultaneous Doeblin condition*, was used by Hordijk [38] in control theory and Federgruen [7] in stochastic games with countably many states. It can also be formulated for general state space stochastic games following pages 192 and 221 in Doob [39].

C4: There is a probability measure ψ on S, a positive integer r, and a positive ϵ, such that

$$q^r(B \mid s, f) \leq 1 - \epsilon \quad \text{for each } s \in S \text{ and } f \in F \text{ if } \psi(B) \leq \epsilon.$$

Moreover, for each $f \in F$, the Markov chain induced by q and f has a single ergodic set and this set contains no cyclically moving subsets.

It turns out that **C1** is equivalent to **C4**; see Chapter V in Doob [39] for details. For a further discussion of several recurrence and ergodicity conditions which have beed used in the theory of Markov decision processes in a general state space consult Hernández-Lerma et al. [30]. Now the main result of Nowak [17] can be formulated.

Theorem 2 *Every nonzero-sum undiscounted stochastic game satisfying (i) – (v) and* **C1** *has a stationary correlated equilibrium with public signals.*

We now mention some special classes of nonzero-sum undiscounted stochastic games, for which there exist Nash equilibria without public signals. First, we consider games satisfying the following separability conditions:

SC1: For each player k and any $s \in S$, $x = (x_1, \dots, x_N) \in A(s)$,

$$r_k(s, x) = \sum_{j=1}^{N} r_{kj}(s, x_j),$$

where each r_{kj} is bounded and $r_{kj}(s, \cdot)$ is continuous on X_j.

SC2: For any $s \in S$, $x = (x_1, \dots, x_N) \in A(s)$,

$$q(\cdot \mid s, x) = \sum_{j=1}^{N} q_j(\cdot \mid s, x_j)/N,$$

where $q(\cdot \mid s, x_j)$ is a transition probability from $S \times X_j$ to S, norm continuous with respect to $x_j \in X_j$.

Himmelberg et al. [21] and Parthasarathy [40] already showed that nonzero-sum β-discounted stochastic games satisfying **SC1** and **SC2** possess stationary Nash equilibria. Their theorem was extended to undiscounted stochastic games in Nowak [17].

Theorem 3 *Every nonzero-sum undiscounted stochastic game satisfying (i) – (v),* **C1** *and separability conditions* **SC1** *and* **SC2** *has a stationary Nash equilibrium without public signals.*

By Theorem 2, the game has a stationary correlated equilibrium, say f^λ. For each player k and any $s \in S$, we define $f_k^*(\cdot \mid s)$ to be the marginal of $f^\lambda(\cdot \mid s)$ on X_k and put $f^* = (f_1^*, \dots, f_N^*)$. It turns out that $(f_1^*, \dots, f_N^*)$ is a Nash equilibrium point for the stochastic game, satisfying **SC1** and **SC2**.

A version of Theorem 3 with a recurrence assumption which is much stronger than the uniform geometric ergodicity was independently proved (by a different method) in Ghosh and Bagchi [15].

Parthasarathy and Sinha [41] showed that β-discounted stochastic games with state independent transitions and finite action spaces have stationary Nash equilibria. An extension of their result to the average payoff stochastic games, obtained in Nowak [17] sounds as follows.

Theorem 4 *Assume that the action spaces X_k are finite sets and $A_k(s) = X_k$ for each $s \in S$. Assume also that the transition probability $q(\cdot \mid s, x)$ depends on x only and is non-atomic for each $x \in X$. If (i), (iv), (v), and* **C1** *are also satisfied, then the nonzero-sum average payoff stochastic game has a stationary Nash equilibrium without public signals.*

We do not know if condition **C1** can be dropped from Theorem 4. When we deal with zero-sum average payoff stochastic games with state independent transitions, then no ergodicity properties of the transition probability q are relevant (cf. Thuijsman [4] for the finite state space case or Theorem 2 in Nowak [42] for general state space games).

The basic idea of the proof of Theorem 2 is rather simple. Let L be any positive number such that $|r_k| \leq L$ for every player k. Then, for every discount factor β, and any stationary correlated equilibrium f_β^λ obtained in Theorem 1, $(1-\beta)\Phi_k(f_\beta^\lambda)(\cdot)$ is in a compact ball $B(L)$ with radius L in $L^\infty(S, \Sigma, \mu)$ space, endowed with the weak-star topology $\sigma(L^\infty, L_1)$. Therefore, it is possible to find a sequence $\{\beta_n\}$ of discount factors which converges to one and $(1-\beta_n)\Phi_k(f_{\beta_n}^\lambda)$ converges to some function $J_k \in B(L)$. Using **C1**, it is shown that J_k are constant equilibrium functions of the players, and $f_{\beta_n}^\lambda$ converges (in some sense) to a stationary correlated equilibrium for the undiscounted game.

As far as two-person zero-sum games are concerned, it is possible to drop the assumption that the transition probability is dominated by some probability measure μ. To prove the existence of stationary optimal strategies of the players, one can use the following assumption (see Nowak [42]).

B: Assume (i)-(iv) and that $q(D|s,\cdot)$ is continuous on $X = X_1 \times X_2$ for each $D \in \Sigma$. Let $v_\beta(\cdot)$ be the value function of the β-discounted game, $\beta \in (0,1)$. We assume that there exists a positive constant L such that

$$|v_\beta(s) - v_\beta(t)| \leq L \text{ for all } s, t \in S \text{ and } \beta \in (0,1).$$

It is easy to see that **C1** implies **B**. Moreover, **B** holds if the transition probability q is independent of the state variable. The main tool in the proof given in [42] is Fatou's lemma for varying probabilities (see Dutta [11] or Schäl [43] for a related approach in dynamic programming). That is the main difference between the proofs contained in [17] and [42]. The existence of value for undiscounted stochastic games in the class of nonstationary strategies is discussed in a paper of Sudderth which is included in this volume.

2 Stopping games

The theory of stopping games started with the paper by Dynkin [44]. He conceived the zero sum game based on optimal stopping problem for discrete time stochastic processes. Let $\{X_n\}_{n=0}^\infty$ be a stochastic sequence defined on some fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Define $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. If each player chooses a strategy, λ, μ respectively both Markov times, the payoff is given by $R(\lambda, \mu) = X_{\lambda \wedge \mu}$. The first player is to maximize the expected value of $R(\lambda, \mu)$ and the other is to minimize. Dynkin [44] assumes a restriction on the moves of the game. Namely, the strategies of the players are such that Player 1 can stop on odd moments n and Player 2 can choose even moments. Under this assumption Dynkin proved the existence of the game value and optimal strategies. Kifer [45] obtained another existence condition. Neveu [46] modified the Dynkin's game changing the payoff function following. There are two preassigned stochastic sequences $\{X_n\}_{n=0}^{\infty}$, $\{Y_n\}_{n=0}^{\infty}$ measurable with respect to some increasing sequence of σ-fields $\mathcal{F}_n$. The players' strategies are stopping times with respect to $\{\mathcal{F}_n\}_{n=0}^{\infty}$. The payoff equals

$$R(\lambda,\mu) = \begin{cases} X_\lambda & \text{on } \{\lambda \le \mu\}, \\ X_\mu & \text{on } \{\lambda > \mu\}, \end{cases} \tag{3}$$

with the condition

$$X_n \le Y_n \text{ for each } n. \tag{4}$$

Under some regularity condition Neveu proved the existence of the game value and ϵ-optimal strategies.

The restriction (4) has been suppressed in some cases by Yasuda [47]. He considers the zero-sum stopping game with payoff equals

$$R(\lambda,\mu) = X_\lambda \mathbb{I}_{\{\lambda \le \mu\}} + W_\lambda \mathbb{I}_{\{\lambda = \mu\}} + Y_n \mathbb{I}_{\{\lambda > \mu\}},$$

where $\mathbb{I}$ is an indicator function. To solve the game the set of strategies has been extended to a class of randomize strategies.

A version of Dynkin's game for Markov chains was considered by Fried [48]. More general version of the stopping game for the discrete time Markov processes was solved by Elbakidze [49]. Let $(X_n, \mathcal{F}_n, \mathbf{P}_x)_{n=0}^{\infty}$ be a homogeneous Markov chain with state space $(\mathbb{E}, \mathcal{B})$, while g, G, e and C are certain $\mathcal{B}$-measurable real valued functions. There are two players. The process can be stopped at any instant $n \ge 0$. If the process is stopped by the first, second or simultaneously by the two players, then the payoffs of the player are $g(X_n)$, $G(X_n)$ and $e(X_n)$, respectively. For an unlimited duration of the game the payoff of the first player equals $\limsup_{n\to\infty} C(X_n)$. The strategies of the first and second player are given by Markov moments relative to $\{\mathcal{F}_n\}_{n=0}^{\infty}$. Let $\mathcal{L}$ denote a class of $\mathcal{B}$-measurable functions f such that $E_x\{\sup_n |f(X_n)|\} < \infty$. It is assumed that

$$g(x) \le e(x) \le G(x),\ g(x) \le C(x) \le G(x),\ x \in \mathbb{E} \text{ and } g, G \in \mathcal{L}$$

Under these assumptions the value of the game and ϵ-optimal strategies are constructed.

Two-person nonzero-sum stopping games is investigated, among others, by Ohtsubo [50]. Let $\{X_n^i\}_{n=0}^{\infty}$, $\{Y_n^i\}_{n=0}^{\infty}$ and $\{W_n^i\}_{n=0}^{\infty}$, $i = 1, 2$, be six sequences of real-valued random variables defined on fixed probability space and adapted to $\{\mathcal{F}_n\}_{n=0}^{\infty}$. It is assumed that

(i) $\min(X_n^i, Y_n^i) \le W_n^i \le \max(X_n^i, Y_n^i)$ for each $i = 1, 2$.

(ii) $E[\sup_n |X_n^i|] < \infty$ and $E[\sup_n |Y_n^i|] < \infty$ for each $i = 1, 2$.

The strategies of the players are stopping times with respect to $\{\mathcal{F}_n\}_{n=0}^{\infty}$. If the first and the second players choose stopping times τ_1 and τ_2, respectively, as their controls, then the i-th player gets the reward

$$\begin{aligned} g_i(\tau_1,\tau_2) &= X^i_{\tau_i}\mathbb{I}_{(\tau_i<\tau_j)} + Y^i_{\tau_j}\mathbb{I}_{(\tau_j<\tau_i)} \\ &\quad W^i_{\tau_i}\mathbb{I}_{(\tau_i=\tau_j<\infty)} + \limsup_n W^i_n\mathbb{I}_{(\tau_i=\tau_j<\infty)}, i,j=1,2, j\neq i \end{aligned}$$

Under the above assumption the Nash equilibrium for the game is constructed. Ohtsubo [50] gave the solution for the version of the game for the Markov processes. Recently, Ferenstein [51] solved the version of the nonzero-sum Dynkin's game with different, special, payoff structure.

Continuous time version of such a game problem was studied by Bensoussan & Friedman [52], [53], Krylov [54], Bismut [55], Stettner [56], Lepeltier & Maingueneau [57] and many others.

We focus our attention to a version of stopping game called the random priority stopping game. The zero-sum version of the problem is considered in Section 2.1 and the nonzero-sum case is presented in Section 2.2.

2.1 Zero-sum random priority stopping game

Let $(X_n,\mathcal{F}_n,\mathbf{P}_x)_{n=0}^N$ be a homogeneous Markov process defined on probability space $(\Omega,\mathcal{F},\mathbf{P})$ with fixed state space $(\mathbb{E},\mathcal{B})$. The decision makers, henceforth called Player 1 and Player 2, observe the process sequentially. They want to accept the most profitable state of the process from their point of view.

