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Pairwise stable and stochastically stable networks in the four-person co-author model

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Abstract

The co-author model is introduced by Jackson and Wolinsky (1996, Journal of Economic Theory) as a typical example of the models of network formation. In this note, we study which network is pairwise stable and/or stochastically stable when the number of players is four.

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1 Introduction

The "co-author model" is introduced by Jackson and Wolinsky (1996) as a typical example of the models of network formation. In this note, we study which network is "pairwise stable" and/or "stochastically stable" when the number of players is four.

2 The model

There are four researchers who get their payoffs by writing joint papers with others. Each paper is written by a pair of researchers. By writing a paper with researcher j, researcher i gets

$$\frac{a}{n_i} + \frac{a}{n_j} + \frac{b}{n_i n_j},$$

where $a > 0$, $b > 0$, and n_i and n_j are the numbers of co-authors for i and j, respectively. If i has no co-author, her payoff is 0. See Figure 1 for the payoffs to researchers in each network.[1]

A network is **pairwise stable** (Jackson and Wolinsky, 1996) if (i) no pair of researchers want to form a new link between them, and (ii) no one wants to sever any single direct link.

Whether a particular network is pairwise stable and/or Pareto optimal depends on the values of a and b, as shown in Table 1.

3 Dynamic process

Let us consider the following discrete-time dynamic process. At each period, a pair of researchers are chosen randomly. If they are already directly linked, they can decide whether to keep the link or sever it. If they are not linked, they can decide whether to form a new link between them. We assume that each researcher is myopic. The resulting transition dynamics is described in Figures 2, 3, and 4.

When there are more than one pairwise stable networks, we can use the above dynamic process to check which one is more likely to be realized than others.

[1] Jackson and Wolinsky (1996) assume $a = b$. So, our model is a slightly generalized version of the original one.

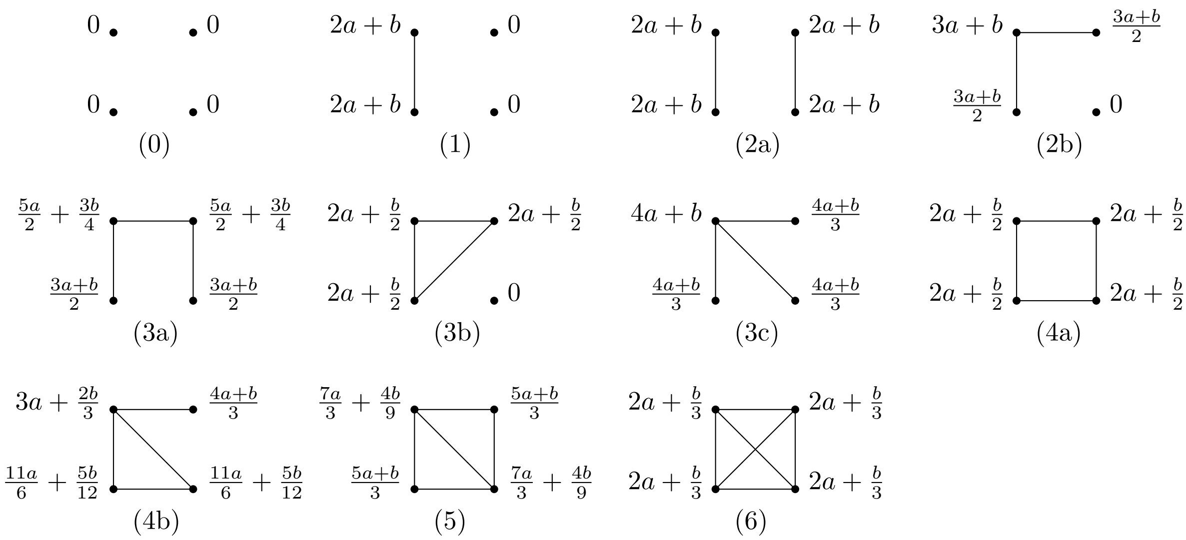

Figure 1: The payoffs to researchers in each network.

Conditions on a and b	Pareto optimal	pairwise stable
$b \leq \frac{5a}{3}$	(2a), (2b), (3a), (3c), (4b), (5)	(6)
$\frac{5a}{3} < b \leq 2a$	(2a), (2b), (3a), (3c), (4b)	(6)
$2a < b \leq 3a$	(2a), (2b), (3c), (4b)	(2a), (6)
$3a < b \leq 6a$	(2a), (2b), (3c)	(2a), (6)
$6a < b$	(2a), (2b), (3c)	(2a), (4a), (6)

Table 1: Pareto optimal networks and pairwise stable networks

Conditions on a and b	pairwise stable	stochastically stable
$b \leq 2a$	(6)	(6)
$2a < b \leq 6a$	(2a), (6)	(6)
$6a < b$	(2a), (4a), (6)	(2a), (4a), (6)

Table 2: Pairwise stable networks and stochastically stable networks

Next, we assume that researchers' decisions are reversed with a small probability ε. This additional assumption generates a Markov chain which has a unique stationary distribution, and the process converges to this distribution from any initial state. The transition dynamics with errors is described in Figures 5, 6, and 7.

