E C O N O M I C S   B U L L E T I N

Testing the assumption of Linearity

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Abstract

The assumption of linearity is tested using five statistical tests for the US and the Canadian unemployment rates. An AR(p) model was used to remove any linear structure from the series. Strong evidence in favour of non−linearity was found in the case of Canada. The result for the US is not so clear cut.

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1. INTRODUCTION

The purpose of this paper is to update and extend the analysis of Brock & Sayers (1988) and Frank & Stengos (1988). These studies followed similar methodology and were amongst the first attempts to test the assumption of linearity in macroeconomic series. Brock & Sayers and Frank & Stengos (1988) employed US and Canadian macroeconomic data respectively.

The assumption of linearity has often been challenged for financial time series, but such challenges are not always successful with macroeconomic time series. Despite the fact that theoretical arguments for linearity rarely exist, the case in favour of non-linearity in macroeconomic time series is not very strong. One explanation might be the lack of long series as well as the convenience of linear forms, which are easy to work and well-understood.

In the original studies by Brock & Sayers and Frank & Stengos (1988), the US and the Canadian unemployment rates were tested using the nonparametric BDS test. For the US quarterly observations from 1949 to 1982 were used and for Canada monthly observations from 1966 to 1984 were used. In the former, strong evidence in favour of non-linearity was found but this did not seem to be the case for Canada. Although this is not the only finding that suggests that the US and Canadian unemployment rates were behaving in a different way, see for instance Dibooglu & Enders (2001), one might argue that the different sample periods as well as the different data frequency might be responsible for this result.

The aim of this econometric exercise is to update and extend this analysis. Five statistical tests are used in order to test for non-linearity. Instead of using only one test (say only the BDS), a battery of tests are employed in order to investigate the univariate properties of a time series, the unemployment rate, that captures some of the most important features of the business cycle. Given the limited sample size, the bootstrap as well as the asymptotic values of the tests are provided. Furthermore, both data sets employed in this paper consist of monthly seasonally adjusted observations from January 1976 to December 2000 (a total of 300 observations). This will facilitate comparison of our results.

It should be noted that our paper does not try to implement a non-linear model in the series - see for example Koop & Potter (1999) where a TAR model, in a Bayesian setting, is used and Dijk et al (2002) where a FI-STAR is suggested – but, following a different strategy, we will use the five different tests for non-linearity in order to investigate the assumption of linearity. Given that the "true" data generating process is not known and the fact that various econometric models have been employed in order to explain the behaviour of the series, we would attempt to assess whether the use of "non-linear" models are justified by our findings. Our argument is reinforced by Potter (1999): "*Successful nonlinear time series modelling would improve forecasts an produce a richer notion of business cycle dynamics than linear time series models allow. For this to happen two conditions are necessary. First, economic time series must contain nonlinearities. Second, we need reliable statistical methods to summarize and understand these nonlinearities suitable for time series of the typical macroeconomic length*".

The outline of the paper is as follows: Section 2 discusses the methodology followed and the tests for non-linearity that are employed. Sections 3 presents the characteristics of the data, the US and the Canadian unemployment rates. The empirical results are discussed in Section 4 and Section 5 concludes.

2. METHODOLOGY

Many statistical tests for non-linear dependence have been proposed in the recent literature. Instead of only using a single statistical test, five different tests are considered in this exercise for detecting non-linear serial dependence. This will allow us on the one hand to obtain a deeper and more detailed insight into the series properties by generating useful information from the various tests and on the other hand to minimise the probability of missing something and thus drawing the wrong conclusion. If our battery of tests displays a unanimous "consensus" in favour of a specific result, we would interpret this "consensus" as strong corroboration of that outcome.

The five tests that are going to be used are the following: McLeod & Li (1983), Engle LM (1982), BDS (1996), Tsay (1986), and Hinich & Patterson (bicovariance) (1995). All these tests share the principle that once any linear structure is removed from the data, any remaining structure should be due to a non-linear data generating mechanism.

The linear structure is removed from the data through a pre-whitening model in the following way: Firstly, we fit an AR(p) model to the sample data for values of p from 0 to 10. The optimal lag length is chosen to minimise the Schwartz criterion (SC). The Schwartz criterion is known to be consistent for AR(p) order determination under the null hypothesis of a linear generating mechanism compared to alternative choices (e.g. AIC; see Judge, et al. 1985, p.246). The residuals of the preferred AR(p), $\{e_t\}$, which are by construction serially uncorrelated, are then tested for non-linear independence using each of the procedures in turn. Other specifications such as ARMA or GARCH could be used as an alternative pre-whitening model, but we note that a GARCH model cannot be used unless the linearity assumption has been rejected.

