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Rate of Arbitrage and Reconciled Beliefs

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Abstract

Any group of risk neutral agents who hold differing beliefs is vulnerable to money pumps (arbitrage). Thus, the agents may wish to reconcile their beliefs into a new joint belief. We propose a criterion for the choice of reconciled belief based on the notion of ``rate of arbitrage." It is shown that there exists a unique belief (probability distribution) that minimizes the maximal expected rate of arbitrage, and an explicit formula for this belief is given.

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1 Introduction

When economic agents possess subjective beliefs over events – sets of states of the world – it is usually assumed that agents' beliefs can be represented by (finitely additive) probability distributions. One of the main justification for this assumption is that any agent whose beliefs do not correspond to a probability distribution can be subjected to a Dutch Book, as was observed by de Finetti (1937).

Even when agents have subjective probability distributions, difficulties may arise from the fact that, axiomatically, no condition is imposed on the *relation* between them, i.e. beliefs are truly subjective. In particular, beliefs may differ when information is public, or be inconsistent in the case of private information (Harsanyi (1967/68)). Differing or inconsistent beliefs cause an effect similar to the Dutch Book effect in the one agent case. As was recently shown by Morris (1994), consistency is necessary for the no-trade result (Milgrom and Stokey (1982) have shown it to be sufficient, based on Aumann (1976)), and, as is shown in Feinberg (2000), inconsistency is essentially equivalent to the existence of a money pump (see also Bonanno and Nehring (1999)) for a formulation in the context of games with incomplete information, and Samet (1998)).

In economic analysis, it is common to assume that beliefs are consistent, i.e., that they originate from a common prior. In situations where there is no private information this simply amounts to assuming that all agents' beliefs are equal. However, it is widely recognized that this assumption might be too restrictive (see Morris (1995) for a discussion of the common prior assumption).

In this paper we take the following approach to the issue: Instead of looking at how beliefs are created we look at how beliefs may change. When considering a change of beliefs the first thing that comes to mind is Bayesian updating, that can only result from new information. Here we will assume that agents may change their beliefs as a result of things other than information. In particular, an agent might change her belief when she enters a market where other agents have differing beliefs (even if she already knew their beliefs before entering the market). Thus agents may wish to change their beliefs when they face a money pump. This approach departs from the standard prior-to-posterior subjective updating, and allows the agents to directly reconcile their beliefs, without information-driven merging (Blackwell and Dubins (1962), Kalai and Lehrer (1993a,b, 1994, 1995), Lehrer and Smorodinsky (1996)). We do not assume that the fact that other agents have differing beliefs constitutes new information for an agent, nor do we deviate from the standard assumption that the knowledge-belief system is common knowledge. We simply assume that no-arbitrage considerations may induce a change in beliefs, even when no new information is introduced. This approach differs from the mainstream literature on Bayesian opinion pooling, where the aim is often to extract information from the different opinions of various experts (see e.g. Genest and Zidek (1986), Dawid, DeGroot, and Mortera (1995)), and also differs from dealing with the issue of arbitrage in the Bayesian framework via the assumption that agents simply make mistakes or face unforeseen contingencies (see Geanakoplos (1989)).

Since this problem occurs even if no asymmetric information issues arise, we concentrate on the more tractable case where agents have differing beliefs over the set of states of the world and have no private information.

The main assumption of this study is that agents *will* reconcile their beliefs. Since we depart from the standard rules of belief updating, there is a priori no constraint on the choice of the new joint belief. We believe that this choice should be based on the reason for reconciling beliefs, i.e., on the existence and nature of money pumps. Thus, we offer a criterion that determines the reconciled belief as a direct function of the money pumps that the agents may face with their original differing beliefs. We base our procedure for reconciling beliefs on the notion of "rate of arbitrage." The rate of arbitrage (with respect to a belief p) is defined as the maximal expected (with respect to p) normalized return that an outsider can extract from the disagreeing risk neutral agents by selling them securities. Our criterion requires that the reconciled belief minimize this rate of arbitrage. The interpretation of this criterion is that the agents will adopt a joint reconciled belief that minimizes the maximal expected rate at which money could be pumped out of them, had they not reconciled their beliefs. The main result of this study is that there exists a unique belief that satisfies this criterion. An explicit formula for this optimal reconciled belief will be given. The idea of using a rate of loss or a rate of profit to measure incoherence of beliefs appears in a one-agent framework in Schervish, Seidenfeld, and Kadane (1998, 2000, 2002). It has been used as a tool to aggregate imprecise probabilities by Nau (2002).

