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Higher-order conditions for strict efficiency revisited*

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Abstract

D. V. Luu and P. T. Kien propose in *Soochow J. Math.* 33 (2007), 17–31, higher-order conditions for strict efficiency of vector optimization problems based on the derivatives introduced by I. Ginchev in *Optimization* 51 (2002), 47–72. These derivatives are defined for scalar functions and in their terms necessary and sufficient conditions can be obtained a point to be strictly efficient (isolated) minimizer of a given order for quite arbitrary scalar function. Passing to vector functions, Luu and Kien lose the peculiarity that the optimality conditions work with arbitrary functions. In the present paper, applying the mentioned derivatives for the scalarized problem and restoring the original idea, optimality conditions for strictly efficiency of a given order are proposed, which work with quite arbitrary vector functions. It is shown that the results of Luu and Kien are corollaries of the given conditions.

Key words: nonsmooth vector optimization, higher-order optimality conditions, strict efficiency, isolated minimizers.

2000 Math. Subject Classification: 90C46, 90C29.

1 Introduction

In this paper $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a given vector function and $C \subset \mathbb{R}^m$ is a closed convex cone with non empty interior. We deal with the local solutions of the problem

$$\min_C f(x), \tag{1}$$

which will be called minimizers for f . The point $x^0 \in \mathbb{R}^n$ is said a w -minimizer (weakly efficient point) if there exists a neighbourhood U of x^0 such that $f(x) \notin f(x^0) - \text{int } C$ for all $x \in U$. The point $x^0 \in \mathbb{R}^n$ is said an i -minimizer (isolated minimizer) of order k if there exists a neighbourhood U of x^0 and a constant $a > 0$ such that $(f(x) + C) \cap B(f(x^0), a\|x - x^0\|^k) = \emptyset$ for all $x \in U \setminus \{x^0\}$. Here $B(y, r)$ stands for the open ball with center y and radius r . In these definitions we assume that both the domain and the image space are supplied with norms. However, since any two norms in a finite dimensional space are equivalent, the defined concepts of minimizers are actually norm-independent. Therefore, any particular norm can be used in the definition. We will use for simplicity Euclidean norms generated by a scalar product $\langle \cdot, \cdot \rangle$.

The isolated minimizers are known also as strictly efficient points. Here we use the notion of *strict efficiency* only in the title to underline the relation of the present paper to [13]. When $m = 1$ and $C = \mathbb{R}_+$ the vector optimization problem (1) is in fact a scalar optimization problem. Hence, the notions of a w -minimizer and an i -minimizer can be applied also for scalar problems.

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D. V. Luu and P. T. Kien [13] propose higher-order conditions for strict efficiency of vector optimization problems based on the derivatives introduced by I. Ginchev [6]. The latter are defined for scalar functions and in their terms necessary and sufficient conditions can be obtained a point to be an i -minimizer of a given order for quite arbitrary scalar function. Passing to vector functions, Luu and Kien lose the peculiarity that the optimality conditions work with arbitrary functions. In the present paper, applying the mentioned derivatives for the scalarized problem and restoring the original idea, optimality conditions for strictly efficiency of a given order are proposed, which work with quite arbitrary vector functions. It is shown that the results of Luu and Kien are corollaries of the given conditions.

To emphasize the main ideas, we confine to functions f with finite dimensional domain and image spaces not restricted by constraints. In [13] the more general case is considered of functions with normed spaces as domain and image spaces and restricted by set constraints.

In Section 2 the vector problem is scalarized. Section 3 recalls the derivatives and the optimality conditions for strict efficiency from [6] for scalar functions. Section 4 deals with the vector problems and establishes Theorem 4.1 which is the main result. Dealing with polyhedral cones, it is shown that as corollaries the results from [13] concerning the finite dimensional case with ordering cone the positive orthant and stated in terms of coordinate functions can be obtained. The gap between the necessary and sufficient conditions is discussed. In Section 5 for problems with regular functions the coincidence of the necessary and sufficient conditions in terms of coordinate functions is shown. Section 6 attempts to generalize the conditions in terms of coordinate functions for arbitrary and not only polyhedral cones. It relates the results to these of [13] concerning arbitrary cones. The final Section 7 offers some discussion.

2 Scalarization of the vector problem

The positive polar cone of C is $C' = \{\xi \in \mathbb{R}^m \mid \langle \xi, y \rangle \geq 0 \text{ for all } y \in C\}$. Since C has non empty interior, C' obeys a compact base Γ . Recall that $\xi \in C'$ is said an extreme direction of C' , if $\xi = \xi^1 + \xi^2$, $\xi^1, \xi^2 \in C'$, implies $\xi^i = \lambda_i \xi$, $i = 1, 2$, for some $\lambda_1, \lambda_2 \in \mathbb{R}_+$. The set of the extreme directions of C' is denoted by $\text{extd } C'$. It holds $C' = \text{cl co extd } C'$ and $\Gamma = \text{cl}(\text{co } \Gamma \cap \text{extd } C')$.

To the vector problem (1) we put into correspondence the scalar problem

$$\min \phi(x), \quad (2)$$

where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is the scalar function defined by

$$\phi(x) = \sup \{ \langle \xi, f(x) - f(x^0) \rangle \mid \xi \in \Gamma \cap \text{extd } C' \} \quad (3)$$

and Γ is a base of C' . The relation between the solutions of problem (1) and (2) is given by the following theorem.

Theorem 2.1 ([12]) *The point $x^0 \in \mathbb{R}^m$ is a w -minimizer or i -minimizer of order k for problem (1) if and only if x^0 is respectively a w -minimizer or i -minimizer of order k for problem (2).*

3 Optimality conditions for the scalar problem

We put $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Let $\phi : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ be a given scalar function.