All the procedures embody the null hypothesis that the series under consideration is an *i.i.d.* process. Details of the tests are as follows.

BDS TEST FOR RANDOMNESS

A powerful test used for independence -and, under certain circumstances, for non-linear dependencies- was developed by Brock, Dechert, and Scheinkman (1996) and is based on the correlation integral. The BDS statistic tests the null hypothesis that the elements of a time series are independently and identically distributed (IID). For a time series which is IID, the distribution of the statistic:

$$W_m(e) = \frac{\sqrt{n}\left\{C_m(e) - C_1(e)^m\right\}}{s_m(e)} \qquad (1)$$

is asymptotically $N(0,1)$.$W_m(e)$ is known as the BDS statistic. $C_m(\varepsilon)$ denotes the fraction of m-tuples in the series, which are within a distance of each other and $s_m(e)$ is an estimate of the standard deviation under the null hypothesis of *IID*. The test statistic is asymptotically standard normal under the null of whiteness. The null is rejected if the test statistic is absolutely large, (say greater than 1.96). If the null hypothesis of *IID* cannot be accepted this implies that the residuals contain some kind of hidden structure, which might be non-linear – or even chaotic. Following the recommendation by Brock, Hsieh & LeBaron (1991, p169) and the suggestions by Brooks & Heravi (1999), we set e/s =0.5 to 2, and m= 2 to 4.

MCLEOD AND LI TEST

The McLeod and Li test (McLeod and Li, 1983) can be used as a portmanteau test of non-linearity. To test for non-linear effects in time series data McLeod and Li have proposed the statistic:

$$Q(m) = \frac{n(n+2)}{n-k} \sum_{k=1}^{m} r_a^2(k) \qquad (2)$$

where

$$r_a^2(k) = \frac{\sum_{t=k+1}^{n} e_t^2 e_{t-k}^2}{\sum_{t=1}^{n} e_t^2} \qquad k = 0,1,\ldots, n\text{-}1 \qquad (3)$$

are the autocorrelations of the squared residuals, e_t^2, obtained from fitting a model to the data. If the series e_t is independently and identically distributed (*IID*) then the asymptotic distribution of $Q(m)$ is χ^2 with m degrees of freedom.

ENGLE LM TEST

This test was suggested by Engle (1982) to detect ARCH disturbances. Bollerslev (1986) suggests that it should also have power against GARCH alternatives. Since it is a Lagrange Multiplier test, the test statistic itself is based on the R^2 of an auxiliary regression, which in this case can be defined as:

$$e_t^2 = a_0 + \sum_{i=1}^{p} a_k e_{t-i}^2 + v_t \qquad (4)$$

Under the null hypothesis of a linear generating mechanism for e_t, NR^2 for this regression is asymptotically $c^2(p)$.

HINICH BICOVARIANCE TEST

This test assumes that $\{e_t\}$ is a realisation from a third-order stationary stochastic process and tests for serial independence using the sample bicovariances of the data. The (r,s) sample bicovariance is defined as :

$$C_3(r,s) = (N-s)^{-1} \sum_{t=1}^{N-s} e_t e_{t+r} e_{t+s} \qquad 0 \le r \le s \qquad (5)$$

The sample bicovariances are thus a generalisation of a skewness parameter. The $C_3(r,s)$ are all zero for zero mean, serially *i.i.d.* data. One would expect non-zero values for the $C_3(r,s)$ from data in which e_t depends on lagged cross-products, such as $e_{t-i}e_{t-j}$ and higher order terms.
Let $G(r,s) = (N\text{-}s)^{1/2} C_3(r,s)$ and define X_3 as

$$X_3 = \sum_{s=2}^{l} \sum_{r=1}^{s-1} [G(r,s)]^2 \qquad (6)$$

Under the null hypothesis that $\{e_t\}$ is a serially *i.i.d.* process, Hinich and Patterson (1995) show that X_3 is asymptotically distributed $c^2(l[l\text{-}1]/2)$ for $l<N^{1/2}$

Based on their simulations, they recommend using $l = N^4$. Under the assumption that $E(x_t^{12})$ exists, the X_3 statistic detects non-zero third order correlations. It can be considered a generalisation of the Box-Pierce portmanteau statistic.

TSAY TEST

The Tsay (1986) test is a generalisation of the Keenan (1985) test. It explicitly looks for quadratic serial dependence in the data.

