Phenomenological and ratio bifurcations of a class of discrete time stochastic processes ☆

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Abstract

Zeeman proposed a classification of stochastic dynamical systems based on the Morse classification of their invariant probability densities; the associated bifurcations are the 'phenomenological bifurcations' of L. Arnold. The classification is however not invariant under diffeomorphisms of the state space. In a recent paper we proposed an alternative classification, based on an invariant that is a ratio of joint and marginal probability density functions, that does not suffer from this defect. This classification entails the concept of what we call 'ratio bifurcations'. In this note it is shown that for a large class of dynamical systems, ratio bifurcations and phenomenological bifurcations actually coincide. Moreover, we link the ratio invariant to the transformation invariant function that Wagenmakers *et al.* obtained for stochastic differential equations. The results are illustrated with numerical applications to stochastic dynamical systems.

1. Introduction

In many applications, families of discrete time Markov chains of the form

$$X_{t+1} = f_\mu(X_t) + \varepsilon_t \tag{1}$$

occur, where $f_\mu : \mathbb{R}^m \to \mathbb{R}^m$ is a parametrised family of C^k maps, $k \geq 2$, and where the sequence $\{\varepsilon_t\}$ consists of independent and identically distributed normal random variables. A specification of the form (1) is often called a stochastic dynamical system. The corresponding stochastic process $\{X_t\}$ is a Markov process, because the future states X_{t+k}, $k \geq 1$, are conditionally independent of the past states, $X_{t-1}, X_{t-2}, \ldots$, given the current state X_t.

There have been several attempts to develop a bifurcation theory for stationary stochastic dynamical systems, all of which have specific limitations. The basic problem encountered is to find a reasonable notion of being 'qualitatively equal', which is usually expressed by an equivalence relation between systems.

Zeeman [4] proposed to use the Morse classification of invariant densities of stochastic dynamical systems as a basis for such an equivalence relation. This approach suffers from the fact that probability densities

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are not invariant under general coordinate transformations; for instance, any absolutely continuous probability distribution on the real line can be mapped to the uniform distribution on the unit interval. This problem was acknowledged by Zeeman, who noted that only the class of isometric transformations could be allowed, but the idea never caught on.

A second approach was developed by L. Arnold [1] and his co-workers. They interpreted the stochastic dynamical system (1) as the skew product of the shift dynamics on the space of outcomes of the process $\{\varepsilon_t\}$ with equation (1), obtaining a deterministic dynamical system on a much larger phase space, and developed a bifurcation theory for these systems. The main objection against this approach is of a practical nature: even if the dynamic equations of such a skew system are known, its state cannot be observed, even approximately, as this entails the knowledge of the complete sequence $\{\varepsilon_t\}$ of noise realisations.

A third approach, related to Zeeman's ideas, was proposed by us in a previous article [2]. It is based on the observation that the dependence ratio, that is, the ratio of the transition density of the process to the invariant density, is invariant under coordinate transformations and hence a geometric invariant of the stochastic process. The Morse classification of these dependence ratios then extends to a classification of the associated stochastic processes. In fact, in the paper a refined classification is developed that takes the behaviour of the ratios at infinity into account. This gives rise to a third notion of bifurcation of stochastic dynamical system, which is based on computable geometrical objects and thus avoids the shortcomings of both Zeeman's and Arnold's proposals. However, of the three notions it is the one that lends itself least to intuitive interpretations.

In the present note, we aim to show that this shortcoming is lessened for the class of systems (1) for which our classification is essentially equivalent to Zeeman's original proposal. For these systems, critical points of the dependence ratio, as well as the critical value and the signature of the Hessian, correspond in a one-to-one fashion to critical points, critical values and signatures of the invariant density. Hence, in this special class, the Morse classification of the invariant density gives indeed rise to a classification of stochastic processes that can easily be computed and interpreted in practical situations. The dependence ratio is then connected to the geometric invariant of Markov diffusions found by Wagenmakers *et al.* [3]. As a consequence, we find that the class for which the invariant density can be used as the basis for a classification is the discrete time analogue of diffusions of constant diffusion strength for which the Wagenmakers invariant also reduces to the invariant density.

Finally, the results are illustrated numerically for two different families of stochastic maps. The first is a family of diffeomorphisms whose deterministic skeleton features a pitchfork bifurcation and, after symmetry breaking, a saddle-node. We obtain numerically that the stochastic system undergoes a stochastic pitchfork bifurcation as the noise intensity increases, and a stochastic saddle-node bifurcation when symmetry is broken. The second example is a family of non-invertible unimodal maps that feature a period doubling cascade. For increasing perturbation strengths we obtain the beginning of a cascade of stochastic saddle node bifurcations.