Let $u \in \mathbb{R}^m \setminus \{0\}$ be a given direction. We define the zero order lower derivative of ϕ at x^0 in direction u by

$$\phi_-^{(0)}(x^0, u) = \liminf_{(t, u') \rightarrow (0^+, u)} \phi(x^0 + tu').$$

For a given positive integer k and a given direction $u \in \mathbb{R}^n \setminus \{0\}$ we accept that the k -th order lower derivative $\phi_-^{(k)}(x^0, u)$ in direction u exists as an element of $\overline{\mathbb{R}}$ if and only if the derivatives $\phi_-^{(i)}(x^0, u)$, $i = 0, 1, \dots, k-1$, exist as elements of \mathbb{R} . We put then

$$\phi_-^{(k)}(x^0, u) = \liminf_{(t, u') \rightarrow (0^+, u)} \frac{k!}{t^k} \left(\phi(x^0 + tu') - \sum_{i=0}^{k-1} \frac{t^i}{i!} \phi_-^{(i)}(x^0, u) \right). \quad (4)$$

Since $\phi_-^{(i)}(x^0, u) \in \mathbb{R}$ for $i = 0, \dots, k-1$, only the term $\phi(x^0 + tu')$ in (4) can eventually take infinite values. Therefore (4) does not contain undefined expressions like $\infty - \infty$.

In the sequel we apply the following conditions:

$$\begin{aligned} \mathbb{S}_-^0(\phi, x^0, u) &: \phi_-^{(0)}(x^0, u) > \phi(x^0), \\ \mathbb{S}_-^k(\phi, x^0, u) &: \phi_-^{(0)}(x^0, u) = \phi(x^0), \phi_-^{(i)}(x^0, u) = 0 \text{ for } i = 1, \dots, k-1, \text{ and } \phi_-^{(k)}(x^0, u) > 0. \end{aligned}$$

Modifying slightly the results from [6], we obtain the following higher-order optimality conditions.

Theorem 3.1 *Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be an arbitrary scalar function.*

(Necessary Conditions) Let x^0 be an i -minimizer of order ν for the function ϕ . Then for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $k = k(u) \leq \nu$ such that all the derivatives $\phi_-^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_-^k(\phi, x^0, u)$ is satisfied.

(Sufficient Conditions) Let for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $k = k(u) \leq \nu$ such that all the derivatives $\phi_-^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_-^k(\phi, x^0, u)$ is satisfied. Then x^0 is an i -minimizer of order ν for the function ϕ .

Usually in optimization the class of functions for which at given point x^0 the necessary conditions are satisfied is larger than the class of functions for which at x^0 the sufficient conditions are satisfied. The set difference of these two classes is called the gap between the necessary and the sufficient conditions at x^0 . Let us underline that Theorem 3.1 clarifies two remarkable properties of the defined derivatives. First, the optimality conditions work for arbitrary scalar functions. Second, due to the coincidence of the necessary and the sufficient conditions, the gap between them is empty.

4 Optimality conditions for the vector problem

Applying for the vector problem (1) the scalarization from Section 2 and Theorem 3.1 we get our main result.

Theorem 4.1 *Consider the vector optimization problem (1). Let $x^0 \in \mathbb{R}^n$, and the scalar function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by (3) where Γ is a base of C' .*

(Necessary Conditions) Let x^0 be an i -minimizer of order ν for the vector function f . Then for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $k = k(u) \leq \nu$ such that all the derivatives $\phi_-^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_-^k(\phi, x^0, u)$ is satisfied.

(Sufficient Conditions) Let for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $k = k(u) \leq \nu$ such that all the derivatives $\phi_-^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_-^k(\phi, x^0, u)$ is satisfied. Then x^0 is an i -minimizer of order ν for the function f .

Like in the scalar case, we observe that the formulated in this theorem conditions work with arbitrary vector functions and the necessary and the sufficient conditions coincide, whence the gap between them is empty.

Next as applications of Theorem 4.1 we establish optimality conditions in terms of coordinate functions. In them some role play also the upper derivatives. Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a given scalar function. Let $x^0 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m \setminus \{0\}$. We define the zero order upper derivative of ϕ at x^0 in direction u by

$$\phi_+^{(0)}(x^0, u) = \limsup_{(t, u') \rightarrow (0^+, u)} \phi(x^0 + tu').$$

Further, for a given positive integer k we accept that the k -th order upper derivative $\phi_+^{(k)}(x^0, u)$ exists as an element of $\overline{\mathbb{R}}$ if and only if the derivatives $\phi_+^{(i)}(x^0, u)$, $i = 0, 1, \dots, k-1$, exist as elements of \mathbb{R} . We put then

$$\phi_+^{(k)}(x^0, u) = \limsup_{(t, u') \rightarrow (0^+, u)} \frac{k!}{t^k} \left(\phi(x^0 + tu') - \sum_{i=0}^{k-1} \frac{t^i}{i!} \phi_+^{(i)}(x^0, u) \right). \quad (5)$$

We apply also the following conditions:

$$\begin{aligned} \mathbb{S}_+^0(\phi, x^0, u) : & \quad \phi_+^{(0)}(x^0, u) > \phi(x^0), \\ \mathbb{S}_+^k(\phi, x^0, u) : & \quad \phi_+^{(0)}(x^0, u) = \phi(x^0), \quad \phi_+^{(i)}(x^0, u) = 0 \text{ for } i = 1, \dots, k-1, \text{ and } \phi_+^{(k)}(x^0, u) > 0. \end{aligned}$$

Recall that C is said polyhedral if C is an intersection of finite number of half-spaces. The cone C having non empty interior is polyhedral if and only if the set $\Gamma \cap \text{extd } C'$ is finite, where Γ is a base of C' . Theorem 4.1 gives the following result for polyhedral cones.

Theorem 4.2 *Consider the vector optimization problem (1) with polyhedral cone C . Let $x^0 \in \mathbb{R}^n$. Suppose that Γ is a base of C' .*

(Necessary Conditions) Let x^0 be an i -minimizer of order ν for the vector function f . Then for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $\xi \in \Gamma \cap \text{extd } C'$ and a positive integer $k = k(u) \leq \nu$ such that for the function $\varphi(x) = \langle \xi, x \rangle$ all the derivatives $\varphi_+^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_+^k(\varphi, x^0, u)$ is satisfied.