In addition to the explicit assumption of risk neutrality we implicitly assume that the agents with differing beliefs will seek to reconcile their beliefs *before* an arbitrage is executed, or, alternatively, before they bet with each other. It is the mere fact that beliefs differ that causes them to abandon their current beliefs. There is an alternative motivation for belief reconciliation of this sort. Namely, assume that the agents serve as consultant to a third party, this third party, whom we shall call a principal, has *no* prior belief at all. The principal wishes to form a belief based solely on the beliefs presented by the agents (serving as consultants). Our criterion suggest that the principal should chose the reconciled belief that minimizes the maximal exploitation of the differing beliefs – yielding a criterion for the cost of the principals mistake. Since in this case the principal plays only a conceptual role, in the rest of the sequel we will only consider the agents with differing beliefs.

The following example sheds some light on the notion of rate of arbitrage and on the criterion used for selecting the reconciled belief. Consider two agents, Alice and Bob, and a state space with two states of the world ω_1, ω_2. Assume that Alice assigns probabilities $1/3, 2/3$ to these states, while Bob assigns probabilities $1/2, 1/2$ respectively. Further assume that this is common knowledge, i.e., this is a complete description of all possible indeterminacies, and Alice and Bob have no private information. Assume that both agents are risk neutral and that the true state of the world will be revealed in the future. In this case a third party, e.g. Carol, can take advantage of these differing beliefs. Carol can sell to Alice a security yielding \$1 if the true state turns out to be ω_2, for the price of \$2/3, and can sell to Bob a security yielding \$1 if the true state is ω_1 for the price of \$1/2. Thus, Carol commits herself to pay exactly \$1 at every state of the world (for sure), but she receives \$7/6 for this commitment: Carol can construct a money pump. Moreover, Carol can extract for sure \$1/6 per each dollar she commits to pay.

Alice and Bob are aware of this situation. They know that they agree to disagree. Thus, there is no room for Bayesian updating, nor do we extend the model to include computational errors or restricted epistemic inference. Alice and Bob simply have *different* subjective beliefs. Recognizing their vulnerability to money pump manipulation, we suggest that Alice and Bob may wish to reconcile their beliefs. Since there is no Bayesian criteria for belief change in this situation, we need to construct a different belief-change mechanism. As we mentioned above, the criterion we suggest for choosing the new joint belief is based on the rate of arbitrage – the maximal expected rate of a money pump. Assume that Alice and Bob change their beliefs to the joint belief $1/2, 1/2$. How will they value an ex-ante arbitrage rate (against their original beliefs) when viewed by their new (ex-post) joint belief? Assume that Carol has sold to Alice a security yielding \$1 if the true state turns out to be ω_2, for the price of \$2/3, and has sold to Bob a security yielding \$3/4 if the true state is ω_1, for the price of \$3/8. Then she would have received \$25/24, but her *expected* commitment according to the new (ex post) joint belief would have been \$7/8, so her rate of arbitrage, according to the new joint belief would have been $(25/24 - 7/8)/(7/8) = 4/21$, i.e., the expected gain per dollar of commitment is more than what she could have previously assured herself. But it turns out that if Alice and Bob adopt the joint belief $3/7, 4/7$, this assures that the ex post expected rate of the ex ante arbitrage will never exceed $1/6$ (which is what Carol can apriori assure), and this is the unique joint belief with this property.