2. The invariants

For a large class of discrete time stochastic processes a geometric invariant, the dependence ratio, has been introduced in [2]. An equivalence relation of stochastic processes is obtained from the Morse classification of their associated dependence ratios. Here we show that for a subclass, the critical points of the invariant densities are linked to the critical points of the associated dependence ratios. This allows to take invariant densities as the basis of a classification of processes, even if they are not geometric invariants of the process.

Itô's lemma suggests a geometric invariant for stochastic processes that are governed by a stochastic differential equation. As for small time steps, a stochastic differential equation yields a discrete time process in the class that we consider, it turns out that the continuous time invariant can be obtained as a limit of the discrete time dependence ratio for small time steps.

2.1. Discrete time systems with additive noise

Consider a Markov chain on $\mathbb{R}^m$ of the form

$$X_{t+1} = f(X_t) + \varepsilon_t, \tag{2}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^2 map, and where the noise process $\{\varepsilon_t\}$ is a sequence of identically and independently distributed random variables, whose distribution can be described by a C^2 probability density function φ. If X_t is distributed according to the probability density function p_t, then X_{t+1} is distributed according to $p_{t+1} = T p_t$, where the transformation T is given as

$$(T p_t)(y) = \int \tau(y|x) p_t(x) \mathrm{d}x, \tag{3}$$

and where the C^2 transition density τ has the form

$$\tau(y|x) = \varphi(y - f(x)). \tag{4}$$

Note that T maps the space of probability densities into itself. If the Markov process $\{X_t\}$ is stationary, there is an invariant probability density p. In [2] it was shown that for stationary Markov processes specified by (2), the *dependence ratio*

$$\rho(x, y) = \frac{\tau(y|x)}{p(y)} \tag{5}$$

is invariant under coordinate transformations, and that it is therefore a suitable geometrical invariant to form the basis of a bifurcation theory of stochastic dynamical systems of the form (2).

In the following, we shall consider noise processes with nondegenerate unimodular probability densities; that is, we shall assume that φ has a unique critical point 0, where it necessarily takes a maximum, such that the symmetric Hessian matrix $D^2\varphi(0)$ is negative definite. More generally, the situation that φ has a unique critical point c could be considered; however, when ε_t is replaced by $c + \varepsilon_t$, the former situation is restored.

Recall that the signature s of a symmetric matrix is the difference $s = n_- - n_+$ between the number n_- of negative and the number n_+ of positive eigenvalues of the matrix. By extension, the signature of a critical point of a C^2 map is the signature of the Hessian matrix of the map at the critical point.

The following result relates the critical points of the invariant density p to the critical points of the dependency ratio ρ for the case that f is a diffeomorphism.

Theorem 1. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a C^2 diffeomorphism and φ a C^2 nondegenerate unimodular probability density with critical point* 0*. Assume that $\varphi(x) > 0$ for all $x \in \mathbb{R}^m$ and that the Markov process defined by* (2) *is strictly stationary with invariant density p.*

Then the point y_c is a critical point of p if and only if the point (x_c, y_c), with x_c determined by

$$y_c = f(x_c), \tag{6}$$

is a critical point of ρ. The critical value is given as

$$\rho(x_c, y_c) = \frac{\varphi(0)}{p(y_c)}.$$

Moreover, the signatures s_ρ of $D^2\rho(x_c, y_c)$ and s_p of $D^2 p(y_c)$ are related as

$$s_\rho = m - s_p. \tag{7}$$

Proof. We have

$$D_x\rho(x,y) = \frac{1}{p(y)} D\varphi\big(a(x,y)\big) D_x a(x,y), \tag{8}$$

$$D_y\rho(x,y) = \frac{1}{p(y)} D\varphi\big(a(x,y)\big) D_y a(x,y) - \frac{\varphi\big(a(x,y)\big)}{p(y)^2} Dp(y), \tag{9}$$

where the argument of φ and its derivatives is

$$a(x,y) = y - f(x).$$

It is clear that if y_c is a critical point of p, and if $y_c = f(x_c)$, then

$$a(x_c, y_c) = 0$$

implying that

$$D\varphi(a(x_c, y_c)) = D\varphi(0) = 0$$

and that (x_c, y_c) is a critical point of ρ.