(Sufficient Conditions) Let for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $\xi \in \Gamma \cap \text{extd } C'$ and a positive integer $k = k(u) \leq \nu$ such that for the function $\varphi(x) = \langle \xi, x \rangle$ all the derivatives $\varphi_-^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_-^k(\varphi, x^0, u)$ is satisfied. Then x^0 is an i -minimizer of order ν for the function f .

Proof. Let $\Gamma \cap \text{extd } C' = \{\xi^1, \dots, \xi^\mu\}$ and write $\phi^j = \langle \xi^j, x \rangle$. Put $J = \{1, \dots, \mu\}$ and $\phi(x) = \max_{j \in J} \{\langle \xi^j, f(x) - f(x^0) \rangle\}$. In the proof of the necessary conditions we make use also of the set $J_k(x^0, u)$ defined for $k = 0$ by

$$J_0(x^0, u) = \{j \in J \mid (\phi^j)_+^{(i)}(x^0, u) = \phi^j(x^0)\},$$

and for $k > 0$ by

$$J_k(x^0, u) = \{j \in J_0(x^0, u) \mid (\phi^j)_+^{(i)}(x^0, u) = 0, \quad i = 1, \dots, k\}.$$

Necessity. Let x^0 be an i -minimizer of order ν . Assume on the contrary, that for some $u \in \mathbb{R}^n \setminus \{0\}$ neither of the conditions $\mathbb{S}_+^k(\phi^j, x^0, u)$, $k = 1, \dots, \nu$, $j \in J$, is satisfied.

Let $\varepsilon > 0$. Fix $j \in J$. By assumption we have $\phi_+^{j(0)}(x^0, u) \leq \phi^j(x^0)$, which gives that there exists $\delta_j > 0$ and a neighbourhood U_j of u such that

$$\langle \xi^j, f(x^0 + tu') \rangle = \phi^j(x^0 + tu') < \phi^j(x^0) + \varepsilon = \langle \xi^j, f(x^0) \rangle + \varepsilon \quad \text{for } 0 < t < \delta_j, \quad u' \in U_j,$$

or equivalently

$$\langle \xi^j, f(x^0 + tu') - f(x^0) \rangle < \varepsilon \quad \text{for } 0 < t < \delta_j, \quad u' \in U_j. \quad (6)$$

Put $\delta = \min\{\delta_1, \dots, \delta_\mu\}$ and $U = U_1 \cap \dots \cap U_\mu$. Then

$$\phi(x^0 + tu') = \max_{j \in J} \langle \xi^j, f(x^0 + tu') - f(x^0) \rangle < \varepsilon \quad \text{for } 0 < t < \delta, u' \in U,$$

whence $\phi_-^{(0)}(x^0, u) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\phi_-^{(0)}(x^0, u) \leq 0 = \phi(0)$. Therefore condition $\mathbb{S}_-^0(\phi, x^0, u)$ is not satisfied, and from the sufficient conditions of Theorem 4.1 it should be $\nu > 0$ and $\phi_-^{(0)}(x^0, u) = \phi(0) = 0$. It holds also $(\phi^j)_+^{(0)}(x^0, u) < \phi^j(x^0)$ for $j \in J \setminus J_0(x^0, u)$. This is consequence from the assumption that condition $\mathbb{S}_+^0(\phi^j, x^0, u)$ does not hold and from the definition of the set $J_0(x^0, u)$. Moreover, we have $J_0(x^0, u) \neq \emptyset$. If this were not the case, we conclude easily that there should be $\varepsilon > 0, \delta > 0$, and a neighbourhood U of u such that

$$\langle \xi^j, f(x^0 + tu') - f(x^0) \rangle \leq -\varepsilon \quad \text{for all } j \in J, 0 < t < \delta, u' \in U,$$

whence $\phi_-^{(0)}(x^0, u) \leq -\varepsilon$, a contradiction.

We prove by induction that for any $k = 0, \dots, \nu - 1$ we have $\phi_-^{(k)}(x^0, u) = 0$ and condition $\mathbb{S}_-^k(\phi, x^0, u)$ is not satisfied. Moreover, the set $J_k(x^0, u)$ is not empty, $J_k(x^0, u) \subset J_{k-1}(x^0, u)$ (for $k \geq 1$), and $(\phi^j)_+^{(k)}(x^0, u) < 0$ (for $k \geq 1$) when $j \in J_{k-1}(x^0, u) \setminus J_k(x^0, u)$.

For $k = 0$ this assertion has been proved above.

Suppose that the assertion is true for $0, \dots, k - 1$. We prove that it is true for k .

Let $\varepsilon > 0$. Fix $j \in J_{k-1}(x^0, u)$. By assumption we have $(\phi^j)_+^{(k)}(x^0, u) \leq 0$, which gives that there exists $\delta_j > 0$ and a neighbourhood U_j of u such that

$$\Delta_+^k \phi^j(x^0, u', t) > \varepsilon \quad \text{for } 0 < t < \delta_j, u' \in U_j,$$

which is equivalent to (6). For brevity here and further for a function $\varphi(x)$, $x \in \mathbb{R}^n$, we put

$$\Delta_\pm^k \varphi(x^0, u', t) = \frac{k!}{t^k} \left(\varphi(x^0 + tu') - \sum_{i=0}^{k-1} \frac{t^i}{i!} \varphi_\pm^{(i)}(x^0, u) \right).$$

Put $\delta = \min\{\delta_1, \dots, \delta_\mu\}$ and $U = U_1 \cap \dots \cap U_\mu$. Then

$$\Delta_-^k \phi(x^0, u', t) = \frac{k!}{t^k} (\phi(x^0 + tu') - \phi(x^0)) < \varepsilon \quad \text{for } j \in J_{k-1}(x^0, u), 0 < t < \delta, u' \in U.$$

Because of the inductive assumption, we see that diminishing eventually δ and U , we can guarantee the above inequality for all $j \in J$. This gives

$$\phi_-^{(k)}(x^0, u) = \liminf_{(t, u') \rightarrow (0^+, u)} \max_{j \in J} \frac{k!}{t^k} (\phi(x^0 + tu') - \phi(x^0)) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\phi_-^{(k)}(x^0, u) \leq 0$. Therefore condition $\mathbb{S}_-^k(\phi, x^0, u)$ is not satisfied.