We do not claim that this is the *right* joint reconciled belief. However, it seems plausible to base the criterion for selecting the new belief on the source of dissatisfaction of the previous differing beliefs.

Our mechanism of belief reconciliation amounts to solving a zero-sum two-person game. A similar approach was considered in the case of one agent by Heath and Sudderth (1972).

The idea of extending de Finetti's no-arbitrage mechanism to a multi-agent framework has been pursued by Nau and McCardle (see Nau and McCardle (1990, 1991), Nau (1992, 1995)). By building on an idea of joint coherence they justify the adoption of a common prior and correlated equilibria. Our analysis differs from theirs in that we assume that the agents start with some (different) subjective probabilities, which are reconciled to avoid money pumps. Our reconciled belief is derived directly from the pre-existing subjective probabilities, which do not exist in Nau and McCardle's framework.

The absence of arbitrage is also central in the modern theory of finance, which implies that agents in financial markets do not use their subjective probabilities (even when they coincide) to evaluate contingent claims, but rather a risk-neutral probability measure whose only relation with the original one is that they have the same null sets. Relations between arbitrage price theory and subjective probability theory have been emphasized by Clark (1993). In our approach the relation between the original probabilities and the reconciled belief is stronger than the relation between the true probability and the risk-neutral measure of mathematical finance. We point out, though, that Platen and Rebolledo (1996) introduce a principle of minimization of increase of arbitrage information, which bears interesting analogies with our idea.

2 Defining the rate of arbitrage and reconciling beliefs

Consider a finite state space $\Omega = \{\omega_1, \ldots, \omega_m\}$ and a finite set $N = \{1, \ldots, n\}$ of agents who have the beliefs $p_1, \ldots, p_n$ over the state space. Assume that all agents are risk neutral and that there is no private information. A *security* f is a real function on Ω. A *portfolio* is defined as an n-tuple $(f_1, \ldots f_n)$ of nonnegative securities. We denote the set of portfolios by $\mathcal{F}$. An outsider can sell a portfolio to the agents, i.e., can sell f_i to agent i, for all $i \in N$. The *return* of a portfolio $\mathbf{f} = (f_1, \ldots f_n)$ at state ω is defined as

$$R(\mathbf{f}, \omega) = \sum_{i \in N} \left[\left(\sum_{\overline{\omega} \in \Omega} f_i(\overline{\omega}) p_i(\overline{\omega}) \right) - f_i(\omega) \right],$$

i.e., it is the sum of the prices the agents are willing to pay for their corresponding securities minus the actual yield of the securities at state ω.

If the outsider sells the portfolio $(f_1, \ldots, f_n)$, she undertakes a commitment. Implicitly it is assumed that the sale is done before the true state of the world ω is revealed, and that after the revelation the outsider is required to make her committed payments, i.e., pay $f_i(\omega)$ to agent i, for all $i \in N$.

Assume that the agents change their beliefs to a new belief $p \in \Delta^\Omega$, where Δ^Ω is the set of probability measures on Ω. How will they evaluate the outsider's return had the sale of the portfolio been carried out? According to a new belief p the expected return from $\mathbf{f} = (f_1, \ldots, f_n)$ is defined as

$$ER(\mathbf{f}, p) = \sum_{\omega \in \Omega} R(\mathbf{f}, \omega) p(\omega).$$

i.e., it is the expected gain from the sale of $\mathbf{f}$, according to the belief p.

