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Comparisons of Heterogeneous Distributions and Dominance Criteria

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Comparaisons de Distributions Hétérogènes et Critères de Dominance [1]

Résumé

Cet article porte sur la comparaison des niveaux de vie lorsque l'on dispose à la fois d'observations sur les revenus et sur la composition des ménages. Nous généralisons l'approche d'Atkinson et Bourguignon (1987) dans le cas où les distributions marginales des besoins peuvent varier entre les populations des ménages considérées. Nous supposons que le planificateur bienveillant a recours à une fonction de bien-être social utilitariste pour classer les distributions de revenus hétérogènes. Dans la mesure où tout individu peut jouer le rôle du planificateur, nous adoptons le point de vue unanimiste suivant lequel les jugements du planificateur doivent se conformer à un certain nombre de principes normatifs élémentaires. Nous imposons des conditions de plus en plus restrictives sur la fonction d'utilité du ménage et examinons leurs conséquences sur le classement des distributions qui en résulte. Ceci nous conduit à proposer quatre critères de dominance permettant de classer sans ambiguïté les distributions de revenu pour des populations hétérogènes.

Mots-clés : Analyse normative, Utilitarisme, Welfarisme, Inégalité et Bien-Être Multidimensionnels, Dominance Stochastique Bidimensionelle, Transformations Égalisantes.

Comparisons of Heterogeneous Distributions and Dominance Criteria [1]

Abstract

We are interested in the comparisons of standard-of-living across societies when observations of both income and household structure are available. We generalise the approach of Atkinson and Bourguignon (1987) to the case where the marginal distributions of needs can vary across the household populations under comparison. We assume that a sympathetic observer uses a utilitarian social welfare function in order to rank heterogeneous income distributions. Insofar as any individual can play the role of the observer, we take the unanimity point of view according to which the planner's judgements have to comply with a certain number of basic normative principles. We impose increasingly restrictive conditions on the household's utility function and we investigate their effects on the resulting rankings of the distributions. This leads us to propose four dominance criteria that can be used for providing an unambiguous ranking of income distributions for heterogeneous populations.

Keywords: Normative Analysis, Utilitarianism, Welfarism, Multidimensional Inequality and Welfare, Bidimensional Stochastic Dominance, Inequality Reducing Transformations.

JEL: D31, D63, I31

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1. Introductory Remarks

It is generally considered that household income constitutes – under a number of conditions – a reasonable indicator of a household's well-being. The welfare or standard of living of a society is then obtained by aggregating the citizens' well-being by means of a social welfare function. [1] The choice of one social welfare function rather than another constitutes a particular value judgement that is not necessarily unanimously approved. In order to limit the arbitrariness of this choice, it is now a well-established tradition to consider a class of social welfare functions rather than a particular social welfare function and to require a unanimity of point of views among the elements in the class. A major difficulty with this approach is the fact that the ranking of situations one obtains – attractive though it is from a normative perspective – is impossible to implement in practice because the class of social welfare functions comprises an infinite number of elements. Seminal work by Kolm (1969) and Atkinson (1970) suggested a simple procedure for checking unanimity when one subscribes to some basic equity requirements and when the distributions under comparison have equal means. Subsequently, Shorrocks (1983) showed that their approach can be extended to the case where the equal mean restriction is dispensed with, provided that one subscribes in addition to some efficiency principle (see also Marshall and Olkin (1979)). Thus, the generalised Lorenz criterion has become the appropriate criterion for comparing income distributions when elementary efficiency and equity principles are agreed upon.

However, such a criterion can be meaningfully appealed to only if the income recipients can be considered identical in all respects other than income. This is taken to mean that, leaving income aside, one makes the judgement that the other attributes according to which individuals might be distinguished should not be taken into account when assessing individuals' well-being. Such an assumption is called into question in most of the practical cases faced by the economist, where the individuals differ in non-income characteristics that are without question considered as important factors influencing their well-being. Such is the case when the income recipients are families or households who differ in size and composition. It is indeed legitimate to assume that size and composition of the households influence their needs, which in turn affect their ability to derive well-being from income. Another example is when comparisons of income distributions are made across populations of persons with different health statuses. Here again it is plausible that health has an important impact on the determination of a person's well-being. In addition, it is no longer possible to invoke the lack of information to justify the exclusive focus on income. The increasing availability of microdata sets provide the researcher with refined information at the individual level which can be exploited for making welfare evaluations. A heterogeneous distribution, which indicates for every household its income and demographic characteristics, is identified with a multidimensional distribution and one may therefore be tempted to use the criteria proposed by Kolm (1977), Atkinson and Bourguignon (1982) and Maasoumi (1986) among others. A difficulty, that makes these criteria inappropriate in the present case, is the (implicit) assumption that the attributes are perfectly divisible, which makes little sense for variables like family size, household composition or health.

Two distinct routes have been taken when one is interested in comparing situations where socio-demographic characteristics affect the ability of households to produce well-being start-

[1] Rigourously speaking, we should use the term *social welfare functional* rather than *social welfare function*. By definition, a social welfare functional has as arguments the list of the agents' utility functions and it therefore takes explicitly into account the informational constraints, while a social welfare function uses the particular utility levels attained by the agents in order to construct the social welfare ordering (see e.g. Sen (1977), d'Aspremont (1985), d'Aspremont and Gevers (2002)).

ing from given incomes. A longstanding practice for taking into account differences in needs when making comparisons of welfare for heterogeneous populations consists in using equivalence scales. After having chosen the reference household type – generally a single adult – one determines for each household its equivalent income obtained by dividing its actual income by a factor reflecting its needs. This procedure generates a fictitious income distribution for a homogeneous population consisting of single adults and one is thus taken back to the unidimensional framework. Taking for granted that a unanimous agreement prevails concerning the choice of the equivalence scale, two problems arise. [2] The first problem is that the equivalence scale adjustment method lacks sound theoretical foundations, so that one may be under the impression that it is more or less an *ad hoc* procedure. [3] A second difficulty has to do with the impact on inequality and social welfare of progressive transfers, i.e., the operation that consists in taking income from a rich individual to give it to a poor one in such a way that their relative positions are unchanged. Glewwe (1991) drew attention to the fact that such transfers that reduce the inequalities of living standards between households do not necessarily translate into a decrease in inequality at the level of society. [4]

A second approach, initiated by Atkinson and Bourguignon (1987), aims precisely at avoiding this arbitrariness by allowing a large diversity of opinions. These authors showed that the so-called sequential generalised Lorenz criterion permits one to unambiguously rank heterogeneous distributions provided one is willing to subscribe to a certain number of elementary value judgements concerning the incidence of needs on the household's standard of living. However, the implementation of this criterion necessitates that a consensus prevails concerning the ranking of needs. If we limit ourselves to the composition of the household, then this might mean for instance that a couple with one child is needier that a couple without children who in turn has more needs that a single adult. While it is illusory to pretend that such a ranking of needs will be unanimously approved, the fact remains that it is a prerequisite for implementing this approach. [5]

Although the sequential generalised Lorenz criterion constitutes without doubt a breakthrough for the comparison of heterogeneous situations, its practical significance is limited to the extent that by definition the populations of households involved must all have the same structure in terms of needs. While this restriction does not pose problems when the focus is on the comparisons of distributions before and after taxation, this is no longer the case when one is interested in comparisons across countries or over time. Jenkins and Lambert (1993) have proposed an adaptation of the sequential generalised Lorenz criterion that resolves this

[2] Actually this unanimity is far from existing as the diversity of scales one finds in the literature can testify (see e.g. Buhmann, Rainwater, Schmaus, and Smeeding (1988)). The fact that the choice of the scale is not universally approved would not be too much a problem if it were shown to have no – or little – impact on the extent of welfare, inequality and poverty for a given population,as well as on the ranking of societies from these different points of view. However, this is far from being the case, as a number of investigations have demonstrated (see in particular Figini (1998)).

[3] It is fair to note that Blackorby and Donaldson (1993) and Blundell and Lewbel (1991) have identified the restrictions to be imposed on the households' utility functions that the econometric estimation of equivalence scales necessitates. Still a certain degree of arbitrariness remains concerning the type of scale – relative or absolute scales, income independent or non-independent scales – to be chosen that cannot be justified exclusively on theoretical grounds (see however Ebert and Moyes (2003)).

[4] This difficulty originates in the choice of the weights associated with the equivalent incomes. Ebert and Moyes (2003) have characterised the adjustment methods that avoid this difficulty.