Note that

$$D_x a(x_c, y_c) = -Df(x_c) \quad \text{and} \quad D_y a(x_c, y_c) = I$$

are both invertible. Therefore, if (x_c, y_c) is a critical point of ρ such that $y_c = f(x_c)$, it follows that

$$D\varphi(a(x_c, y_c)) = 0.$$

Substitution in (9) yields that then $Dp(y_c) = 0$, as φ, and by extension p, is everywhere positive.

At a critical point (x_c, y_c) of ρ, the second derivative of ρ takes the form

$$D^2\rho = \frac{1}{p} \begin{pmatrix} Df^T D^2\varphi Df & -Df^T D^2\varphi \\ -D^2\varphi Df & D^2\varphi - \frac{\varphi}{p} D^2 p \end{pmatrix} \tag{10}$$

where the arguments of f, p and φ and their derivatives are respectively x_c, y_c and c.

Let $u, v \in \mathbb{R}^m$ and introduce $w \in \mathbb{R}^{2m}$ as

$$w = \begin{pmatrix} Df(x_c)^{-1}(u+v) \\ v \end{pmatrix}.$$

Compute

$$\begin{aligned} &\left\langle w, pD^2\rho\, w \right\rangle \\ &= \left\langle u+v, D^2\varphi(u+v) \right\rangle - 2\left\langle v, D^2\varphi(u+v) \right\rangle + \left\langle v, D^2\varphi\, v \right\rangle - \frac{\varphi}{p} \left\langle v, D^2 p\, v \right\rangle \\ &= \left\langle u, D^2\varphi\, u \right\rangle - \frac{\varphi(0)}{p} \left\langle v, D^2 p\, v \right\rangle \end{aligned}$$

This is a sum of two independent quadratic forms: the signature of the first term is m, while the signature of the second term is $-s_p$. This completes the proof of the theorem.

We have the following easy corollary to the proof of the theorem for mappings that are not diffeomorphisms.

Theorem 2. *Let the conditions of theorem 1 hold, excepting the assumption that $f : \mathbb{R}^m \to \mathbb{R}^m$ is invertible. If y_c is a critical point of p, and if there is $x_c \in \mathbb{R}^m$ such that*

$$y_c = f(x_c), \tag{11}$$

then (x_c, y_c) is a critical point of ρ.

2.2. Stochastic differential equations

A stochastic process of the form (2) on the real line arises naturally as the Euler approximation of a stochastic differential equation. For such equations, a geometric invariant has been given by [3]; it turns out that this invariant is related to the dependence ratio.

2.2.1. Invariants of continuous time processes

More generally, consider a process X_t on $\mathbb{R}^m$, governed by the stochastic differential equation

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \tag{12}$$

where B_t is an m-dimensional Brownian motion. Assume that X_t is ergodic, and that $\det \sigma(x) \neq 0$ for any $x \in \mathbb{R}^m$. Denote the invariant density by p. Let the function $\theta : \mathbb{R}^m \to \mathbb{R}$ be defined as

$$\theta(x) = p(x)|\det \sigma(x)|.$$

A C^2 diffeomorphism $\psi : \mathbb{R}^m \to \mathbb{R}^m$ defines an associated process $\tilde{X}_t = \psi(X_t)$, which, by Itô's lemma, is of the form

$$\mathrm{d}\tilde{X}_t = \tilde{f}(X_t)\mathrm{d}t + \tilde{\sigma}(X_t)\mathrm{d}B_t,$$

where in particular

$$\tilde{\sigma}(x) = D\psi(\psi^{-1}(x))\sigma(\psi^{-1}(x)) \tag{13}$$

The invariant density $\tilde{p}$ of $\tilde{X}_t$ has the form

$$\tilde{p}(x) = p(\psi^{-1}(x))|\det D\psi^{-1}(x)|. \tag{14}$$

Combining (13) and (14) yields the following result, due to [3].

Theorem 3. *The function θ is an invariant of the process X_t, that is*

$$\tilde{\theta}(\psi(x)) = \theta(x)$$

for all $x \in \mathbb{R}^m$.

