E C O N O M I C S   B U L L E T I N

Efficiency Tradeoffs in Estimating the Linear Trend Plus Noise Model

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Abstract

This paper presents the results of a Monte Carlo comparison of feasible GLS estimators of the trend parameter in the linear trend plus noise model, where the noise component may or may not be a unit root process. We include an FGLS estimator that estimates the noise component using a median−unbiased estimator.

Citation: Falk, Barry and Anindya Roy, (2006) "Efficiency Tradeoffs in Estimating the Linear Trend Plus Noise Model." *Economics Bulletin,* Vol. 3, No. 6 pp. 1−9
Submitted: January 17, 2006. **Accepted:** February 22, 2006.
URL: http://www.economicsbulletin.com/2006/volume3/EB−06C20001A.pdf

1. Introduction

Many economic time series display a clear tendency to grow over time. However, quantifying the growth path of a series is complicated by the fact that most trending series behave somewhat erratically, seemingly subject to the transitory and/or permanent effects of external shocks. In other words, the observed time series is composed of a long-run (i.e., trend or permanent) component and a short-run (i.e., noise or transitory) component, which must somehow be untangled in order to quantify the trend behavior of the series.

This paper provides a Monte Carlo comparison of the precision of several closely-related estimators of the average growth rate of a time series that has been generated by the widely used "trend plus noise" model. In particular, it considers the implications of recent developments in estimating the noise component for estimation of the average growth rate of the series. We will show that more precise estimators of the parameters of the noise component do not necessarily translate into more precise estimators of the parameters of the trend component. That is, over a certain (and practically important) range of parameter values, there may be a tradeoff between the precision of estimating the parameters of the two components of the model. The tradeoff we highlight has not been previously recognized in the literature.

Let Y_t denote the series of interest or, as if often the case in practice, the logarithm of the series. According to the (linear form of the) trend plus noise model, Y_t is assumed to be generated as follows:

$$Y_t = a + bt + y_t \tag{1}$$

where

$$y_t = \rho y_{t-1} + \varepsilon_t \tag{2}$$

and ε_t is a white noise (i.e., zero-mean, serially uncorrelated, and constant variance) process. Note that, for expositional convenience and in anticipation of our Monte Carlo analysis, we are assuming that the noise component, y_t, is an AR(1) process rather than a more general AR(p) or ARMA(p,q) process.

Two important cases are typically distinguished in practice. If, $-1 < \rho < 1$, the noise component y_t is stationary and Y_t is trend stationary. That is, Y_t, which is the sum of a deterministic linear trend $(a + bt)$ and a stationary component (y_t), grows linearly with time while being subject to external shocks whose effects are purely transitory. If $\rho = 1$, the noise component y_t is nonstationary, although it is stationary in its first-differenced form. In this case, Y_t is nonstationary but its first-differenced form is stationary and, therefore, Y_t is difference stationary. In both cases, the parameter b is the average growth of Y_t, i.e., it is the slope of the trend path. However, in the difference stationary case the level or intercept of the trend line will be stochastically shifting over time, reflecting the permanent effects of the external shocks, so that Y_t has a stochastic trend.

If $\rho \varepsilon (-1,1)$, the Grenander and Rosenblatt (1957) result implies that the ordinary least squares (OLS) estimator of (a,b) in (1) is asymptotically equivalent to the generalized least squares (GLS) estimator of (a,b) using (1) and (2). If $\rho = 1$, the parameter a is not identified and although the OLS estimator of b is consistent, it is not asymptotically efficient. In this case, the sample mean of ΔY_t is an asymptotically efficient estimator of b, being equivalent to the GLS estimator.

Of course, in practice we do not know *a priori* whether ρ is equal to or less than one. The main focus of this paper is estimating the parameter b, the average growth of Y_t (or, the average growth rate of the original series if Y_t is its logged form), when it is not known, *a priori*, whether Y_t is a trend stationary or difference stationary time series.

The remainder of the paper is organized as follows. Section 2 provides a brief literature regarding estimation of the trend-plus-noise model and median-unbiased estimators. Section 3 provides the setup and results of our Monte Carlo experiment. Section 4 provides an explanation of the results. Concluding comments are given in Section 5.

2. Literature Review

Canjels and Watson (1997) recently studied this problem. They used standard and local-to-unity asymptotic distribution theory, along with Monte Carlo simulations, to evaluate the OLS estimator, the first-difference estimator, and several feasible GLS estimators of b when the researcher does not know ρ or commit *a priori* to either the trend-stationary or difference-stationary representation of y_t. They conclude that the feasible Prais-Winsten (FPW) estimator of b is the preferred estimator.