[5] If all households where composed only of perfectly identical adults, then such a ranking in terms of increasing needs would be similar to the one obtained by considering the number of persons in the household. Things are complicated by the fact that the household's members differ in many respects such as their age, their social status or their health, many variables that make difficult the construction of a unique complete ordering over the set of household types.

difficulty. However, their criterion, which involves the joint distributions of income and needs, has the property that the marginal distributions of needs play little role in the determination of the ranking of situations. Suppose that there are only two types: single adults and households composed of two adults. Consider two societies S^1 and S^2 such that the distributions of income for each type are identical in both situations and such that there are twice as many couples and half as many singles in society S^1 than in society S^2. Then, the application of Jenkins and Lambert (1993)'s criterion indicates that societies S^1 and S^2 have the same living standards, while one intuitively expects that S^2 would dominate S^1. Furthermore, the corresponding class of social welfare functions is not perfectly identified since the restrictions placed on the utility functions are only shown to be sufficient for utilitarian unanimity to imply their criterion. The fact that these restrictions are also necessary for the application of unanimity to be consistent with the Jenkins-Lambert criterion has been demonstrated by Chambaz and Maurin (1998). Nevertheless, the criticism which concerns the independence of the ranking of situations with respect to the marginal distributions of needs still applies.

The main purpose of this paper is to reveal the implicit value judgements that underlie the different dominance criteria one can propose for comparing situations where households can be distinguished on the basis of increasing needs. We will suppose throughout that the evaluation of social welfare is performed by means of a utilitarian social welfare function so that social welfare is simply equal to the sum of the households' well-beings. The well-being or living standard of a household will depend both on its income and needs, which implies that one is always able (i) to fully order needs on an increasing scale, and (ii) to associate to each household a position on this ordinal scale. Implicitly, we assimilate neediness with a handicap with the convention that, for a given income, the production of well-being by the household decreases with its size. Rather than reasoning in terms of neediness – increasing with the number of persons in the household – one can equally argue in terms of ability which is decreasing with family size. This approach, based on the notions of neediness or of ability, does not take into consideration the decisions for a couple to have children or not. It is a paternalistic approach where a sympathetic ethical observer (i) evaluates in a first stage the well-being of each household on the basis of its income and family composition according to the principle of extended sympathy, and (ii) uses this information in a second stage in order to determine the society's welfare by means of the utilitarian rule. The value judgements of our ethical observer constitute as many conditions imposed on the household utility function as it is conceived by her. This allows one to distinguish different classes of admissible utility functions and then in conformity with the principle of unanimity applied to the utilitarian rule, to identify the corresponding dominance criteria.

We introduce in Section 2 the general framework and the main notation that we will use throughout the paper. Unlike our predecessors we identify a situation – or equivalently a heterogeneous distribution – with a joint distribution of income and ability. This way of proceeding will find its justification later on when we introduce the dominance criteria. The following two sections contain the main results and present the characterisations of the different dominance criteria that generalise the usual stochastic dominance criteria. Following the approach of Hardy, Littlewood, and Pólya (1952), we will in a first stage identify those modifications of the situations – the so-called elementary transformations – which from the point of view of our ethical observer improve social welfare. We will then look for the conditions that the household utility function must satisfy in order to ensure that the application of the utilitarian rule concludes that social welfare has increased as a result of such transformations. These conditions will allow us to define classes of household utility functions and we will show that the application of utilitarian unanimity to each of these classes permits one to characterise multidimensional stochastic dominance criteria. The ordinal or cardinal character of the concepts of neediness and ability will prove to play a crucial role for the characterisation of the dominance criteria. We suppose in Section 3 that the notion of needs is purely ordinal, while we admit in Section 4 that making comparisons of differences of ability between households of different types has some meaning. We discuss in Section 5 the approach advocated by Jenkins and Lambert (1993) and we show that the dominance criteria they proposed violate a minimal consistency requirement quite close in spirit to the condition of independence of irrelevant alternatives in social choice theory. Section 6 concludes the paper by summarising our main findings, pointing at limitations and suggesting avenues for further research. In order to keep technicalities to a minimum in the course of the exposition, the proofs of the main results are relegated to an Appendix.

2. Preliminary Definitions and Notation

We consider a *population* or *society* $S := \{1, 2, \ldots, n\}$ comprising n households, where each household is described by two attributes: *income* and family composition or *household type*. We assume that there exists a finite number of household types H $(2 \leqq H \leqq n)$ and we denote by $\mathscr{H} := \{1, 2, \ldots, H\}$ the set of possible types. In this paper, we interpret the index $h \in \mathscr{H}$ as an (ordinal) indicator of household ability, which is decreasing with family size. As a matter of illustration, one might conceive of the following correspondence between ability, needs and household composition:

Household Composition	A	C	$C+1$	$C+2$	$C+3$	$C+4$
Ability h	6	5	4	3	2	1
Neediness $H-h+1$	1	2	3	4	5	6

where the symbols "A" indicate a household comprising a single adult, "C" a household constituted of a married couple, and "$C+K$" a household constituted of a married couple with K children. [6] A *heterogeneous distribution* or *situation* is a partitioned vector $\mathbf{s} \equiv (\mathbf{x}; \mathbf{a}) := (x_1, \ldots, x_n; a_1, \ldots, a_n)$, where $x_i \in \mathscr{D} := [\underline{v}, \overline{v}] \subset \mathbb{R}$ and $a_i \in \mathscr{H}$ represent respectively the *income* and *ability* of household i. We denote as

$$\text{(2.1)} \qquad \mathscr{X} := \{\mathbf{s} \equiv (\mathbf{x}; \mathbf{a}) \mid x_i \in \mathscr{D} \text{ and } a_i \in \mathscr{H}, \text{ for all } i \in S\}$$

the set of heterogeneous distributions. It is convenient to conceive of distribution $\mathbf{s} \equiv (\mathbf{x}; \mathbf{a}) \in \mathscr{X}$ as a matrix with n rows and 2 columns

$$\text{(2.2)} \qquad \mathbf{s} \equiv (\mathbf{x}; \mathbf{a}) := \begin{bmatrix} x_1 & a_1 \\ \vdots & \vdots \\ x_i & a_i \\ \vdots & \vdots \\ x_n & a_n \end{bmatrix}.$$

If we limit ourselves to a fixed population of households – or equivalently populations of the same size – then a situation $\mathbf{s} \in \mathscr{X}$ can be visualised as a scatterplot in the space $\mathscr{D} \times \mathscr{H}$

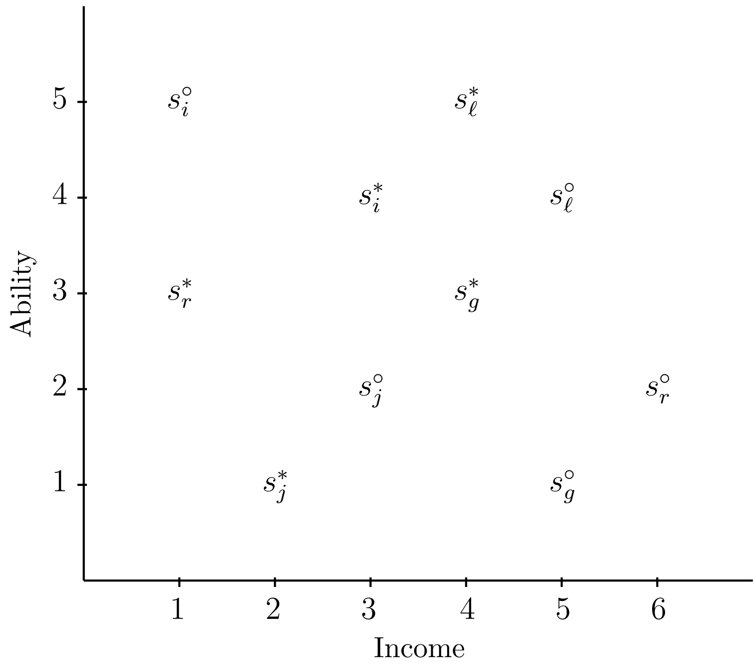
[6] This way of proceeding amounts to identifying family size with a handicap for the household in the same way that poor health constitutes a handicap for a person. This point of view, which, for instance, does not take into consideration the decision for a couple to have children is with no doubt disputable. An approach that would take explicitly into account the well-being that the parents can derive from the presence of children lies outside of the scope of the present paper.

where the point $s_i \equiv (x_i; a_i)$ identifies household i. Consider for instance the population $S = \{i, j, g, \ell, r\}$ involving five households and the two following situations:

$$\mathbf{s}^\circ \equiv (\mathbf{x}^\circ; \mathbf{a}^\circ) = \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 5 & 1 \\ 5 & 4 \\ 6 & 2 \end{bmatrix}; \quad \mathbf{s}^* \equiv (\mathbf{x}^*; \mathbf{a}^*) = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 4 & 3 \\ 4 & 5 \\ 1 & 3 \end{bmatrix}, \tag{2.3}$$

where we have made use of the conventions mentioned above concerning the correspondence between ability and family composition. Denoting by i° and i^* the situations of household i

Figure 2.1: Graphical representation of situations $\mathbf{s}^*$ and $\mathbf{s}^\circ$



in states $\mathbf{s}^\circ$ and $\mathbf{s}^*$, respectively, we have the situation depicted in Figure 2.1.