We adopt the zero-sum game model for the problem. In view of this approach, the preferences of each player are described by gain function $f:\mathbb{E}\times\mathbb{E}\to\Re$. The function depends on the state chosen by both players. It would be natural to consider the stopping times with respect to $(\mathcal{F}_n)_{n=0}^N$ as the strategies of the player if the players could obtain the state which they want. Since there is only one random sequence $(X_n)_{n=0}^N$ on a trial, therefore at each moment n only one player can obtain realization x_n of X_n. The problem of assigning an object to the players when both want to accept the same one at the same moment is solved by adopting the random mechanism i.e. a lottery chooses the player who benefits. The player chosen by the lottery obtains realization x_n and the player thus deprived of the acceptance of x_n at $n<N$ may select any later realization. The realization can only be accepted when it appears. No recall is allowed. We can think about the decision process as an investigation of objects with characteristics described by the Markov process. Both players together can accept at most two objects.

The above described decision model is a generalization of the problems considered by Szajowski [58] and Radzik & Szajowski [59]. The related questions, when Player 1 has permanent priority, have been considered by many authors in the zero-sum game or the non-zero sum game setting. One can mention, for example, the papers of Ano [60], Enns & Ferenstein [61], Ferenstein [62] and Sakaguchi [63]. Many papers on the subject were inspired by the secretary problem (see the papers by Enns and Ferenstein [64], Fushimi [65], Majumdar [66], Sakaguchi [67, 63], Ravindran and Szajowski [68] and Szajowski [69] where non-zero sum versions of the games have been investigated). Sakaguchi [63] considered the nonzero-sum two-person game related to the full information best choice problem with random priority. A review of these problems one can find in Ravindran and Szajowski [68]. For the original secretary problem and its extension the reader is referred to Gilbert & Mosteller [70], Freeman [71], Rose [72] and Ferguson [73]. We recall the best choice problem in Section 2.1.2.

A formal model of the random priority is derived. The lottery is taken into account in the sets of the strategies of the players. The very interesting question concerns the influence of the level of priority on the value of the problem or the probability of obtaining the required state of the process (or, in other words, the required object). The tip of the problem is shown by the example related to the secretary problem. The simplest problem with asymmetric aims of the players is considered. The first player's aim is to choose the best applicant (BA) and the second player wants to accept the best or the second best (BOS) but a better one than the opponent. The numerical solution provides that the game is fair when Player 1 has priority $p \cong 0.7579$ (in the limiting case when $N \to \infty$). More examples and further considerations can be found in [74].

2.1.1 Random priority and stopping the Markov process

Let a homogeneous Markov chain $(X_n, \mathcal{F}_n, \mathbf{P}_x)_{n=0}^N$ be defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a state space $(\mathbb{E}, \mathcal{B})$ and let $f : \mathbb{E} \times \mathbb{E} \to \Re$ be a $\mathcal{B} \times \mathcal{B}$ real valued measurable function. Horizon N is finite. The players observe the Markov chain and they try to accept the "best realization" according to function f and a possible selection of another player. Each realization x_n of X_n can be accepted by only one player and each player can accept at most one realization. If the players have not accepted previous realizations, they evaluate the state of the Markov chain at instant n and they have two options, either to accept the observed state of the process at moment n or to reject it. If both players want to accept the same realization, the following random priority mechanism is applied. Let $\xi_1, \xi_2, \ldots, \xi_N$ be a sequence of i.i.d. r.v. with the uniform distribution on $[0, 1]$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ be a given vector of real numbers, $\alpha_i \in [0, 1]$. When both players want to accept realization x_n of X_n, then Player 1 obtains x_n if $\xi_n \le \alpha_n$, otherwise Player 2 benefits. If Player 1 rejects the applicant, then Player 2 turns to exercise one of his options which also consists in accepting the observed state of the Markov chain or rejecting it. If one of the players accepts realization x_n of X_n, then the other one is informed about it and he continues to play alone. If, in the above decision process, Player 1 and Player 2 have accepted states x and y, respectively, then Player 2 pays $f(x, y)$ to Player 1. When only Player 1 (Player 2) accepts state x (y) then Player 1 obtains $f_1(x) = \sup_{y \in \mathbb{E}} f(x, y)$ $(f_2(y) = \inf_{x \in \mathbb{E}} f(x, y))$ by assumption. If both players finish the decision process without any accepted state, then they gain 0. The detail construction of the model is given in [74]. A brief is presented below.

Let $\mathcal{S}^N$ be the aggregation of Markov times with respect to $(\mathcal{F}_n)_{n=0}^N$. We admit that $\mathbf{P}_x(\tau \leq N) < 1$ for some $\tau \in \mathcal{S}^N$ (i.e. there is a positive probability that the Markov chain will not be stopped). The elements of $\mathcal{S}^N$ are possible strategies for the players with the restriction that Player 2 cannot stop at the same moment than Player 1. If the players declare willingness to accept the same object, the random device decide which player is endowed. Let us formalize these consideration. Denote $\mathcal{S}_k^N = \{\tau \in \mathcal{S}^N : \tau \geq k\}$. Let Λ_k^N and M_k^N be copies of $\mathcal{S}_k^N$ ($\mathcal{S}^N = \mathcal{S}_0^N$). One can define set of strategies $\tilde{\Lambda}^N = \{\lambda, \{\sigma_n^1\}) : \lambda \in \Lambda^N, \sigma_n^1 \in \Lambda_{n+1}^N$ for every $n\}$ and $\tilde{M}^N = \{(\mu, \{\sigma_n^2\}) : \mu \in M^N, \sigma_n^2 \in M_{n+1}^N$ for every $n\}$ for Player 1 and 2, respectively. Denote $\tilde{\mathcal{F}}_n = \sigma(\mathcal{F}_n, \xi_1, \xi_2, \ldots, \xi_n)$ and let $\tilde{\mathcal{S}}^N$ be the set of stopping times with respect to $(\tilde{\mathcal{F}}_n)_{n=0}^N$. Define $\tau_1 = \lambda \mathbb{I}_{\{\lambda<\mu\}} + (\lambda \mathbb{I}_{\{\xi_\lambda \leq \alpha_\lambda\}} + \sigma_\mu^1 \mathbb{I}_{\{\xi_\lambda > \alpha_\lambda\}}) \mathbb{I}_{\{\lambda=\mu\}} + \sigma_\mu^1 \mathbb{I}_{\{\lambda>\mu\}}$ and $\tau_2 = \mu \mathbb{I}_{\{\lambda>\mu\}} + (\mu \mathbb{I}_{\{\xi_\mu>\alpha_\mu\}} + \sigma_\lambda^2 \mathbb{I}_{\{\xi_\mu \leq \alpha_\mu\}}) \mathbb{I}_{\{\lambda=\mu\}} + \sigma_\lambda^2 \mathbb{I}_{\{\lambda \leq \mu\}}$.

Lemma 1 *Random variables τ_1 and τ_2 are Markov times with respect to $(\tilde{\mathcal{F}}_n)_{n=0}^N$ and $\tau_1 \neq \tau_2$.*

Let $E_x f_1^+(X_n) < \infty$ and $E_x f_2^-(X_m) < \infty$ for $n, m = 0, 1, \ldots, N$ and $x \in \mathbb{E}$. Let $s \in \tilde{\Lambda}^N$ and $t \in \tilde{M}^N$. Define $\tilde{R}(x, s, t) = E_x f(X_{\tau_1}, X_{\tau_2})$ as the expected gain of Player 1. In this way the normal form of the game $(\tilde{\Lambda}^N, \tilde{M}^N, \tilde{R}(x, s, t))$ is defined. This game is denoted by $\mathcal{G}$. The game $\mathcal{G}$ is a model of the considered bilateral stopping problem for the Markov process.

Definition 1 *Pair (s^*, t^*), $s^* \in \tilde{\Lambda}^N$, $t^* \in \tilde{M}^N$ is an equilibrium point in the game $\mathcal{G}$ if for every $x \in \mathbb{E}$, $s \in \tilde{\Lambda}^N$ and $t \in \tilde{M}^N$ we have*

$$\tilde{R}(x, s, t^*) \leq \tilde{R}(x, s^*, t^*) \leq \tilde{R}(x, s^*, t).$$

The aim is to construct the equilibrium pair (s^*, t^*). To this end, the following auxiliary game $\mathcal{G}_a$.

Define $s_0(x, y) = S_0(x, y) = f(x, y)$ and

$$\begin{aligned} s_n(x, y) &= \inf_{\tau \in \mathcal{S}^n} E_y f(x, X_\tau), \\ S_n(x, y) &= \sup_{\tau \in \mathcal{S}^n} E_x f(X_\tau, y) \end{aligned}$$

for all $x, y \in \mathbb{E}$, $n = 1, 2, \ldots, N$. By the theory of optimal stopping for the Markov processes [75], the function $s_n(x, y)$ ($S_n(x, y)$) can be constructed by the recursive procedure as $s_n(x, y) = Q_{\min}^n f(x, y)$ ($S_n(x, y) = Q_{\max}^n f(x, y)$), where $Q_{\min} f(x, y) = f(x, y) \wedge T_2 f(x, y)$ ($Q_{\max} f(x, y) = f(x, y) \vee T_1 f(x, y)$) and $T_2 f(x, y) = E_y f(x, X_1)$ ($T_1 f(x, y) = E_x f(x, y)$). ($\wedge$, $\vee$ denote minimum and maximum, respectively). Operations $\wedge$ and T_2 ($\vee$ and T_1) preserve measurability. This can be proved in a standard way. Hence $s_n(x, y)$ ($S_n(x, y)$) are $\mathcal{B} \otimes \mathcal{B}$

measurable (cf. [76]). If Player 1 is the first to accept x at moment n, then his expected gain is

$$h(n,x) = E_x s_{N-n-1}(x, X_1), \tag{5}$$

for $n = 0, 1, ..., N-1$ and $h(N,x) = f_1(x)$. When Player 2 is the first then the expected gain of Player 1 is

$$H(n,x) = E_x S_{N-n-1}(X_1, x), \tag{6}$$

for $n = 0, 1, ..., N-1$ and $H(N,x) = f_2(x)$. Functions $h(n,x)$ and $H(n,x)$ are well defined. They are $\mathcal{B}$-measurable of the second variable, $h(n, X_1)$ and $H(n, X_1)$ are integrable with respect to $\mathbf{P}_x$. Let Λ^N and M^N be sets of strategies in $\mathcal{G}_a$ for Player 1 and Player 2, respectively. For $\lambda \in \Lambda^N$ and $\mu \in M^N$, define payoff function

$$r(\lambda,\mu) = \begin{cases} h(\lambda, X_\lambda)(\mathbb{I}_{\{\lambda<\mu\}} + \mathbb{I}_{\{\lambda=\mu,\xi_\lambda\le\alpha_\lambda\}}) & \\ +H(\mu, X_\mu)(\mathbb{I}_{\{\lambda>\mu\}} + \mathbb{I}_{\{\lambda=\mu,\xi_\mu>\alpha_\mu\}}) & \text{if } \lambda \le N \text{ or } \mu \le N, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $\mathbb{I}_A$ is a characteristic function of set A. As a solution of the game we search for equilibrium pair (λ^*, μ^*) such that

$$R(x, \lambda, \mu^*) \le R(x, \lambda^*, \mu^*) \le R(x, \lambda^*, \mu) \qquad \text{for all} \qquad x \in \mathbb{E}, \tag{8}$$

where $R(x,\lambda,\mu) = E_x r(\lambda,\mu)$. By (7) we can observe that $\mathcal{G}_a$ with the sets of strategies Λ^N and M^N is equivalent to the Neveu's stopping problem [46] considered by Yasuda [47] if the sets of strategies are extended to the set of stopping times not greater than $N+1$ and the payoff function is (7). The monotonicity of gains are not fulfilled here, but the solution is still in pure strategies. Because the Markov process is observed here, one can define a sequence $v_n(x),\ n = 0, 1, \ldots, N+1$ on $\mathbb{E}$ by setting $v_{N+1}(x) = 0$ and

$$v_n(x) = \mathrm{val}\begin{bmatrix} h(n,x)\alpha_n + (1-\alpha_n)H(n,x) & h(n,x) \\ H(n,x) & Tv_{n+1}(x) \end{bmatrix} \tag{9}$$

for $n = 0, 1, \ldots, N$, where $Tv.(x) = E_x v.(X_1)$ and val A denotes a value of the two person zero-sum game with payoff matrix A (see [77], [47]).