A network is **stochastically stable** (Jackson and Watts, 2002) if it has a positive support in the limiting stationary distribution when $\varepsilon \to +0$.

It is known that the set of stochastically stable networks is a subset of the set of pairwise stable networks. It is also known that, when there are more than one pairwise stable networks, there is a method to find out which one is stochastically stable. Basically, this method uses a picture such as Figures 2, 3, and 4 and counts the number of "mutations" which is required to go from one pairwise stable network to others.[2] The results are presented in Table 2. When $2a < b \leq 5a$, although both (2a) and (6) are pairwise stable, only (6) is stochastically stable. When $6a < b$, however, the set of pairwise stable networks coincides with that of stochastically stable networks. In such a case, one way to find out which network is more likely to be realized than others is to calculate the limiting stationary distribution explicitly, and this is what we do in the rest of the note.

Suppose that $6a < b$. Given $\varepsilon > 0$, the stationary distribution of the corresponding Markov chain is defined by the following equations:

$$
\begin{aligned}
p_0 &= \varepsilon p_0 + \frac{\varepsilon}{6} \cdot p_1, \\
p_1 &= (1-\varepsilon)p_0 + \frac{1+4\varepsilon}{6} \cdot p_1 + \frac{2\varepsilon}{6} \cdot p_{2\mathrm{a}} + \frac{2\varepsilon}{6} \cdot p_{2\mathrm{b}}, \\
p_{2\mathrm{a}} &= \frac{1-\varepsilon}{6} \cdot p_1 + (1-\varepsilon)p_{2\mathrm{a}} + \frac{1-\varepsilon}{6} \cdot p_{3\mathrm{a}},
\end{aligned}
$$

[2]See Jackson and Watts (2002) or Young (1998) for the details of this method.

$$
\begin{aligned}
p_{2b} &= \frac{4(1-\varepsilon)}{6}\cdot p_1 + \frac{2(1+\varepsilon)}{6}\cdot p_{2b} + \frac{2\varepsilon}{6}\cdot p_{3a} + \frac{3\varepsilon}{6}\cdot p_{3b} + \frac{3\varepsilon}{6}\cdot p_{3c},\\
p_{3a} &= \frac{4\varepsilon}{6}\cdot p_{2a} + \frac{2(1-\varepsilon)}{6}\cdot p_{2b} + \frac{4-2\varepsilon}{6}\cdot p_{3a} + \frac{4\varepsilon}{6}\cdot p_{4a} + \frac{2(1-\varepsilon)}{6}\cdot p_{4b},\\
p_{3b} &= \frac{1-\varepsilon}{6}\cdot p_{2b} + \frac{3}{6}\cdot p_{3b} + \frac{\varepsilon}{6}\cdot p_{4b},\\
p_{3c} &= \frac{1-\varepsilon}{6}\cdot p_{2b} + \frac{3}{6}\cdot p_{3c} + \frac{\varepsilon}{6}\cdot p_{4b},\\
p_{4a} &= \frac{1-\varepsilon}{6}\cdot p_{3a} + (1-\varepsilon)p_{4a} + \frac{1-\varepsilon}{6}\cdot p_5,\\
p_{4b} &= \frac{2\varepsilon}{6}\cdot p_{3a} + \frac{3(1-\varepsilon)}{6}\cdot p_{3b} + \frac{3(1-\varepsilon)}{6}\cdot p_{3c} + \frac{2(1+\varepsilon)}{6}\cdot p_{4b} + \frac{4\varepsilon}{6}\cdot p_5,\\
p_5 &= \frac{2\varepsilon}{6}\cdot p_{4a} + \frac{2(1-\varepsilon)}{6}\cdot p_{4b} + \frac{4-2\varepsilon}{6}\cdot p_5 + \varepsilon p_6,\\
p_6 &= \frac{1-\varepsilon}{6}\cdot p_5 + (1-\varepsilon)p_6,\\
1 &= p_0 + p_1 + p_{2a} + p_{2b} + p_{3a} + p_{3b} + p_{3c} + p_{4a} + p_{4b} + p_5 + p_6.
\end{aligned}
$$