We have $J_k(x^0, u) \subset J_{k-1}(x^0, u)$ by definition.

It holds $(\phi^j)_+^{(k)}(x^0, u) < 0$ for $j \in J_{k-1}(x^0, u) \setminus J_k(x^0, u)$, a consequence from the assumption that condition $\mathbb{S}_+^k(\phi^j, x^0, u)$ does not hold and from the definition of the set $J_k(x^0, u)$. Moreover, we have $J_k(x^0, u) \neq \emptyset$. Otherwise, like in the case $k = 0$ we would derive easily a contradiction with the condition $\mathbb{S}_-^k(\phi^j, x^0, u)$ which on the base of Theorem 4.1 should be true for some $k < \bar{k} \leq \nu$.

Thus, from the proved assertion and Theorem 4.1 we get that condition $\mathbb{S}_-^\nu(x^0, u)$ should be satisfied ($\mathbb{S}_-^k(x^0, u)$ should hold for at least one $k \leq \nu$, but the inductive assertion says that this

is not true for $k = 0, \dots, \nu - 1$). However, repeating again the reasoning from the inductive step, we see that at the same time condition $\mathbb{S}_-^\nu(x^0, u)$ cannot be satisfied, a contradiction.

Sufficiency. Suppose that condition $\mathbb{S}_-^k(\phi^j, x^0, u)$ is true. We prove that then $\mathbb{S}_-^{\bar{k}}(\phi, x^0, u)$ is true with some $\bar{k} \leq k$.

Let $k = 0$. Condition $\mathbb{S}_-^0(\phi^j, x^0, u)$ gives $(\phi^j)_-^{(0)}(x^0, u) > \phi(x^0)$. Then

$$\begin{aligned} \phi_-^{(0)}(x^0, u) &= \liminf_{(t, u') \rightarrow (0^+, u)} \max_{j \in J} \langle \xi^j, f(x^0 + tu') - f(x^0) \rangle \\ &\geq \liminf_{(t, u') \rightarrow (0^+, u)} \langle \xi^j, f(x^0 + tu') - f(x^0) \rangle = \liminf_{(t, u') \rightarrow (0^+, u)} \phi^j(x^0 + tu') - \phi^j(x^0) \\ &= (\phi^j)_-^{(0)}(x^0, u) - \phi(x^0) > 0 = \phi(x^0). \end{aligned}$$

Therefore condition $\mathbb{S}_-^{\bar{k}}(\phi, x^0, u)$ is true with $\bar{k} = 0 = k$.

Let $k > 0$. Now $(\phi^j)_-^{(0)}(x^0, u) = \phi(x^0)$ and slightly modifying the above reasoning we get $\phi_-^{(0)}(x^0, u) \geq 0 = \phi(x^0)$. If $\phi_-^{(0)}(x^0, u) > 0$ then $\mathbb{S}_-^{\bar{k}}(\phi, x^0, u)$ is true with $\bar{k} = 0 < k$. If $\phi_-^{(0)}(x^0, u) = 0$ then the following two cases could have place.

1⁰. $\phi_-^{(i)}(x^0, u) = 0$ for $i < \bar{k} \leq k$ and $\phi_-^{(\bar{k})}(x^0, u) \neq 0$. Then

$$\begin{aligned} \phi_-^{(\bar{k})}(x^0, u) &= \liminf_{(t, u') \rightarrow (0^+, u)} \frac{\bar{k}!}{t^{\bar{k}}} \max_{j \in J} \langle \xi^j, f(x^0 + tu') - f(x^0) \rangle \\ &\geq \liminf_{(t, u') \rightarrow (0^+, u)} \frac{\bar{k}!}{t^{\bar{k}}} \langle \xi^j, f(x^0 + tu') - f(x^0) \rangle \\ &= \liminf_{(t, u') \rightarrow (0^+, u)} \frac{\bar{k}!}{t^{\bar{k}}} (\phi^j(x^0 + tu') - \phi^j(x^0)) = (\phi^j)_-^{(\bar{k})}(x^0, u) \geq 0. \end{aligned}$$

Hence $\phi_-^{(\bar{k})}(x^0, u) > 0$ and condition $\mathbb{S}_-^{\bar{k}}(\phi, x^0, u)$ holds.

2⁰. $\phi_-^{(i)}(x^0, u) = 0$ for $i \leq k$. Then repeating the above reasonings, we get

$$\phi_-^{(k)}(x^0, u) \geq (\phi^j)_-^{(k)}(x^0, u) > 0.$$

The contradiction shows that this case is impossible.

Resuming, we have shown, that for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $\bar{k} = \bar{k}(u) \leq \nu$ such that condition $\mathbb{S}_-^{\bar{k}}(\phi, x^0, u)$ is true. The sufficient conditions of Theorem 4.1 are satisfied, whence x^0 is an i -minimizer of order ν . \square

In the particular case when $C = \mathbb{R}_+^m$ we get the following result, in which the necessary condition is similar to Theorem 5.1 in [13], and the sufficient conditions coincide with Theorem 5.2 in [13].

Theorem 4.3 *Consider the vector optimization problem (1) where $f = (f_1, \dots, f_m)$, and with ordering cone $C = \mathbb{R}_+^m$. Let $x^0 \in \mathbb{R}^n$.*

(Necessary Conditions) Let x^0 be an i -minimizer of order ν for the vector function f . Then for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists an index $j = j(u)$ and a positive integer $k = k(u) \leq \nu$ such that for the coordinate function $f_j(x)$ all the derivatives $(f_j)_+^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_+^k(f_j, x^0, u)$ is satisfied.

(Sufficient Conditions) Let for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists an index $j = j(u)$ and a positive integer $k = k(u) \leq \nu$ such that all the derivatives $f_{j-}^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_-^k(f_j, x^0, u)$ is satisfied. Then x^0 is an i -minimizer of order ν for the function f .