One may assume that underlining the construction is the fact that the outsider has to post a collateral against her commitments, or to compare the gain from selling the portfolio to other possible investments. In any case we assume that measuring and comparing portfolios and joint beliefs has to be done on a *normalized* basis. Thus the *normalized expected return* from a portfolio $\mathbf{f}$ according to the belief p is defined as follows

$$\overline{ER}(\mathbf{f}, p) = \begin{cases} 0, & \text{if } ER(\mathbf{f}, p) = 0, \\ \dfrac{ER(\mathbf{f}, p)}{\sum_{i \in N} \sum_{\omega \in \Omega} f_i(\omega) p(\omega)}, & \text{otherwise.} \end{cases}$$

Note that if $ER(\mathbf{f}, p) > 0$ and $\sum_{i \in N} \sum_{\omega \in \Omega} f_i(\omega) p(\omega) = 0$ then $\overline{ER}(\mathbf{f}, p) = \infty$. Furthermore, the normalized expected return $\overline{ER}$ satisfies

$$\overline{ER}((\alpha f_1, \ldots, \alpha f_n), p) = \overline{ER}((f_1, \ldots, f_n), p)$$

for all $\alpha > 0$, for all $(f_1, \ldots, f_n)$, and for all p. This property allows us to compare securities in a bounded environment. Finally, notice that $\overline{ER}$ is not necessarily continuous.

We, or the agents in N, assume that the outsider's objective is to maximize the normalized expected return, given a belief p, i.e., the objective function is

$$\sup_{\mathbf{f} \in \mathcal{F}} \overline{ER}(\mathbf{f}, p).$$

We call this (finite or infinite) quantity the *rate of arbitrage with respect to p.*

Since the agents have no private information, the existence of an arbitrage (money pump) opportunity has no Bayesian effect on their beliefs. However, it seems plausible that all the agents may consider revising their beliefs. This change of beliefs does not stem from an informational argument, and there is no reason to assume that any of the agents made a mistake. We might simply be facing a situation of different subjective beliefs.

We suggest that the agents may choose a new joint belief that relates to their previous beliefs by minimizing the previous arbitrage opportunity when viewed from the new belief point of view. Thus, we assume that the agents will adopt a new belief $p \in \Delta^\Omega$ that minimizes the maximal expected normalized yield of an arbitrage against their previous beliefs. The new joint belief will be used to measure the expected rate of arbitrage made against the agents' previous differing beliefs. It is the fundamental assumption of this paper that the agents choose a joint belief in this way. Their target is therefore

$$r := \inf_{p \in \Delta^\Omega} \sup_{\mathbf{f} \in \mathcal{F}} \overline{ER}(\mathbf{f}, p).$$

If r is finite, then we call it the *rate of arbitrage*. If there exist a rate of arbitrage r and a belief p^* such that

$$\sup_{\mathbf{f} \in \mathcal{F}} \overline{ER}(\mathbf{f}, p^*) = r,$$

we say that p^* is an *optimal reconciled belief*, i.e., when such a belief is adopted, no more than the rate of arbitrage can be obtained from any portfolio.

3 The existence and uniqueness of an optimal reconciled belief

In this section we show that the rate of arbitrage always exists and that it is positive if and only if the beliefs of the agents are not all equal, i.e., they were not reconciled, to begin with. We prove that an optimal reconciled belief exists and moreover it is unique (in particular, if all the original beliefs are equal then it coincides with them). We give an explicit expression for the rate of arbitrage, the unique optimal reconciled belief and an optimal portfolio for the outsider, all as a function of the agents' original beliefs.

The existence and uniqueness of the reconciled beliefs demonstrates that our criterion for reconciliation of beliefs is decisive. Our results are summarized in the following theorem.

Theorem 3.1. *The rate of arbitrage r exists and it is given by*

$$r = \left[\sum_{\omega \in \Omega} \max_{i \in N} p_i(\omega) \right] - 1.$$

An optimal reconciled belief p^* exists and it is given by

$$p^*(\omega) = \frac{\max_{i \in N} p_i(\omega)}{r+1},$$

p^* is a unique *reconciled belief in the sense that for every $p \neq p^*$ there exists a portfolio $\mathbf{f}^p \in \mathcal{F}$ such that*

$$\overline{ER}(\mathbf{f}^p, p) > r = \sup_{\mathbf{f} \in \mathcal{F}} \overline{ER}(\mathbf{f}, p^*).$$

There exists a portfolio $\mathbf{f}^*$ which always yields the rate of arbitrage, i.e.,

$$\overline{ER}(\mathbf{f}^*, p) = r, \qquad \forall p \in \Delta^\Omega.$$

Note that what the theorem states is that the zero-sum-game in which player I chooses a portfolio $\mathbf{f}$, player II chooses a belief p, and then payoffs are given by $\overline{ER}(\mathbf{f}, p)$, has a value and a unique optimal strategy for player II.

