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The hierarchical structure of a firm: a geometric approach

Leopoldo Yanes
School of Economics, University of Queensland

Kam Ki Tang
School of Economics, University of Queensland

Evelyn Ng
School of Economics, University of Queensland

Rodney Beard
ngevie@yahoo.com

Abstract

This paper develops a novel, geometric approach to modelling a firm's hierarchical structure. We model the firm's hierarchy as the sector of a circle, in which the radius represents the height of the hierarchy and the angle of the sector represents the width of the hierarchy. The firm then chooses the height and angle in order to maximise profit. We analyse the impacts of changes in economies of scale, input substitutability and labour productivity on the firm's hierarchical structure. We find that the firm will unambiguously become more hierarchical as specialisation of its workers increases or as its output price increases. The effect of changes in scale economies is contingent on the level of task specialisation and output price.

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The Hierarchical Structure of a Firm: A Geometric Approach

1. Introduction

There is wide variation in the hierarchical structure of firms. Typically, firms in heavy manufacturing industries have tall hierarchies (e.g., General Electric). Conversely, firms in the services industry, such as consulting firms, are relatively flat and less hierarchical (e.g., Ernst & Young). Moreover, the hierarchical structure of a firm rarely stays constant over time. Consider Google Inc. as an example. When the company was founded in 1998 it had four employees other than its two founders. By 2001, it had grown to 200 employees. Currently, it has almost ten organisational departments.

These observations lead to the following research questions: (1) What are the economic factors that influence the hierarchical structure of firms? (2) How does the hierarchical structure respond to a change in these factors?

The literature on hierarchy can be broadly divided into two branches. One studies the "shape" of the firm, while the other focuses on the "distribution of authority". The distribution of authority attempts to explain the *internal* structure of a hierarchy, that is, how workers are organised according to the influence one has over others. A set theoretic framework based on simple games, namely, command and control games, has been used to model the authority structure. The Shapley-Shubik power index is widely used in this branch of the literature as a quantification of the authority distribution and as a measure of each worker's task responsibility. Research in this branch includes the works of Shapley and Palamara (2000), and Hu and Shapley (2003a; 2003b).

In contrast, understanding the shape of a hierarchy gives an idea of its *external* structural design. The shape is determined by the number of levels in the hierarchy and by the span of control of a superior. The approaches in this branch are diverse, ranging from theories of knowledge, promotion tournaments, contracting, costs and optimisation problems to stage games. Works in this branch include the papers by Garicano (2000), Mookherjee and Reichelstein (2001), Qian (1994), Yang (2002), and Dubey and Haimanko (2003).

We aim to contribute to the second branch of the literature in two ways. Firstly, the paper adapts the neoclassical methodologies of profit maximisation to explain the hierarchical structure of the firm. However, instead of assuming the span of control and number of levels, we specify production in terms of the hierarchy itself. In doing so, our model combines certain elements from neoclassical theory as an attempt to shed light on the age-old question of "how does a firm organize its workers for production?".

Secondly, we use a geometric approach to pose the optimisation problem. More specifically, we model the shape of the hierarchy using the sector of a circle. This allows us to employ the geometric formula of the sector in the construction of the cost function, which keeps the analysis simple and tractable. The simplicity of the modelling method provides room for further extensions and captures the essential features of the firm's decision process, albeit in an abstract form.

2. The Model

Consider the sector of a circle as an abstract representation of the hierarchical structure of a firm, as depicted in Figure 1. The height of the firm is represented by *H*, and can be interpreted as the total number of levels in the hierarchy. The angle, θ, can be interpreted as span of control from any supervisor's perspective. In the figure, *h* represents the distance from the highest ranking worker. Thus *h* is inversely related to authority, with $h=0$ representing the highest ranking worker (namely the CEO, who is located at the tip of the sector), and $h=H$ representing the lowest echelon of the hierarchy. *b* represents the number of workers in a particular hierarchical level (*h*). *b* can be interpreted as the width of the firm at level *h*. Using the arc length formula, we have that $b=h\theta$.

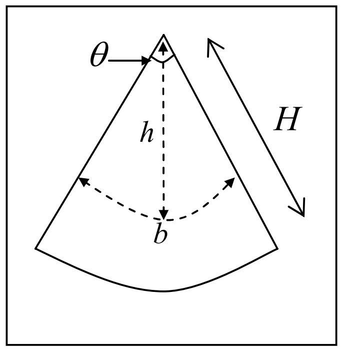

Figure 1. A Sector of a Circle Representation of a Firm's Hierarchy

With this abstract representation, we reduce the problem of designing the hierarchical structure of the firm to one of choosing the height (H) and the angle (θ) of the sector. In treating the firm's design problem as one of choosing the dimensions of a geometric sector, we make the assumption that the hierarchy features a continuum of workers, both horizontally and vertically. Thus, the area of the sector represents the total number of workers in the firm. This specification helps keep the model tractable, whilst allowing us to capture some essential features of the design of a firm's hierarchy.