2.2.2. Connection to dependence ratio

Again, consider the process X_t on $\mathbb{R}^m$ given by (12), as well as its invariant density p and its geometric invariant θ. The step-h Euler approximation Y^h_{t+1} of $X_{h(t+1)}$ conditional on $Y^h_t = X_{ht}$ is given as

$$Y^h_{t+1} = Y^h_t + hf(Y^h_t) + \sqrt{h}\sigma(Y^h_t)\varepsilon_t, \tag{15}$$

where the $\{\varepsilon_t\}$ are independently distributed standard normal variables. Note that (15) defines a discrete Markov chain with transition density

$$\tau_h(y|x) = \frac{1}{|\det \sigma(x)|\sqrt{2\pi h}} \exp\left(-\frac{1}{2h}\left\langle y - x - hf(x), \Sigma^{-1}(x)(y - x - hf(x))\right\rangle\right)$$

where $\Sigma(x) = \sigma(x)^T\sigma(x)$. Let p_h denote the invariant density of the Euler approximation (15). The dependence ratio ρ_h of this Markov chain is given as

$$\rho_h(x, y) = \frac{\tau_h(y|x)}{p_h(x)}.$$

Let $i : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ denote the embedding $i(x) = (x, x)$. Note that the function

$$\Delta_h(x) = (i^*\rho_h)(x) = \frac{\tau(x|x)}{p_h(x)}$$

is also a geometric invariant of the Euler process.

The following result connects the invariant θ of the continuous time process (12) to the invariant Δ_h, and by extension ρ_h, of the discrete time process (15).

Theorem 4. *Assume that $p_h(x) \to p(x)$ as $h \to 0$. Then*

$$\sqrt{2\pi h}\,\Delta_h(x) \to \frac{1}{\theta(x)}$$

as $h \to 0$.

Proof. We have

$$\sqrt{2\pi h}\,\Delta_h(x) = \frac{1}{\theta(x)} \exp\left(-\frac{h}{2}\left\langle f(x), \Sigma(x)^{-1} f(x)\right\rangle\right) \to \frac{1}{\theta(x)} \quad \text{as} \quad h \to 0.$$

2.2.3. Interpretation

As noted, systems of the form (2) are obtained as the Euler approximations of stochastic differential equations (12) for which the diffusion strength is independent of x, that is, $\sigma(x) = \sigma_0$ for all x. We find that then the geometric invariant reads as

$$\theta(x) = c\,p(x),$$

where $c = |\det \sigma_0|$. That is: systems from the class of Markov diffusions (12) with constant diffusion strength can be invariantly classified by the Morse classification of the invariant densities.

3. Bifurcations

Theorem 1 is illustrated using the stochastic hyperbolic tangent map.

Example 1. Consider the stochastic hyperbolic tangent map given by

$$x_t = \tanh(a\, x_{t-1}) + b + \varepsilon_t, \tag{16}$$

where $\{\varepsilon_t\}$ is a sequence of independent $N(0, \sigma^2)$-distributed random variables. As σ varies for fixed values of a and b, stochastic bifurcations occur. Since by Theorem 1 the ratio bifurcations and phenomenological bifurcations coincide for this system, changes in the number of critical points of the marginal density occur exactly at the ratio bifurcation points. Figures 1 and 2 show the critical points of the marginal density as a function of σ, for $a = 1.3$, $b = 0$ (symmetric case) and $b = 0.01$ (asymmetric case), respectively.

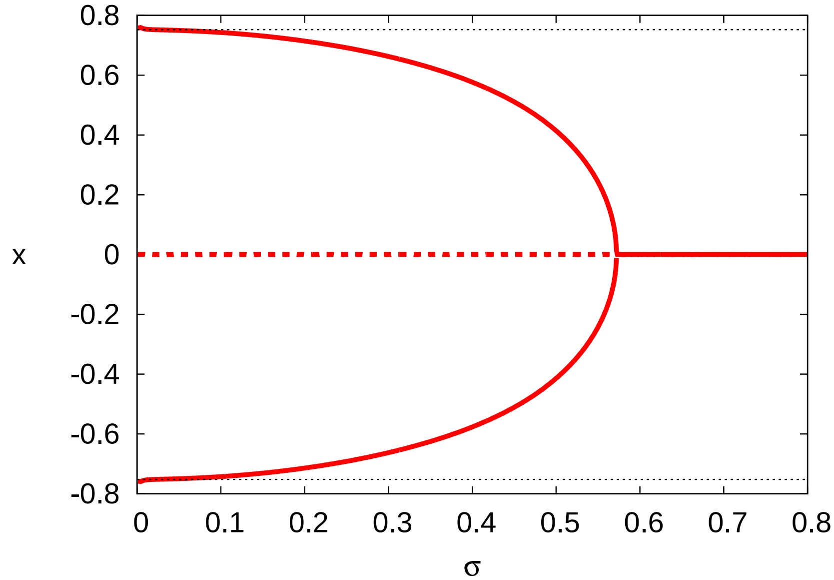

Figure 1: Critical points of the invariant density as a function of the noise level σ for the stochastic hyperbolic map (16), for the case $a = 1.3$, $b = 0$. Local maxima are indicated by solid lines, while local minima are indicated by dashed lines. The thin dashed lines indicate the fixed points for the deterministic map.