4 Conclusion

The main purpose of this study was to provide a criterion for a change of beliefs when disagreement prevails. We feel that the strengths of the criterion presented here is its direct formulation in terms of a major problematic feature of differing beliefs: The existence of a money pump.

In order to emphasize that no informational issues are a priori involved when analyzing differing beliefs, we made the assumption that there is no private information.

One possible extension of the analysis presented here is the case where agents have asymmetric private information. As shown by Morris (1994), the existence of a money pump or trade is a consequence of the lack of a common prior. Thus one can use the rate of arbitrage to construct a reconciled common prior.

A limiting property of our formulation is the assumption that the arbitrage is generated by an outsider who is selling only nonnegative securities. Other plausible alternative formulations surely exist. However the goal of this study is to promote the mere consideration of criteria for reconciling beliefs rather than advocate solely the particular criterion presented here – plausible as it may be.

A natural extension of our formulation is to an infinite state space. Other than some technical difficulties we see no major difference in the essence of the problem or in the nature of the results as long as there is no private information. The general asymmetric case could pose other difficulties because the converse to Aumann (1976) theorem need not hold (see Feinberg (2000)).

We conclude by presenting a possible extension of this study based on the addition of "objective" probabilities (probabilities that all the agents agree on and would not want to change): These act as constraints to our formulation. Even if all agents agree on the probability of a certain state or event, the optimal reconciled belief may give it a different value. This suggests that one may wish to consider the following variation of the model. One can impose that for some states or events, in a given situation, the value of a reconciled belief should be fixed a priori. Such constraints would emerge if one considered the probabilities of some events to be "objective" probabilities. The results generated by these constraints might have different qualitative properties.

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A Appendix

Proof of Theorem 3.1. We begin by showing that

$$\inf_{p\in\Delta^\Omega}\sup_{\mathbf{f}\in\mathcal{F}}\overline{ER}(\mathbf{f},p)$$

exists and is equal to

$$r=\left[\sum_{\omega\in\Omega}\max_{i\in N}p_i(\omega)\right]-1,$$

and that there exists a portfolio $\mathbf{f}^*$ such that $\overline{ER}(\mathbf{f}^*,p)\geq r$ for all $p\in\Delta^\Omega$.

Let χ_E denote the characteristic function the set E, i.e.,

$$\chi_E(\omega)=\begin{cases}0, & \omega\notin E,\\ 1, & \omega\in E,\end{cases}$$

and let $\chi_\varnothing$ be the zero function.

Let us now construct the portfolio $\mathbf{f}^*$. We define inductively a partition $E_1,\ldots,E_n$ of Ω. Let

$$E_1=\{\omega\in\Omega\big|p_1(\omega)\geq p_j(\omega),\forall j\neq 1\},$$

and, for $i > 1$, define

$$E_i = \{\omega \in \Omega \big| p_i(\omega) \geq p_j(\omega), \forall j \neq i\} \setminus \bigcup_{k<i} E_k,$$

i.e., E_i contains the states to which p_i assigns the highest probability, with an arbitrary choice when the maximal probability is assigned by more than one agent. Note that E_i might be empty. We define $\mathbf{f}^* = (\chi_{E_1}, \chi_{E_2}, \ldots, \chi_{E_n})$. For every $p \in \Delta^\Omega$ we have that