Proof. This result follows immediately from Theorem 4.2 with the account that now $C' = \mathbb{R}_+^m$, and choosing $\Gamma = \text{co}\{e^1, \dots, e^m\}$. Here $e^j = (0, \dots, 1, \dots, 0)$ (the only unit is on j -th place). \square

In Theorem 4.3 (and in some sense similarly in Theorem 4.2) the optimality conditions concern the coordinates of the vector function f . From this point of view they might look more convenient than those of Theorem 4.1. But in fact they are essentially weaker. Below Example 4.1 shows this fact for the sufficient conditions and Example 4.2 for the necessary conditions.

Example 4.1 Let $n = 1$, $m = 2$, $C = \mathbb{R}_+^2$ and $f : \mathbb{R} \rightarrow \mathbb{R}^2$ be given by

$$f(x) = \begin{cases} (-|x|, |x|), & x \in \mathbb{Q}, \\ (|x|, -|x|), & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

where \mathbb{Q} stands for the set of rational numbers. Then $x^0 = 0$ is an i -minimizer of order 1, which can be established by the sufficient conditions of Theorem 4.1, but cannot be established by the sufficient conditions of Theorem 4.3.

In this example the coordinate functions are

$$f_1(x) = \begin{cases} -|x|, & x \in \mathbb{Q}, \\ |x|, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases} \quad f_2(x) = \begin{cases} |x|, & x \in \mathbb{Q}, \\ -|x|, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We have $f(x^0) = (0, 0)$. Put $\xi^1 = (1, 0)$, $\xi^2 = (0, 1)$, and $\Gamma = \text{co}\{\xi^1, \xi^2\}$. Then the function (3) is $\phi(x) = |x|$. For each $u \in \mathbb{R} \setminus \{0\}$ we have

$$\phi_-^{(0)}(x^0, u) = 0, \quad \phi_-^{(1)}(x^0, u) = |u| > 0,$$

whence condition $\mathbb{S}_-^1(\phi, x^0, u)$ is satisfied. However for the coordinate functions we have

$$(f_1)_-^{(0)}(x^0, u) = (f_2)_-^{(0)}(x^0, u) = 0, \quad (f_1)_-^{(1)}(x^0, u) = (f_2)_-^{(1)}(x^0, u) = -|u| < 0,$$

whence neither $\mathbb{S}_-^1(f_1, x^0, u)$ nor $\mathbb{S}_-^1(f_2, x^0, u)$ is true.

Observe that we have also

$$(f_1)_+^{(0)}(x^0, u) = (f_2)_+^{(0)}(x^0, u) = 0, \quad (f_1)_+^{(1)}(x^0, u) = (f_2)_+^{(1)}(x^0, u) = |u| > 0,$$

that is both $\mathbb{S}_+^1(f_1, x^0, u)$ and $\mathbb{S}_+^1(f_2, x^0, u)$ are true. Hence for the point x^0 the function f from this example is in the gap between the necessary and the sufficient conditions of Theorem 4.3, that is neither the sufficient conditions assert nor the necessary conditions reject x^0 as an i -minimizer of order 1. With regard to this observation we find strange the discussion in [13] affirming that “the gap (between Theorems 5.1 and 5.2 in [13]) is the vertex of the cone $-\mathbb{R}_+^m$ ”.

Example 4.2 Let $n = 1$, $m = 2$, $C = \mathbb{R}_+^2$ and $f : \mathbb{R} \rightarrow \mathbb{R}^2$ be given by

$$f(x) = \begin{cases} (-|x|, -|x|), & x \in \mathbb{Q}, \\ (|x|, |x|), & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $x^0 = 0$ is not an i -minimizer of order 1 (it is not even a w -minimizer), which can be established by the necessary conditions of Theorem 4.1, but cannot be established by the necessary conditions of Theorem 4.3.

In this example $f(x^0) = (0, 0)$. The coordinate functions and the function ϕ in (3) (with $\Gamma = \text{co} \{(1, 0), (0, 1)\}$) are

$$f_1(x) = f_2(x) = \phi(x) = \begin{cases} -|x|, & x \in \mathbb{Q}, \\ |x|, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

For $u \in \mathbb{R} \setminus \{0\}$ this gives

$$\phi_-^{(0)}(x^0, u) = 0, \quad \phi_-^{(1)}(x^0, u) = -|u| < 0.$$

Thus, condition $\mathbb{S}_-^1(\phi, x^0, u)$ fails, whence from the necessary conditions of Theorem 4.1 (for $\nu = 1$) we can reject x^0 as an i -minimizer of order 1. This cannot be done on the base of Theorem 4.3. Indeed,

$$(f_1)_+^{(0)}(x^0, u) = (f_2)_+^{(0)}(x^0, u) = 0, \quad (f_1)_+^{(1)}(x^0, u) = (f_2)_+^{(1)}(x^0, u) = |u| > 0,$$

whence both conditions $\mathbb{S}_+^1(f_1, x^0, u)$ and $\mathbb{S}_+^1(f_2, x^0, u)$ are satisfied.

Observe that for the point x^0 also the function f from this example belongs to the gap between the necessary and the sufficient conditions of Theorem 4.3 (and in the gap between Theorems 5.1 and 5.2 in [13]).

5 Optimality conditions for regular functions

Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a given scalar function. For $u \in \mathbb{R}^n \setminus \{0\}$ we define the zero order directional derivative of ϕ at x^0 in direction u by

$$\phi^{(0)}(x^0, u) = \lim_{(t, u') \rightarrow (0^+, u)} \phi(x^0 + tu')$$

provided the limit exists as an element of $\overline{\mathbb{R}}$. By induction, for a given positive integer k we define the k -th order directional derivative as an element of $\overline{\mathbb{R}}$ by

$$\phi^{(k)}(x^0, u) = \lim_{(t, u') \rightarrow (0^+, u)} \frac{k!}{t^k} \left(\phi(x^0 + tu') - \sum_{i=0}^{k-1} \frac{t^i}{i!} \phi^{(i)}(x^0, u) \right),$$

provided the derivatives $\phi^{(i)}(x^0, u)$, $i < k$, exist as elements of \mathbb{R} and the limit exists as an element of $\overline{\mathbb{R}}$. It is clear, that $\phi^{(k)}(x^0, u)$ exists if and only if exist and coincide the derivatives $\phi_-^{(i)}(x^0, u) = \phi_+^{(i)}(x^0, u)$, $i \leq k$, and then it coincides with the common value of the of the lower and upper derivative of order k .