To prove the correctness of the construction let us observe that the payoff matrix in (9) has the form

$$A = \begin{array}{c c} & \begin{array}{cc} \mathrm{s} & \mathrm{f} \end{array} \\ \begin{array}{c} \mathrm{s} \\ \mathrm{f} \end{array} & \begin{bmatrix} (a-b)\alpha + b & a \\ b & c \end{bmatrix} \end{array}, \tag{10}$$

where a, b, c, α are real numbers and $\alpha \in [0,1]$. By direct checking we have

Lemma 2 *The two person zero-sum game with payoff matrix A given by (10) has an equilibrium point* (ϵ, δ) *in pure strategies, where*

$$(\epsilon, \delta) \quad = \quad \begin{cases} (s,s) & \text{if } a \geq b, \\ (s,f) & \text{if } c \leq a < b, \\ (f,s) & \text{if } a < b \leq c, \\ (f,f) & \text{if } a < c < b. \end{cases}$$

Notice that v_{N+1} is measurable. Let us assume that functions v_i, $i = N, N-1, \ldots, n+1$ are measurable. Denote

$$\begin{aligned} A_n^{\rm ss} &= \{x \in \mathbb{E} : h(n,x) \geq H(n,x)\} \\ A_n^{\rm sf} &= \{x \in \mathbb{E} : h(n,x) < H(n,x), h(n,x) \geq Tv_{n+1}(x)\} \\ A_n^{\rm fs} &= \{x \in \mathbb{E} : h(n,x) < H(n,x), H(n,x) \leq Tv_{n+1}(x)\} \end{aligned}$$

and

$$A_n^{\rm ff} \quad = \quad \mathbb{E} \setminus (A_n^{\rm ss} \cup A_n^{\rm sf} \cup A_n^{\rm fs}).$$

By sets $A_n^{\rm ss}, A_n^{\rm sf}, A_n^{\rm fs} \in \mathcal{B}$ and Lemma 2 we have

$$\begin{aligned} v_n(x) \quad = \quad & [(h(n,x) - H(n,x))\alpha_n + H(n,x)]\mathbb{I}_{A_n^{\rm ss}}(x) + h(n,x)\mathbb{I}_{A_n^{\rm sf}}(x) \\ & + H(n,x)\mathbb{I}_{A_n^{\rm fs}(x)} + Tv_{n+1}(x)\mathbb{I}_{A_n^{\rm ff}(x)}, \end{aligned}$$

hence $v_n(x)$ is $\mathcal{B}$-measurable.

Define $\lambda^* = \inf_n\{X_n \in A_n^{\rm ss} \cup A_n^{\rm sf}\}$ and $\mu^* = \inf_n\{X_n \in A_n^{\rm ss} \cup A_n^{\rm fs}\}$.

Theorem 5 *([74]) Game* $\mathcal{G}_a$ *with payoff function (7) and sets of strategies* Λ^N *and* M^N *for Player 1 and 2, respectively, has a solution. Pair* (λ^*, μ^*) *is the equilibrium point and* $v_0(x)$ *is the value of the game.*

Let us construct an equilibrium pair for game $\mathcal{G}$. Define (see [76])

$$\sigma_n^{1^*} \quad = \quad \inf\{m > n : S_{N-m}(X_m, X_n) = f(X_m, X_n)\}, \tag{11}$$

$$\sigma_n^{2^*} \quad = \quad \inf\{m > n : s_{N-m}(X_n, X_m) = f(X_n, X_m)\}. \tag{12}$$

Let (λ^*, μ^*) be an equilibrium point in $\mathcal{G}_a$.

Theorem 6 *([74]) Game* $\mathcal{G}$ *has a solution. Pair* (s^*, t^*) *such that* $s^* = (\lambda^*, \{\sigma_n^{1^*}\})$ *and* $t^* = (\mu^*, \{\sigma_n^{2^*}\})$ *is the equilibrium point. The value of the game is* $v_0(x)$*.*

Proof. Let

$$\tau_1^* \quad = \quad \lambda^* \mathbb{I}_{\{\lambda^* < \mu^*\}} + (\lambda^* \mathbb{I}_{\{\xi_{\lambda^*} \leq \alpha_{\mu^*}\}} + \sigma_{\mu^*}^{1^*} \mathbb{I}_{\{\xi_{\lambda^*} > \alpha_{\lambda^*}\}})\mathbb{I}_{\{\lambda^* = \mu^*\}} + \sigma_{\mu^*}^{1^*} \mathbb{I}_{\{\lambda^* > \mu^*\}}$$

and

$$\tau_2^* = \mu^*\mathbb{I}_{\{\lambda^*>\mu^*\}} + (\mu^*\mathbb{I}_{\{\xi_{\lambda^*}>\alpha_{\lambda^*}\}} + \sigma^{2^*}_{\lambda^*}\mathbb{I}_{\{\xi_{\lambda^*}\leq\alpha_{\lambda^*}\}})\mathbb{I}_{\{\lambda^*=\mu^*\}} + \sigma^{2^*}_{\lambda^*}\mathbb{I}_{\{\lambda^*<\mu^*\}}.$$

We obtain by the properties of conditional expectation and by (11) and (12)

$$\begin{aligned}
\tilde{R}(x,s^*,t^*) &= E_x f(X_{\tau_1^*},X_{\tau_2^*}) = E_x[\mathbb{I}_{\{\lambda^*<\mu^*\}\cup\{\lambda^*=\mu^*,\xi_{\lambda^*}\leq\alpha_{\mu^*}\}} f(X_{\lambda^*},X_{\sigma^{2*}_{\lambda^*}}) \\
&\quad + \mathbb{I}_{\{\lambda^*>\mu^*\}\cup\{\lambda^*=\mu^*,\xi_{\lambda^*}>\alpha_{\lambda^*}\}} f(X_{\sigma^{1*}_{\mu^*}},X_{\mu^*})] \\
&= E_x\mathbb{I}_{\{\lambda^*<\mu^*\}\cup\{\lambda^*=\mu^*,\xi_{\lambda^*}\leq\alpha_{\mu^*}\}} E_{X_{\lambda^*}} f(X_{\lambda^*},X_{\sigma^{2*}_{\lambda^*}}) \\
&\quad + E_x\mathbb{I}_{\{\lambda^*>\mu^*\}\cup\{\lambda^*=\mu^*,\xi_{\lambda^*}>\alpha_{\lambda^*}\}} E_{X_{\mu^*}} f(X_{\sigma^{1*}_{\mu^*}},X_{\mu^*}) \\
&= R(x,\lambda^*,\mu^*).
\end{aligned}$$

Let $t=(\mu,\{\sigma_n^2\})\in\tilde{M}^N$. We obtain

$$\begin{aligned}
\tilde{R}(x,s^*,t^*) &= R(x,\lambda^*,\mu^*) \leq R(x,\lambda^*,\mu) \\
&= E_x[\mathbb{I}_{\{\lambda^*<\mu\}\cup\{\lambda^*=\mu,\xi_{\lambda^*}\leq\alpha_\mu\}} h(\lambda^*,X_{\lambda^*}) \\
&\quad + \mathbb{I}_{\{\lambda^*>\mu\}\cup\{\lambda^*=\mu,\xi_{\lambda^*}>\alpha_{\lambda^*}\}} H(\mu,X_\mu)] \\
&= E_x[\mathbb{I}_{\{\lambda^*<\mu\}\cup\{\lambda^*=\mu,\xi_{\lambda^*}\leq\alpha_\mu\}} E_{X_{\lambda^*}} f(X_{\lambda^*},X_{\sigma^{2*}_{\lambda^*}}) \\
&\quad + \mathbb{I}_{\{\lambda^*>\mu\}\cup\{\lambda^*=\mu,\xi_{\lambda^*}>\alpha_{\lambda^*}\}} H(\mu,X_\mu)] \\
&\leq E_x[\mathbb{I}_{\{\lambda^*<\mu\}\cup\{\lambda^*=\mu,\xi_{\lambda^*}\leq\alpha_\mu\}} E_{X_{\lambda^*}} f(X_{\lambda^*},X_{\sigma^2_{\lambda^*}}) \\
&\quad + \mathbb{I}_{\{\lambda^*>\mu\}\cup\{\lambda^*=\mu,\xi_{\lambda^*}>\alpha_{\lambda^*}\}} E_{X_\mu} f(X_{\sigma^{1*}_\mu},X_\mu)] \\
&= E_x[\mathbb{I}_{\{\lambda^*<\mu\}\cup\{\lambda^*=\mu,\xi_{\lambda^*}\leq\alpha_\mu\}} f(X_{\lambda^*},X_{\sigma^2_{\lambda^*}}) \\
&\quad + \mathbb{I}_{\{\lambda^*>\mu\}\cup\{\lambda^*=\mu,\xi_{\lambda^*}>\alpha_{\lambda^*}\}} f(X_{\sigma^{1*}_\mu},X_\mu)] \\
&= E_x f(X_{s^*},X_t) = \tilde{R}(x,s^*,t)
\end{aligned}$$

Similarly one can show that for every $s\in\tilde{\Lambda}^N$ we have $\tilde{R}(x,s,t^*)\leq\tilde{R}(x,s^*,t^*)$. Hence (s^*,t^*) is the equilibrium pair for $\mathcal{G}$.

□

2.1.2 The best vs the best or the second best game

Two employers, Player 1 and Player 2, are to view a group of N applicants for a vacancies in their enterprises sequentially. Each of the applicant has some characteristic unknown to the employer. Let $\mathbb{K}=\{x_1,x_2,\ldots,x_N\}$ be the set of characteristics, assuming that the values are different. The employer observes a permutation $\eta_1,\eta_2,\ldots,\eta_N$ of the elements of $\mathbb{K}$ sequentially. We assume that all permutations are equally likely. Let Z_k denote the absolute rank of the object with the characteristics η_k , i.e.

$$Z_k = \min\{\ r : \eta_k = \bigwedge_{1\leq i_1<\ldots<i_r\leq N}\ \bigvee_{1\leq j\leq r}\eta_{i_j}\},$$

($\bigwedge$, $\bigvee$ denote minimum and maximum, respectively). The object with the smallest characteristics has the rank 1. The decisions of the employer at each time n are based on the relative ranks $Y_1, Y_2, ..., Y_N$ of the applicants and the previous decisions of the opponent, where

$$Y_k = \min\{\ r : \eta_k = \bigwedge_{1\leq i_1<\ldots<i_r\leq k} \bigvee_{1\leq j\leq r} \eta_{i_j}\}.$$

The random priority assignement model is applied when both players want to accept the same applicant. We assume that $\alpha_n = p, p \in [0,1]$ for every n. If the applicant is viewed the employer must either accept or reject her. Once accepted the applicant cannot be rejected, once rejected cannot be reconsidered. Each employer can accept at most one applicant. The aim of Player 1 is to accept BA and Player 2 is to accept BOS but a better one than that chosen by the opponent. Both players together can accept at most two objects. It makes the problem resembling to the optimal double stop of Markov process (cf. [78], [79], [76]). It is a generalization of the optimal choice problem. We adopt the following payoff function here. The player obtains $+1$ from another if he has chosen the required applicant, -1 when the opponent has done it and 0 otherwise.

