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Some equivalence results between mixed strategy Nash equilibria and minimax regret in 2x2 games

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Abstract

We show that in any 2x2 game in which a unique mixed strategy Nash equilibrium exists, the probability distribution that this equilibrium assigns to player i is either the same or the mirror image of the distribution that the minimax regret criterion defines for player j. Sharper results that connect the two distributions for the same player are then established for the class of symmetric games.

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1 Introduction

This note identifies the precise relationship that connects the $m$ixed $s$trategy Nash $e$quilibria (shortened in what follows to $msNe$) with the $m$ixed version of the $m$ini$m$ax $r$egret criterion ($mmmr$) in 2x2 games. Nash equilibrium is a fundamental concept in game theory. Minimax regret is an important tool in decision theory.

It is shown that, whenever a well-defined $msNe$ exists, the probability that this equilibrium assigns to player i playing a certain strategy and the probability that the $mmmr$ attaches to player j playing the correspondent strategy are either identical or they are complement to one. In other words the probability distribution defined by the two concepts is always the same, but the order with which the probabilities are associated with the two strategies may be different. Consider for instance a game in which generic player $i \in \{A, B\}$ can choose between strategies T_i and B_i. If the $msNe$ of player A is given by $\lambda T_A + (1-\lambda)B_A$ with $\lambda \in [0,1]$ then the $mmmr$ of player B will be either $\lambda T_B + (1-\lambda)B_B$ or $(1-\lambda)\, T_B + \lambda B_B$. We provide sufficient conditions for distinguishing between these two cases as well as more precise results for the class of symmetric games. Before moving to the proof of the claim we briefly review the two concepts we are interested in.

In a mixed strategy Nash equilibrium (Nash, 1951) each player randomizes over his pure strategies according to a probability distribution that makes his opponent indifferent as to what to play. It follows that each player is mutually (weakly) best responding and therefore the two distributions identify a Nash equilibrium.

Minimax regret has been originally proposed by Savage (1951) as a criterion to deal with choices under uncertainty. An axiomatic characterization of minimax regret appears in Milnor (1954). As the name suggests, this criterion indicates the (pure or mixed) strategy an agent should adopt in order to minimize the maximum regret he may suffer. The regret is defined as the difference between the payoff stemming from the actual choice of the player and the payoff associated with the optimal choice conditional on the realized state of Nature. As already mentioned, minimax regret is mainly used in decision theory (Acker, 1997, Bossert and Peters, 2001) but it also finds applications in pure (Rustichini, 1999) and behavioral (Gallice, 2007) game theory as well as in econometrics (Manski, 2007) and AI design (Brafman and Tennenholtz, 2000).

2 The case of a generic game

Consider the following 2x2 game which we indicate with G_1.

		(q_B)	$(1-q_B)$
		T_B	B_B
$G_1)$	T_A	$a, \cdot$	$b, \cdot$
	B_A	$c, \cdot$	$d, \cdot$

Notice that only the payoffs of player A appear in the game matrix. The payoffs of player B are left unspecified for simplicity and without loss of generality.

The game does not have to be symmetric or zero sum. The only restriction on the payoffs is that they are such that no dominant strategies exist. In 2x2 games the absence of strictly dominant strategies is in fact a necessary and sufficient condition to ensure the existence of a $msNe$. Still, in order to ensure its uniqueness, also weakly dominant strategies have to be ruled out. In other words we focus on non-degenerate games.[1] On the other hand a unique minimax regret (pure or mixed) strategy exists for any game in normal form, in the sense that the criterion always selects a distribution defined by probabilities belonging to the interval $[0, 1]$.

Ruling out the possibility that in Game G_1 player A may have a dominant strategy, we are left with two possible cases:

Case 1) $a, b, c, d : (a > c \ \wedge \ b < d)$

Case 2) $a, b, c, d : (a < c \ \wedge \ b > d)$

Depending on B's payoffs, the partial structure of Game G_1 is thus compatible with Matching Pennies (both cases 1 and 2), Hawk-Dove (case 2) and various kinds of coordination games (case 1): Pure Coordination, Stag Hunt, Battle of the Sexes.