The different dominance criteria that we will introduce later on generalise the usual stochastic dominance quasi-orderings and will therefore involve the joint distribution functions. Let us associate to situation $\mathbf{s} \equiv (\mathbf{x}; \mathbf{a}) \in \mathscr{X}$ its *joint distribution function* $F(y, h)$ which indicates the proportion of households whose ability is no greater than h and whose income does not exceed y. Formally,

$$F(y, h) := \sum_{r=1}^{h} \int_{\underline{v}}^{y} f(\xi, r)\, d\xi, \ \forall\ h \in \mathscr{H}, \ \forall\ y \in \mathscr{D}, \tag{2.4}$$

where $f(y, h)$ is the *joint density function* associated to situation $\mathbf{s} \equiv (\mathbf{x}; \mathbf{a}) \in \mathscr{X}$. [7] The corresponding *marginal distribution of income* is indicated by $F(y, H)$, while $F(\overline{v}, h)$ denotes

[7] Since the heterogeneous distributions are discrete, this representation is not perfectly correct to the extent that the distribution function presents discontinuities for some income values. However, this presentation can be justified on the basis of arguments developed in Fishburn and Vickson (1978).

the *marginal distribution of ability.* [8] Given two situations $\mathbf{s}^\circ \equiv (\mathbf{x}^\circ; \mathbf{a}^\circ), \mathbf{s}^* \equiv (\mathbf{x}^*; \mathbf{a}^*) \in \mathscr{X}$, we will henceforth use f° and f^* (resp. F° and F^*) to indicate their density (resp. distribution) functions.

Following Atkinson and Bourguignon (1987), we assume that the sympathetic ethical observer relies on the (*average*) *utilitarian social welfare function* in order to compare the distributions of well-being between different societies. The social welfare in situation $\mathbf{s} \equiv (\mathbf{x}; \mathbf{a}) \in \mathscr{X}$ is then given by

$$W_U(\mathbf{s}) := \sum_{i=1}^{n} \left(\frac{1}{n}\right) U\left(x_i, a_i\right) \equiv \sum_{h=1}^{H} \int_{\underline{v}}^{\overline{v}} U(y,h)\, f(y,h)\, dy, \tag{2.5}$$

where $U(y,h)$ is – from the point of view of the ethical observer – the utility obtained by a household with ability h and income y. We interpret $U(y,h)$ as a measure of the *well-being* of a household of type h, which we identify with the maximum utility a representative member of the household gets when the household is given income y. It therefore depends on the way the household's members distribute resources among them. For instance, if the household maximises a symmetric, monotone and quasi-concave function of the utilities of its members and if the latter have the same *individual* utility function, then $U(y,h)$ is the utility level that all persons in the household will get. A discussion of the underlying model goes far beyond the scope of the present paper and we refer the interested reader to Blackorby and Donaldson (1993), Bourguignon (1989), and more recently Ebert and Moyes (2009), where these questions are discussed in more detail. Because the list of abilities is fixed throughout, we may interpret $U(\,\cdot\,,h)$ as the utility function of a type h-household. To simplify things, we assume that the household utility function is continuous and differentiable to the required order with respect to income and we denote by $\mathscr{U}$ the set of such functions. The social welfare function being decided once and for all, the choice of the utility function U defines a (complete) ordering of the situations under comparison such that:

$$\forall\ \mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X} : \mathbf{s}^* \geq_U \mathbf{s}^\circ \Longleftrightarrow W_U(\mathbf{s}^*) \geqq W_U(\mathbf{s}^\circ). \tag{2.6}$$

It must be noted that the ranking defined above is invariant with respect to specific transformations of the households' utility functions. More precisely, if $V(x_i, a_i) = \alpha_i + \beta\, U(x_i, a_i)$ ($\beta > 0$), for all $i = 1, 2, \ldots, n$, then

$$\forall\ \mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X} : \mathbf{s}^* \geq_U \mathbf{s}^\circ \Longleftrightarrow \mathbf{s}^* \geq_V \mathbf{s}^\circ. \tag{2.7}$$

This informational restriction – known as *cardinal unit comparability* – only requires that comparisons of utility differences between households are meaningful. Initially, this informational requirement was introduced in the social choice framework where individuals – here households – have possibly distinct utility functions (see e.g. Sen (1977), d'Aspremont (1985)). Since we implicitly assume in (2.5) that $U_i(x_i, a_i) = U(x_i, a_i)$, for all $i = 1, 2, \ldots, n$ and all $\mathbf{s} \equiv (\mathbf{x}; \mathbf{a})$ – all households have the same utility function – we must have $\alpha_i = \alpha$, for all $i = 1, 2, \ldots, n$. Thus, in our specific context, *cardinal unit comparability* reduces to *cardinal full comparability* and differences of utilities as well as levels of utilities are comparable between households. [9]

[8] It is possible to use the *conditional distribution function of income* $F(y \mid h)$ rather than the joint distribution functions $F(y,h)$ as our predecessors have done. However, the recourse to the joint distribution functions comes naturally in a context where the distributions of household types vary from one population to another. Furthermore, the statistical tests that will allow one to implement our criteria will involve the joint distribution functions.

[9] As we will see later on, the possibility of making comparisons of utility levels across households is a crucial step towards the construction of our dominance criteria.

In order to limit the arbitrariness inherent in the choice of the utility function, we will suppose that the ethical observer bases her assessment of the different situations under comparison not on a particular utility function but rather on the basis of a set of utility functions that one can consider as reasonable. Letting $\mathscr{U}^*$ indicate such a class, we have the following (partial) ranking of situations:

$$\forall\ \mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X} : \mathbf{s}^* \geq_{\mathscr{U}^*} \mathbf{s}^\circ \Longleftrightarrow W_U(\mathbf{s}^*) \geqq W_U(\mathbf{s}^\circ),\ \forall\ U \in \mathscr{U}^*. \tag{2.8}$$

This ranking, which represents the unanimity point of view when we limit ourselves to the class $\mathscr{U}^*$, will quite naturally depend on the restrictions that the class $\mathscr{U}^*$ imposes on its elements. This paper aims at introducing successively more stringent restrictions on the household utility function and at examining the implications for the ranking of the situations under comparison as defined in (2.8). Finally, we would like to emphasise that we can with no loss of generality restrict attention to comparisons of situations for populations of equal sizes. Indeed, by definition, the social welfare function (2.5) satisfies to the *principle of population* according to which welfare does not change under a replication of the population. [10] The application of this principle allows us to make comparisons of well-being across populations of distinct sizes by replicating in a suitable way the corresponding heterogeneous distributions and by applying to the resulting equal sized situations the unanimity criterion defined in (2.8).

3. Dominance Criteria with Ordinal Abilities

One can enter into endless debates about the question of whether ability is an ordinal or a cardinal concept just as one might discuss the ordinal or cardinal meaning of IQ. More relevant is the fundamental question of the meaning of the notion of ability or equivalently of that of needs. Actually, the notion of ability only makes sense when the objective of the household is one of producing well-being. The idea is that one household type is more able – or equivalently has less needs – than another if, at any given income level, the utility it manages to achieve is higher than the utility attained by the other type. Ability is then an ordinal variable if we consider that only comparisons of utility levels between household types make sense. [11] In other words, while it makes sense to say that a couple with one child is needier than a childless couple, claiming that the difference of needs between these two households is greater than the difference of needs between a couple with no children and a single person is meaningless.

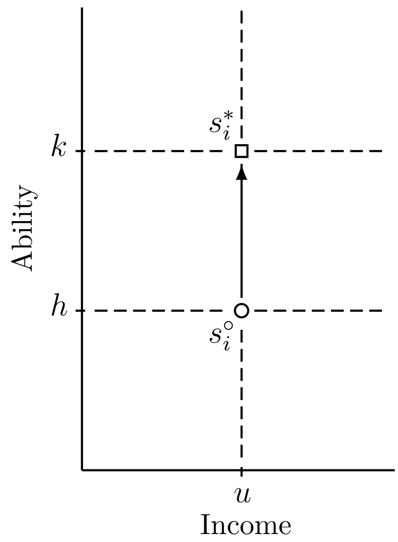
Definition 3.1 Given two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$, we will say that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of an *ability increment* if there exists a household i such that (i) $a_i^* > a_i^\circ$; (ii) $x_i^* = x_i^\circ$; and (iii) $s_r^* = s_r^\circ$, for all $r \in S$ $(r \neq i)$. Equivalently, we would say that $\mathbf{s}^\circ$ is obtained from $\mathbf{s}^*$ by means of an *ability decrement*.

The notion of an ability increment is illustrated in Figure 3.1, where we depict schematically two situations $\mathbf{s}^\circ$ and $\mathbf{s}^*$ that differ only in the ability level attained by household h which increases from h to k when going from situation $\mathbf{s}^\circ$ to situation $\mathbf{s}^*$. The notion of an ability increment is a convenient fiction, which strictly speaking is only meaningful in the case of distinct populations, like different countries for instance. Identifying modifications of family composition in the course of time with such transformations is disputable to the extent that these changes were chosen by the household – one may think of decisions like marrying or

[10] We will say that situation $\mathbf{s}^*$ is a q-replication of situation $\mathbf{s}^\circ$ if there exists $q \geqq 2$ such that $\mathbf{s}^* = (\mathbf{s}^\circ; \dots; \mathbf{s}^\circ) \in \mathscr{X}^q$. Then, the principle of population states: for all $\mathbf{s}^\circ, \mathbf{s}^* \in \bigcup_{m=1}^\infty \mathscr{X}^m$, if $\mathbf{s}^*$ is a q-replication of situation $\mathbf{s}^\circ$, then $W_U(\mathbf{s}^*) = W_U(\mathbf{s}^\circ)$.