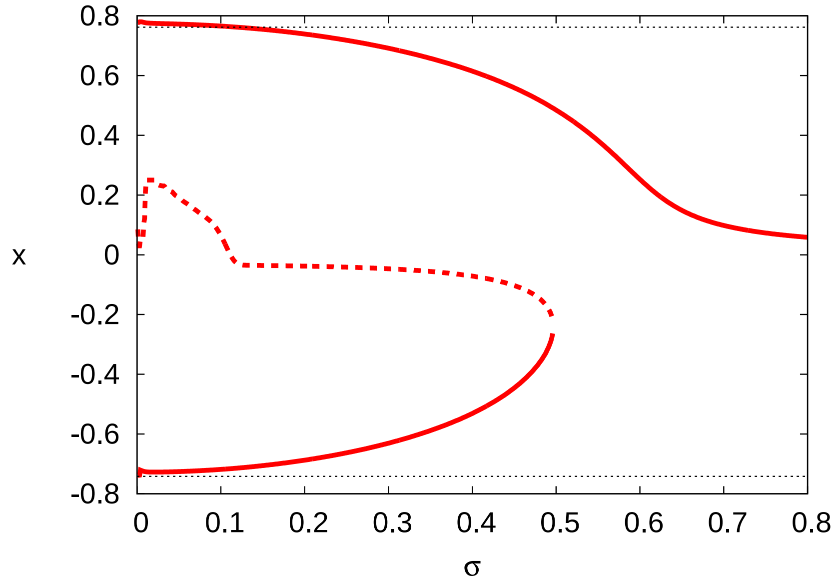

Figure 2: Critical points of the invariant density as a function of the noise level σ for the stochastic hyperbolic map (16), for the case $a = 1.3$, $b = 0.01$. Other details are as for Fig. 1.

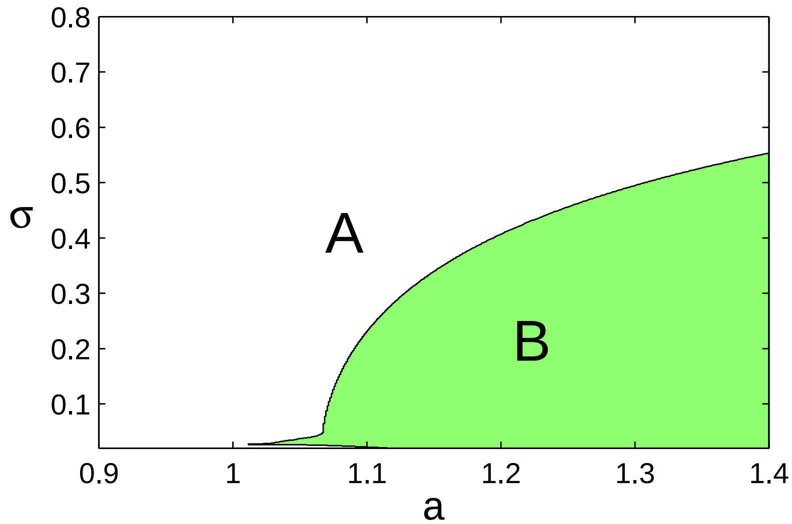

Figure 3: Phase diagram in the (a, σ)-plane. In the region labelled 'A' the invariant distribution is unimodal, while it is bimodal in the region labelled 'B'.

In both figures it can be observed that for small noise levels the modes of the bimodal distribution are close to the two fixed points of the noise-free dynamical system ($\sigma = 0$). Note that in the asymmetric case ($b = 0.01$) the point where the minimum of the marginal density is attained displays a rather erratic dependence on σ for small σ. For the symmetric ($b = 0$) as well as the asymmetric case ($b = 0.01$), increasing σ sufficiently leads to a phenomenological bifurcation, in which the invariant density changes from bimodal to unimodal.