Let us describe the mathematical model of the problem. With sequential observation of the applicants we connect the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The elementary events are a permutation of the elements of $\mathbb{K}$ and the probability measure $\mathbf{P}$ is the uniform probability on Ω. The observable sequence of relative ranks $Y_k,\ k = 1, 2, \ldots, N$ defines a sequence of σ-fields $\mathcal{F}_k = \sigma(Y_1, \ldots, Y_k)$, $k = 1, 2, \ldots, N$. The random variables Y_k are independent and $\mathbf{P}(Y_k = i) = 1/k$. Denote by $\mathcal{S}^N$ the set of all Markov times τ with respect to the σ-fields $\{\mathcal{F}_k\}_{k=1}^N$. The problem considered can be formulated as follows. For $s \in \tilde{\Lambda}^N$ and $t \in \tilde{M}^N$ denote $A_1 = \{\omega : X_{\tau_1} = 1\}$ and $A_2 = \{\omega : X_{\tau_2} = 1\} \cup \{\omega : X_{\tau_2} = 2, X_{\tau_1} \neq 1\}$. Define the payoff function $g(s,t) = \mathbb{I}_{A_1} - \mathbb{I}_{A_2}$ and the expected payoff $G(s,t) = Eg(s,t)$. We are looking for (s^*, t^*) such that for every $s \in \tilde{\Lambda}^N$ and $t \in \tilde{M}^N$

$$G(s, t^*) \leq G(s^*, t^*) \leq G(s^*, t).$$

It is obvious that the essential decisions of the players can be taken when applicants with relative rank 1 or 2 have appeared. We will call them candidates. For further consideration it is convenient to define the following random sequence $(W_k)_{k=1}^N$. Let $W_1 = (1, Y_1) = (1, 1),\ \rho_1 = 1$. Define

$$\rho_t = \inf\{r > \rho_{t-1} : Y_r \in \{1, 2\}\},\ t > 1,$$

($\inf \emptyset = \infty$) and $W_t = (\rho_t, Y_{\rho_t})$. If $\rho_t = \infty$ then we put $W_t = (\infty, \infty)$. Markov chain $(W_t, \mathcal{G}_t, \mathbf{P}_{(1,1)})_{t=1}^N$ with state space $\mathbb{E} = \{(s,l) : l \in \{1,2\}, s = 1, 2, ..., N\} \cup \{(\infty, \infty)\}$ and $\mathcal{G}_t = \sigma(W_1, W_2, ..., W_t)$ is homogeneous. One step transition probabilities are following.

$$\begin{aligned} p(r,s) &= \mathbf{P}\{W_{t+1}=(s,l_s) \mid W_t=(r,l_r)\} \\ &= \begin{cases} \frac{1}{2} & \text{if } r=1,\ s=2, \\ \frac{r(r-1)}{s(s-1)(s-2)} & \text{if } 2 \le r < s, \\ 0 & \text{if } r \ge s \text{ or } (r=1,\ s \ne 2), \end{cases} \end{aligned} \tag{14}$$

$p(\infty,\infty)=1$, $p(r,\infty)=1-2\sum_{s=r+1}^{N} p(r,s)$ for $l_s, l_r \in \{1,2\}$ and $1 \le r \le s \le N$. We will call this Markov chain the auxiliary Markov chain (**AMC**).

The solution of the two decision makers problem will use partially the solution of the problem of choosing BOS (see [70], [80], [68]). The problem can be treated as an optimal stopping problem for **AMC** with the following payoff function

$$f_{\mathrm{BOS}}(r,l_r) = \begin{cases} \frac{r(2N-r-1)}{N(N-1)} & \text{if} \quad l_r = 1, \\ \frac{r(r-1)}{N(N-1)} & \text{if} \quad l_r = 2. \end{cases} \tag{15}$$

Let $\mathcal{T}^N = \{\tau \in \mathcal{S}^N : \tau = r \Rightarrow Y_r \in \{1,2\}\}$. It is a set of stopping times with respect to $\mathcal{G}_t$, $t=1,2,\ldots,N$. We search $\tau^* \in \mathcal{S}^N$ such that

$$\mathbf{P}\{Z_{\tau^*} \in \{1,2\}\} = \sup_{\tau \in \mathcal{S}^N} \mathbf{P}\{Z_\tau \in \{1,2\}\} = \sup_{\sigma \in \mathcal{T}^N} E_{(1,1)} f_{\mathrm{BOS}}(W_\sigma).$$

Denote $\Gamma(r,s) = \{(t,l_t) : t > r,\ l_t = 1\} \cup \{(t,l_t) : t > s,\ l_t = 2\}$. Let $r < s$ and $c(r,s) = E_{(r,l_r)} f_{\mathrm{BOS}}(W_\sigma)$, where $\sigma = \inf\{t : W_t \in \Gamma(r,s)\}$. Denote $c(r) = E_{(r,l_r)} f_{\mathrm{BOS}}(W_{\sigma_1}) = 2\frac{r(N-r)}{N(N-1)}$, where $\sigma_1 = \inf\{t : W_t \in \Gamma(r,r)\}$. We have

$$c(r,s) = \frac{r}{N(N-1)} \sum_{i=r+1}^{s-1} \frac{2N-i-1}{i-1} + \frac{r}{s-1} c(s-1) \tag{16}$$

for $r < s$, $r,s = 1,2,\ldots,N$ ($\sum_r^s = 0$ if $s < r$). Define $r_a = \inf\{1 \le r \le N : f_{\mathrm{BOS}}(r,2) \ge c(r,r)\}$ and $r_b = \inf\{1 \le r \le r_a : f_{\mathrm{BOS}}(r,1) \ge c(r,r_a)\}$. Denote

$$\tilde{c}_{\mathrm{BOS}}(r,l_r) = \sup_{\tau \in \mathcal{S}_{r+1}^N} \mathbf{P}\{Z_\tau \in \{1,2\} \mid Y_r = l_r\}.$$

We have

$$\tilde{c}_{\mathrm{BOS}}(r,l_r) = \tilde{c}_{\mathrm{BOS}}(r) = \begin{cases} c(r) & \text{if} \quad r_a \le r \le N, \\ c(r,r_a) & \text{if} \quad r_b \le r < r_a, \\ c(r_b-1,r_a) & \text{if} \quad 1 \le r < r_b. \end{cases} \tag{17}$$

The optimal stopping time for the one decision maker problem of choosing BOS is $\sigma^* = \inf\{t : W_t \in \Gamma(r_b,r_a)\} \in \mathcal{T}^N$ or $\tau^* = \inf\{r : (r,Y_r) \in \Gamma(r_b,r_a)\} \in \mathcal{S}^N$.

We have $a = \lim_{N\to\infty} \frac{r_a}{N} = \frac{2}{3}$, $b = \lim_{N\to\infty} \frac{r_b}{N} \cong 0.3470$ and $\lim_{N\to\infty} \tilde{c}_{\mathrm{BOS}}(1) \cong 0.5736$ (cf. [81], [80], [82]).

To solve the two person competitive stopping problem described at the beginning of the section let us perform a strategy of the players when one of them accepts some observation at moment r with relative rank $Y_r = l_r$. Since the aims of the players are different we have to consider independently the situation when Player 1 has stopped as the first and when Player 2 has done it. We introduce useful denotation

$$h_{ik}(r, l_r) = \mathbf{P}(A_k | \tau_i = r, \tau_j > \tau_i, Y_r = l_r)$$

for $k, i, j = 1, 2$, $i \neq j$, $r = 1, 2, \ldots, N$, $l_r = 1, 2$.

Let Player 1 stop the process as the first at the moment r on the object with $Y_r = l_r$. As he wants to accept the object with the absolute rank 1, it is obvious that he will stop on the relatively first object. He will also accept probably, in some circumstances, the relatively second objects to disturb Player 2 in the realization of his aims. We will see that this supposition is true. Player 2 staying alone will use a strategy $\sigma_r^{2^*} = \varsigma^*(r, l_r)$ defined by

$$h(r, l_r) = E_{(r,l_r)} g((r, \sigma_\mu^1), (\mu, \varsigma^*(r, l_r))) = \inf_{\sigma \in \mathcal{S}_{r+1}^N} E_{(r,l_r)} g((r, \sigma_\mu^1), (\mu, \sigma)), \quad (18)$$

where the expectation is taken with respect to $\mathbf{P}_{(r,l_r)}$ of **AMC** . To perform strategy $\varsigma^*(r, l_r)$ let us consider the possible essential situations. Let $W_t = (r, l_r)$. Since Player 2 minimizes his expected loss (cf. (18)), he can do it by stopping on some object with relative rank 1 or 2. If $l_r = 1$ then he cannot change the payoff stopping on the objects with relative rank 2 before another one having relative rank 1 has appeared. Let $W_s = (m, 1)$ and $W_u = (n, 2)$ for $u = t + 1, t + 2, \ldots, s - 1$. Player 1 can be the winner in this case if $W_{s+1} = (\infty, \infty)$ and Player 2 does not accept the m-th object. We see that it is the first moment after the accepting decision of Player 1 when Player 2 can change the gain of Player 1. We want to know if it is optimal to stop at $(m, 1)$ for Player 2. If he stops, he has -1 with the probability $f_{\mathrm{BOS}}(m, 1)$ (see (15)). When he passes over and he will behave optimally in future, he has -1 with probability $\tilde{c}_{\mathrm{BOS}}(m)$. Since he minimizes his loss therefore his optimal strategy in $(m, 1)$ is the same as in the mentioned one player problem. If it happens that $n < m < r_b$, then according to the optimal strategy in the one player choosing BOS problem, m-th object will not be accepted and Player 2 will behave according to σ^*. It means Player 1 will have $+1$ if n-th object is the best or it happens that his candidate is the absolutely second and the best one will not be chosen by Player 2 (because she has appeared before r_b). Hence, by (14), (17) and (15) we have

$$\begin{aligned} h_{12}(r, l_r) &= \mathbf{P}\{A_2 \mid \tau_1 = r, \tau_2 > \tau_1, Y_r = l_r\} \\ &= \begin{cases} \sum_{s=r+1}^{N} \frac{r}{s(s-1)} \max\left\{\frac{s(2N-s-1)}{N(N-1)}, \tilde{c}_{\mathrm{BOS}}(s)\right\} & \text{if } l_r = 1, \\ \tilde{c}_{\mathrm{BOS}}(r) & \text{if } l_r = 2, \end{cases} \end{aligned}$$

where $\sigma_n^{i^*} = \varsigma^*(n, Y_n)$, $s = (\lambda, \{\sigma_n^{1^*}\})$ and $t = (\mu, \{\sigma_n^{2^*}\})$. The optimal strategy ς^*, after the first acceptance has been done at moment r on $Y_r = l_r$ has the form

$$\varsigma^*(r, l_r) = \begin{cases} \begin{cases} \vartheta_r & \text{if } \vartheta_r \geq r_b, \\ \sigma^*_{\vartheta_r} & \text{if } \vartheta_r < r_b \end{cases} & \text{for } l_r = 1, \\ \sigma^*_r & \text{for } l_r = 2, \end{cases} \tag{19}$$

where $\vartheta_r = \inf\{s > r : Y_s = 1\}$ and $\sigma^*_r = \inf\{s > r : (s, Y_s) \in \Gamma(r_b, r_a)\}$. Consequently,

$$h(r, l_r) = h_{11}(r, l_r) - h_{12}(r, l_r),$$

where we have $h_{11}(r, l_r) = \frac{r}{N}$ for $l_r = 1$ and 0 otherwise.