With the scalar function ϕ for which $\phi^{(k)}(x^0, u)$ exists we associate condition $\mathbb{S}^k(\phi, x^0, u)$ introduced as follows:

$$\begin{aligned} \mathbb{S}^0(\phi, x^0, u) : & \quad \phi^{(0)}(x^0, u) > \phi(x^0), \\ \mathbb{S}^k(\phi, x^0, u) : & \quad \phi^{(0)}(x^0, u) = \phi(x^0), \quad \phi^{(i)}(x^0, u) = 0 \text{ for } i = 1, \dots, k-1, \text{ and } \phi^{(k)}(x^0, u) > 0. \end{aligned}$$

Let ν be a non negative integer. We call the scalar function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ν -regular at $x^0 \in \mathbb{R}^n$ if for any $u \in \mathbb{R}^n \setminus \{0\}$ there exist all the derivatives $\phi^{(i)}(x^0, u)$, $i \leq \nu$. We call the vector function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ν -regular at $x^0 \in \mathbb{R}^n$ if the scalar functions $\langle \xi, f(x) \rangle$ are ν -regular for all $\xi \in \text{extd } C'$.

For ν -regular functions Theorem 4.2 gives the following result.

Theorem 5.1 Consider the vector optimization problem (1) with polyhedral cone C . Let $x^0 \in \mathbb{R}^n$ and suppose that f is ν -regular at x^0 , where ν is a non negative integer. Suppose also that Γ is a base of C' . Then x^0 be an i -minimizer of order ν for the vector function f if and only if for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $\xi \in \Gamma \cap \text{extd } C'$ and a positive integer $k = k(u) \leq \nu$ such that for the function $\varphi(x) = \langle \xi, x \rangle$ condition $\mathbb{S}^k(\varphi, x^0, u)$ is satisfied.

Proof. If the vector function $f(x)$ is ν -regular, so are the scalar functions $\varphi(x) = \langle \xi, x \rangle$, $\xi \in \Gamma \cap \text{extd } C'$. Therefore conditions $\mathbb{S}_-^k(\varphi, x^0, u)$ and $\mathbb{S}_+^k(\varphi, x^0, u)$ coincide with $\mathbb{S}^k(\varphi, x^0, u)$, and the thesis is an immediate reformulation of Theorem 4.2. \square

Also Theorem 4.3 for ν -regular functions admits a reformulation.

Theorem 5.2 Consider the vector optimization problem (1) with ν -regular vector function $f = (f_1, \dots, f_m)$, and with ordering cone $C = \mathbb{R}_+^m$.

Then $x^0 \in \mathbb{R}^n$ is an i -minimizer of order ν for the vector function f if and only if for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists an index $j = j(u)$ and a positive integer $k = k(u) \leq \nu$ such that for the coordinate function $f_j(x)$ condition $\mathbb{S}^k(f_j, x^0, u)$ is satisfied.

Since the necessary and sufficient conditions in Theorems 5.1 and 5.2 coincide, the gap at x^0 between them (in the class of ν -regular functions) is empty.

6 Problems with non polyhedral cones

In this section we discuss whether Theorems 4.2 is true for arbitrary and not only for polyhedral cones (saying arbitrary cone we mean within the general assumptions a closed convex cone with nonempty interior).

The sufficient conditions of Theorem 4.2 are generalized for arbitrary cones immediately.

Theorem 6.1 Consider the vector optimization problem (1) with arbitrary cone C . Let $x^0 \in \mathbb{R}^n$ and Γ be a base of C' . Suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $\xi \in \Gamma \cap \text{extd } C'$ and a positive integer $k = k(u) \leq \nu$ such that for the function $\varphi(x) = \langle \xi, x \rangle$ all the derivatives $\varphi_-^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_-^k(\varphi, x^0, u)$ is satisfied. Then x^0 is an i -minimizer of order ν for the function f .

Proof. Actually the proof of the sufficient conditions of Theorem 4.2 applies for arbitrary and not only for polyhedral cone. \square

The necessary conditions of Theorem 4.2 are generalized for arbitrary cones only in special cases.

Theorem 6.2 Consider the vector optimization problem (1) with arbitrary cone C . Suppose that Γ is a base of C' . Let $x^0 \in \mathbb{R}^n$ be an i -minimizer of order ν for f , and let there exist constants m and ρ such that

$$\|f(x) - f(x^0)\| \leq m\|x - x^0\|^\nu \quad \text{for } 0 \leq \|x - x^0\| \leq \rho. \quad (7)$$

Then for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $\xi \in \Gamma \cap \text{extd } C'$ and a positive integer $k = k(u) \leq \nu$ such that for the function $\varphi(x) = \langle \xi, x \rangle$ all the derivatives $\varphi_+^{(i)}(x^0, u)$, $i = 0, \dots, k$, exist and condition $\mathbb{S}_+^k(\varphi, x^0, u)$ is satisfied.

Proof. Let $0 < \alpha < 1$. Consider the cone $C(\alpha) = \{y \in \mathbb{R}^m \mid d(y, C) \leq \alpha \|y\|\}$ where $d(y, C) = \{\|y - c\| \mid c \in C\}$ is the distance from y to C . We make the following observations.