In what follows we indicate with $\hat{q}_i^n$ the unique probability that the mixed strategy Nash equilibrium attaches to player i playing strategy T_i. Subscript $i \in \{A, B\}$ thus indicates the player while superscript $n \in \{1, 2\}$ refers to the two possible cases described above. In a similar way we indicate with $\hat{p}_i^n$ the probability that the mixed minimax regret assigns to player i playing strategy T_i. Given that both the $msNe$ and the $mmmr$ are defined over a discrete support of just two elements (the two pure strategies T_i and B_i), it follows that $\hat{q}_i^n$ and $\hat{p}_i^n$ are enough to identify the complete distributions.

Going back to Game G_1, player A's payoffs are the "ingredients" needed in order to compute the $msNe$ of player B. The mixed Nash equilibrium of generic player i is in fact only a function of the payoffs of the other player j. In the literature this peculiar feature of the $msNe$ is called the "no own payoff effect".[2]

In the case of Game G_1 the $msNe$ of player B is computed imposing the equality $aq_B + b(1 - q_B) = cq_B + d(1 - q_B)$ and solving it for q_B to get:

$$\hat{q}_B^1 = \hat{q}_B^2 = \frac{d - b}{a - c + d - b}$$

If player B adopts the mixed strategy $\hat{q}_B^n T_B + (1 - \hat{q}_B^n)\, B_B$ with $n = \{1, 2\}$ then player A is indifferent to playing any of his two pure strategies (or any convex combination of them) since they both lead to an expected payoff of $\frac{ad-bc}{a-c+d-b}$. Notice that the

[1] A 2x2 game is defined as degenerate if some player has two pure best responses to a pure strategy of the opponent.

[2] Incidentally the empirical evidence of such an effect is seriously questioned by experimental results (see for instance Goeree et al., 2002, for the case of Matching Pennies games).

structure of the $msNe$ is not affected by the ranking of the payoffs. In fact its formula remains the same both in Case 1 $\left(\hat{q}_B^1\right)$ and in Case 2 $\left(\hat{q}_B^2\right)$.

The payoffs of player A can also be used to compute A's minimax regret. The first step for doing so consists in constructing the regret matrix $r_{G_1}^n$, in our context one for each case $n \in \{1, 2\}$. In fact, contrary to the $msNe$, the formula for the minimax regret does depend on the ranking of the payoffs. Each cell of the two tables below displays the value of the regret. The regret is defined as the (non negative) difference between the payoff that player A got and the best payoff he could have got if he had known in advance the move of his opponent. The regret matrix of any 2x2 game always contains at least two zeros, a feature which makes the calculation of the minimax regret very easy.

$r_{G_1}^1$)

		T_B	B_B
(p_A^1)	T_A	$0, \cdot$	$d-b, \cdot$
$(1-p_A^1)$	B_A	$a-c, \cdot$	$0, \cdot$

$r_{G_1}^2$)

		T_B	B_B
(p_A^2)	T_A	$c-a, \cdot$	$0, \cdot$
$(1-p_A^2)$	B_A	$0, \cdot$	$b-d, \cdot$

Consider the regret matrix $r_{G_1}^1$ which refers to Case 1 defined above. Pure strategy T_A attains minimax regret if $d-b < a-c$. At the opposite, if $d-b > a-c$, strategy B_A is the one selected by the (pure) minimax regret criterion. Allowing for mixed strategies (a possibility already considered in Milnor, 1954), player A can randomize with probabilities p_A^1 and $(1-p_A^1)$ over T_A and B_A such as to minimize the expected regret he may suffer.

In the situation of Game G_1, the $mmmr$ of player A takes the following values:

Case 1, matrix $r_{G_1}^1$.
The minimax regret mixed strategy is given by $\hat{p}_A^1 T_A + (1-\hat{p}_A^1)B_A$ where $\hat{p}_A^1$ solves $(d-b)p_A^1 = (a-c)(1-p_A^1)$, i.e.:

$$\hat{p}_A^1 = \frac{a-c}{a-c+d-b}$$

Case 2, matrix $r_{G_1}^2$.
The minimax regret mixed strategy is given by $\hat{p}_A^2 T_A + (1-\hat{p}_A^2)B_A$ where $\hat{p}_A^2$ solves $(c-a)p_A^2 = (b-d)(1-p_A^2)$, i.e.:

$$\hat{p}_A^2 = \frac{d-b}{a-c+d-b}$$

Having computed the $msNe$ of player B and the $mmmr$ of player A we are now able to state the first result of the paper.