[11] This indirect way of defining the ordinal or cardinal nature of ability arises from the relationship between household well-being and ability.

Figure 3.1: An ability increment



having children – and therefore lie outside the domain of evaluation of the ethical observer. One can easily check that it is sufficient for welfare not to decrease as the result of an ability increment that the utility function is non-decreasing with ability. More precisely:

C1 $$U(y,h) \leqq U(y,h+1),\ \forall\, y \in \mathscr{D},\ \forall\, h = 1,2,\dots,H-1.$$

Indeed, let us assume that $\mathbf{s}^*$ follows from $\mathbf{s}^\circ$ by means of an ability increment as it is defined in Figure 3.1. Then, welfare will not diminish when we go from situation $\mathbf{s}^\circ$ to situation $\mathbf{s}^*$ if

(3.1) $$\Delta W_U := W_U(\mathbf{s}^*) - W_U(\mathbf{s}^\circ) = \frac{1}{n}\sum_{r=0}^{k-h-1}[U(u,h+r+1) - U(u,h+r)] \geqq 0,$$

which follows from condition C1. [12]

Definition 3.2 Given two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$, we will say that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of an *income increment* if there exists a household i such that (i) $x_i^* > x_i^\circ$; (ii) $a_i^* = a_i^\circ$; and (iii) $s_r^* = s_r^\circ$, for all $r \in S$ $(r \neq i)$. Equivalently, we would say that $\mathbf{s}^\circ$ is obtained from $\mathbf{s}^*$ by means of an *income decrement*.

We have represented an income increment in Figure 3.2 and, here again, it is immediate that social welfare will not decrease as the result of an income increment provided that the utility function is non-decreasing with income, and this holds whatever the ability level. Formally, this condition is stated as:

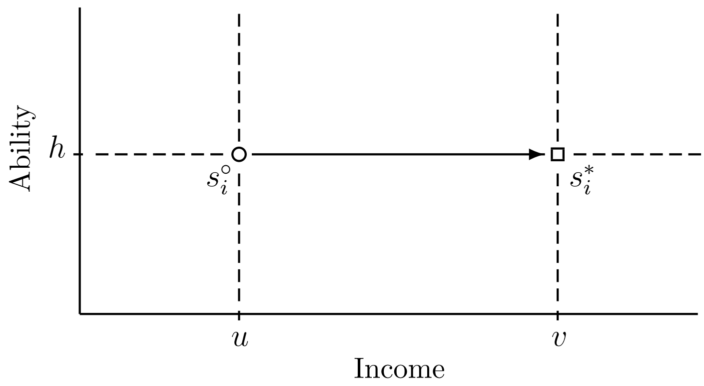
C2 $$U_y(y,h) \geqq 0,\ \forall\, y \in \mathscr{D},\ \forall\, h \in \mathscr{H},$$

where $U_y(y,h)$ denotes the first (partial) derivative of $U(y,h)$with respect to income. Suppose that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of the income increment depicted in Figure 3.2. Then, social welfare will not decrease if and only if

(3.2) $$\Delta W_U = \frac{1}{n}[U(v,h) - U(u,h)] = \frac{1}{n}\int_0^\Delta U_y(u+\xi,h)\,d\xi \geqq 0,$$

[12] Condition C1 would not be appropriate if $U(y,h)$ were interpreted as being total household utility and h is inversely related to household size: for more on this see e.g. Fleurbaey, Hagneré, and Trannoy (2007).

Figure 3.2: An income increment



where $\Delta := v - u > 0$, and condition C2 is sufficient for (3.2) to hold.

The preceding conditions do not incorporate any consideration of distributive justice, like for instance the pursuit of greater equality. While the notion of inequality reduction is well understood for homogeneous populations, this is far from being the case in our heterogeneous context where households differ both in income and family composition. The elementary transformation we introduce below is the foundation of the approach of Atkinson and Bourguignon (1982, 1987).

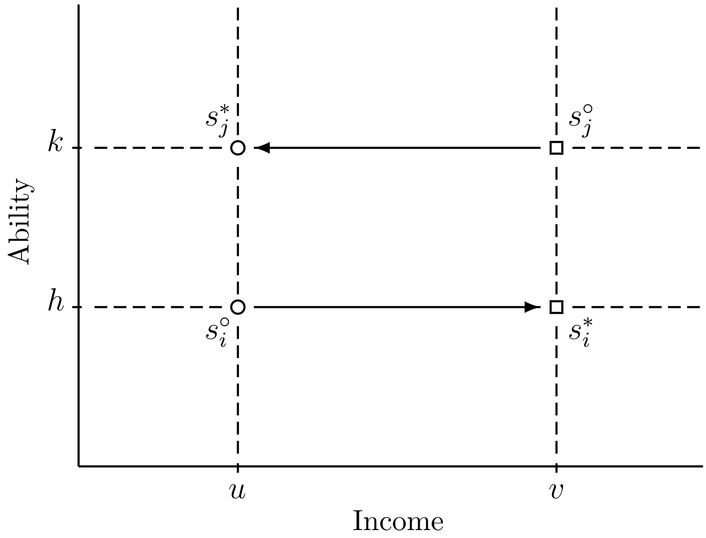
Definition 3.3 Given two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$, we will say that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of a *favourable permutation* if there exists two households $i, j \in S$ $(i \neq j)$ such that: (i) $x_i^\circ = x_j^* < x_i^* = x_j^\circ$; (ii) $a_i^\circ = a_i^* < a_j^* = a_j^\circ$; and (iii) $s_r^\circ = s_r^*$, for all $r \neq i, j$. Equivalently, we would say that $\mathbf{s}^\circ$ is obtained from $\mathbf{s}^*$ by means of an *unfavourable permutation*.

Intuitively, a favourable permutation contributes to a reduction in the inequalities between the two households involved. We verify in Figure 3.3 that in situation $\mathbf{s}^\circ$ household i is at the same time poorer and needier than household j. It follows that, from the ethical observer's point of view, household i's well-being will always be smaller – and in any case not greater – than household j's well-being as long as the utility functions verify conditions C1 and C2. A favourable permutation consists in using income in order to compensate for differences in well-being of the two households by permuting the incomes of the rich and poor households as indicated in Figure 3.3. Such a transformation actually combines an income increment and an income decrement of the same magnitude – equal to the income difference between the rich and the poor households – with the additional constraint that the income of the beneficiary household after the transformation is equal to the income of donor household before the transformation. This is the interpretation given by Atkinson and Bourguignon (1987) when the demographic structure of the population is fixed: income is the only attribute whose distribution can be altered in order to reduce the differences in well-being between the households. In our less restrictive framework, one can equally interpret a favourable permutation as an exchange of abilities between households i and j.

One can easily check that social welfare will not decrease as the result of a favourable permutation provided that the marginal utility of income is non-increasing with ability, which can be formally stated as:

C3 $$U_y(y, h) \geqq U_y(y, h+1),\ \forall\ y \in \mathscr{D},\ \forall\ h = 1, 2, \ldots, H-1.$$

Figure 3.3: A favourable permutation



Suppose that social welfare does not decrease as the result of a favourable permutation as described in Figure 3.3, in which case one has

$$\Delta W_U = -\frac{1}{n} \sum_{r=0}^{k-h-1} \int_0^{\Delta} [U_y(u+\xi, h+r+1) - U_y(u+\xi, h+r)]\, d\xi \geqq 0, \tag{3.3}$$

where $\Delta := v - u > 0$. Clearly, condition C3 is sufficient for the weak inequality (3.3) to hold. It is convenient to introduce the following class of utility functions:

$$\mathscr{U}_{11} := \{U \in \mathscr{U} \mid \text{ conditions C1, C2 and C3 are fulfilled}\}. \tag{3.4}$$

We will say that situation $\mathbf{s}^*$ results from situation $\mathbf{s}^\circ$ by means of a T_{11}-*transformation* if $\mathbf{s}^*$ can be obtained from $\mathbf{s}^\circ$ by means of an ability increment and/or an income increment and/or a favourable permutation. Then, we have:

Remark 3.1 Social welfare does not decrease as the result of a T_{11}-transformation if and only if $U \in \mathscr{U}_{11}$. [13]

Proof. We have already shown that $U \in \mathscr{U}_{11}$ is a sufficient condition for social welfare not to decrease as the result of a T_{11}-transformation. In order to prove that it is also necessary, we will argue a contrario and show that, if either condition C1, C2 or C3 is not fulfilled, then it is possible to find two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$ such that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of a T_{11}-transformation and $W_U(\mathbf{s}^*) < W_U(\mathbf{s}^\circ)$. Suppose for example that condition C3 is violated, so that there exists $y \in \mathscr{D}$ and $h \in \{1, 2, \ldots, H-1\}$ such that $U_y(y,h) < U_y(y,h+1)$. Consider then the situations $\mathbf{s}^\circ$ and $\mathbf{s}^*$ defined in Figure 3.3 with $k = h+1$. Choosing $\Delta := v - u > 0$ arbitrarily small, we get

$$\lim_{\Delta \to 0} \left[\frac{\Delta W_U}{\Delta}\right] = U_y(u,h) - U_y(u,h+1) < 0, \tag{3.5}$$