Let us assume that Player 2 has stopped the process as the first on some object at moment r with relative rank $Y_r = l_r$. Player 1 will use a strategy $\sigma_r^{1^*} = \delta^*(r, l_r)$. The strategy $\delta^*(r, l_r)$ is such that

$$H(r, l_r) = E_{(r,l_r)} g_1((\lambda, \delta^*(r, l_r)), (r, \sigma^2_\lambda)) = \sup_{\sigma \in \mathcal{S}^N_{r+1}} E_{(r,l_r)} g_1((\lambda, \sigma), (r, \sigma^2_\lambda))$$

.

Let $W_t = (r, l_r)$. Since Player 1 maximizes his expected gain and he would like to choose the best object he can do it by stopping on some object with relative rank 1. Denote $\tilde{c}_{\mathrm{BA}}(r) = \sup_{\tau \in \mathcal{S}^N_{r+1}} \mathbf{P}\{Z_\tau = 1 | Y_r = l_r\}$, $r_c = \inf\{1 \leq r \leq N : \sum_{i=r+1}^N \frac{1}{i-1} \leq 1\}$ and $\tau^*_r = \inf\{s > r : Y_s = 1, s \geq r_c\}$. The optimal strategy δ^* of Player 1, after the first acceptance done at the moment r on $Y_r = l_r$ by Player 2, has the form

$$\delta^*(r, l_r) = \begin{cases} \begin{cases} \vartheta_r & \text{if } \vartheta_r \geq r_c, \\ \tau^*_{\vartheta_r} & \text{if } \vartheta_r < r_c \end{cases} & \text{for } l_r = 1, \\ \tau^*_r & \text{for } l_r = 2, \end{cases} \tag{20}$$

where ϑ_r is the first moment after r when $Y_r = 1$. We have

$$H(r, l_r) = h_{21}(r, l_r) - h_{22}(r, l_r),$$

where

$$h_{21}(r, l_r) = \sum_{s=r+1}^N p(r, s)[\max\{\frac{s}{N}, \tilde{c}_{\mathrm{BA}}(r)\} + \tilde{c}_{\mathrm{BA}}(r)] = \tilde{c}_{\mathrm{BA}}(r)$$

and

$$\begin{aligned} h_{22}(r, l_r) &= \mathbf{P}\{A_2 \mid \tau_2 = r,\ \tau_1 > \tau_2,\ Y_r = l_r\} \\ &= \begin{cases} \frac{r}{N} + \begin{cases} 0 & \text{if } r \geq r_c \\ \sum_{s=r+1}^{r_c-1} \frac{r}{s(s-1)} \frac{s(s-1)}{N(N-1)} & \text{if } r < r_c \end{cases} & \text{for } l_r = 1, \\ \frac{r(r-1)}{N(N-1)} & \text{for } l_r = 2 \end{cases} \end{aligned}$$

Denote $h_p(r,l_r) = ph(r,l_r) + (1-p)H(r,l_r)$. Define $r_d = \min\{1 \leq r \leq N : h(r,2) \geq H(r,2)\}$ and $r_\ell = \min\{1 \leq r \leq r_d : h(r,1) \geq H(r,1)\}$. During the recursive construction of $\tilde{v}(r,l_r;p)$ and the strategy according to Theorem 5 and 6 (see also (9)) for a large N we get that there exist $r_{\nu(p)} = \min\{r < r_d : H(r,2) \leq \tilde{v}(r;p)\}$ and $\tilde{p}_1 = \min\{0 \leq p \leq 1 : h(\ell,1) < \tilde{v}(\ell;p)\}$. For $p \geq \tilde{p}_1$ there exists $r_{\kappa(p)} = \min\{r \leq r_\ell : H(r,1) \leq \tilde{v}(r;p)\}$ and for $p < \tilde{p}_1$ there exists $r_{\kappa(p)} = \min\{r \leq r_\ell : h(r,1) \geq \tilde{v}(r;p)\}$. These points r_d, r_ℓ, $r_{\nu(p)}$, $r_{\kappa(p)}$ are such that

$$v(r,l_r;p) = \begin{cases} h_p(r,l_r) & \text{if } (r,l_r) \in B_{r_\ell N}(1) \cup B_{r_d N}(2), \\ H(r,l_r) & \text{if } (r,l_r) \in B_{r_{\nu(p)} r_d - 1}(2), \\ H(r,l_r)\mathbb{I}_{\{p \geq \tilde{p}_1\}} & \\ \ + h(r,l_r)\mathbb{I}_{\{p < \tilde{p}_1\}} & \text{if } (r,l_r) \in B_{r_{\kappa(p)} r_\ell - 1}(1), \\ \tilde{v}(r;p) & \text{if } (r,l_r) \in B_{1 r_{\kappa(p)} - 1}(1) \cup B_{1 r_{\nu(p)} - 1}(2), \end{cases} \tag{21}$$

where

$$\tilde{v}(r;p) = Tv(r,l_r;p) = \begin{cases} w(r, r+1, r+1, r+1; p) & \text{if } r_d \leq r \leq N, \\ w(r, r+1, r+1, r_d; p) & \text{if } r_{\nu(p)} \leq r < r_d, \\ w(r, r+1, r_{\nu(p)}, r_d; p) & \text{if } r_\ell \leq r < r_{\nu(p)}, \\ w(r, r_\ell, r_{\nu(p)}, r_d; p) & \text{if } r_{\kappa(p)} \leq r < r_\ell, \\ w(r_{\kappa(p)}, r_\ell, r_{\nu(p)}, r_d; p) & \text{if } 1 \leq r < r_{\kappa(p)} \end{cases} \tag{22}$$

and

$$\begin{aligned} w(r,s,t,u;p) \quad &= \quad \sum_{j=r+1}^{s-1} \frac{r}{j(j-1)}[H(j,1)\mathbb{I}_{\{p \geq \tilde{p}_1\}} + h(j,1)\mathbb{I}_{\{p < \tilde{p}_1\}}] \\ &\quad + \sum_{j=s}^{t-1} \frac{r}{j(j-1)} h_p(j,1) + \sum_{j=t}^{u-1} \frac{r(t-2)}{j(j-1)(j-2)}[h_p(j,1) + H(j,2)] \\ &\quad + \sum_{j=u}^{N} \frac{r(t-2)}{j(j-1)(j-2)}[h_p(j,1) + h_p(j,2)] \end{aligned}$$

for $r \leq s \leq t \leq u$. The optimal first stop strategy is given by sets $A_t^{ss} = B_{r_\ell N}(1) \cup B_{r_d N}(2)$, $A_t^{fs} = (\mathbb{I}_{\{p \geq \tilde{p}_1\}} B_{r_{\kappa(p)} r_\ell - 1}(1)) \cup B_{r_{\nu(p)} r_d - 1}(2)$, $A_t^{sf} = \mathbb{I}_{\{p < \tilde{p}_1\}} B_{r_{\kappa(p)} r_\ell - 1}(1)$, $A_t^{ff} = \mathbb{E} \setminus (A_t^{ss} \cup A_t^{fs} \cup A_t^{sf})$, $t = 1,2,\ldots,N$. Here we adopt convention that for every set A we have $1 \cdot A = A$ and $0 \cdot A = \emptyset$, where $\emptyset$- the empty set.

The function $w(r,s,t,u;p)$ depends also on r_b and r_c. Let $r \leq s \leq r \leq t \leq u$. When $N \to \infty$ and $\frac{r}{N} \to x_1$, $\frac{s}{N} \to x_2$, $\frac{t}{N} \to y_1$, $\frac{u}{N} \to y_2$ we get

$$\begin{aligned} \hat{w}(x_1,x_2,y_1,y_2;p) \quad &= \quad \lim_{N \to \infty} w(r,s,t,u;p) \\ &= \quad \hat{w}_{21}(x_1,x_2,y_1,y_2;p)\mathbb{I}_{\{p \geq p_1\}} + \hat{w}_{22}(x_1,x_2,y_1,y_2;p)\mathbb{I}_{\{p < p_1\}}, \end{aligned}$$

where

$$\begin{aligned}\hat{w}_{21}(x_1,x_2,y_1,y_2;p) &= x_1[(\ln\frac{x_1}{x_2}-\frac{1}{2}((\ln x_2)^2-(\ln x_1)^2)\mathbb{I}_{\{x_1>c\}}\\ &\quad -(\frac{1}{2}+\ln x_2+\frac{1}{2}(\ln x_2)^2)\mathbb{I}_{\{x_1\le c\}}]\\ &\quad +\frac{x_1\mathbb{I}_{\{x_1>c\}}+c\mathbb{I}\{x_1\le c\}}{x_2}\hat{w}_1(x_2,y_1,y_2;p),\end{aligned}$$

$$\begin{aligned}\hat{w}_{22}(x_1,x_2,y_1,y_2;p) &= x_1[((\ln x_2)^2-(\ln x_1)^2+x_1-x_2+2\ln\frac{x_2}{x_1})\mathbb{I}_{\{x_1>b\}}\\ &\quad +((4-2b+2\ln b)\ln\frac{b}{x_1}+(2-b)(x_1-b)\\ &\quad +(\ln x_2)^2-(\ln b)^2-x_2+b+2\ln\frac{x_2}{b})\mathbb{I}_{\{x_1\le b\}}]\\ &\quad +\frac{x_1\mathbb{I}_{\{x_1>b\}}+b\mathbb{I}_{\{x_1\le b\}}}{x_2}w_1(x_2,y_1,y_2;p)\end{aligned}$$

and

$$\begin{aligned}\hat{w}_1(x_2,y_1,y_2;p) &= x_2[(3p-1)\ln\frac{y_1}{x_2}+\frac{3p-1}{2}((\ln y_1)^2-(\ln x_2)^2)-p(y_1-x_2)]\\ &\quad +x_2y_1[3(2p-1)(\frac{1}{y_1}-\frac{1}{y_2})+(3p-2)(\frac{\ln y_1}{y_1}-\frac{\ln y_2}{y_2})\\ &\quad +(1+p)\ln\frac{y_1}{y_2}]+y_1^2[(p-1)(\frac{1}{y_2}-1)\\ &\quad +(4p-2)(\frac{\ln y_2}{y_2}+\frac{1}{y_2}-1)-(2p-1)\ln y_2].\end{aligned}$$

Parameter p_1 is asymptotic equivalent of $\tilde{p}_1$. The value of p_1 can be determined as the solution of some equation which will be given later.

Let $d=\lim_{N\to\infty}\frac{r_d}{N}\cong 0.7587$ and $\ell=\lim_{N\to\infty}\frac{r_\ell}{N}\cong 0.4237$. We have $\nu(p)=\lim_{N\to\infty}\frac{r_{\nu}(p)}{N}$ is the solution of the equation $\hat{w}_1(\nu,\nu,d;p)=\hat{H}(\nu,2)$ in $[\ell,d]$. Now we can determine p_1 as the solution of the equation $\hat{w}_1(\ell,\nu(p),d;p)=\hat{H}(\ell,1)$ with respect to p in $[0,1]$. Such solution exists since $\hat{w}_1(\ell,\nu(p),d;p)$ is non-decreasing function of p and $\hat{H}(\ell,\nu(1),d;1)<\hat{w}_1(\ell,\nu(1),d;1)$. We have $p_1\cong 0.5659$.