1⁰. *There exists a finite set $\Xi \in \Gamma_0 := \Gamma \cap \text{extd } C'$ such that $C_\Xi \subset C(\alpha)$.*

Define the sets $S = \{y \in \mathbb{R}^n \mid \|y\| = 1\}$, $F = C \cap S$, and $G = \{y \in \mathbb{R}^m \mid d(y, F) < \alpha\}$. For $\xi \in C'$ define also the half-space $H_\xi = \{y \in \mathbb{R}^m \mid \langle \xi, y \rangle \geq 0\}$. Since $C = \bigcap \{H_\xi \mid \xi \in \Gamma_0\}$, for the complements we have $C^c = \bigcup \{H_\xi^c \mid \xi \in \Gamma_0\}$. Therefore $S \subset G \cup \bigcup \{H_\xi^c \mid \xi \in \Gamma_0\}$. Since the unit sphere S is compact and the sets on the right hand side of this inclusion are open, there exists a finite set $\Xi \subset \Gamma_0$ such that $S \subset G \cup \bigcup \{H_\xi^c \mid \xi \in \Xi\}$, or in compliments $(S \setminus G) \cap \bigcap \{H_\xi \cap S \mid \xi \in \Xi\} = \emptyset$. Therefore $\bigcap \{H_\xi \cap S \mid \xi \in \Xi\} \subset G$. Taking the conic hull we get $C_\Xi = \bigcap \{H_\xi \mid \xi \in \Xi\} \subset \text{cone } G \subset C(\alpha)$.

2⁰. *For some α , $0 < \alpha < 1$, the point x^0 is an i -minimizer of order ν for the function f minimized with the cone $C(\alpha)$.*

Since x^0 is an i -minimizer of order ν , diminishing eventually ρ , we can find $a > 0$ for which

$$d(f(x^0) - f(x), C) > a \|x - x^0\|^\nu \quad \text{for } 0 < \|x - x^0\| \leq \rho.$$

Now we see that

$$\inf \left\{ \frac{d(f(x^0) - f(x), C)}{\|f(x^0) - f(x)\|} \mid 0 < \|x - x^0\| \leq \rho \right\} \geq \frac{a}{m}.$$

This inequality shows that $f(x^0) - f(x) \notin C(a/m)$. Let $\alpha, \beta > 0$ and $\alpha + \beta = a/m$. From the inclusion $C(\alpha)(\beta) \subset C(\alpha + \beta)$ (see e. g. [4], Lemma1, p. 93) it follows that

$$f(x^0) - f(x) \notin C(\alpha)(\beta),$$

or equivalently

$$d(f(x^0) - f(x), C(\alpha)) \geq \beta \|f(x^0) - f(x)\| > \beta a \|x - x^0\|^\nu \quad \text{for } 0 < \|x - x^0\| \leq \rho.$$

This shows that x^0 is an i -minimizer of order ν for f minimized with the cone $C(\alpha)$.

3⁰. *The thesis is true.*

According to 2⁰ there exists $\alpha > 0$ such that the point x^0 is an i -minimizer of order ν for f minimized with the cone $C(\alpha)$. According to 1⁰ there exists a finite set $\Xi \subset \Gamma \cap \text{extd } C'$ such that $C_\Xi \subset C(\alpha)$. This inclusion implies that x^0 is also an i -minimizer of order ν for f minimized with C_Ξ . The cone C_Ξ is however polyhedral. Applying now necessary conditions of Theorem 4.2 we get the thesis. \square

The following example shows that without condition (7) Theorem 6.2 is not true.

Example 6.1 *Let $n = 1$, $m = 3$,*

$$C = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_3^2 \geq y_1^2 + y_2^2, y_3 \leq 0\},$$

and let $f : \mathbb{R} \rightarrow \mathbb{R}^3$ be given by $f(x) = (x \cos x, x \sin x, x - \gamma x^3)$ where $0 < \gamma < 1/2$ is a constant. Then $x^0 = 0$ is an i -minimizer of order 3. But neither of the conditions $\mathbb{S}_+^i(x^0, 1)$, $i = 0, 1, 2, 3$ has place.

Here we have $f(x^0) = (0, 0, 0)$ and $d(f(x^0) - f(x), C) \geq \gamma |x|^3 / \sqrt{2}$ (for $x \geq 0$ we have an equality), whence x^0 is an i -minimizer of order 3.

We have $C' = C$. The cone C' has a base $\Gamma = \{y \in \mathbb{R}^3 \mid y_1^2 + y_2^2 \leq 1, y_3 = -1\}$ for which

$$\Gamma \cap \text{extd } C' = \{y \in \mathbb{R}^3 \mid y_1^2 + y_2^2 = 1, y_3 = -1\} = \{(\cos \alpha, \sin \alpha, -1) \mid -\pi < \alpha \leq \pi\}.$$

For the function $\varphi(x) = \langle \xi, f(x) \rangle$, where $\xi = (\cos \alpha, \sin \alpha, -1) \in \Gamma \cap \text{extd } C'$ we get

$$\varphi(x) = x \cos(x - \alpha) - x + \gamma x^3.$$

We have

$$\begin{aligned} \varphi_+^{(0)}(x^0, 1) &= \varphi^{(0)}(x^0, 1) = 0 = \varphi(x^0), \\ \varphi_+^{(1)}(x^0, 1) &= \varphi^{(1)}(x^0, 1) = \varphi'(0) = \cos \alpha - 1 \leq 0. \end{aligned}$$

These equalities show that neither condition $\mathbb{S}_+^0(x^0, 1)$ nor $\mathbb{S}_+^1(x^0, 1)$ is satisfied. The second equality shows that for $\alpha \neq 0$ the inequality is strict, hence also neither conditions $\mathbb{S}_+^2(x^0, 1)$ nor $\mathbb{S}_+^3(x^0, 1)$ can have place. For $\alpha = 0$ we have

$$\begin{aligned} \varphi_+^{(2)}(x^0, 1) &= \varphi^{(2)}(x^0, 1) = \varphi''(0) = 0, \\ \varphi_+^{(3)}(x^0, 1) &= \varphi^{(3)}(x^0, 1) = \varphi'''(0) = 6 \left(\gamma - \frac{1}{2} \right) < 0. \end{aligned}$$

These equalities show that neither condition $\mathbb{S}_+^2(x^0, 1)$ nor $\mathbb{S}_+^3(x^0, 1)$ is satisfied.

For a regular function Theorems 6.1 and 6.2 give immediately the following result.