[13] This result depends crucially on the definition of the T_{11}-transformation. We would like to draw attention to the fact that one cannot conclude that, if social welfare does not decrease for all utility functions $U \in \mathscr{U}_{11}$, then $\mathbf{s}^*$ can be derived from $\mathbf{s}^\circ$ by means of a finite sequence of ability increments, income increments and/or favourable permutations. This is – to the best of our knowledge – a conjecture that has not yet been proven, even though there are results in the literature that are making steps in this direction (see e.g. Epstein and Tanny (1980), Gravel and Moyes (2010)).

which indicates that social welfare decreases as the result of a favourable permutation and, therefore, as the result of a T_{11}-transformation. We do not reproduce the proofs when conditions C1 and C2 are not satisfied since they are similar in all respects. □

We will assume that the ethical observer subscribes to the value judgements embedded in conditions C1, C2 and C3. In order to avoid as much arbitrariness as possible, we will conform to the unanimity principle according to which situation $\mathbf{s}^*$ is considered better than situation $\mathbf{s}^\circ$ if, whatever the utility function verifying conditions C1, C2 and C3, the utilitarian rule never ranks $\mathbf{s}^*$ below $\mathbf{s}^\circ$. Attractive as it is, this way of proceeding raises however the practical question of its implementation as it is impossible to verify directly if it applies or not. The following result proposes a simple procedure in order to decide whether one situation is preferable or not to another when attention is restricted to utility functions in the class $\mathscr{U}_{11}$.

Proposition 3.1 *Consider two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{S}$. The following two statements are equivalent:*

(a) $W_U(\mathbf{s}^*) \geqq W_U(\mathbf{s}^\circ),\ \forall\, U \in \mathscr{U}_{11}$.

(b) $F^*(y,h) \leqq F^\circ(y,h),\ \forall\, y \in \mathscr{D},\ \forall\, h \in \mathscr{H}$.

Condition P3.1b is nothing more than a condition of multidimensional stochastic dominance in the particular case where one of the variables – here ability – is discrete and ordinal. We refer the reader to Atkinson and Bourguignon (1982) or Appendix B for a presentation of multidimensional stochastic dominance criteria in the case of continuous and cardinal attributes. Whatever the ability and income levels, the proportion of households with abilities and incomes not greater than these levels is no larger in situation $\mathbf{s}^*$ than in situation $\mathbf{s}^\circ$. This implies in particular that the distribution of income for the whole population in situation $\mathbf{s}^*$ must stochastically dominate to the first order the corresponding distribution in situation $\mathbf{s}^\circ$. Indeed, for $h = H$, condition P3.1b reduces to $F^*(y,H) \leqq F^\circ(y,H)$, for all $y \in \mathscr{D}$. Making use of the relationship between stochastic dominance and poverty measurement (see Foster and Shorrocks (1988)), this means that, whatever the poverty line, there must – all types of households combined – be proportionally no more poor households in $\mathbf{s}^*$ than in situation $\mathbf{s}^\circ$. Similarly, setting $y = \overline{v}$, we obtain $F^*(\overline{v},h) \leqq F^\circ(\overline{v},h)$, for all $h \in \mathscr{H}$. In other words, whatever the ability level, the proportion of households whose ability falls below that level in situation $\mathbf{s}^*$ cannot exceed the proportion of households in situation $\mathbf{s}^\circ$ who are in the same position. While dominance in terms of the marginal distributions of income and ability are necessary for condition P3.1b to hold, they are far from being sufficient. [14]

It is standard practice in the case of homogeneous populations to consider that, other things equal, a transfer of income from a rich individual to a poorer one contributes to an increase in social welfare. The so-called *Pigou-Dalton condition* or *principle of transfers* constitutes the cornerstone of the theory of inequality and welfare measurement in a homogeneous framework. It is quite natural to require that a multidimensional measure of welfare also verifies this condition and to this aim we introduce the following adaptation of the notion of a progressive transfer in our context.

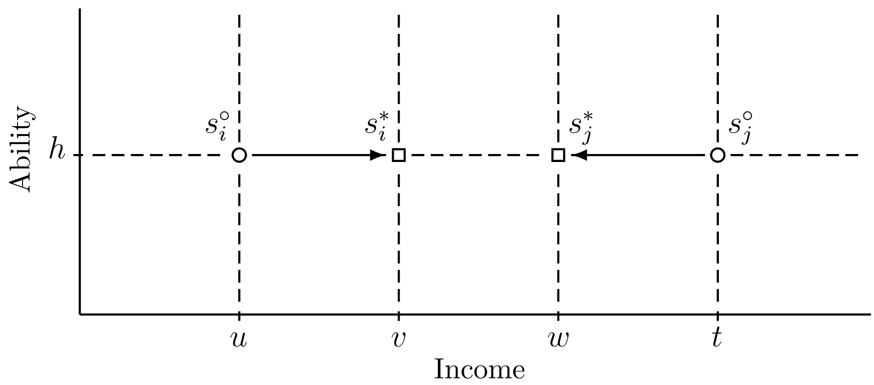
Definition 3.4 Given two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{S}$, we will say that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of an *progressive income transfer* if there exists two households $i, j \in S$ $(i \neq j)$ such

[14] There is however one case where the conditions on the marginal distributions are necessary and sufficient for utilitarian unanimity to hold for all utility functions in $\mathscr{U}_{11}$. It is when the utility functions are separable in the sense that $U(y,h) := \phi(y) + \psi(h)$, for all $y \in \mathscr{D}$ and all $h \in \mathscr{H}$. One can easily check in this case that conditions C1, C2 and C3 are satisfied as long as the functions ϕ and ψ are non-decreasing. However, one gets in this particular case $U_y(y,h) = U_y(y,h+1)$, for all $y \in \mathscr{D}$ and all $h = 1, 2, \ldots, H-1$, which implies that favourable – as well as unfavourable – permutations have no impact on social welfare.

that: (i) $x_i^\circ < x_i^* < x_j^\circ$, $x_i^\circ < x_j^* < x_j^\circ$; (ii) $x_i^* - x_i^\circ = x_j^\circ - x_j^*$; (iii) $a_i^\circ = a_i^* = a_j^* = a_j^\circ$; and (iv) $s_r^\circ = s_r^*$, for all $r \neq i, j$. Equivalently, we would say that $\mathbf{s}^\circ$ is obtained from $\mathbf{s}^*$ by means of an *regressive income transfer*.

The notion of progressive income transfer is illustrated in Figure 3.4, where $v-u = t-w$. It is well-known that the concavity of the utility function is a sufficient (and necessary) condition for welfare not to diminish as the result of a progressive transfer in the homogeneous case. In our multidimensional setting, this condition naturally translates into the requirement that

Figure 3.4: A progressive income transfer



the household utility function is concave in income for any given level of ability, which, upon making use of the differentiability with respect to income of the household utility function, is stated as:

C4 $$U_{yy}(y,h) \leqq 0,\ \forall\, y \in \mathscr{D},\ \forall\, h \in \mathscr{H},$$

where $U_{yy}(y,h)$ is the second derivative of $U(y,h)$ with respect to income. For a proof suppose that social welfare does not decrease as the result of a progressive income transfer as defined in Figure 3.4 so that

(3.6) $$\Delta W_U = -\frac{1}{n}\int_0^\Delta \int_0^\epsilon U_{yy}(u+\vartheta+\xi, h)\, d\vartheta\, d\xi \geqq 0,$$

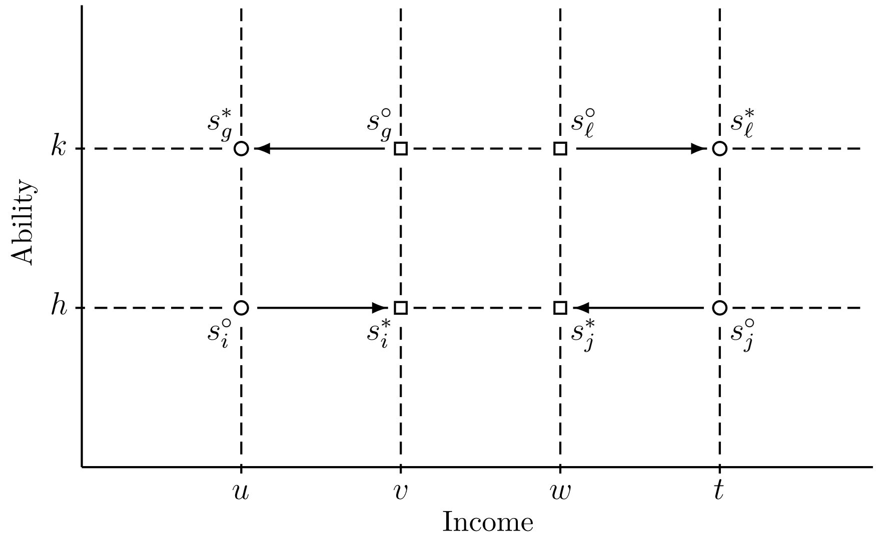
where $\Delta := v-u = t-w > 0$ and $\epsilon := w-u > 0$. Clearly, condition C4 is sufficient for the preceding inequality to hold.