Let $K=k(k-1)/2$ column vectors $V_1,\ldots,V_k$ contain all of the possible cross-products of the form $e_{t-i}e_{t-j}$, where i Î $[1,k]$ and j Î $[i,k]$. Thus, $v_{t,1} = e_{t-1}^2$, $v_{t,2}=e_{t-1}e_{t-2}$, $v_{t-3}=e_{t-1}e_{t-3}$, $v_{t,k+1}=e_{t-2}e_{t-3}$, $v_{t,k+2}=e_{t-2}e_{t-4},..$, $v_{t,k} = e_{t-k}^2$ .And let $\hat{v}_{t,j}$ denote the projection of $v_{t,i}$ on the orthogonal subspace $e_{t-1},\ldots,e_{t-k}$, (i.e. the residuals from a regression of $v_{t,j}$ on $e_{t-1},\ldots,e_{t-k}$.

The parameters $g_1,\ldots,g_k$ are then estimated by applying OLS to the regression equation

$$e_t = g_0 + \sum_{i=1}^{K} g_i \hat{v}_{t,i} + h_t \qquad (7)$$

Note that the jth regressor in this equation is $\hat{v}_{t,j}$, the period t fitting error from a regression of $v_{t,j}$ on $e_{t-1},\ldots,e_{t-k}$. So long as p exceeds K, this projection is unnecessary for the dependent variable $\{e_t\}$ if it is pre-whitened using an AR(p) model. The Tsay test statistic then is just the usual F statistic for testing the null hypothesis that $g_1,\ldots,g_k$ are all zero.

The reader is also referred to the detailed discussion of these tests in Patterson & Ashley (2000). In line with other studies (e.g. Brock, Hsieh and LeBaron, 1991), they conclude that the BDS test is the most powerful one. However, two simulation studies by Brooks & Heravi (1999) and Brooks & Henry (2000) revealed that the BDS test can sometimes confuse different types of non-linear structure (such as threshold autoregressive and GARCH-type models) and has small power in detecting neglected asymmetries in conditional variance models. Both problems are present when a GARCH filter is used and the data are generated from a non-linear DGP. In the context of this econometric exercise, a linear AR filter is employed instead. Additionally, the followed methodology is employed in order to investigate the linearity assumption and not to differentiate between different non-linear models.

All the estimations in our exercise are carried out using Nonlinear Toolkit 4.6 by Patterson & Ashley (2000) and EViews 4.1.

3. DATA

The data employed in this paper consists of monthly observations of the US and the Canadian unemployment rate from 1976:1 to 2000:12. These are seasonally adjusted observations taken from the US Bureau of Labor Statistics (www.bls.gov) and the Statistics Canada (www.statcan.ca). Figure 1 presents the two series and Table 1 provides their basic statistics.

Note that unit root analysis is not carried out. The argument is that the unemployment rate is a bounded time series and as a result the focus of our analysis is the levels and not the differences.

4. RESULTS

Firstly, we determine the pre-whitening model, AR(p). The values of p from 0 (regress on a constant) up to 10 are considered and the one with the minimum SC criterion is chosen. Table 2 present the "best" AR model for each series. This is an AR(5) model for the US unemployment rate and an AR(4) for the Canadian.

The next step of the exercise is to save the residuals of the best linear AR(p) model and test for any remaining serial dependence. Tables 3 and 4 present the results for the US and Canada. The employed tests are, like most econometric procedures, only asymptotically justified. Given the limited sample available, the tests are estimated using both the asymptotic theory and the bootstrap. The values under "asymptotic theory" are based on the large sample distributions of the relevant test statistics. For the "Bootstrap" results, 1000 new samples are independently drawn from the empirical distribution of the pre-whitened data. Each new sample is used to calculate a value for the test statistic under the null hypothesis of serial independence. The obtained fraction of the 1000 test statistics, which exceeds the sample value of the test statistic from the original data, is then reported as the significance level at which the null hypothesis can be rejected (for a detailed discussion on the sample size, the asymptotic theory and the bootstrap see Patterson & Ashley 2000).