Theorem 6.3 *Consider the vector optimization problem (1) with arbitrary cone C . Let $x^0 \in \mathbb{R}^n$ and suppose that f is ν -regular at x^0 , where ν is a non negative integer. Suppose also that Γ is a base of C' . Then in order that x^0 be an i -minimizer of order ν for the vector function f it is sufficient, and under condition (7) also necessary, that for each $u \in \mathbb{R}^n \setminus \{0\}$ there exists $\xi \in \Gamma \cap \text{extd } C'$ and a positive integer $k = k(u) \leq \nu$ such that for the function $\varphi(x) = \langle \xi, x \rangle$ condition $\mathbb{S}^k(\varphi, x^0, u)$ is satisfied.*

The sufficient conditions of Theorem 6.3 state a bit more general assertion than Theorem 4.1 in [13], the latter concern differentiable of order ν in sense of [13] functions. Actually the vector function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable of order ν in sense of [13] if it is ν -regular and for all $\xi \in \Gamma \cap \text{extd } C'$ the function $\varphi(x) = \langle \xi, f(x) \rangle$ has a finite derivative $\varphi^{(\nu)}(x^0, u)$ (the finiteness does not occur in Theorem 6.3).

The necessary conditions of Theorem 6.3 are similar to that of Theorem 3.1 in [13] for differentiable of order ν in sense of [13] functions. Theorem 3.1 in [13] concerns efficiency, that is w -minimizers. It states that in order that x^0 be a w -minimizer for such a function it is necessary that for all $u \in \mathbb{R}^n \setminus \{0\}$ and all $\xi \in \Gamma \cap \text{extd } C'$ the function $\varphi(x) = \langle \xi, f(x) \rangle$ satisfies conditions $\mathbb{N}^i(\varphi, x^0, u)$, $i = 0, \dots, n$. These conditions are defined as follows:

$$\begin{aligned} \mathbb{N}^0(\varphi, x^0, u) : & \quad \varphi^{(0)}(x^0, u) \geq \varphi(x^0), \\ \mathbb{N}^k(\varphi, x^0, u) : & \quad \text{If } \varphi^{(0)}(x^0, u) = \varphi(x^0) \text{ and } \varphi^{(i)}(x^0, u) = 0, \text{ for } i = 1, \dots, k-1, \\ & \quad \text{then } \varphi^{(k)}(x^0, u) \geq 0. \end{aligned}$$

Observe, that condition of type (7) does not occur in the necessary conditions for w -minimizers. Here we gave the conditions $\mathbb{N}^k(\varphi, x^0, u)$ to introduce some imagination about the manner in which the proved in this paper necessary conditions should be changed, when w -minimizers instead of i -minimizers are considered.

7 Comments

We accept the name of an *isolated minimizers* after Auslender [1] where scalar problems are considered. In Ginchev [7] the concept is generalized under the same name for vector problems. In the literature this concept is known also under other names, for instance *strictly efficient point* for vector problems.

Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a scalar function, $x^0 \in \mathbb{R}^n$, $u \in \mathbb{R}^n \setminus \{0\}$. In Demyanov, Rubinov [5] the derivatives

$$\phi'_H(x^0, u) = \lim_{(t, u') \rightarrow (0^+, u)} \frac{1}{t} (f(x^0 + tu') - f(x^0)) , \quad \phi'_D(x^0, u) = \lim_{t \rightarrow 0^+} \frac{1}{t} (f(x^0 + tu) - f(x^0))$$

are called respectively Hadamard and Dini derivatives (obviously the same name is attributed to the respective lower and upper derivative). The type of passing to a limit is what distinguishes the two derivatives. We try to keep the name as far as possible also for other type derivatives defined with the help of the respective convergence.

Passing to higher orders we get more possibilities to generalize a derivative. For instance the Hadamard second-order derivative can be generalized in the following ways called here Hadamard derivatives of type I and type II respectively:

$$\begin{aligned} \phi''_{H_1}(x^0, u) &= \lim_{(t, u') \rightarrow (0^+, u)} \frac{2}{t^2} (f(x^0 + tu') - f(x^0) - t \phi'_{H_1}(x^0, u)) , \\ \phi''_{H_2}(x^0, u) &= \lim_{(t, u') \rightarrow (0^+, u)} \frac{2}{t^2} (f(x^0 + tu') - f(x^0) - t \phi'_{H_2}(x^0, u')) . \end{aligned}$$

The essential difference between the two of them are the directions u and u' respectively used by the first-order derivative.

The Hadamard derivatives of type I is used in Ginchev [6] to derive higher-order optimality conditions. They possess the remarkable property that in their terms optimality conditions for quite arbitrary functions can be established. An attempt to generalize the results of [6] to vector functions is undertaken in Ginchev [7], and thereafter in Luu, Kien [13]. The first of these works deals only with the positive orthant as ordering cone. Both these works demonstrate some difficulties when a vector optimization problem is attempted to be treated directly. As we see in [13] the arbitrariness of the optimized functions has been lost. The present paper, treating the problems through scalarization, restores this feature (see Theorem 4.1 which works for quite arbitrary vector functions).

The second (and higher) order Hadamard derivatives of type I show some inconsistency with the classical derivatives, see comments in [6] and [12]. For this reason the author had abandoned the idea for further development, toward constrained problems for instance. The return to the subject was caused by the paper [13]. Actually [6] can be considered as a certain scheme to formulate optimality conditions. In Ginchev, Guerraggio [8] this scheme within second-order theory has been stated in a generic form, and as concrete applications the optimality conditions of Ben-Tal, Zowe [2] and Chaney [3] are compared. While the Hadamard derivatives of type I show inconsistency with the classical derivatives, this is not the case of Hadamard derivatives of type II. They are introduced in Studniarski [14] but thereafter are not consequently used there for the stated optimality conditions. So, in the author's opinion it is an open question to determine the larger class of scalar functions for which the generic optimality conditions scheme works with Hadamard derivatives of type II, and to look further for vector generalizations. As for the Dini derivatives, as several works on vector optimization of Ginchev, Guerraggio, Rocca show [9], [12], [10], [11], both for unconstrained and for constrained problems, optimality conditions of order ν should work well with the class of $C^{\nu-1,1}$ functions (and may be also with more general classes).

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