Building upon the idea of compensation once again, one might argue that other things equal a progressive income transfer between two households of the same type will increase social welfare all the more if the households involved are needier. The following transformation will help make this idea more precise:

Definition 3.5 Given two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$, we will say that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of a *favourable composite income permutation* if there exist four households $i, j, g, \ell \in S$ ($i \neq j \neq g \neq \ell$) such that: (i) $x_i^\circ = x_g^* < x_i^* = x_g^\circ$; (ii) $x_j^* = x_\ell^\circ < x_j^\circ = x_\ell^*$; (iii) $x_g^\circ - x_i^\circ = x_j^\circ - x_\ell^\circ$ and $x_i^\circ < x_\ell^\circ$; (iv) $a_i^\circ = a_i^* = a_j^* = a_j^\circ < a_g^* = a_g^\circ = a_\ell^\circ = a_\ell^*$; and (v) $s_r^\circ = s_r^*$, for all $r \neq i, j, g, \ell$. Equivalently, we would say that $\mathbf{s}^\circ$ is obtained from $\mathbf{s}^*$ by means of an *unfavourable composite income permutation*.

The preceding definition might look abstruse at first glance and Figure 3.5 shows what a favourable composite income permutation looks like. It clearly shows that a favourable composite income permutation combines a favourable permutation and a unfavourable permutation of the same magnitude, the former involving lower incomes than the latter. The idea is that the positive effect on social welfare of the favourable permutation more than offsets the negative impact of the unfavourable permutation. But Figure 3.5 also suggests

Figure 3.5: A favourable composite income permutation



another interpretation: a favourable composite income permutation can be decomposed into (i) a progressive income transfer between households i and ℓ who are of type h and (ii) a regressive income transfer between households j and g of type k, where both transfers are of the same magnitude. The reduction in inequality resulting from the progressive transfer of income between the households of low ability outweighs the increase in inequality caused by the regressive transfer taking place between the households of higher ability.

It can be shown that social welfare never diminishes following a favourable composite income permutation if the second partial derivative of the household utility function with respect to income is non-decreasing with ability, or more precisely:

C5 $$U_{yy}(y,h) \leqq U_{yy}(y,h+1), \ \forall\, y \in \mathscr{D}, \ \forall\, h = 1,2,\dots,H-1.$$

Suppose that situation $\mathbf{s}^*$ is obtained from situation $\mathbf{s}^\circ$ by means of a favourable composite income permutation as described in Figure 3.5 and that this results in a social welfare improvement. We have

(3.7) $$\Delta W_U = -\frac{1}{n}\sum_{r=0}^{k-h-1}\int_0^{\Delta}\int_0^{\epsilon}[U_{yy}(u+\vartheta+\xi,h+r+1)-U_{yy}(u+\vartheta+\xi,h+r)]\,d\vartheta d\xi \geqq 0,$$

where $\Delta := v-u = t-w > 0$ and $\epsilon := w-u > 0$. Clearly, condition C5 is sufficient for inequality (3.7) to hold. It is convenient to introduce the following class of utility functions:

(3.8) $$\mathscr{U}_{12} := \{U \in \mathscr{U} \mid \text{conditions C1, C2, C3, C4 and C5 are fulfilled}\}.$$

We will say that situation $\mathbf{s}^*$ results from situation $\mathbf{s}^\circ$ by means of a T_{12}-*transformation* if $\mathbf{s}^*$ can be obtained from $\mathbf{s}^\circ$ by means of an ability increment and/or an income increment and/or a favourable permutation and/or a progressive income transfer and/or a favourable composite income permutation. Then, we have:

Remark 3.2 Social welfare does not decrease as the result of a T_{12}-transformation if and only if $U \in \mathscr{U}_{12}$.

Proof. It follows from the discussion above that $U \in \mathscr{U}_{12}$ is a sufficient condition for social welfare not to decrease as the result of a T_{12}-transformation. In order to prove that it is also necessary, we argue a contrario and show that, if any of the conditions C1, C2, C3, C4 or C5 is not fulfilled, then there exists two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$ such that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of a T_{12}-transformation but $W_U(\mathbf{s}^*) < W_U(\mathbf{s}^\circ)$. Suppose that condition C5 does not hold, in which case there exists $y \in \mathscr{D}$ and $h \in \{1, 2, \ldots, H-1\}$ such that $U_{yy}(y, h) > U_{yy}(y, h+1)$. Consider then the situations $\mathbf{s}^\circ$ and $\mathbf{s}^*$ defined in Figure 3.5 with $k = h+1$. Choosing $\Delta := v - u > 0$ and $\epsilon := w - u > 0$ arbitrarily small, we get

$$\lim_{\Delta, \epsilon \to 0} \left[\frac{\Delta W_U / \Delta}{\epsilon} \right] = U_{yy}(u, h+1) - U_{yy}(u, h) < 0, \tag{3.9}$$

which demonstrates that social welfare decreases as the result of a favourable composite income permutation and by a way of consequence as the result of a T_{12}-transformation. We would arrive at the same result when any of the conditions C1 to C4 does not hold by choosing appropriately the situations $\mathbf{s}^\circ$ and $\mathbf{s}^*$. □

Here again we assume that the ethical observer shares the ethical value judgments captured by conditions C1 to C5 and ranks the situations under comparison on the basis of the utilitarian unanimity principle. The result below identifies a simple procedure for deciding when one situation is preferable to another when attention is restricted to utility functions in the class $\mathscr{U}_{12}$.

Proposition 3.2 *Consider two situations* $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$. *Statements* (a) *and* (b) *below are equivalent:*

(a) $W_U(\mathbf{s}^*) \geqq W_U(\mathbf{s}^\circ), \ \forall\, U \in \mathscr{U}_{12}$.

(b1) $F^*(\overline{v}, h) \leqq F^\circ(\overline{v}, h), \ \forall\, h = 1, 2, \ldots, H-1;$ *and*

(b2) $\int_{\underline{v}}^{y} F^*(\xi, h)\, d\xi \leqq \int_{\underline{v}}^{y} F^\circ(\xi, h)\, d\xi, \ \forall\, y \in \mathscr{D}, \ \forall\, h \in \mathscr{H}$.

Condition P3.2b2 actually corresponds to the sequential generalised Lorenz dominance criterion introduced by Atkinson and Bourguignon (1987) in the particular case where *the marginal distribution of ability is fixed*. It requires that the distribution of income for those households whose ability does not exceed h in situation $\mathbf{s}^*$ stochastically dominates to the second order the corresponding income distribution in situation $\mathbf{s}^\circ$, and this holds whatever the level h of ability considered. But, contrary to Atkinson and Bourguignon (1987)'s criterion, the proportions of households whose abilities are no greater than h may be different in the two situations. However, while it is necessary, condition P3.2b2 is not sufficient for guaranteeing a non-ambiguous increase in social welfare, given the conditions maintained for the household utility function. One has actually to add a condition on the marginal distribution of ability, namely condition P3.2b1: whatever the level of ability, the proportion of households with abilities no greater than this level is no larger in situation $\mathbf{s}^*$ than in situation $\mathbf{s}^\circ$.

Drawing upon Shorrocks and Foster (1987), it would be possible to impose additional restrictions on the household utility function in order to increase the discriminatory power of the dominance criteria. One would then obtain conditions similar to those in Proposition 3.2 where statement P3.2b2 is replaced by

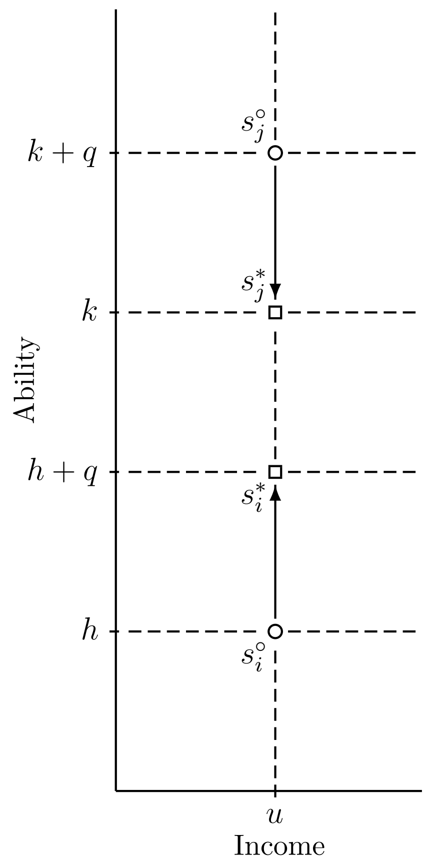
$$\int_{\underline{v}}^{y} \int_{\underline{v}}^{\xi} F^*(\vartheta, h)\, d\vartheta\, d\xi \leq \int_{\underline{v}}^{y} \int_{\underline{v}}^{\xi} F^\circ(\vartheta, h)\, d\vartheta\, d\xi, \ \forall\, y \in \mathscr{D}, \ \forall\, h \in \mathscr{H}, \tag{3.10}$$

and where one requires in addition that the third derivative of the utility function with respect to income is non-negative and non-increasing with ability. Lambert and Ramos (2002) obtained a related result but without imposing the condition on the marginal distributions of ability as we have done in Proposition 3.2. This is achieved at the cost of replacing condition C1 with a stronger property which actually raises difficulties as we will argue in Section 5. Insofar as the corresponding properties of the utility function are not that transparent, we would rather explore another direction by introducing a certain degree of cardinalism into the analysis.