To establish the dependence of this pehnomenon on the parameter a, Fig. 3 shows the regions in the (a, σ)-plane where the invariant distribution is unimodal and bimodal, respectively. It can be seen that if $a > a^*$, where $a^* = 1$ is the classical bifurcation value of a for the noise-free case $\sigma = 0$, increasing σ leads to a phenomenological bifurcation (and hence a corresponding ratio bifurcation).

To illustrate the fact that these phenomenological bifurcations coincide with ratio bifurcations, Fig. 4 shows the level sets (top panels) of the dependence ratio for noise levels below, at, and above the bifurcation value, which is approximately $\sigma = 0.5$. The lower panels show the invariant densities for the corresponding values of σ. By comparing the upper and lower panel figures it can be readily verified that the x-values for which the invariant density reaches a local maximum or minimum, correspond with values y_c of critical points (x_c, y_c) of the dependence ratios.

Example 2. To illustrate the case of a non-invertible map, we consider

$$x_t = x_{t-1} - \log(\exp(3x_{t-1}) + \mu) + \varepsilon_t, \tag{17}$$

where, as before, $\varepsilon_t \sim N(0, \sigma^2)$. The deterministic skeleton is conjugated, via $x_t = \log z_t$, to the system $z_t = z_{t-1}/(z_{t-1}^3 + \mu)$. Fig. 5 shows a bifurcation diagram for the noise-free dynamical system ($\sigma = 0$). It can be observed that the deterministic skeleton shows a cascade of period doubling bifurcations for decreasing values of μ. As shown by the Lyapunov exponent in Fig. 6, the skeleton becomes chaotic for even smaller values of μ.

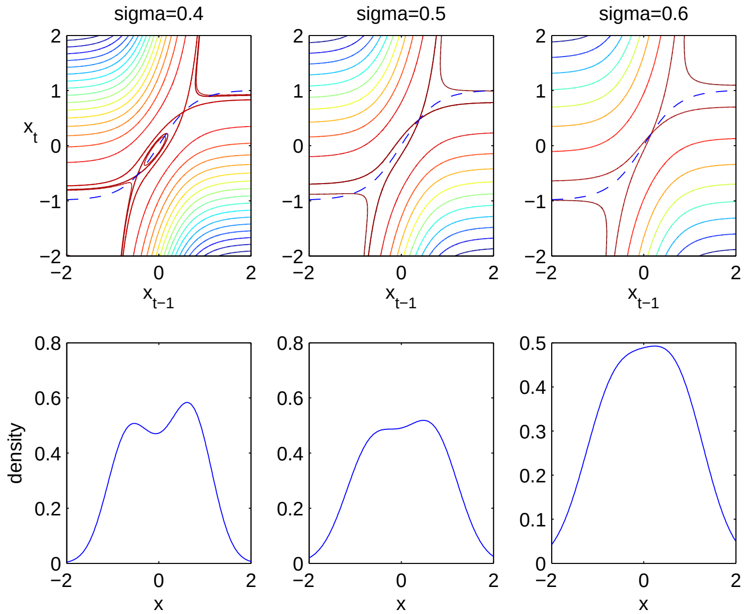

Figure 4: Ratio bifurcation for increasing σ, coinciding with the phenomenological bifurcation ($a = 1.3$ and $b = 0.01$). The top panels show contour plots of the dependence ratio, and the lower panels the invariant density. The dashed line is the graph of the function.

Fig. 7 shows phenomenological bifurcations in the (μ, σ)-plane. The number of m of local maxima of the invariant density is indicated for $m = 1, 2, 4$. In our numerical implementation the computational time (or inaccuracy) was seen to increase fast for smaller values of σ and μ, which is why attention is limited to the parameter region shown here.

Fig. 8 shows the local maxima and minima, respectively, of the invariant density for $\mu = 0.1$ as σ varies. The deterministic skeleton has a stable period-4 cycle for this parameter value. It can be observed that decreasing the noise level σ gives rise to a bifurcation scenario in which the number of local maxima increases from one, to two, and finally four.

To illustrate Theorem 2, according to which we expect the phenomenological bifurcations in this cascade to coincide with ratio bifurcations, Fig. 9 shows the level sets (top panels) of the dependence ratio, as well as the invariant densities (lower panels). The parameter μ is fixed at 0.1, while the σ-values are 0.05, 0.2 and 0.4, respectively. For these parameter values of σ the invariant density has 4, 2 and 1 local maxima, respectively. Again, it can be readily verified from this figure that the x-values for which the invariant density reaches a local maximum or minimum, correspond with the values y_c of the critical points (x_c, y_c) of the dependence ratio.