By solving these equations, we have

$$
\begin{aligned}
p_0 &= \frac{4\varepsilon^2 - 4\varepsilon^3 + 3\varepsilon^4 + 36\varepsilon^5 - 25\varepsilon^6 + 58\varepsilon^7}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184},\\
p_1 &= \frac{24\varepsilon - 48\varepsilon^2 + 42\varepsilon^3 + 198\varepsilon^4 + 366\varepsilon^5 + 498\varepsilon^6 - 348\varepsilon^7}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184},\\
p_{2a} &= \frac{105\varepsilon^2 - 204\varepsilon + 594\varepsilon^3 - 1374\varepsilon^4 + 1380\varepsilon^5 - 735\varepsilon^6 + 174\varepsilon^7 + 60}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184},\\
p_{2b} &= \frac{24\varepsilon + 120\varepsilon^2 - 204\varepsilon^3 - 36\varepsilon^4 + 780\varepsilon^5 - 1380\varepsilon^6 + 696\varepsilon^7}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184},\\
p_{3a} &= \frac{336\varepsilon - 816\varepsilon^2 - 276\varepsilon^3 + 3132\varepsilon^4 - 4548\varepsilon^5 + 2868\varepsilon^6 - 696\varepsilon^7}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184},\\
p_{3b} &= \frac{8\varepsilon + 36\varepsilon^2 + 32\varepsilon^3 - 136\varepsilon^4 - 240\varepsilon^5 + 532\varepsilon^6 - 232\varepsilon^7}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184},\\
p_{3c} &= \frac{8\varepsilon + 36\varepsilon^2 + 32\varepsilon^3 - 136\varepsilon^4 - 240\varepsilon^5 + 532\varepsilon^6 - 232\varepsilon^7}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184},\\
p_{4a} &= \frac{9\varepsilon^2 - 288\varepsilon + 1230\varepsilon^3 - 2388\varepsilon^4 + 2124\varepsilon^5 - 951\varepsilon^6 + 174\varepsilon^7 + 90}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184},\\
p_{4b} &= \frac{12\varepsilon + 420\varepsilon^2 - 576\varepsilon^3 - 1536\varepsilon^4 + 3756\varepsilon^5 - 2772\varepsilon^6 + 696\varepsilon^7}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184},
\end{aligned}
$$

$$
\begin{aligned}
p_5 &= \frac{204\varepsilon - 372\varepsilon^2 - 858\varepsilon^3 + 3114\varepsilon^4 - 3534\varepsilon^5 + 1794\varepsilon^6 - 348\varepsilon^7}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184}, \\
p_6 &= \frac{662\varepsilon^3 - 81\varepsilon^2 - 96\varepsilon - 1108\varepsilon^4 + 888\varepsilon^5 - 357\varepsilon^6 + 58\varepsilon^7 + 34}{28\varepsilon - 587\varepsilon^2 + 674\varepsilon^3 - 267\varepsilon^4 + 36\varepsilon^5 + 4\varepsilon^6 + 184}.
\end{aligned}
$$

Thus, in the limiting distribution, we have $p_{2a} = \frac{60}{184} \approx 0.326$, $p_{4a} = \frac{90}{184} \approx 0.489$, $p_6 = \frac{34}{184} \approx 0.185$, and all other probabilities are zero. So, one can say that among these three stochastically stable networks, network (4a) is more likely to be realized than others.

References

[1] Matthew O. Jackson and Alison Watts. "The evolution of social and economic networks." *Journal of Economic Theory*, 106:265–295, 2002.

[2] Matthew O. Jackson and Asher Wolinsky. "A strategic model of social and economic networks." *Journal of Economic Theory*, 71:44–74, 1996.

[3] H. P. Young. *Individual Strategy and Social Structure: An Evolutionary Theory of Institutions.* Princeton UP, 1998.