The FPW estimator proceeds in two steps. In the first step, the OLS residuals from (1) are fit to (2) to obtain an estimate of ρ. In the second step, the full GLS estimator is applied to (1) using the estimated ρ in place of its actual value.

Typically, in the first step of the FPW estimator, the residuals from (1) are fit to (2) using the OLS estimator. Although the OLS estimator of ρ is consistent for any ρ in the parameter space, it is downward-biased with respect to both the mean and median. Further, the bias is of the same order as the standard deviation of the estimator for values of ρ close to or equal to one.

Andrews (1993) proposed using the median-unbiased principle (i.e., choose an estimator such that the median of the distribution of the estimator is equal to ρ) to construct an improved estimator of ρ for model (1)-(2) when the errors are normally distributed. Andrews and Chen (1994) extended the estimator to obtain a nearly median-unbiased estimator when the errors are not normally distributed (and the order of the autoregressive component is greater than one). Estimators of ρ which are nearly median-unbiased have also been suggested by Rudebusch (1992), Fuller (1996), Roy and Fuller (2001) and Roy, Falk, and Fuller (2004).

These (approximately) median-unbiased estimators of ρ improve upon the OLS estimator of ρ for values of ρ close to or equal to one by reducing the (mean and median) bias and the mean-squared error of the estimator. Table I compares the simulated medians, means, and mean squared-errors of the distributions of the ordinary least squares estimator and Andrews's (1993) exact median unbiased estimator of ρ for sample size 100 . Even for values of ρ substantially less than one, where the bias in the OLS estimator is quite small, the median unbiased estimator seems to perform at least as well as the OLS estimator.

In light of these results, it would seem that the finite-sample performance of the FPW estimator of the trend coefficient b in (1) could be improved by using a median-unbiased estimator of ρ in place of the OLS estimator in the final step of the FPW procedure. This view seems to be implicit in Andrews and Chen (1994). Although they do not report a direct comparison between their estimator of b and the estimator of b based upon the OLS estimator of ρ, they do report simulation comparisons of estimators of α and β in the following model that is an alternative representation of (1)-(2):

$$Y_t = \alpha + \beta t + \rho Y_{t-1} + \varepsilon_t. \qquad (3)$$

The parameters α and β in (3) are related to the parameters in (1)-(2) according to

α = a(1-ρ)+ bρ and β = b(1-ρ). They conclude that the OLS estimator of β conditioned upon the estimate of ρ derived from their median-unbiased estimation procedure, is more accurate (in the mean-square sense) than the unconditional OLS estimator of β from (3).

3. Monte Carlo Experiment

We conducted a simulation experiment to investigate the finite-sample performance of the FPW estimator of the trend parameter b, using an exact median unbiased estimator of ρ. Five thousand realizations of $Y_1,\ldots,Y_{100}$ were generated by (1)-(2) with a = b = 0 and the ε_t's independently drawn from the standard normal distribution. The initial value y_0 was set equal to zero when $\rho = 1$. Otherwise, y_0 was determined by drawing from its stationary distribution.

For each simulated realization of the y's, the following estimators of b were constructed:

1. Apply the OLS estimator to (1) to obtain $\hat{b}_{\text{OLS}}$.
2. Apply the exact GLS estimator to (1)-(2) to obtain $\hat{b}_{\text{GLS}}$.
3. Apply the FPW estimator to (1)-(2) using the OLS estimator of ρ to obtain $\hat{b}_{\text{FPW}}(\hat{\rho}_{\text{OLS}})$.
4. Apply the FPW estimator to (1)-(2) using Andrews's exact median-unbiased estimator of ρ to obtain $\hat{b}_{\text{FPW}}(\hat{\rho}_{\text{MU}})$.

The results are presented in Table II.

Notice that for each value of ρ considered, the MSE of $\hat{b}_{\text{OLS}}$ is greater than the MSE of $\hat{b}_{\text{GLS}}$. The MSE's of the two feasible Prais-Winsten estimators, $\hat{b}_{\text{FPW}}(\hat{\rho}_{\text{OLS}})$ and $\hat{b}_{\text{FPW}}(\hat{\rho}_{\text{MU}})$ always fall between MSE($\hat{b}_{\text{OLS}}$) and MSE($\hat{b}_{\text{GLS}}$). For ρ equal to 1, .99, and .975, the MSE of $\hat{b}_{\text{FPW}}(\hat{\rho}_{\text{OLS}})$ is greater than the MSE of $\hat{b}_{\text{FPW}}(\hat{\rho}_{\text{MU}})$ while the MSE of $\hat{b}_{\text{FPW}}(\hat{\rho}_{\text{OLS}})$ is less than the MSE of $\hat{b}_{\text{FPW}}(\hat{\rho}_{\text{MU}})$ for ρ equal to .95, .90, .85, and .80. The MSE's of the four estimators are about equal to one another for smaller values of ρ.