The Tsay test, which is powerful against Threshold Autoregressive Processes (TAR), produces the lowest p-values in both cases and rejects the linearity assumption. The same conclusion- rejection of linearity - can be drawn based on the Bicovariance test since both in the US and in the Canadian case p-values of less that *0.05* are obtained. The McLeod-Li and the Engle test are both tests for GARCH effects. Although the picture is not clear, one could argue that GARCH effects are present in the Canadian data. Additionally, the BDS test statistic rejects the assumption of linearity with regard to the Canadian unemployment rate. All the p-values are zero suggesting that some kind of hidden structure is contained in the residuals of the pre-whitened model. Surprisingly, some evidence in favour of linearity emerges for the US data. The results of the Engle LM test are marginal for the US data and provide some evidence in favour of linearity. This is reinforced by the BDS test statistic, which accepts linearity in most cases. Although the assumption of linearity cannot be unambiguously accepted, the rejection is not as clear cut as it is in the case of Canada. An alternative possible explanation might be the low power (or the failure) of the BDS test statistic against some form of asymmetry.
Overall, evidence in favour of non-linearity is found in both series. However, this evidence is strongest in the case of Canadian unemployment rate.

5. CONCLUSIONS

The assumption of linearity in the case of the US and the Canadian unemployment rate is tested. In the light of previous similar studies, the results are not the expected ones. Strong evidence of non-linearity is found in the behaviour of the Canadian unemployment rate. This finding is in contrast to the finding by Frank & Stengos (1988) in which, using a different time period and frequency, evidence in favour of linearity emerged. For the US, the evidence did not provide with a strong case in favour of non-linearity. Although three tests supported the hypothesis that significant non-linearities are contained in the series, the verdict is not unanimous. In particular using the BDS (and the Engle LM) test statistic we are able to accept the linearity assumption in some cases; this implies that a naïve AR model is satisfactory. This finding is not in line with Brock & Sayers (1988).

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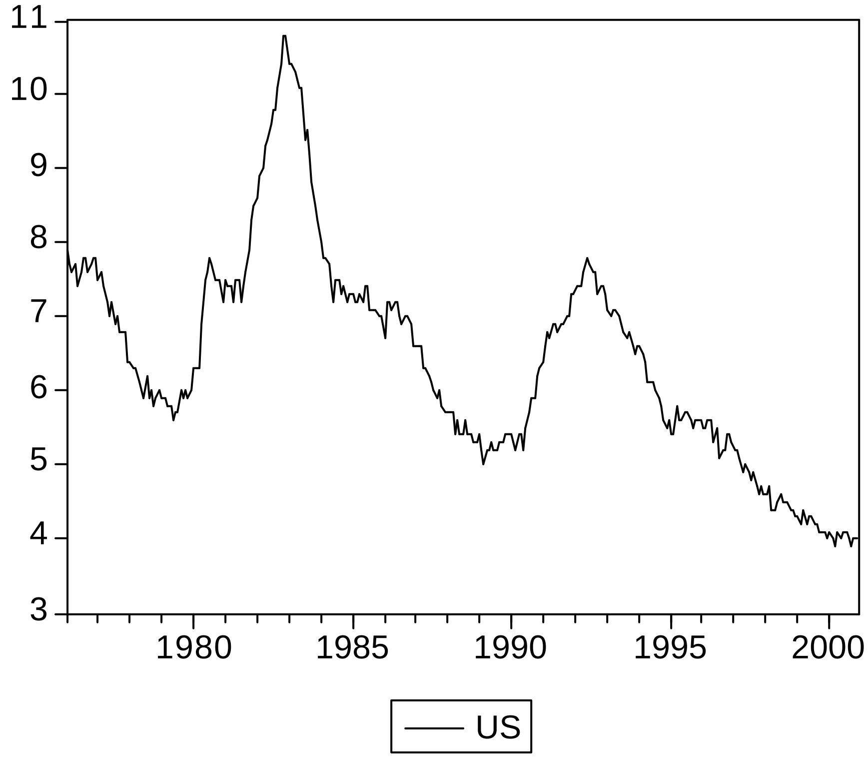
FIGURE 1 : US AND CANADIAN UNEMPLOYMENT RATES, 1976:01 – 2000:12