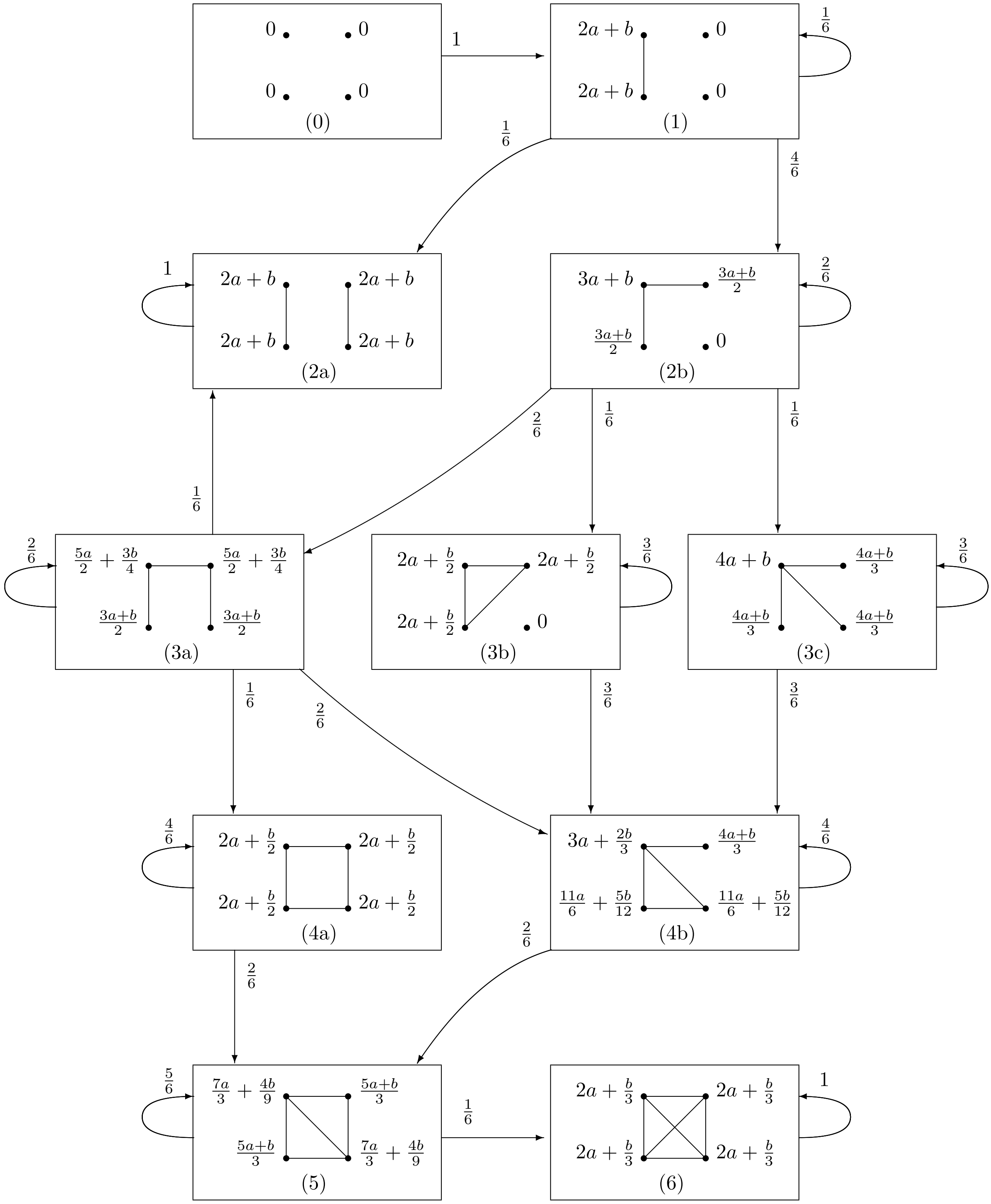

Figure 2: The dynamics in the case of $2a < b \leq 4a$.