Therefore, with respect to estimating the trend parameter b in model (1)-(2), we draw the following conclusions. First, as is well known, the feasible Prais-Winsten estimator outperforms the OLS estimator. But, second, using a more precise (in the MSE sense) estimator of ρ is not necessarily desirable. In particular, the FPW estimator of b based on the OLS estimator of ρ has a smaller MSE than the FPW estimator of b based on Andrews's (1993) median-unbiased estimator of ρ for an intermediate range of values of ρ. This occurs even though the latter is a better (with respect to mean bias, median bias, and MSE) estimator of ρ and the parameter β in (3).

4. Discussion

We propose the following heuristic argument to help explain these results. Let $\hat{\rho}$ denote the estimator of ρ obtained from the detrended data, $\hat{y}_t = Y_t - \hat{a} - \hat{b}t$, where $\hat{a}$ and $\hat{b}$ are the OLS estimators of a and b, respectively. In addition, for ease of calculation, let the time index run from t = 0 through t = T.

The FGLS estimator of b, $\hat{b}_{FGLS}$, is the ordinary least squares estimator of b from the regression:

$$(1-\hat{\rho}^2)^{1/2} Y_0 = (1-\hat{\rho}^2)^{1/2} a^* + \varepsilon_0,$$

$$Y_t - \hat{\rho} Y_{t-1} = a^*(1-\hat{\rho}) + b[(t-\bar{t}) - \hat{\rho}(t-1-\bar{t})] + \varepsilon_t, \quad t = 1,...,T$$

where $\bar{t}$ = 0.5(T+1) and $a^* = a + b\bar{t}$. The error in estimation in the FGLS estimator of b is:

$$\hat{b}_{FGLS} - b = \frac{(1-\hat{\rho}^2)\varepsilon_0 + \sum_{t=1}^{T} [(t-\bar{t})(1-\hat{\rho}) + \hat{\rho}][y_t - \hat{\rho} y_{t-1}]}{\sum_{t=1}^{T} [(t-\bar{t})(1-\hat{\rho}) + \hat{\rho}]^2}.$$

Most of the common estimators of ρ are even functions of the ε_t's as they are based on the sufficient statistics $y_0^2, \sum_{t=1}^{T} y_{t-1}^2$, and $\sum_{t=1}^{T} y_t y_{t-1}$, which are even functions of the innovations. Therefore, $\hat{b}_{FGLS}$ is typically an odd function of the ε_t's. So for symmetrically distributed ε_t's the FGLS estimator of b will be unbiased and the MSE will be solely due to the variance of the estimator.

Now, consider the estimation error more carefully. Ignoring smaller order terms, we obtain

$$\hat{b}_{FGLS} - b = \frac{(1-\hat{\rho})^{-1} \sum_{t=1}^{T} (t-\bar{t})[\varepsilon_t - (\hat{\rho}-\rho) y_{t-1}]}{\sum_{t=1}^{T} (t-\bar{t})^2}$$

$$= (1-\hat{\rho})^{-1} T^{-3/2} [\sum_{t=1}^{T} w_t \varepsilon_t - (\hat{\rho}-\rho) \sum_{t=1}^{T} w_t y_{t-1}]$$

where $w_t = [\sum_{t=1}^{T} (t-\bar{t})^2]^{-1/2} (t-\bar{t})$. Note that the term $(1-\hat{\rho})^{-1}$ is increasing in the estimation error $\hat{b}_{FGLS} - b$ while the term $T^{-3/2}[\sum_{t=1}^{T} w_t \varepsilon_t - (\hat{\rho}-\rho) \sum_{t=1}^{T} w_t y_{t-1}]$ is generally decreasing in the estimation error since $\sum_{t=1}^{T} w_t \varepsilon_t$ and $\sum_{t=1}^{T} w_t y_{t-1}$ are positively correlated random variables.. Put slightly differently, suppose we restrict our attention to samples for which $\hat{\rho} < \rho$, which is a significantly large proportion of samples since the OLS and median-unbiased estimators have a negative bias. Then, on average, increasing $\hat{\rho}$ (and reducing its downward bias) will decrease the magnitude of $T^{-3/2}[\sum_{t=1}^{T} w_t \varepsilon_t - (\hat{\rho}-\rho) \sum_{t=1}^{T} w_t y_{t-1}]$ but will increase the magnitude of $(1-\hat{\rho})^{-1}$.