Determine $\kappa(p)=\lim_{N\to\infty}\frac{r_{\kappa(p)}}{N}$. The decision point $\kappa(p)$ is the solution of the equation $\hat{w}(\kappa,\ell,\nu(p);p)=\hat{h}(\kappa,1)\mathbb{I}_{\{p<p_1\}}+\hat{H}(\kappa,1)\mathbb{I}_{\{p\ge p_1\}}$.

$$\hat{v}(x;p)=\lim_{N\to\infty}\tilde{v}(r;p)=\begin{cases}\hat{w}(x,x,x,x;p) & \text{if } d\le x\le 1,\\ \hat{w}(x,x,x,d;p) & \text{if } \nu(p)\le x<d,\\ \hat{w}(x,x,\nu(p),d;p) & \text{if } \ell\le x<\nu(p),\\ \hat{w}(x,\ell,\nu(p),d;p) & \text{if } \kappa(p)\le x<\ell,\\ \hat{w}(\kappa(p),\ell,\nu(p),d;p) & \text{if } 0\le x<\kappa(p).\end{cases} \tag{23}$$

We can formulate

Theorem 7 *For a large N, in the competitive two person problem of choosing the best vs the best or the second best applicant but a better than the opponent, the asymptotically optimal strategy of the first stop is described by the sets* A_t^{ss}, A_t^{fs}, A_t^{sf} *and* A_t^{ff}. *The second stop is according to* ς^* *given by (19) for Player 2 and* δ^* *given by (20) for Player 1. The value function of the problem is given by (21), the expected value with respect to* $\mathbf{P}_{(r,l_r)}$ *of* **AMC** *by (22) and its limit by (23).*

2.2 Nonzero-sum random priority stopping game

A construction of Nash equilibria for a random priority finite horizon two-person non-zero sum game with stopping of Markov process is considered in this section. Let $(X_n, \mathcal{F}_n, \mathbf{P}_x)_{n=0}^N$ be a homogeneous Markov

process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a state space $(\mathbb{E}, \mathcal{B})$. At each moment $n = 1, 2, ..., N$ the decision makers (henceforth called Player 1 and Player 2) are able to observe the Markov chain sequentially. Each player has his utility function $g_i : \mathbb{E} \to \Re$, $i = 1, 2$, and at each moment n each decides separately if he accepts or rejects the realization x_n of X_n. We admit g_i are measurable and bounded. If it happens that both players have selected the same moment n to accept x_n , then the similar random assignment mechanism, as in the zero-sum game model described in Section 2.1, is applied. If a player has not chosen any realization of Markov process he gets $g_i^* = \inf_{x\in\mathbb{E}} g_i(x)$. The aim of each player is to choose a realization which maximizes his expected utility. In fact, the problem will be formulated as a two person non-zero sum game with the concept of the Nash equilibrium as the solution. The problem with permanent priority for Player 1 (i.e. $\alpha_n = 1,\ n = 1, 2, \ldots$) has been solved by Ferenstein [62]. This game is also strictly connected with optimal stopping of stochastic processes. The ideas of Kuhn [83] and Rieder [84] as well as Yasuda [47] and Ohtsubo [50] will be adopted to this random priority game model. Based on this approach, as an example, we deal with non-cooperative two-person time sequential non-zero-sum game version of the best choice problem (*the secretary problem*). The example is a generalization of a game model of the problem considered by Fushimi [65].

In non-cooperative non-zero sum games one of possible definitions of solution is Nash equilibrium. This approach gives very often many different solutions in stopping games (pairs of strategies for the players) with various gains for the players. It is important to have knowledge about all possible solutions of the game. Investigation of alternative solutions is also an interesting theoretical problem. Consideration in this direction for the matrix games can be found, for instance, in Moulin [85]. A tip from two person non-zero sum generalized secretary problem with fixed priority has been given in Szajowski [69] and from the random priority game in Sakaguchi [63]. A very interesting illustration of the problem in stopping games are models of two person best choice problems considered by Fushimi [65]. One of them is generalized in this paper and can be described as follows.

Two companies (Player 1 and Player 2) interview a sequence of applicants one by one (as in the best choice problem which has been recalled above) every morning independently of the other company, and the results of the interviews are communicated to the applicant in the afternoon. If only one of the companies decides to accept the applicant, she agrees to this offer at once, the other company is informed of this fact and continues the interviewing process. If, on the other hand, both companies decide to accept the applicant, she selects one of them with equal probabilities and the other company can continue interviewing and employ another applicant. In Fushimi (1981) the threshold strategies for the players were admitted. It was shown that equilibrium strategies for players in the model are different. One of the players should behave more hastily than in the original secretary problem and he should start solicitation at .2865 for the limiting version of the problem. There are two Nash equilibria in the considered set of strategies for this game with values $(.2865, .2963)$ and $(.2963, .2865)$, respectively.

The following generalization of the above problem has been considered in Section 2.2.2. It is assumed that if both companies want to accept the same applicant, Player 1 is selected with fixed probability α, Player 2 with probability $1-\alpha$, $\alpha \in [0,1]$, and the player who has not been chosen continues interviewing and employs another applicant. Also a more general set of strategies is admitted. This particular game problem is presented as interesting *per se*. The mathematical model of the above formulated problem will be presented and equilibria for each α will be derived. The problem need modified set of strategies with respect to those applied in the zero-sum random priority game (see Section 2.1). More details are in [86].

2.2.1 The payoff functions and strategies

In the problem of optimal stopping the basic class of strategies $\mathcal{T}^N$ are Markov times with respect to σ-fields $\{\mathcal{F}_n\}_{n=1}^N$. We admit that $\mathbf{P}(\tau \leq N) < 1$ for some $\tau \in \mathcal{T}^N$. The class of strategies described in Section 2.1 is not sufficient in the nonzero-sum stopping game. To extend the class of strategies we consider a class of randomized stopping times. It is assumed that the probability space is rich enough to admit the following constructions.

Definition 2 *(see Yasuda [47]) A strategy for each player is a random sequence $p = (p_n) \in \mathcal{P}^N$ or $q = (q_n) \in \mathcal{Q}^N$ such that, for each n,*

(i) $p_n,\ q_n$ are adapted to $\mathcal{F}_n$;

(ii) $0 \leq p_n,\ q_n \leq 1$ a.s. .

If each random variables equals either 0 or 1 we call it a pure strategy.

Let $A_1, A_2, \ldots, A_N$ and $B_1, B_2, \ldots, B_N$ be i.i.d.r.v. of the uniform distribution on $[0,1]$ and independent of Markov process $(X_n, \mathcal{F}_n, \mathbf{P}_x)_{n=0}^N$. Let $\mathcal{H}_n$ be the σ-field generated by $\mathcal{F}_n$, $\{A_1, A_2, ..., A_n\}$ and $\{B_1, B_2, ..., B_n\}$. A randomized Markov time $\lambda(p)$ for strategy $p = (p_n) \in \mathcal{P}^N$ and $\mu(q)$ for strategy $q = (q_n) \in \mathcal{Q}^N$ are defined by $\lambda(p) = \inf\{N \geq n \geq 1 : A_n \leq p_n\}$ and $\mu(q) = \inf\{N \geq n \geq 1 : B_n \leq q_n\}$, respectively. We denote by Λ^N and M^N the sets of all randomized strategies of Player 1 and Player 2. Clearly, if each p_n is either zero or one, then the strategy is pure and $\lambda(p)$ is in fact an $\{\mathcal{F}_n\}$-Markov time. In particular an $\{\mathcal{F}_n\}$- Markov time λ corresponds to the strategy $p = (p_n)$ with $p_n = \mathbb{I}_{\{\lambda = n\}}$, where $\mathbb{I}_A$ is an indicator function of the set A.

Denote $\mathcal{T}_k^N = \{\tau \in \mathcal{T}^N : \tau \geq k\}$. One can define the set of strategies $\tilde{\Lambda}^N = \{(p, \{\sigma_n^1\}) : p \in \mathcal{P}^N, \{\sigma_n^1\} \in \mathcal{T}_{n+1}^N$ for every $n\}$ and let $\tilde{M}^N = \{(q, \{\sigma_n^2\}) : q \in \mathcal{Q}^N, \{\sigma_n^2\} \in \mathcal{T}_{n+1}^N$ for every $n\}$ for Player 1 and Player 2, respectively.

Let $\xi_1, \xi_2, ...$ be i.i.d.r.v. uniformly distributed on $[0,1]$ and independent of $\bigvee_{n=1}^N \mathcal{H}_n$ and the lottery is given by $\bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_N)$. Denote $\tilde{\mathcal{H}}_n = \sigma\{\mathcal{H}_n, \xi_1, \xi_2, \ldots, \xi_n\}$ and let $\tilde{\mathcal{T}}^N$ be the set of Markov times with respect to $(\tilde{\mathcal{H}}_n)_{n=0}^N$. For every pair (s,t) such that $s \in \tilde{\Lambda}^N$, $t \in \tilde{M}^N$ we define $\tau_1(s,t) = \lambda(p)\mathbb{I}_{\{\lambda(p)<\mu(q)\}} + (\lambda(p)\mathbb{I}_{\{\xi_{\lambda(p)} \leq \alpha_{\lambda(p)}\}} + \sigma^1_{\mu(q)}\mathbb{I}_{\{\xi_{\lambda(p)} > \alpha_{\lambda(p)}\}})\mathbb{I}_{\{\lambda(p)=\mu(q)\}} + \sigma^1_{\mu(q)}\mathbb{I}_{\{\lambda(p)>\mu(q)\}}$ and $\tau_2(s,t) = \mu(q)\mathbb{I}_{\{\lambda(p)>\mu(q)\}} + (\mu(q)\mathbb{I}_{\{\xi_{\mu(q)} > \alpha_{\mu(q)}\}} + \sigma^2_{\lambda(p)}\mathbb{I}_{\{\xi_{\mu(q)} \leq \alpha_{\mu(q)}\}})\mathbb{I}_{\{\lambda(p)=\mu(p)\}} + \sigma^2_{\lambda(p)}\mathbb{I}_{\{\lambda(p)<\mu(q)\}}$. The random variables $\tau_1(s,t)$, $\tau_2(s,t) \in \tilde{\mathcal{T}}^N$ for every $s \in \tilde{\Lambda}$ and $t \in \tilde{M}$.

Definition 3 *The Markov times $\tau_1(s,t)$ and $\tau_2(s,t)$ are selection times of Player 1 and Player 2 when they use strategies $s \in \tilde{\Lambda}$ and $t \in \tilde{M}$, respectively, and the lottery is $\bar{\alpha}$.*

For each $(s,t) \in \tilde{\Lambda}^N \times \tilde{M}^N$ and given $\bar{\alpha}$ the payoff function for the i-th player is defined as $f_i(s,t) = g_i(X_{\tau_i(s,t)})$. Let $\tilde{R}_i(x,s,t) = E_x f_i(s,t) = E_x g_i(X_{\tau_i(s,t)})$ be the expected gain of i-th player if the players use (s,t). We have defined the game in normal form $(\tilde{\Lambda}^N, \tilde{M}^N, \tilde{R}_1, \tilde{R}_2)$. This random priority game will be denoted $\mathcal{G}_{rp}$.

Definition 4 *A pair (s^*, t^*) of strategies such that $s^* \in \tilde{\Lambda}^N$ and $t^* \in \tilde{M}^N$ is called a Nash equilibrium in $\mathcal{G}_{rp}$ if for all $x \in \mathbb{E}$*

$$\begin{aligned} v_1(x) &= \tilde{R}_1(x, s^*, t^*) \geq \tilde{R}_1(x, s, t^*) \text{ for every } s \in \tilde{\Lambda}^N, \\ v_2(x) &= \tilde{R}_2(x, s^*, t^*) \geq \tilde{R}_2(x, s^*, t) \text{ for every } t \in \tilde{M}^N. \end{aligned}$$

The pair $(v_1(x), v_2(x))$ will be called the Nash value.