The profit function of the firm is specified as follows:

$$\pi = PY - \int_0^H b(h,\theta)w(h)dh \qquad (1)$$

where P is output price, Y is output, $b(h,\theta) = h\theta$ is the width of the hierarchy (i.e., arc length) at any hierarchical level *h*, $w(h)$ is the wage rate (which is a convex and decreasing function of the hierarchical level, *h*).

The hierarchy is incorporated into the firm's production function rather than being treated as an 'outside' factor, thereby capturing the forces shaping the hierarchical structure.

The major departure of our model from the standard neoclassical model of firm is that inputs are measured in terms of H and θ in the form of a constant elasticity of substitution (CES) production function:

$$Y = \left(H^{\rho} + \theta^{\rho}\right)^{\frac{\beta}{\rho}}; \quad H > 0, \theta > 0\,,\ \rho < 1, \rho \neq 0 \text{ and } \beta > 0 \qquad (2)$$

The capital-labour ratio is assumed to be exogenously determined, thus we need only consider the distribution and quantity of labour within the hierarchy (which would then be associated to a certain level of capital)[1]. The structure of production implies that, for a given number of workers, it is the organization of the hierarchy which determines output.

As argued by Erich Gutenberg (1951), a firm can be seen as a 'productive relationship' whereby productivity depends more on the organisation and complementarity of factor inputs rather than just the potential productivity of each input (Albach et al., 1999). This concept of productivity is atypical in the sense that the firm's productivity is not only determined by neoclassical input-output relations but is also influenced by the organisation of workers. Several other authors such as Greenan (2003), Carlson (2001) and Foray (2004) have also pointed out that the organisation of workers (combined with other factors) influences how successful a firm is in achieving its objective. In other words, the firm can be seen as a system where the way in which its elements (in our case, workers) are organised will determine the extent to which it can achieve its objective. These authors argue that characteristics such as teamwork and the monitoring of workers are therefore crucial to the workings of the firm. In short, this strand of the literature affirms that a firm's production depends on its hierarchical structure.

In (2), β measures economies of scale where

$$\beta \begin{cases} < 3: \text{decreasing returns to scale} \\ = 3: \text{constant returns to scale} \\ > 3: \text{increasing returns to scale} \end{cases} \qquad (3)$$

To see this, note that the total number of workers is equal to the area of a sector, $\theta H^2/2$. So, if both θ and H increase by factor *g*, the total input will be equal to $g^3\theta H^2/2$, while output will be equal to $Y(g\theta, gH) = g^{\beta}Y(\theta, H)$. Therefore, returns to scale depend on whether β is bigger than, equal to, or less than 3.

The elasticity of substitution between H and θ is measured by $\frac{1}{1-\rho}$. If workers at different levels perform different tasks, then an increase in ρ indicates a decrease in the task specialisation of workers within the firm. In other words, there is a decrease in intra-firm specialisation. An indicator of intra-firm specialisation is whether subordinates are able to fill in their superior's role when she/he is absent, and vice versa. For example, intra-firm specialisation is expected to be high in the IT sector, and low in fast food catering: In the IT sector, qualifications of subordinates are not too different from supervisors', while the opposite may hold in the fast food industry. In particular, as H and θ become more substitutable, for example, the firm would be more inclined to give up one superior for another lower-level worker since that worker is cheaper to hire.

[1] Qian (1994) makes use of a similar assumption to dispatch the problem of substitution between capital and workers in order to make his model tractable. The extension of the model to consider capital we will be the topic of future research.

The wage rate schedule is exogenously determined (presumably by the CEO), and is uniform across each level but increasing convexly as we move up the hierarchy. The total wage bill of workers at level h for a given θ is given by $b(h,\theta)w(h)$. Therefore, the total wage bill for the firm is the integral of $b(h,\theta)w(h)$ from $h=0$ (the CEO) to $h=H$ (the lowest level worker). The wage schedule is specified as:

$$w = h^{-\phi} \tag{4}$$

where ϕ is (the absolute value of) the wage elasticity with respect to h.

The firm's problem is to choose H and θ to maximise profits, just like a neoclassical firm chooses capital and labour to maximise profits. Solving the firm's problem yields the following solutions

$$\theta^{**} = \left[P\beta \frac{(3-\phi)^{\frac{\beta-\rho}{\rho}}}{(2-\phi)^{\frac{2-\phi-\rho}{\rho}}} \right]^{\frac{1}{3-\phi-\beta}} \tag{5}$$

$$H^{**} = \left[P\beta \frac{(3-\phi)^{\frac{\beta-\rho}{\rho}}}{(2-\phi)^{\frac{\beta-\rho-1}{\rho}}} \right]^{\frac{1}{3-\phi-\beta}} \tag{6}$$

In order to obtain tractable comparative statics results, we restrict the wage elasticity, ϕ, to one. This simpler model allows us to pinpoint precisely how the firm's hierarchical structure responds to changes in the economic parameters, and also enables a more stylized analysis. In this simpler setting, the solutions in (5) and (6) become:

$$H^{*} = \theta^{*} = \left[P\beta \cdot 2^{\frac{\beta-\rho}{\rho}} \right]^{\frac{1}{2-\beta}} \tag{7}$$

Lemma 1. If $\phi = 1$, the second-order conditions for profit-maximisation require $0 < \beta < 2$ and $\rho < 0$.