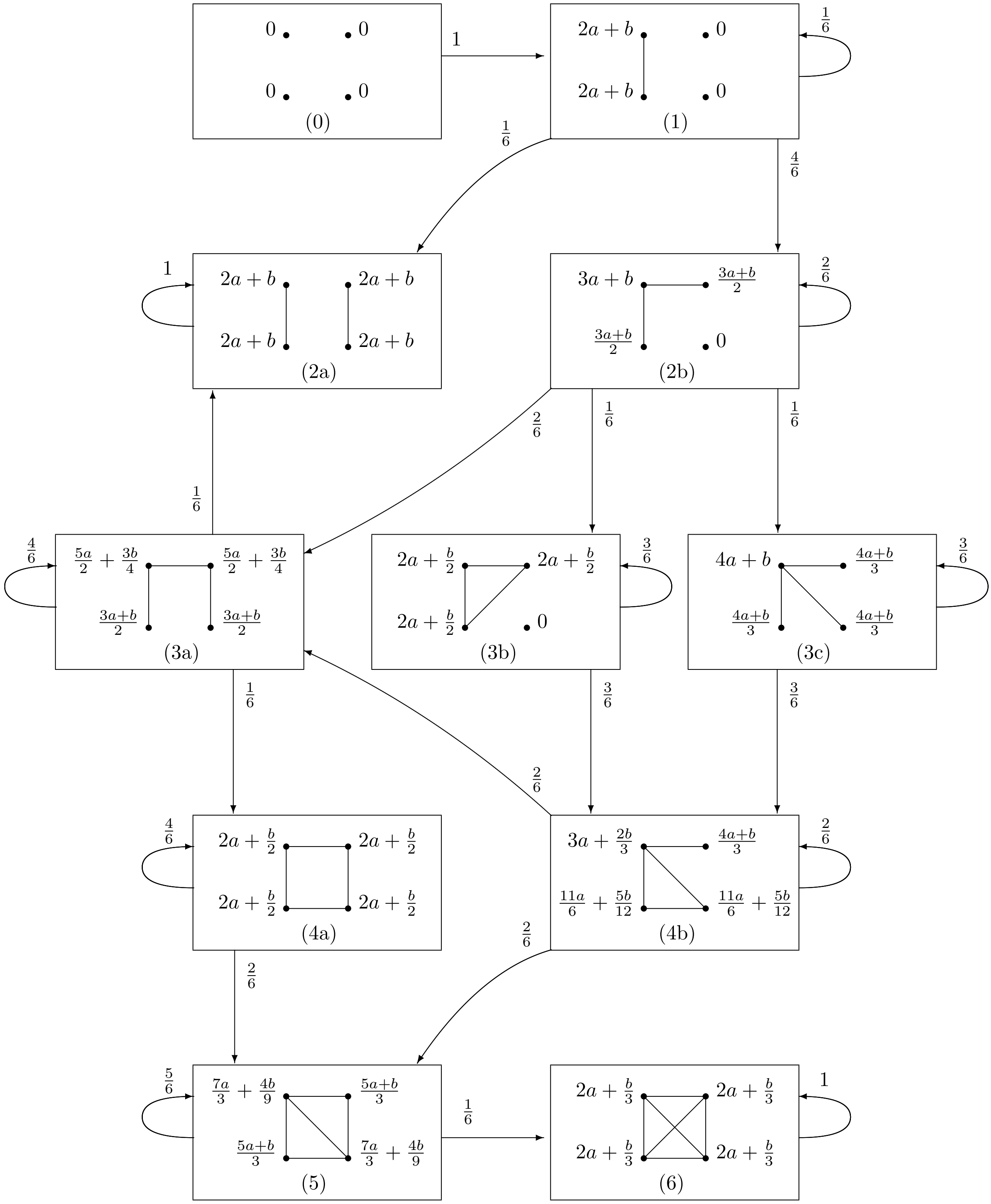

Figure 3: The dynamics in the case of $4a < b \leq 6a$.