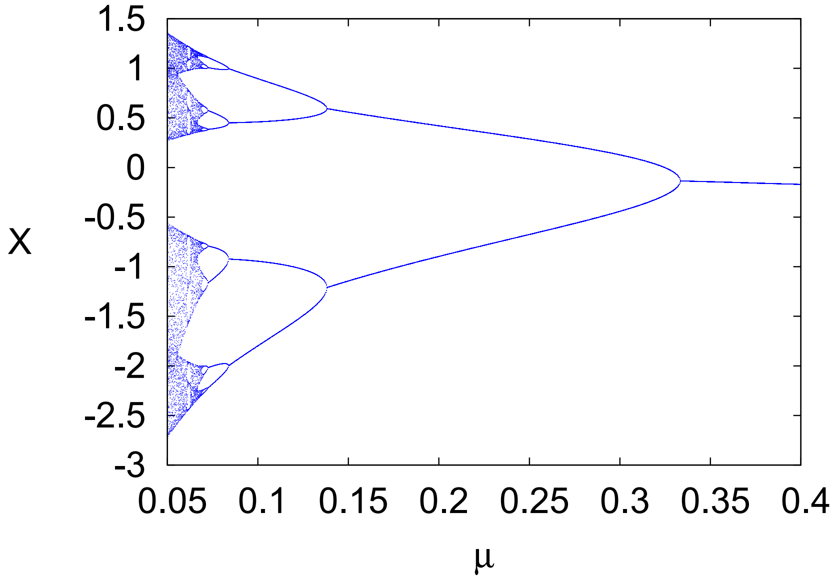

Figure 5: Bifurcation diagram of the deterministic skeleton of system (17).

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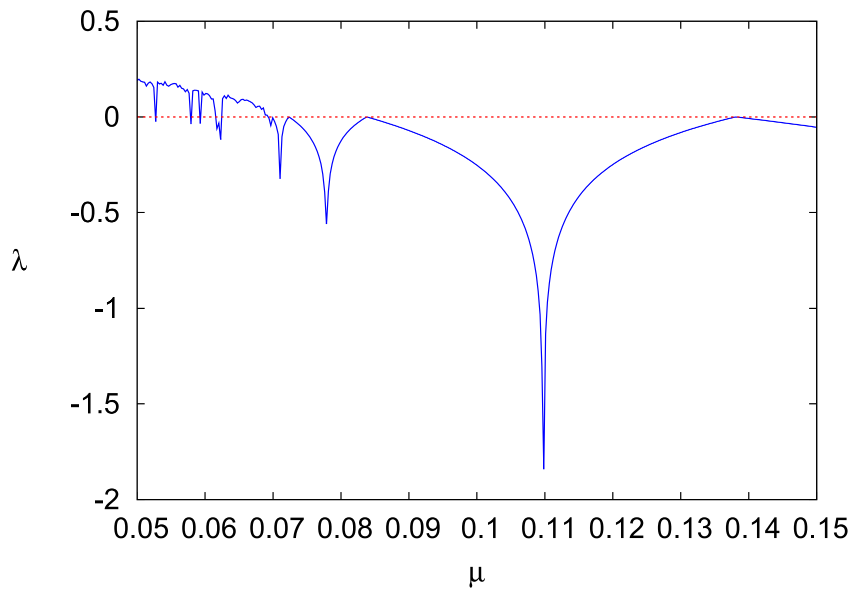

Figure 6: Lyapunov exponent for the deterministic skeleton of system (17), as a function of μ.