Theorem 1 *In any 2x2 game with no dominant strategies the probability distributions implied by the msNe of player j and by the mmmr of player i are either mirror images* ($\hat{q}_j = 1 - \hat{p}_i$) *or they are the same* ($\hat{q}_j = \hat{p}_i$).

Proof. Compare $\hat{q}_B^n$ (the probability the *msNe* attaches to player B playing strategy T_B) with $\hat{p}_A^n$ (the probability the *mmmr* assigns to player A playing T_A) and notice that:

$$\text{Case 1)} \quad \hat{q}_B^1 = 1 - \hat{p}_A^1 \qquad\qquad \text{Case 2)} \quad \hat{q}_B^2 = \hat{p}_A^2$$

Given that a single probability (respectively $\hat{q}_B^n$ and $\hat{p}_A^n$) is enough to define the complete distributions of the Nash equilibrium and of the minimax regret, the equalities above prove the relation between the *msNe* of player B and the *mmmr* of player A. An analogous demonstration proves the link between the *msNe* of player A and the *mmmr* of player B. ∎

A sufficient condition for having $\hat{q}_j = 1 - \hat{p}_i$ is the one which defines Case 1. In particular, payoffs of A have to be such that $a > c \ \wedge \ b < d$ and payoffs of B have to obey a similar condition. On the other hand if the condition $a < c \ \wedge \ b > d$ holds then $\hat{q}_j = \hat{p}_i$ will be the case.

The reason for the equivalence result of Theorem 1 relies on the fact that both the mixed strategy Nash equilibrium and the minimax regret are based on some indifference conditions. In a mixed strategy Nash equilibrium a player randomizes over his two pure strategies in such a way to make his opponent indifferent as to what to play. Under minimax regret the player mixes in order to be indifferent with respect to the regret the two strategies may lead to. In the simplified framework of 2x2 games these two probability distributions happen to be related according to the statement of Theorem 1.[3]

3 The case of symmetric games

If the 2x2 game is symmetric then more accurate results can be stated. Symmetric games are games in which the payoff matrix of player j is the transpose of the payoff matrix of player i. In other words each of the two players faces exactly the same situation so that there are no differences between being a row or a column player. An example of a symmetric game which is built on the initial partial game of the previous section is given by Game G_2 below:

[3] Still notice that this does not happen in the case of another well known concept with a long tradition both in game theory (von Neumann and Morgenstern, 1944) and in decision theory (Wald, 1950, Milnor, 1954), namely the maximin criterion which selects the strategy that maximizes the minimum payoff a player can get. In the context of Game G_1 the maximin mixed strategy of player A is defined by $\hat{r}_A^n T_A + (1 - \hat{r}_A^n) B_A$ with $\hat{r}_A^n = \frac{d-c}{a-c+d-b}$ for $n \in \{1, 2\}$ and it is thus different from the distributions defined by the *msNe* and the *mmmr* whenever $b \neq c$.

$G_2)$		T_B	B_B
	T_A	a, a	b, c
	B_A	c, b	d, d

It is a basic result of game theory that every symmetric game has at least a symmetric Nash equilibrium. A symmetric equilibrium is an equilibrium in which the two players adopt the same strategy. In particular the mixed equilibrium, if it exists, is always symmetric $\left(\hat{q}_i^n = \hat{q}_j^n\right)$. Given the structure of symmetric games it is clear that also the minimax regret distributions of the two players are always identical $\left(\hat{p}_i^n = \hat{p}_j^n\right)$. Combining these last two equalities with the results of Theorem 1 we can thus establish a precise relationship between the $msNe$ and the *mmmr* of the *same* player.

Theorem 2 refers to the class of symmetric Coordination games. These games are characterized by the presence of three symmetric Nash equilibria: the mixed equilibrium and the two equilibria in pure strategies. In the case of Game G_2, these last two equilibria would obviously be (T_A, T_B) and (B_A, B_B).