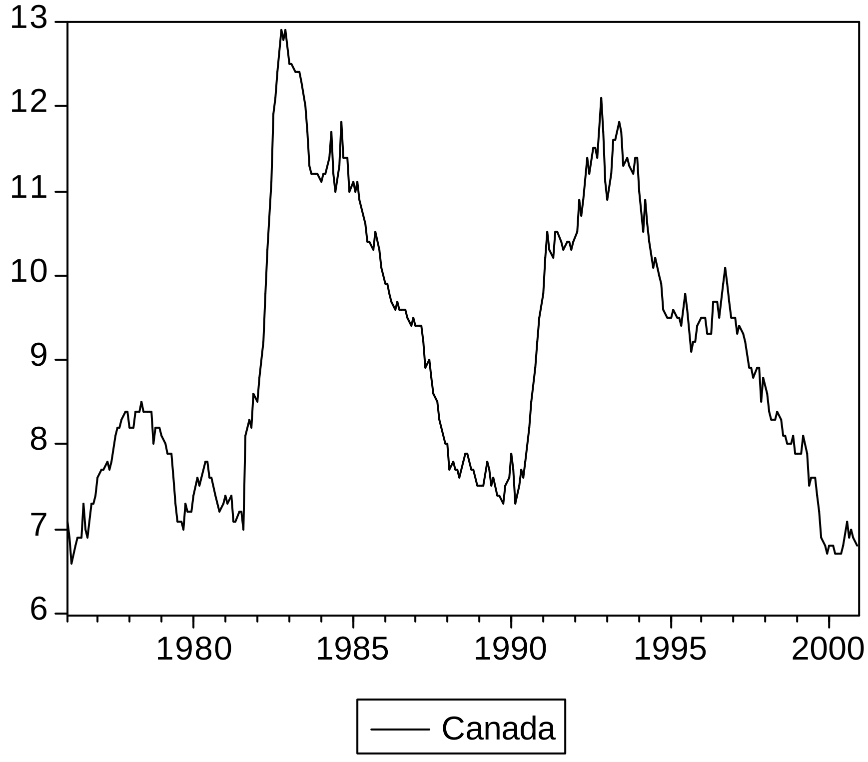


TABLE 1: SUMMARY STATISTICS OF UNEMPLOYMENT RATES

	US	CAN
Mean	6.44	9.07
Median	6.30	8.80
Maximum	10.80	12.90
Minimum	3.90	6.60
Std. Dev.	1.45	1.61
Skewness	0.57	0.43
Kurtosis	3.36	2.07
Jarque-Bera	17.56	20.00
Probability	0.00	0.00
Sum	1,932.60	2,720.80
Sum Sq. Dev.	631.41	777.46
Observations	300	300

TABLE 2: THE ORDER OF THE AR(P) PRE-WHITENING MODEL

	US Unemployment Rate	Canadian Unemployment Rate
Lag	AR(5) Coefficient	AR(4) Coefficient
1	0.944	1.135
	(16.387)	(19.845)
2	0.175	-0.108
	(2.201)	(1.245)
3	0.100	0.166
	(1.255)	(1.914)
4	-0.045	-0.209
	(0.573)	(3.676)
5	-0.184	
	(3.205)	
SC	*-3.494*	*-2.910*
$\bar{R}^2$	*0.987*	*0.980*

Note: In the parenthesis the t-ratio for each coefficient is reported. SC is the Schwartz Criterion. $\bar{R}^2$ is the adjusted R^2 .

TABLE 3: TESTS FOR NONLINEAR SERIAL DEPENDENCE

	US Unemployment Rate		Canadian Unemployment Rate	
	BOOTSTRAP	ASYMPTOTIC THEORY	BOOTSTRAP	ASYMPTOTIC THEORY
MCLEOD-LI TEST				
USING UP TO LAG 20	0.040	0.035	0.042	0.017
USING UP TO LAG 24	0.053	0.048	0.056	0.039
BICOVARIANCE TEST				
UP TO LAG 9	0.023	0.013	0.002	0.000
ENGLE TEST				
USING UP TO LAG 1	0.035	0.054	0.015	0.015
USING UP TO LAG 2	0.082	0.115	0.031	0.030
USING UP TO LAG 3	0.150	0.183	0.035	0.035
USING UP TO LAG 4	0.114	0.136	0.049	0.064
TSAY TEST	0.001	0.001	0.000	0.000

Note: Only p-values are reported, under the null hypothesis that the time series is a serially *i.i.d.* process.

TABLE 4: BDS TEST STATISTICS

	US Unemployment Rate			Canadian Unemployment Rate		
BDS			BOOTSTRAP			
Dimension	EPS=0.50	EPS=1.00	EPS=2.00	EPS=0.50	EPS=1.00	EPS=2.00
2	0.155	0.172	0.025	0.001	0.000	0.000
3	0.144	0.203	0.023	0.000	0.000	0.000
4	0.103	0.196	0.018	0.002	0.000	0.000
			ASYMPTOTIC THEORY			
2	0.123	0.159	0.013	0.000	0.000	0.000
3	0.101	0.193	0.017	0.000	0.000	0.000
4	0.044	0.175	0.012	0.000	0.000	0.000