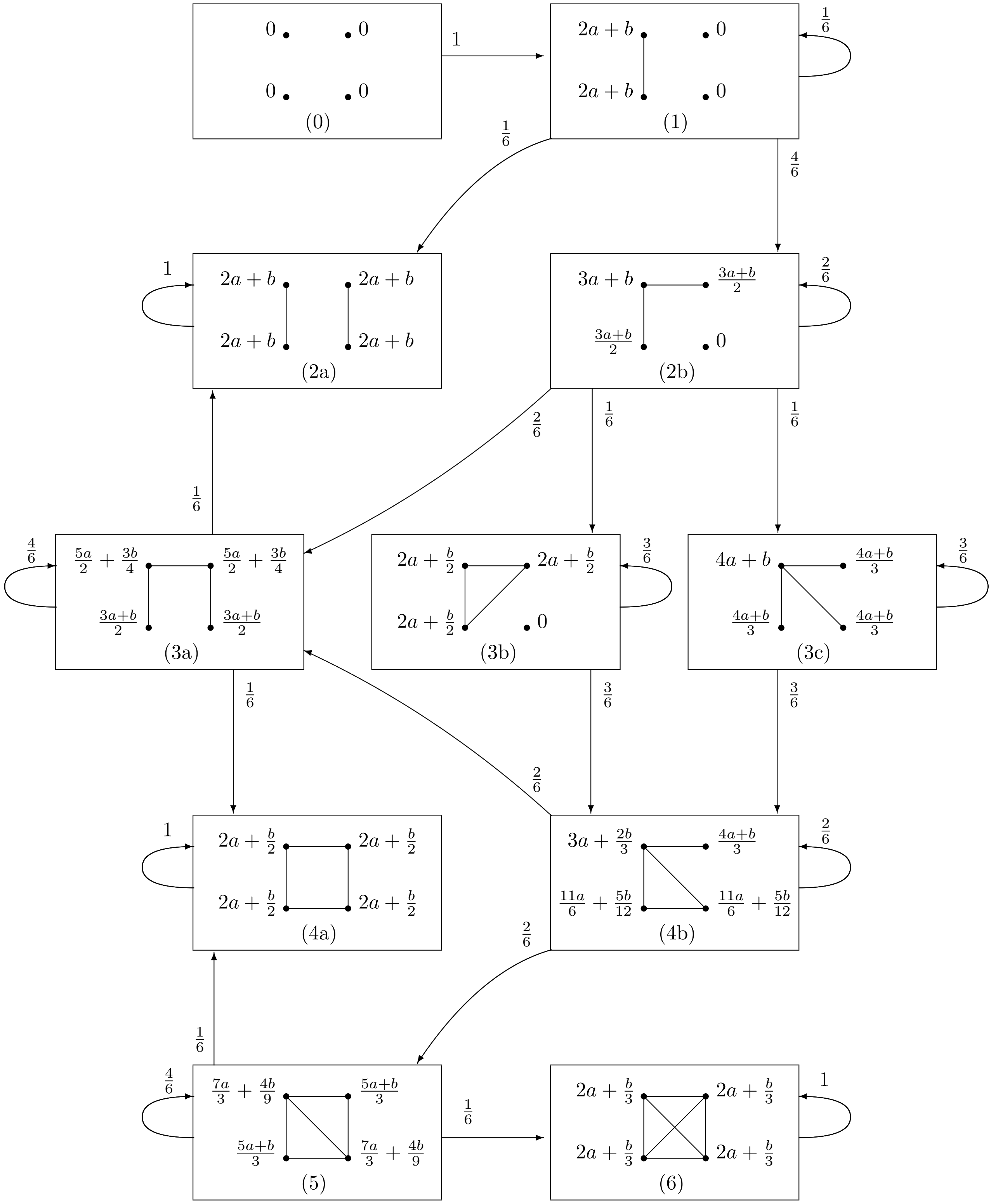

Figure 4: The dynamics in the case of $6a < b$.