4. Dominance Criteria with Cardinal Abilities

So far ability has been treated as a purely ordinal concept so that comparisons of ability differences or ratios were meaningless. While avoiding any recourse to any equivalence scale, one might think of introducing some degree of cardinalism into the analysis. The idea is that, for a given income, making comparisons of utility differences between households of different types is not totally nonsensical. For instance, we might consider that, for a given income, the difference in utility between a household of type $h+1$ and a household of type h is smaller the difference in utility between households of types $h+2$ and $h+1$. This value judgement – assuming that the list of abilities is fixed once for all and that abilities are totally ordered – introduces a degree of cardinalism into the measurement of household well-being and implicitly

Figure 4.1: A progressive ability transfer



in the concept of ability. [15]

Definition 4.1 Given two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$, we will say that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of an *progressive ability transfer* if there exist two households $i, j \in S$ ($i \neq j$) such that: (i) $a_i^\circ < a_i^* < a_j^\circ$, $a_i^\circ < a_j^* < a_j^\circ$; (ii) $a_i^* - a_i^\circ = a_j^\circ - a_j^*$; (iii) $x_i^\circ = x_i^* = x_j^* = x_j^\circ$; and (iv) $s_r^\circ = s_r^*$, for all $r \neq i, j$. Equivalently, we would say that $\mathbf{s}^\circ$ is obtained from $\mathbf{s}^*$ by means of an *regressive ability transfer*.

The notion of a progressive ability transfer is illustrated in Figure 4.1. Making the value judgement that a progressive ability transfer results in a social welfare improvement amounts to imposing the following condition on the household utility function:

$$\text{C6} \qquad U(y, h+2) - 2\,U(y, h+1) + U(y, h) \leqq 0,\ \forall\, y \in \mathscr{D},\ \forall\, h = 1, 2, \ldots, H-2.$$

Indeed, one can easily verify that social welfare does not decrease as the result of a progressive ability transfer if condition C6 is fulfilled. Consider the situations $\mathbf{s}^*$ and $\mathbf{s}^\circ$ depicted in Figure 4.1 and suppose further that

$$(4.1)\ \ \Delta W_U = -\frac{1}{n} \sum_{p=0}^{q-1} \sum_{r=0}^{k-h-1} [U(u, h+p+r+2) - 2\,U(u, h+p+r+1) + U(u, h+p+r)] \geqq 0.$$

Clearly, inequality (4.1) will hold if condition C6 is fulfilled. Condition C6 – non-increasing utility differences with respect to ability – is the counterpart for ability of condition C4. [16]

Definition 4.2 Given two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$, we will say that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of a *favourable composite ability permutation* if there exist four households $i, j, g, \ell \in S$ ($i \neq j \neq g \neq \ell$) such that: (i) $a_i^\circ = a_g^* < a_i^* = a_g^\circ$; (ii) $a_j^* = a_\ell^\circ < a_j^\circ = a_\ell^*$; (iii) $a_g^\circ - a_i^\circ = a_j^\circ - a_\ell^\circ$ and $a_i^\circ < a_\ell^\circ$; (iv) $x_i^\circ = x_i^* = x_j^* = x_j^\circ < x_g^* = x_g^\circ = x_\ell^\circ = x_\ell^*$; and (v) $s_r^\circ = s_r^*$, for all $r \neq i, j, g, \ell$. Equivalently, we would say that $\mathbf{s}^\circ$ is obtained from $\mathbf{s}^*$ by means of a *unfavourable composite ability permutation*.

The preceding definition is illustrated in Figure 4.2 which shows that a favourable composite ability permutation is the result of the combination of a progressive ability transfer and of a regressive ability transfer where the households involved in the first transfer have a lower income than those taking part in the second. But Figure 4.2 suggests also another interpretation: a favourable composite ability permutation can be decomposed into a favourable permutation involving types h and $h+q$ and an unfavourable permutation between types k and $k+q$. The idea, at the very foundation of this kind of transformation, is that the increase in social welfare resulting from the progressive ability transfer (resp. favourable permutation) overcompensates for the decrease in social welfare caused by the regressive ability transfer (resp. unfavourable permutation).

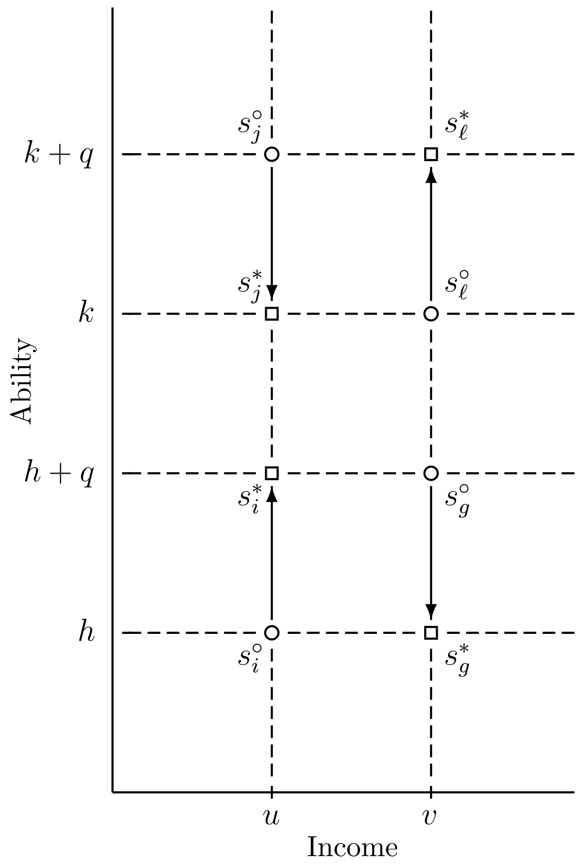
As we will show, the following property of the household utility function is closely related to progressive ability transfers:

$$\text{C7} \qquad U_y(y, h+2) - 2\,U_y(y, h+1) + U_y(y, h) \geqq 0,\ \forall\, y \in \mathscr{D},\ \forall\, h = 1, 2, \ldots, H-2.$$

[15] The condition that the marginal utility of income is non-increasing with household size – or equivalently non-decreasing with ability – is inseparable from the way the households types and the corresponding list of abilities are defined.

[16] The condition of non-increasing utility differences with respect to ability is reminiscent of the notion of concavity in integers (see Marshall and Olkin (1979, Chapter 16)).

Figure 4.2: A favourable composite ability permutation



Indeed, suppose we want social welfare not to decrease as the result of a progressive ability transfer as it is defined in Figure 4.2. Then, we must have:

$$\Delta W_U = \frac{1}{n}\sum_{p=0}^{q-1}\sum_{r=0}^{k-h-1}\int_0^{\Delta}\left\{\begin{array}{r} U_y(u+\xi, h+p+r+2) \\ -\,2\,U_y(u+\xi, h+p+r+1) \\ +\,U_y(u+\xi, h+p+r) \end{array}\right\} d\xi \geqq 0, \tag{4.2}$$

where $\Delta := v - u > 0$, which establishes sufficiency of condition C7. For later reference we introduce

$$\mathscr{U}_{21} := \{U \in \mathscr{U} \mid \text{ conditions C1, C2, C3, C6 and C7 are fulfilled}\}. \tag{4.3}$$

The classes $\mathscr{U}_{21}$ and $\mathscr{U}_{12}$ have in common the fact that the utility functions that constitute them are non-decreasing in both ability and income and have marginal utility of income non-increasing with ability. Both also involve progressive transfers and favourable permutations, but differ with respect to the attribute that is involved in these transformations: income in the case of $\mathscr{U}_{12}$ and ability in the case of $\mathscr{U}_{21}$.

We will say that situation $\mathbf{s}^*$ results from situation $\mathbf{s}^\circ$ by means of a T_{21}-*transformation* if $\mathbf{s}^*$ can be obtained from $\mathbf{s}^\circ$ by means of an ability increment and/or an income increment and/or a favourable permutation and/or a progressive ability transfer and/or a favourable composite ability permutation. Then, we have:

Remark 4.1 Social welfare does not decrease as the result of a T_{21}-transformation if and only if $U \in \mathscr{U}_{21}$.