$$\begin{aligned}
ER(\mathbf{f}^*, p) &= \sum_{i \in N} \left[\sum_{\omega \in \Omega} \chi_{E_i}(\omega) p_i(\omega) - \sum_{\omega \in \Omega} \chi_{E_i}(\omega) p(\omega) \right] \\
&= \left(\sum_{i \in N} p_i(E_i) \right) - \left(\sum_{i \in N} p(E_i) \right) \\
&= \left(\sum_{i \in N} p_i(E_i) \right) - 1,
\end{aligned}$$

since E_i constitute a partition. But

$$\left(\sum_{i \in N} p_i(E_i) \right) - 1 = \left(\sum_{\omega \in \Omega} \max_{i \in N} p_i(\omega) \right) - 1 = r,$$

by the definition of the partition $E_1, \ldots, E_n$.

Since

$$\sum_{i \in N} \sum_{\omega \in \Omega} \chi_{E_i}(\omega) p(\omega) = 1,$$

we have $\overline{ER}(\mathbf{f}^*, p) = r$ for all $p \in \Delta^\Omega$. Thus we have shown that $\mathbf{f}^*$ guarantees that the rate of arbitrage is at least r (Note that, if $p_1 = p_2 = \cdots = p_n$, then $ER(\mathbf{f}, p_1) = 0$ for all $\mathbf{f} \in \mathcal{F}$ and $r = 0$).

Define p^* as

$$p^*(\omega) = \frac{\max_{i \in N} p_i(\omega)}{r + 1} = \frac{\max_{i \in N} p_i(\omega)}{\sum_{\omega \in \Omega} \max_{i \in N} p_i(\omega)}.$$

Clearly p^* is a probability distribution.

Let $\mathbf{f} \in \mathcal{F}$. Then

$$\begin{aligned}
\overline{ER}(\mathbf{f}, p^*) &= \frac{ER(\mathbf{f}, p)}{\sum_{i \in N} \sum_{\omega \in \Omega} f_i(\omega) p^*(\omega)} \\
&= \frac{\sum_{i \in N} \sum_{\omega \in \Omega} f_i(\omega) p_i(\omega) - \sum_{i \in N} \sum_{\omega \in \Omega} f_i(\omega) p^*(\omega)}{\sum_{i \in N} \sum_{\omega \in \Omega} f_i(\omega) p^*(\omega)}.
\end{aligned}$$

Now, if $\sum_{i\in N}\sum_{\omega\in\Omega} f_i(\omega)p^*(\omega)=0$, then $\sum_{\omega\in\Omega}\left(\sum_{i\in N} f_i(\omega)\right)p^*(\omega)=0$, which means that for every $\omega\in\Omega$ such that $\sum_{i\in N} f_i(\omega)>0$, we must have that $p^*(\omega)=0$, which, by the definition of p^*, implies that $p_i(\omega)=0$ for all $i\in N$, and therefore we must have that $\sum_{i\in N}\sum_{\omega\in\Omega} f_i(\omega)p_i(\omega)=0$, and, by the definition of $\overline{ER}$, we get $\overline{ER}(\mathbf{f},p^*)=0\le r$, as required. If $\sum_{i\in N}\sum_{\omega\in\Omega} f_i(\omega)p^*(\omega)\neq 0$, we have that

$$\begin{aligned}
\overline{ER}(\mathbf{f},p^*) &= \frac{\sum_{i\in N}\sum_{\omega\in\Omega} f_i(\omega)p_i(\omega)}{\sum_{i\in N}\sum_{\omega\in\Omega} f_i(\omega)p^*(\omega)}-1\\
&= \frac{\sum_{\omega\in\Omega}\sum_{i\in N} f_i(\omega)p_i(\omega)}{\sum_{\omega\in\Omega}\sum_{i\in N} f_i(\omega)p^*(\omega)}-1\\
&\le \frac{\sum_{\omega\in\Omega}\sum_{i\in N} f_i(\omega)\max_{j\in N} p_j(\omega)}{\sum_{\omega\in\Omega}\sum_{i\in N} f_i(\omega)p^*(\omega)}-1\\
&= \frac{\sum_{\omega\in\Omega}\sum_{i\in N} f_i(\omega)\max_{j\in N} p_j(\omega)}{\sum_{\omega\in\Omega}\sum_{i\in N} f_i(\omega)(\max_{j\in N} p_j(\omega)/(r+1))}-1\\
&= (r+1)\frac{\sum_{\omega\in\Omega}\sum_{i\in N} f_i(\omega)\max_{j\in N} p_j(\omega)}{\sum_{\omega\in\Omega}\sum_{i\in N} f_i(\omega)\max_{j\in N} p_j(\omega)}-1\\
&= r,
\end{aligned}$$