Proof: The Lagrangian for the profit maximization problem is $\mathcal{L} = P\left(H^{\rho} + \theta^{\rho}\right)^{\frac{\beta}{\rho}} - \theta H$ when $\phi = 1$. It can be shown that $\mathcal{L}_{HH}^{*} = \mathcal{L}_{\theta\theta}^{*} = (\beta+\rho-2)/2$ and $\mathcal{L}_{H\theta}^{*} = \mathcal{L}_{\theta H}^{*} = (\beta-2-\rho)/2$. The second-order conditions for H^{*} and θ^{*} to be profit maximising values are $\mathcal{L}_{HH}^{*} < 0, \mathcal{L}_{\theta\theta}^{*} < 0$ and $(\mathcal{L}_{HH}^{*})(\mathcal{L}_{\theta\theta}^{*}) > (\mathcal{L}_{\theta H}^{*})^{2}$, which simplify to $0 < \beta < 2$ and $\rho < 0$. **Q.E.D.**

The upper bound of β set by the second order condition (i.e. 2) is in fact equal to the constant returns to scale value of β (i.e. 3) minus the height elasticity of the wage schedule (i.e. $\phi = 1$). From (3), it can be seen that the second order conditions imply the firm operates under decreasing returns to scale: Given that the wage is a decreasing function of h, the workers at the bottom will be paid progressively less as the firm's hierarchy extends. If workers' productivity does not decrease fast enough down the hierarchy, then a profit-maximising firm will expand vertically indefinitely. The intuition for restricting the substitutability between H and θ (i.e. ρ < 0) can be explained by similar reasoning. If H and θ are almost perfectly substitutable (i.e. $\rho \to$ 1), then the firm can save labour costs by continuously expanding vertically and shrinking horizontally. In summary, $0 < \beta < 2$ and $\rho < 0$ are conditions that prevent the firm from expanding infinitely in the vertical direction.

The following propositions summarise the comparative statics properties of the model:

PROPOSITION 1. The firm will expand vertically and horizontally when output price rises.

Proof: Let $Z^* = H^* = \theta^*$. Then $\frac{\partial Z^*}{\partial P} = \frac{Z^*}{P(2-\beta)} > 0$, by the second-order condition: $\beta < 2$. **Q.E.D.**

As output price rises, marginal revenue increases and becomes greater than marginal cost. The firm takes advantage of this by expanding its hierarchy. In doing so, its marginal benefit decreases while its marginal cost increases until they equate again. The result suggests that, in industries in which output prices are protected (e.g. by regulation), firms would be 'bigger' in terms of the number of workers they hire.

PROPOSITION 2. The firm becomes less hierarchical as intra-firm specialisation decreases.

Proof. A decrease in intra-firm specialisation is represented by an increase in substitutability between H^* and θ^*, i.e., an increase in ρ. $\frac{\partial Z^*}{\partial \rho} = -\frac{Z^*}{\rho^2}\left(\frac{\beta}{2-\beta}\ln 2\right) < 0$, by the second-order conditions: $\beta < 2$ and $\rho < 0$. **Q.E.D.**

Given that ρ must be less than zero, an increase in ρ causes marginal revenue to decrease. Therefore, the optimal response for the firm is to structure its hierarchy to cut costs until marginal cost equates marginal revenue again. The result implies that firms would be smaller (in terms of the size of their labour force) in industries in which division of labour happens to a lesser extent.

PROPOSITION 3. An increase in scale economies leads to a hierarchical expansion if either output price is sufficiently large for given input substitutability, or intra-firm specialisation is sufficiently high for a given output price.

Proof: $\frac{\partial Z^*}{\partial \beta} = \frac{Z^*}{(2-\beta)}\left[\frac{1}{\beta} + \frac{1}{\rho}\ln 2 + \ln Z^*\right]$. For this to be positive, we require either $P > \frac{e^{\frac{\beta-2}{\beta}} \cdot 2^{\frac{\rho-2}{\rho}}}{\beta}$ or , equivalently, $\rho < \frac{2\beta \ln 2}{[1 + \ln 2 - \ln(P\beta)]\beta - 2}$. **Q.E.D.**

3. Concluding Remarks

In this paper, we have introduced a geometric approach to model the hierarchy of firms. A simple and stylised model has enabled us to examine how the firm's hierarchical structure changes with certain economic factors, including output price, economies of scale in production and intra-firm specialisation. By introducing a small modification in the standard neoclassical structure, the model sheds light on possible causes of variation in hierarchical structure across firms and possibly even within a given firm through time. We have taken a completely different perspective by directly incorporating the hierarchical structure into the production process. The model described in the paper is (purposefully) a basic test case of this geometric approach. The specification of the model can be relaxed in many directions to explain more complex issues. Possible extensions include relaxing the assumption that the elasticity of the wage schedule is equal to one, relaxing the assumption of a fixed capital to labour ratio, while letting the firm choose capital for each hierarchical level, as well as generalizing the functional forms for the production function and wage schedule.

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