Denote $h_i(n, X_n) = \operatorname*{esssup}_{\tau \in \mathcal{T}_n^N} E_{X_n} g_i(X_\tau)$ and σ^{*i} a stopping time such that $h_i(0,x) = E_x g_i(X_{\sigma^{*i}})$ for every $x \in \mathbb{E}$, $i = 1, 2$. Let $\Gamma_n^i = \{x \in \mathbb{E} : h_i(n,x) = g_i(x)\}$. We have $\sigma^{*i} = \inf\{n : X_n \in \Gamma_n^i\}$ (cf. Shiryaev (1978)). Denote

${\sigma^*}_k^i = \inf\{n > k : X_n \in \Gamma_n^i\}$. Taking into account the above definition of $\mathcal{G}_{rp}$ one can conclude that the Nash values of this game are the same as in the auxiliary game $\mathcal{G}_{wp}$ with the sets of strategies of the players $\mathcal{P}^N$, $\mathcal{Q}^N$ and payoff functions (cf. Yasuda (1985))

$$\begin{aligned}
\varphi_1(p,q) &= g_1(X_{\lambda(p)})\mathbb{I}_{\{\lambda(p)<\mu(q)\}} + \tilde{h}_1(\mu(q), X_{\mu(q)})\mathbb{I}_{\{\lambda(p)>\mu(q)\}} \qquad (24)\\
&\quad + \left[g_1(X_{\lambda(p)})\alpha_{\lambda(p)} + \tilde{h}_1(\lambda(p), X_{\lambda(p)})(1-\alpha_{\lambda(p)})\right]\mathbb{I}_{\{\lambda(p)=\mu(q)\}},\\
\varphi_2(p,q) &= g_2(X_{\mu(q)})\mathbb{I}_{\{\mu(q)<\lambda(p)\}} + \tilde{h}_2(\lambda(p), X_{\lambda(p)})\mathbb{I}_{\{\mu(q)>\lambda(p)\}} \qquad (25)\\
&\quad + \left[g_2(X_{\lambda(p)})(1-\alpha_{\lambda(p)}) + \tilde{h}_2(\lambda(p), X_{\lambda(p)})\alpha_{\lambda(p)}\right]\mathbb{I}_{\{\lambda(p)=\mu(q)\}},
\end{aligned}$$

for each $p \in \mathcal{P}$, $q \in \mathcal{Q}$, where $\tilde{h}_i(n, X_n) = \operatorname*{esssup}_{\tau \in \mathcal{T}_{n+1}^N} E_{X_n} g_i(X_\tau) = E_{X_n} h_i(n+1, X_{n+1})$. Denote $R_i(x,p,q) = E_x\varphi_i(p,q)$ for every $x \in \mathbb{E}$, $i = 1, 2$.

Let $\mathcal{P}_n^N = \{p = (p_n) \in \mathcal{P} : p_1 = \ldots = p_{n-1} = 0, p_N = 1\}$ and $\mathcal{Q}_n^N = \{q = (q_n) \in \mathcal{Q} : q_1 = \ldots = q_{n-1} = 0, q_N = 1\}$. We will use the following convention: if $p \in \mathcal{P}^N$ then (p_n, p) is the strategy belonging to $\mathcal{P}^N$ in which the n-th coordinate is changed to p_n.

Definition 5 *A pair $(p^*, q^*) \in \mathcal{P}_n^N \times \mathcal{Q}_n^N$ is called an equilibrium point of $\mathcal{G}_{wp}$ at n if*

$$\begin{aligned}
v_1(n, X_n) &= E_{X_n}\varphi_1(p^*, q^*) \geq E_{X_n}\varphi_1(p, q^*) \text{for every } p \in \mathcal{P}_n^N,\ \mathbf{P}_x\text{-a.s.},\\
v_2(n, X_n) &= E_{X_n}\varphi_2(p^*, q^*) \geq E_{X_n}\varphi_2(p^*, q) \text{for every } q \in \mathcal{Q}_n^N,\ \mathbf{P}_x\text{-a.s.}\ .
\end{aligned}$$

A Nash equilibrium point at $n = 0$ is a solution of $\mathcal{G}_{wp}$. The pair $(v_1(0,x), v_2(0,x))$ of values is a Nash value corresponding to $(p^*, q^*) \in \mathcal{P}^N \times \mathcal{Q}^N$.

Theorem 8 *([86]) There exists a Nash equilibrium (p^*, q^*) in the game $\mathcal{G}_{wp}$. The Nash value and an equilibrium point can be calculated recursively.*

Proof. At moment N the players play the following bimatrix game

$$\begin{pmatrix} (\tilde{g}_1(N, X_N), \tilde{g}_2(N, X_N)) & (g_1(X_N), g_2^*) \\ (g_1^*, g_2(X_N)) & (g_1^*, g_2^*) \end{pmatrix}$$

where $\tilde{g}_1(n,x) = \alpha_n g_1(x) + (1-\alpha_n)\tilde{h}_1(n,x)$ and $\tilde{g}_2(n,x) = (1-\alpha_n)g_2(x) + \alpha_n\tilde{h}_2(n,x)$. This game always has an equilibrium in pure or randomized strategies on $\{\omega : X_N = x\}$ for every $x \in \mathbb{E}$. We denote a Nash equilibrium in $\mathcal{P}_N^N \times \mathcal{Q}_N^N$ by (p_N^*, q_N^*) and the corresponding Nash value by $(v_1(N,x), v_2(N,x))$. Let us assume that an equilibrium $(p^*, q^*) \in \mathcal{P}_{n+1}^N \times \mathcal{Q}_{n+1}^N$ has been constructed and $(v_1(n+1,x), v_2(n+1,x))$ is the Nash value corresponding to this strategy on $\{\omega : X_n = x\}$. We consider the following bimatrix game

$$\begin{pmatrix} (\tilde{g}_1(n, X_n), \tilde{g}_2(n, X_n)) & (g_1(X_n), \tilde{h}_2(n, X_n)) \\ (\tilde{h}_1(n, X_n), g_2(X_n)) & (\tilde{v}_1(n, X_n), \tilde{v}_2(n, X_n) \end{pmatrix} \qquad (26)$$

where $\tilde{v}_j(n,x)$ is such that $\tilde{v}_j(n,X_n) = E_{X_n} v_j(n+1,X_{n+1}),\ j=1,2$. On the set $\{\omega : X_n = x\}$ there is at least one equilibrium point in pure or randomized strategies in this bimatrix game. By measurability of $g_i(x)$ there exists (p_n^*, q_n^*) such that $p_n^*, q_n^* \in \mathcal{F}_n$ and (p_n^*, q_n^*) is a Nash equilibrium in the above bimatrix game. We are now in a position to show that $((p_n^*, p^*), (q_n^*, q^*))$ is an equilibrium of $\mathcal{G}_{wp}$ in $\mathcal{P}_n^N \times \mathcal{Q}_n^N$. Let $(p_n, p) \in \mathcal{P}_n^N$, where $p \in \mathcal{P}_{n+1}^N$. By properties of conditional expectation and induction assumption we have $\mathbf{P}_x$-a.s.

$$\begin{aligned}
E_{X_n}\varphi_1((p_n,p),(q_n^*,q^*)) \quad = \quad & p_n q_n^* \tilde{g}_1(n,X_n) + p_n(1-q_n^*)g_1(X_n) \\
& + (1-p_n)q_n^*\tilde{h}_1(n,X_n) \\
& + (1-p_n)(1-q_n^*)E_{X_n}E_{X_{n+1}}\varphi_1(p,q^*) \\
& \le p_n^* q_n^* \tilde{g}_1(n,X_n) + p_n^*(1-q_n^*)g_1(X_n) \\
& + (1-p_n^*)q_n^*\tilde{h}_1(n,X_n) \\
& + (1-p_n^*)(1-q_n^*)E_{X_n}v_1(n+1,X_{n+1}) = v_1(n,X_n).
\end{aligned}$$

for each $x \in \mathbb{E}$. The same is valid for Player 2. This proves the theorem.

□

The solution of the game $\mathcal{G}_{rp}$ can be constructed based on the solution (p^*, q^*) of the corresponding game $\mathcal{G}_{wp}$.

Theorem 9 *([86]) Game $\mathcal{G}_{rp}$ has a solution. The pair (s^*, t^*), where $s^* = (p^*, \{{\sigma^*}_n^1\}) \in \tilde{\Lambda}^N$ and $t^* = (q^*, \{{\sigma^*}_n^2\}) \in \tilde{M}^N$, is an equilibrium point. The value of the game is $(v_1(0,x), v_2(0,x))$.*

In fact, the players play optimally $\mathcal{G}_{rp}$ using a Nash equilibrium strategy from $\mathcal{G}_{wp}$. If the strategy of both players indicates stopping at moment n and neither player has stopped earlier, then the lottery chooses one of them. The player who has not been selected will accept any future realization according to the adequate optimal strategy in the optimization problem.

2.2.2 Two person best choice problem with random priority

The solution of the best choice problem (one player game), described in Section 2.1.1, is auxiliary in the solution of the two person game with random priority. It is used in further consideration. Let us consider the two person nonzero-sum game with random priority described in Section 2.2 related to the secretary problem. We admit that both players observe Markov chain $W_t,\ t=1,2,...$ and their utility functions $g_j(r) = f(r),\ j=1,2,\ r \in \mathbb{E}$. Let lottery $\bar{\alpha}$ be constant, i.e. $\alpha_i = \alpha,\ i=1,2,...,N$. Denote $\tilde{c}(r) = \tilde{c}_{\mathrm{BA}}(r)$ defined in Section 2.1.1 and $r_a = \inf\{1 \le r \le N : \sum_{i=r+1}^N \frac{1}{i-1} \le 1\}$ and $\tau_r^* = \inf\{s > r : Y_s = 1, s \ge r_a\}$. We have $\tilde{g}_1(r) = \alpha f(r) + (1-\alpha)\tilde{c}(r)$, $\tilde{g}_2(r) = (1-\alpha)f(r) + \alpha\tilde{c}(r)$ and $g_i^* = 0$. Our aim is to determine the equilibria which give the highest and lowest value for Player 1. At first, we construct the highest value Nash equilibrium for Player 1.

By analysis of the matrices (26) we have that $p_r^* = q_r^* = 1$ is an equilibrium point for $r \geq r_a$. We have then

$$\begin{aligned}
\tilde{v}_1(r) &= \sum_{i=r+1}^{N} p(r,i)\tilde{g}_1(i) \\
\tilde{v}_2(r) &= \sum_{i=r+1}^{N} p(r,i)\tilde{g}_2(i)
\end{aligned}$$

for $j = 1, 2$. For $r = r_a - 1$ we have two pure equilibria in (26) in this case: $(1, 0)$ and $(0, 1)$ and one in randomized strategies. Since for $r < r_a$ we have $f(r) < \tilde{c}(r)$ henceforth we can choose $(1, 0)$ at $r = r_a - 1$ and assume for induction that the same strategy is optimal for $r < r_a$. Under this assumption

$$\begin{aligned}
\tilde{v}_1(r) &= \sum_{i=r+1}^{r_a-1} p(r,i)g_1(i) + \sum_{i=r_a}^{N} p(r,i)\tilde{g}_1(i) \\
\tilde{v}_2(r) &= \sum_{i=r+1}^{r_a-1} p(r,i)\tilde{c}(i) + \sum_{i=r_a}^{N} p(r,i)\tilde{g}_2(i).
\end{aligned}$$

Since $f(r)$ is increasing and $\tilde{c}(r)$ is constant for $r < r_a$ the strategy $(1, 0)$ can be used as equilibrium in $r_b \leq r \leq r_a$, where $r_b = \inf\{r < r_a : \tilde{v}_1(r) \leq g_1(r)\}$. Denote $r_{b'} = \inf\{r < r_a : \tilde{v}_2(r) \leq g_2(r)\}$. For large N we have $r_b < r_{b'}$ if $\alpha < \alpha_0 = \min\{\alpha \in [0, 1] : \frac{2}{2+\alpha} \geq e^{-\frac{1-\alpha}{2}}\} \cong 0.5299$. Denote

$$\begin{aligned}
w_1(r, s, \alpha) &= \sum_{i=r+1}^{s-1} p(r,i)f(i) + \sum_{i=s}^{N} p(r,i)\tilde{g}_1(i) \\
w_2(r, s, \alpha) &= \sum_{i=r+1}^{s-1} p(r,i)\tilde{c}(i) + \sum_{i=s}^{N} p(r,i)\tilde{g}_2(i).
\end{aligned}$$