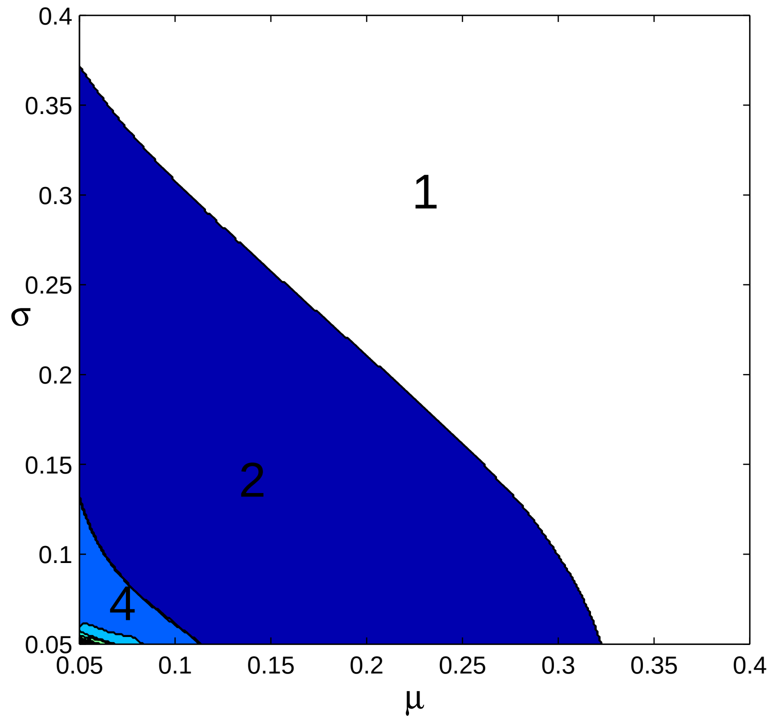

Figure 7: Phase diagram in the (μ, σ)-plane for the system (17). The numbers 1, 2 and 4 in the largest regions correspond to the number of local maxima of the invariant density.

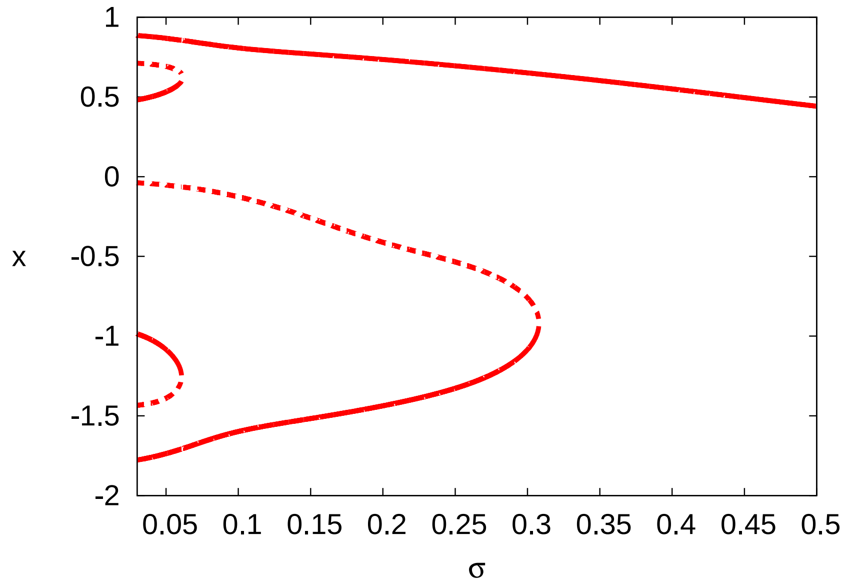

Figure 8: Critical points of the invariant density as a function of the noise level σ for the map (17) with $\mu = 0.1$.

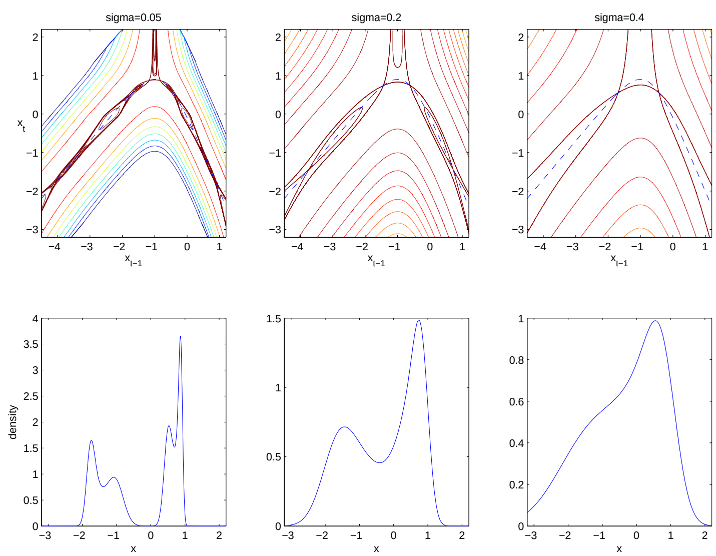

Figure 9: Ratio bifurcation for increasing σ for the map (17), $\mu = 0.1$. The top panels show contour plots of the dependence ratio, and the lower panels the invariant density. The dashed line is the graph of the function.