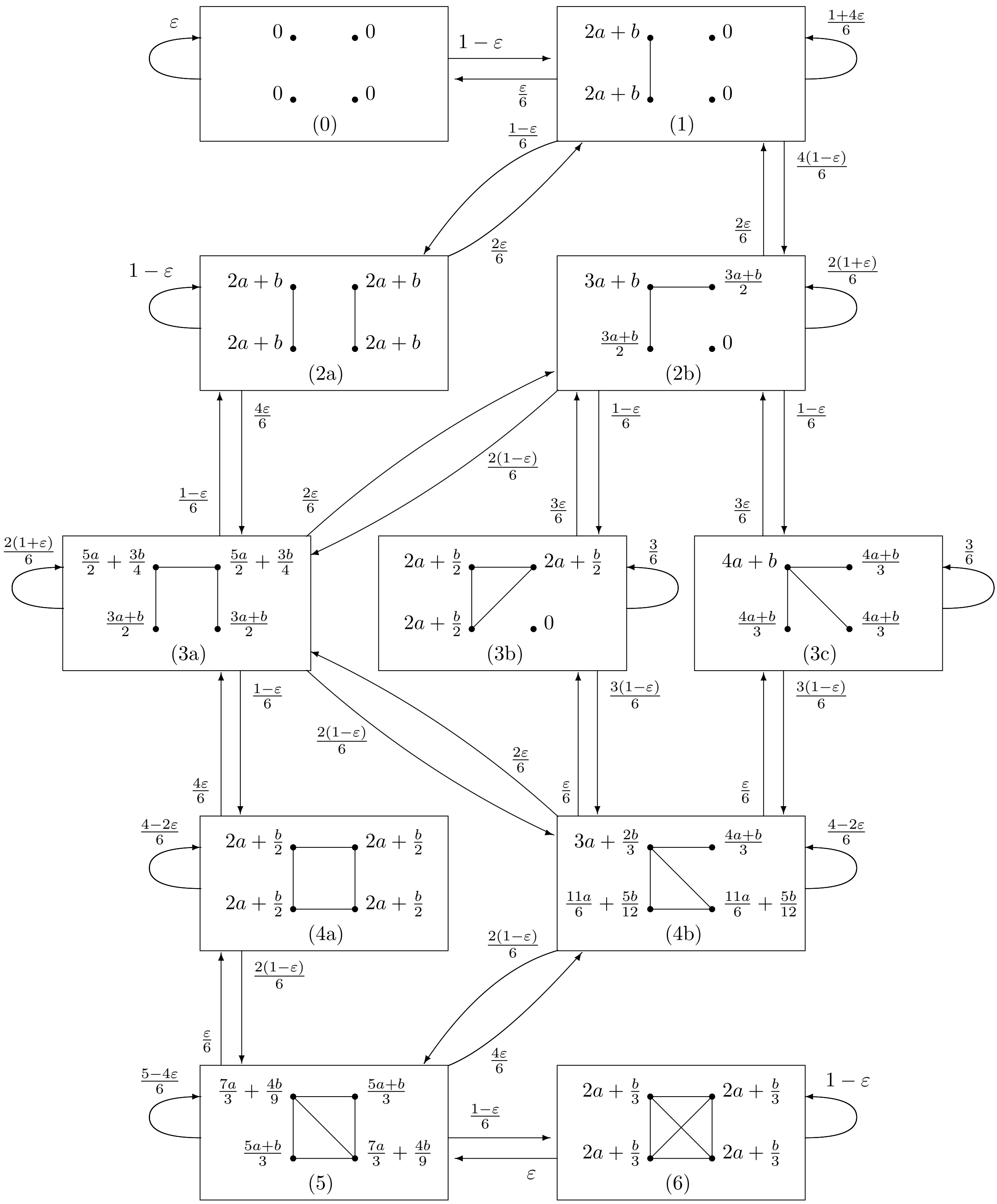

Figure 5: The dynamics with errors in the case of $2a < b \leq 4a$.

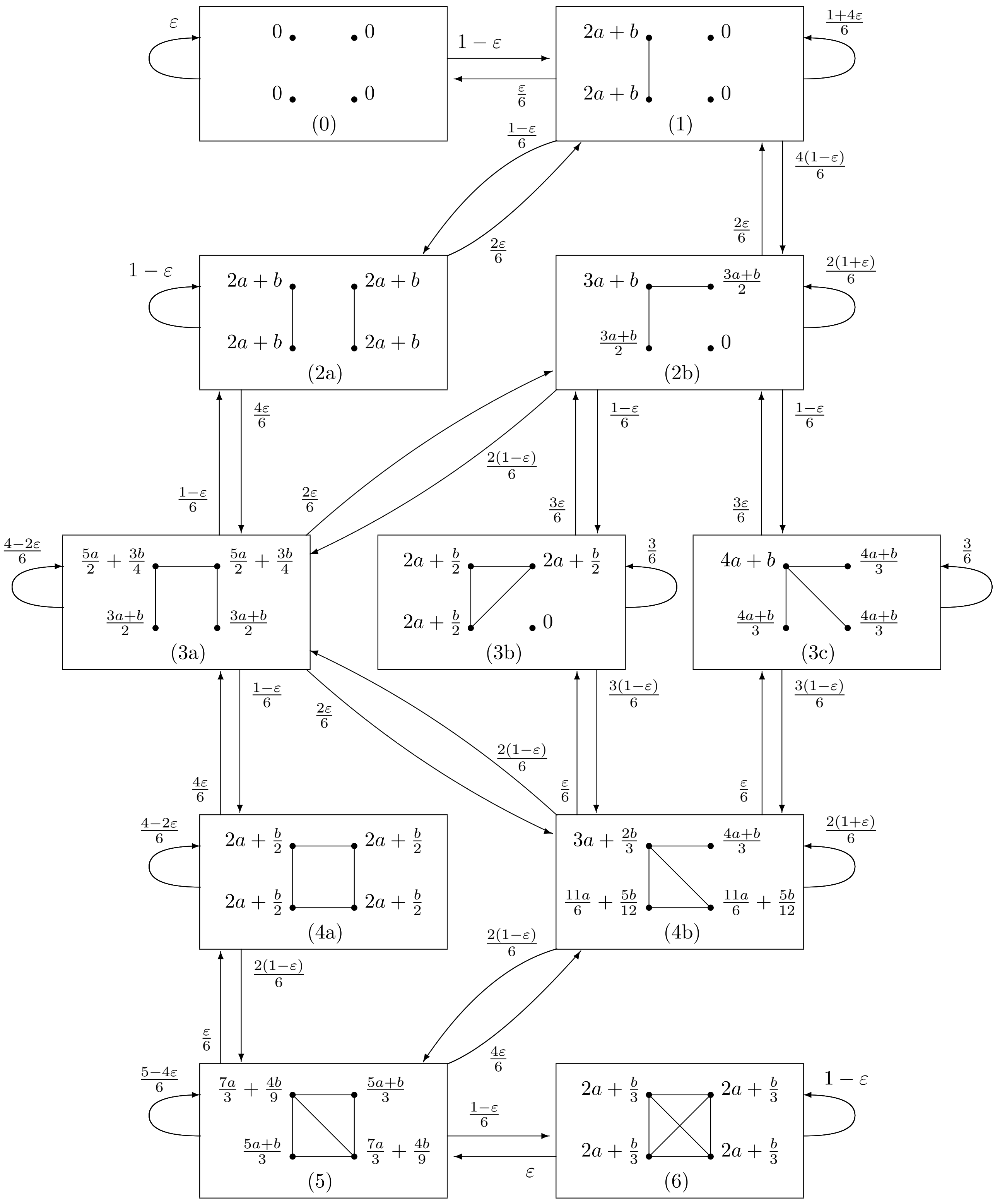

Figure 6: The dynamics with errors in the case of $4a < b \leq 6a$.