5. Concluding Remarks

There have been important recent developments in the estimation of the parameters of the noise component of the linear trend plus noise model. In particular, a class of exact or near-median unbiased estimators of the autoregressive component of this model has been developed. These estimators reduce both the median and mean bias as well as the mean-squared error of the estimator in comparison to the OLS estimator over the entire parameter space, including the unit root extreme. It might then be expected that condition the feasible GLS estimator of the trend parameter characterizing the average growth rate of the series on median-unbiased estimates of the autoregressive parameters would be preferred to the feasible GLS estimator conditioning on the OLS esitmator of the autoregressive parameters. Our Monte Carlo results and subsequent discussion suggest otherwise. For a practically important range of values for the autoregressive parameters, the feasible GLS estimator of the average growth constructed using the OLS estimator of the autoregressive parameters actually has a lower mean-squared error than the feasible GLS estimator constructed based upon the median-unbiased estimator of the autoregressive parameters.

We conclude with the following remarks. First, note that both estimators of b, $\hat{b}_{FPW}(\hat{\rho}_{OLS})$ and $\hat{b}_{FPW}(\hat{\rho}_{MU})$, are asymptotically efficient so that the issue we are raising is strictly a finite sample concern. Second, note that the relatively poor performance of $\hat{b}_{FPW}(\hat{\rho}_{MU})$ occurs when ρ is in a neighborhood bounded *away* from unity so that local-to-unity asymptotic theory will not be helpful here. Third, note that this issue will arise with any of the recently proposed median-unbiased estimators of ρ so that it is not specific to Andrews's (1993) estimator, which is used here only as an illustrative example.

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TABLE I

Empirical Properties of the OLS and Median Unbiased Estimators of the AR Coefficient ρ in the AR(1) Model with Trend (T=100)

	Median of:		Mean of:		100 x MSE of:	
ρ	$\hat{\rho}_{OLS}$	$\hat{\rho}_{MU}$	$\hat{\rho}_{OLS}$	$\hat{\rho}_{MU}$	$\hat{\rho}_{OLS}$	$\hat{\rho}_{MU}$
1	0.910	0.999	0.900	0.959	1.33	0.480
0.98	0.904	0.980	0.894	0.955	1.07	0.404
0.95	0.885	0.951	0.875	0.938	0.910	0.435
0.90	0.844	0.900	0.834	0.892	0.847	0.525
0.80	0.751	0.800	0.744	0.793	0.823	0.578
0.70	0.654	0.698	0.648	0.692	0.914	0.710
0.60	0.560	0.600	0.555	0.594	0.925	0.788
0.40	0.463	0.399	0.363	0.395	1.02	0.975
0.20	0.174	0.200	0.172	0.200	1.05	1.05
0.00	-0.020	0.000	-0.020	0.000	1.01	1.05

Notes:
$\hat{\rho}_{OLS}$ = OLS estimator and $\hat{\rho}_{MU}$ = Andrews's (1993) median unbiased estimator.

For each ρ, the results are derived from 10,000 samples of size 100 generated by model (1)-(2) with k = 1, a = b = 0, and $\varepsilon_t \sim N(0,1)$.

TABLE II

1000 x MSE of Estimators of the Trend Coefficient b in the Linear Trend Model with AR(1) Errors

ρ	$\hat{b}_{OLS}$	$\hat{b}_{FPW}(\hat{\rho}_{OLS})$	$\hat{b}_{FPW}(\hat{\rho}_{MU})$	$\hat{b}_{GLS}$
1	11.75	10.44	10.09	9.950
0.99	8.673	7.409	7.120	6.956
0.975	5.004	4.173	4.027	3.864
0.95	2.497	2.097	2.139	2.004
0.90	0.870	0.749	0.805	0.732
0.85	0.436	0.393	0.405	0.386
0.80	0.254	0.232	0.235	0.229
0.50	0.045	0.044	0.044	0.044
0.30	0.023	0.023	0.023	0.023

Note: $\hat{b}_{OLS}$ = OLS estimator of b, $\hat{b}_{FPW}(\hat{\rho}_{OLS})$ = Prais-Winsten estimator of b using the OLS estimator of ρ, $\hat{b}_{FPW}(\hat{\rho}_{MU})$ = Prais-Winsten estimator of b using Andrews's (1993) median-unbiased estimator of ρ, $\hat{b}_{GLS}$ = GLS estimator of b.