Theorem 2 *In any 2x2 symmetric Coordination game the probability distributions defined by the $msNe$ and by the mmmr for generic player i are mirror images: $\hat{q}_i = 1-\hat{p}_i$.*

Proof. The condition which defines Case 1 ($a > c \ \wedge \ b < d$) is automatically satisfied by symmetric Coordination games. Because of Theorem 1, the $msNe$ and the *mmmr* are thus linked by the relation $\hat{q}_j = 1 - \hat{p}_i$. Moreover, being the game symmetric, we know that $\hat{q}_i = \hat{q}_j$. It follows that $\hat{q}_i = 1 - \hat{p}_i$. ■

Symmetric Pure Coordination and Stag Hunt games are examples to which Theorem 2 applies. A Stag Hunt game, together with the associated $msNe$ and *mmmr* distributions, appears in Game G_3 below:

$G_3)$		T_B	B_B
	T_A	$2, 2$	$3, 0$
	B_A	$0, 3$	$4, 4$

$msNe:$ $\left(\frac{1}{3}T_i + \frac{2}{3}B_i\right)$ for $i \in \{A, B\}$

$mmmr:$ $\left(\frac{2}{3}T_i + \frac{1}{3}B_i\right)$ for $i \in \{A, B\}$

There are other symmetric games with an equilibrium in mixed strategies. So called Hawk-Dove games have in fact a symmetric $msNe$ as well as two asymmetric pure equilibria ((B_A, T_B) and (T_A, B_B) in the case of game G_2). Theorem 3 refers to this class of games.

Theorem 3 *In any 2x2 symmetric Hawk-Dove game the probability distributions defined by the $msNe$ and by the mmmr for generic player i are equal: $\hat{q}_i = \hat{p}_i$.*

Proof. The condition which defines Case 2 ($a < c \ \wedge \ b > d$) is embedded in the structure of Hawk-Dove games. Theorem 1 states that $\hat{q}_j = \hat{p}_i$. Given symmetry we also have $\hat{q}_i = \hat{q}_j$. Therefore $\hat{q}_i = \hat{p}_i$ holds. ■

An example of a symmetric Hawk-Dove game is given by Game G_4:

G_4)

	T_B	B_B
T_A	$0,0$	$5,3$
B_A	$3,5$	$4,4$

$$msNe: \quad \left(\tfrac{1}{4}T_i + \tfrac{3}{4}B_i\right) \text{ for } i \in \{A,B\}$$
$$mmmr: \quad \left(\tfrac{1}{4}T_i + \tfrac{3}{4}B_i\right) \text{ for } i \in \{A,B\}$$

A simple counterexample is enough to prove that the relations identified by the theorems presented in this paper only apply to 2x2 games. In fact similar equivalence results do not necessarily hold for games with more than two strategies. In these situations the Nash equilibrium and the minimax regret probability distributions are often totally unrelated as is the case in Game G_5.

G_5)

	T_B	M_B	B_B
T_A	$0,0$	$0,1$	$2,0$
M_A	$1,0$	$0,0$	$0,1$
B_A	$0,2$	$1,0$	$0,0$

$$msNe: \quad \left(\tfrac{2}{5}T_i + \tfrac{2}{5}M_i + \tfrac{1}{5}B_i\right) \text{ for } i \in \{A,B\}$$
$$mmmr: \quad \left(\tfrac{3}{7}T_i + \tfrac{2}{7}M_i + \tfrac{2}{7}B_i\right) \text{ for } i \in \{A,B\}$$

4 Discussion

This note explored the relationship which links the mixed strategy Nash equilibrium with the mixed version of the minimax regret criterion. In any 2x2 game, to know the probability distribution of one of these two concepts means to know the probability distribution of the other one as well. Results are sharper for the case of symmetric games as stated by theorems 2 and 3.

Whenever interpreting the data of experiments involving 2x2 symmetric games it is important to keep in mind these results. For instance experimental data that are in line with the Nash equilibrium distribution may be consistent with the minimax regret hypothesis as well. In such a situation credit for capturing players' behavior cannot be given to any of the two concepts without further discriminating between the two alternatives.

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