as required.

We have shown that

$$\overline{ER}(\mathbf{f}^*,p)=r=\sup_{\mathbf{f}\in\mathcal{F}}\overline{ER}(\mathbf{f},p^*),\qquad \forall p.$$

Hence p^* guarantees that the rate of arbitrage is not higher than r, and p^* is therefore a reconciled belief.

We need to show that p^* is unique, in the sense that for every $p\neq p^*$ there exists a portfolio $\mathbf{f}^p$ such that $\overline{ER}(\mathbf{f}^p,p)>r$, i.e., no other belief can guarantee r. Let $p\in\Delta^\Omega$ be such that $p\neq p^*$. Then there exists $\overline{\omega}\in\Omega$ such that $p^*(\overline{\omega})<p(\overline{\omega})$. Let $j\in N$ be such that $\overline{\omega}\in E_j$, where the sets E_i are as defined above. We define

$$\mathbf{f}^p=(\chi_{E_1},\ldots,\chi_{E_{j-1}},\chi_{E_j}-\chi_{\{\overline{\omega}\}},\chi_{E_{j+1}},\ldots,\chi_{E_n}).$$

Now

$$\begin{aligned}
\overline{ER}(\mathbf{f}^p,p) &= \frac{\sum_{i\in N}\sum_{\omega\in\Omega} f_i^p(\omega)p_i(\omega)}{\sum_{i\in N}\sum_{\omega\in\Omega} f_i^p(\omega)p(\omega)}-1\\
&= \frac{\sum_{i\in N}\sum_{\omega\in\Omega}\chi_{E_i}(\omega)p_i(\omega)-p_j(\overline{\omega})}{\sum_{i\in N}\sum_{\omega\in\Omega}\chi_{E_i}(\omega)p(\omega)-p(\overline{\omega})}-1\\
&= \frac{\sum_{\omega\in\Omega}\max_{i\in N} p_i(\omega)-p_j(\overline{\omega})}{1-p(\overline{\omega})}-1\\
&= \frac{r+1-p_j(\overline{\omega})}{1-p(\overline{\omega})}-1.
\end{aligned}$$

Since $\overline{\omega} \in E_j$, we have $p_j(\overline{\omega}) = \max_{i \in N} p_i(\overline{\omega})$ and therefore $p_j(\overline{\omega}) = p^*(\overline{\omega})(r+1)$, by the definition of p^*. Hence

$$\begin{aligned}\overline{ER}(\mathbf{f}^p, p) &= \frac{(r+1) - p^*(\overline{\omega})(r+1)}{1 - p(\overline{\omega})} - 1 \\ &= (r+1)\left(\frac{1 - p^*(\overline{\omega})}{1 - p(\overline{\omega})}\right) - 1.\end{aligned}$$

Since $p^*(\overline{\omega}) < p(\overline{\omega})$, we have

$$\frac{1 - p^*(\overline{\omega})}{1 - p(\overline{\omega})} > 1,$$

and therefore

$$\overline{ER}(\mathbf{f}^p, p*) = (r+1)\left(\frac{1 - p^*(\overline{\omega})}{1 - p(\overline{\omega})}\right) - 1 > r + 1 - 1 = r,$$

which means that there exists a unique optimal reconciled belief p^*, as required. $\square$