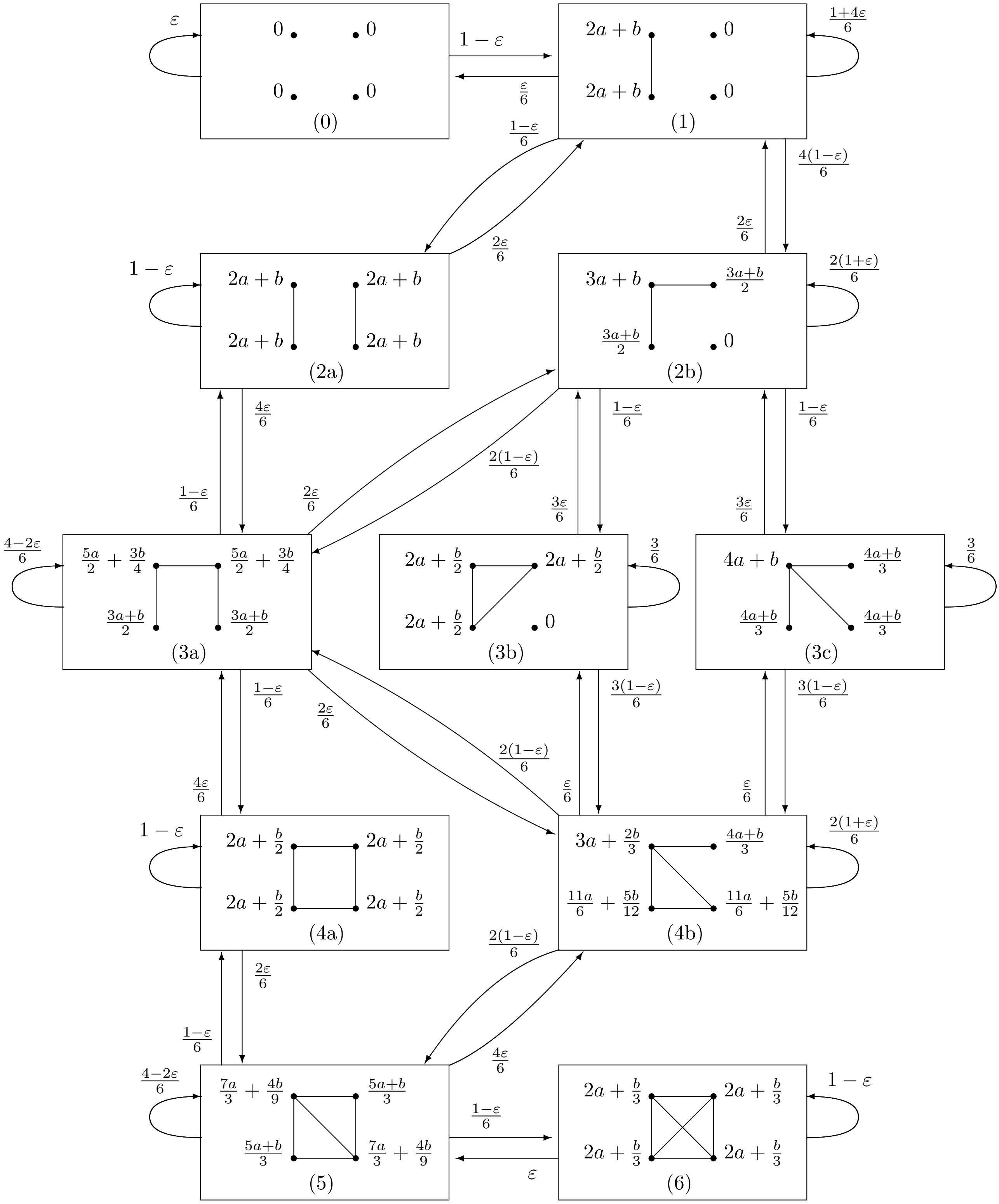

Figure 7: The dynamics with errors in the case of $6a < b$.