For $\alpha < \alpha_0$ we have

$$(p_r^*, q_r^*) = \begin{cases} (1,1) & \text{if } r \geq r_a, \\ (1,0) & \text{if } r_b \leq r < r_a, \\ (0,0) & \text{if } 1 \leq r < r_b, \end{cases} \tag{27}$$

and

$$v_j(r) = \begin{cases} w_j(r, r+1, \alpha) & \text{if } r \geq r_a, \\ w_j(r, r_a, \alpha) & \text{if } r_b \leq r < r_a, \\ w_j(r_b - 1, r_a, \alpha) & \text{if } 1 \leq r < r_b, \end{cases} \tag{28}$$

$j = 1, 2$. The value of the game is $(v_1, v_2) = (v_1(1), v_2(1))$. When $N \to \infty$ such that $\frac{r}{N} \to x$ we obtain

$$\hat{v}_j(x) = \lim_{N\to\infty} v_j(r) = \begin{cases} \hat{w}_j(x, x, \alpha) & \text{if } x \geq a, \\ \hat{w}_j(x, a, \alpha) & \text{if } b \leq r < a, \\ \hat{w}_j(b, a, \alpha) & \text{if } 0 \leq r < b, \end{cases}$$

where

$$\begin{aligned}\hat{w}_1(x,y,\alpha) &= -x\ln x+(1-\alpha)x(\ln y+\frac{(\ln y)^2}{2}),\\ \hat{w}_2(x,y,\alpha) &= y-x-(1-\alpha)x\ln y+\alpha x\frac{(\ln y)^2}{2}\end{aligned}$$

and $b=\lim_{N\to\infty}\frac{r_b}{N}=e^{-\frac{3-\alpha}{2}}$. The asymptotic value of the game in this equilibrium is

$$(\hat{v}_1,\hat{v}_2)=(e^{-\frac{3-\alpha}{2}},e^{-1}-\frac{\alpha}{2}e^{-\frac{3-\alpha}{2}}). \tag{29}$$

Let $\alpha\geq\alpha_0$. Denote

$$\begin{aligned}u_1(r,s,t,\alpha) &= \sum_{i=r+1}^{s-1}p(r,i)\tilde{c}(i)+\sum_{i=s}^{t-1}p(r,i)f(i)+\sum_{i=t}^{N}p(r,i)\tilde{g}_1(i),\\ u_2(r,s,t,\alpha) &= \sum_{i=r+1}^{s-1}p(r,i)f(i)+\sum_{i=s}^{t-1}p(r,i)\tilde{c}(i)+\sum_{i=r_a}^{N}p(r,i)\tilde{g}_2(i).\end{aligned}$$

Similar analysis as above leads to conclusion that

$$(p_r^*,q_r^*)=\begin{cases}(1,1) & \text{if } r\geq r_a,\\ (1,0) & \text{if } r_b\leq r<r_a,\\ (0,1) & \text{if } r_c\leq r<r_b,\\ (0,0) & \text{if } 1\leq r<r_c,\end{cases} \tag{30}$$

and

$$v_j(r)=\begin{cases}u_j(r,r+1,r+1,\alpha) & \text{if } r\geq r_a,\\ u_j(r,r+1,r_a,\alpha) & \text{if } r_b\leq r<r_a,\\ u_j(r,r_b,r_a,\alpha) & \text{if } r_c\leq r<r_b,\\ u_j(r_c-1,r_b,r_a,\alpha) & \text{if } 1\leq r<r_c,\end{cases} \tag{31}$$

$j=1,2$, where $r_c=\inf\{r<r_b:\tilde{v}_2(r)\leq g_2(r)\}$. When $N\to\infty$ such that $\frac{r}{N}\to x$ we have

$$\hat{v}_j(x)=\lim_{N\to\infty}v_j(r)=\begin{cases}\hat{u}_j(x,x,x,\alpha) & \text{if } x\geq a,\\ \hat{u}_j(x,x,a,\alpha) & \text{if } b\leq r<a,\\ \hat{u}_j(x,b,a,\alpha) & \text{if } c\leq r<b,\\ \hat{u}_j(c,b,a,\alpha) & \text{if } 0\leq r<c,\end{cases}$$

where $\hat{u}_1(x,y,z,\alpha)=z-x\frac{z}{y}+\frac{x}{y}\hat{w}_1(y,z,\alpha)$ and $\hat{u}_2(x,y,z,\alpha)=x\ln\frac{y}{x}+\frac{x}{y}\hat{w}_2(y,z,\alpha)$. The asymptotic value of the game for this equilibrium point is

$$(\hat{v}_1,\hat{v}_2)=(e^{-1}+e^{-\frac{5}{2}+e^{\frac{1-\alpha}{2}}}(1-e^{\frac{1-\alpha}{2}}),e^{-\frac{5}{2}+e^{\frac{1-\alpha}{2}}}). \tag{32}$$

Theorem 10 *In the random priority two person non-zero sum game of choosing the best applicant the Nash equilibrium which gives the maximal probability of success for Player 1 is given by (27) for $\alpha < \alpha_0$ and by (30) for $\alpha \geq \alpha_0$. The Nash value for the equilibrium is (28) and (31), respectively. For the limiting case the Nash value is given by (29) and (32), respectively.*

Now, we construct the Nash equilibrium with the lowest probability of success for Player 1. The same arguments as above suggest that one can choose $(0,1)$ in $r_a - 1$. Using backward induction procedure as long as possible we minimize the Nash value of Player 1. In such a way we obtain the following equilibrium strategy. For $\alpha \geq 1 - \alpha_0$

$$(p_r^*, q_r^*) = \begin{cases} (1,1) & \text{if } r \geq r_a, \\ (0,1) & \text{if } r_d \leq r < r_a, \\ (0,0) & \text{if } 1 \leq r < r_d, \end{cases} \tag{33}$$

and the Nash value

$$v_j^*(r) = \begin{cases} w_j^*(r, r+1, \alpha) & \text{if } r \geq r_a, \\ w_j^*(r, r_a, \alpha) & \text{if } r_d \leq r < r_a, \\ w_j^*(r_d - 1, r_a, \alpha) & \text{if } 1 \leq r < r_d, \end{cases} \tag{34}$$

where $w_1^*(r,s,\alpha) = w_2(r,s,1-\alpha)$, $w_2^*(r,s,\alpha) = w_1(r,s,1-\alpha)$ and $r_d = \inf\{r < r_a : \tilde{v}_1^*(r) \leq g_1(r)\}$.

For $\alpha < 1 - \alpha_0$ we have

$$(p_r^*, q_r^*) = \begin{cases} (1,1) & \text{if } r \geq r_a, \\ (0,1) & \text{if } r_d \leq r < r_a, \\ (1,0) & \text{if } r_f \leq r < r_d, \\ (0,0) & \text{if } 1 \leq r < r_f, \end{cases} \tag{35}$$

and

$$v_j^*(r) = \begin{cases} u_j^*(r, r+1, r+1, \alpha) & \text{if } r \geq r_a, \\ u_j^*(r, r+1, r_a, \alpha) & \text{if } r_d \leq r < r_a, \\ u_j^*(r, r_d, r_a, \alpha) & \text{if } r_f \leq r < r_d, \\ u_j^*(r_f - 1, r_d, r_a, \alpha) & \text{if } 1 \leq r < r_f, \end{cases} \tag{36}$$

where $u_1^*(r,s,t,\alpha) = u_2^*(r,s,t,1-\alpha)$, $u_2^*(r,s,t,\alpha) = u_1^*(r,s,t,1-\alpha)$ and $r_f = \inf\{r < r_d : \tilde{v}_1(r) \leq g_1(r)\}$. When $N \to \infty$ such that $\frac{r}{N} \to x$ we obtain $\frac{r_d}{N} \to d = e^{-\frac{2+\alpha}{2}}$ and $\frac{r_f}{N} \to f = e^{-\frac{5}{2} + e^{\frac{\alpha}{2}}}$. The asymptotic value of the game in this equilibrium is

$$(\hat{v}_1^*, \hat{v}_2^*) = \begin{cases} (e^{-\frac{5}{2}+e^{\frac{\alpha}{2}}}, e^{-1} + e^{-\frac{5}{2}+e^{\frac{\alpha}{2}}}(1 - e^{\frac{\alpha}{2}})) & \text{if } \alpha < 1 - \alpha_0 \\ (e^{-1} - \frac{1-\alpha}{2} e^{-\frac{2+\alpha}{2}}, e^{-\frac{2+\alpha}{2}}) & \text{if } \alpha \geq 1 - \alpha_0. \end{cases} \tag{37}$$

Theorem 11 *In the random priority two person non-zero sum game of choosing the best applicant the Nash equilibrium which gives the lowest probability of success for Player 1 is given by (33) for $\alpha \geq 1 - \alpha_0$ and by (35) for $\alpha < 1 - \alpha_0$. The Nash value for the equilibrium is (34) and (36), respectively. For limiting case the Nash value is given by (37).*

Remark 1 *These solutions do not exhaust all Nash points in considered game. The other pure Nash equilibria can be obtained, roughly speaking, by more often "switches" between $(1,0)$ and $(0,1)$ strategy (when both strategies are the Nash equilibria in bimatrix game (26)). This idea is used in Remark 3 to construct Nash equilibria, for $\alpha \in [1-\alpha_0, \alpha_0]$, with equal Nash values for both players.*

Remark 2 *For $\alpha \in (0,1)$ and for r such that $\tilde{v}_1(r) \leq f(r) < \tilde{c}(r)$ and $\tilde{v}_2(r) \leq f(r) < \tilde{c}(r)$ we can use the randomized Nash equilibrium (see Moulin (1986) for details)*

$$(p_r^*, q_r^*) = \left(\frac{\tilde{v}_2(r) - f(r)}{\tilde{v}_2(r) - f(r) + (1-\alpha)(f(r) - \tilde{c}(r))}, \frac{\tilde{v}_1(r) - f(r)}{\tilde{v}_1(r) - f(r) + \alpha(f(r) - \tilde{c}(r))} \right).$$

Table 1. Decision points for selected α.

α	d	b
α_0	.3677	.2908
.525	.3432	.2936
.520	.3335	.2966
.515	.3264	.2998
.510	.3205	.3032
.505	.3154	.3069
.500	.3109	.3109

Remark 3 *Let $\alpha \in (1-\alpha_0, \alpha_0)$. We have at least two Nash equilibria with the same Nash values for both players equal* $\exp(-(3-\alpha_0)/2)$ *(in the limiting case). The first pair of strategies is (30) and the second pair is (35) with $c = f \cong .2908$ and b, d, chosen in an appropriate way. The values of b and c one can obtain as solution of the system of equation $\hat{u}_1(c,b,a,\alpha) = \hat{u}_2(c,b,a,\alpha) =$* $\exp(-(3-\alpha_0)/2)$. *Similarly, d and f is solution of the system of equation $\hat{u}_1^*(f,d,a,\alpha) = \hat{u}_2^*(f,d,a,\alpha) =$* $\exp(-(3-\alpha_0)/2)$. *The values of b and d for selected α are given in Table 1.*

It seems that the concept of correlated equilibria applied to nonzero-sum stopping games would be an interesting and important approach.

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