Proof. It follows from the discussion above that $U \in \mathscr{U}_{21}$ is a sufficient condition for social welfare not to decrease as the result of a T_{21}-transformation. In order to prove that it is also necessary, we argue a contrario and show that, if any of the conditions C1, C2, C3, C6 or C7 is not fulfilled, then there exist two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$ such that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of a T_{21}-transformation but $W_U(\mathbf{s}^*) < W_U(\mathbf{s}^\circ)$. Suppose for instance that condition C7 is violated in which case there exists $y \in \mathscr{D}$ and $h \in \{1, 2, \dots, H-1\}$ such that $U_y(y, h+2) - 2\, U_y(y, h+1) + U_y(y, h) < 0$. Consider then the situations $\mathbf{s}^\circ$ and $\mathbf{s}^*$ depicted in Figure 4.2, and choose $\Delta := v - u > 0$ arbitrarily small, $k = h+1$ and $q = 1$. We get

$$\lim_{\Delta \longrightarrow 0} \left[\frac{\Delta W_U}{\Delta} \right] = \frac{1}{n} \left[U_y(u, h+2) - 2\, U_y(u, h+1) + U_y(u, h) \right] < 0, \tag{4.4}$$

and we conclude that social welfare has decreased as the result of a favourable composite ability permutation and, as a consequence as the result of a T_{21}-transformation. We would arrive at the same result when any of the conditions C1 to C3 and C6 does not hold by choosing appropriately the situations $\mathbf{s}^\circ$ and $\mathbf{s}^*$. □

Suppose that the utilitarian ethical observer subscribes to the value judgements embedded in conditions C1, C2, C3, C6 and C7. The next result identifies a simple procedure in order to decide when one situation is better than another according to utilitarian unanimity restricted to the class $\mathscr{U}_{21}$.

Proposition 4.1 *Consider two situations* $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$. *Statements* (a) *and* (b) *below are equivalent:*

(a) $W_U(\mathbf{s}^*) \geqq W_U(\mathbf{s}^\circ),\ \forall\, U \in \mathscr{U}_{21}$.

(b1) $F^*(y, H) \leqq F^\circ(y, H),\ \forall\, y \in \mathscr{D}$; *and*

(b2) $\sum_{r=1}^{h} F^*(y, r) \leqq \sum_{r=1}^{h} F^\circ(y, r),\ \forall\, y \in \mathscr{D},\ \forall\, h \in \mathscr{H}$.

Condition P4.1b1 means that – considering all household types combined – the distribution of income in situation $\mathbf{s}^*$ stochastically dominates to the first order the distribution of income in situation $\mathbf{s}^\circ$. Making use of the relationship between poverty and stochastic dominance (see Foster and Shorrocks (1988)), we conclude that, whatever the poverty line, there are no more poor households in situation $\mathbf{s}^*$ than in situation $\mathbf{s}^\circ$. Condition P4.1b2, which concerns the joint distributions of income and ability, is more difficult to interpret. It requires that the distribution of ability for those households whose income does not exceed y in situation $\mathbf{s}^*$ stochastically dominates to the second order the corresponding distribution of ability in situation $\mathbf{s}^\circ$, and this holds whatever the level y of income considered. In the particular case where $y = \overline{v}$, it amounts to a second degree stochastic dominance ranking of the marginal distributions of ability.

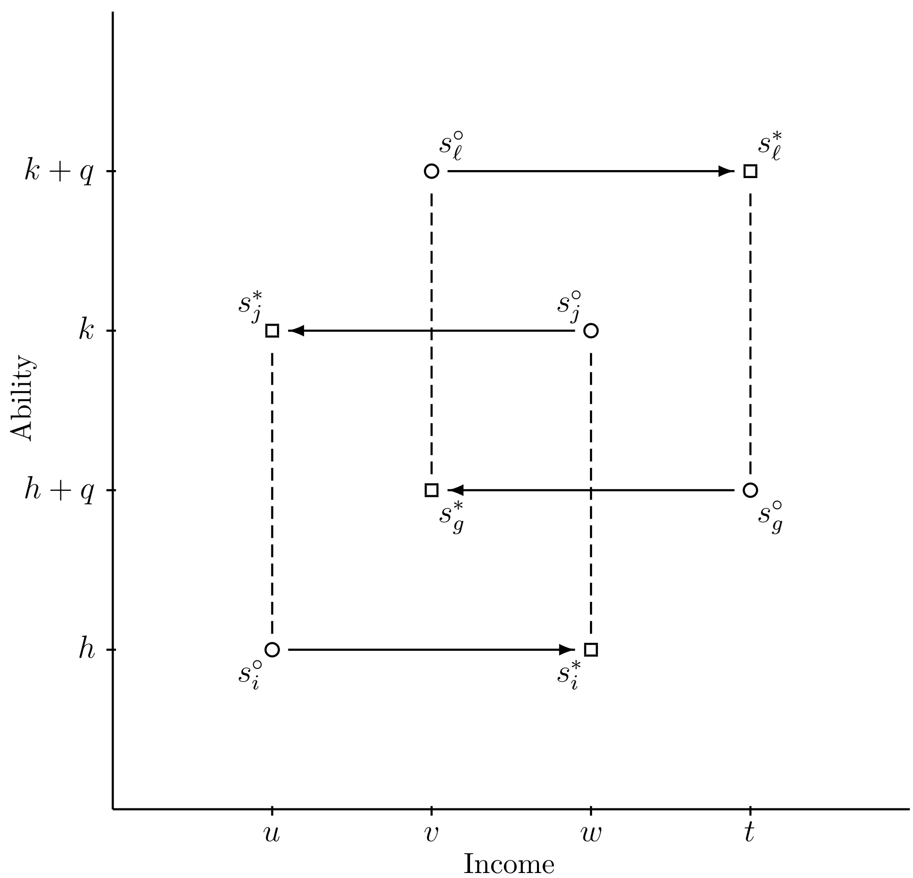
Proposition 3.2 and Proposition 4.1 actually mirror each other: both results impose a first order stochastic dominance condition and a (partial) second order stochastic dominance condition. In the case of Proposition 3.2 the first order stochastic dominance test concerns the marginal distributions of ability, while in the case of Proposition 4.1 it is applied to the marginal distributions of income. Similarly, the second order stochastic dominance test applies to the distributions of income evaluated at every level of ability in the case of Proposition 3.2, while it is the distributions of ability evaluated at every income level that are subjected to the test in the case of Proposition 4.1.

The composite permutations we have examined up to now are permutations in only one dimension: *either* income, *or* ability. This restricts the possibilities open for transforming the situations and our next transformation is freed from this constraint.

Definition 4.3 Given two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{L}$, we will say that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of a *favourable composite permutation* if there exist four households $i, j, g, \ell \in S$ $(i \neq j \neq g \neq \ell)$ such that: (i) $x_i^\circ = x_j^* < x_i^* = x_j^\circ$, $a_i^\circ = a_i^* < a_j^* = a_j^\circ$; (ii) $x_g^* = x_\ell^\circ < x_g^\circ = x_\ell^*$, $a_g^* = a_g^\circ < a_\ell^\circ = a_\ell^*$; (iii) $x_j^\circ - x_i^\circ = x_g^\circ - x_\ell^\circ$, $a_j^\circ - a_i^\circ = a_\ell^\circ - a_g^\circ$; (iv) $x_i^\circ < x_g^*$, $a_i^\circ < a_g^*$; and (v) $s_r^\circ = s_r^*$, for all $r \neq i, j, g, \ell$. Equivalently, we would say that $\mathbf{s}^\circ$ is obtained from $\mathbf{s}^*$ by means of a *unfavourable composite permutation.*

A typical favourable composite permutation is represented in Figure 4.3, which shows that such a transformation can be interpreted as the combination of a favourable permutation and a unfavourable permutation of the same magnitude, where the favourable permutation involves households who are at the same time poorer and less able than the households taking part in the unfavourable permutation. Actually, all three of the favourable composite permutations we

Figure 4.3: A favourable composite permutation



have considered combine favourable and unfavourable permutations, but it is the favourable composite permutation which imposes the least restrictions on the way the permutations have to be combined.

The following property of the household utility function ensures that social welfare cannot decrease as the result of a favourable composite permutation:

C8 $\quad U_{yy}(y, h+2) - 2\,U_{yy}(y, h+1) + U_{yy}(y, h) \leqq 0,\ \forall\ y \in \mathscr{D},\ \forall\ h = 1, 2, \ldots, H-2.$

To show this, suppose that situation $\mathbf{s}^*$ is obtained from situation $\mathbf{s}^\circ$ by means of a favourable composite permutation as described in Figure 4.3. For social welfare not to decrease, we need to have

$$\Delta W_U = -\frac{1}{n}\sum_{p=0}^{q-1}\sum_{r=0}^{k-h-1}\int_0^{\Delta}\int_0^{\epsilon}\left\{\begin{array}{r} U_{yy}(u+\vartheta+\xi, h+p+r+2) \\ -\,2\,U_{yy}(u+\vartheta+\xi, h+p+r+1) \\ +\,U_{yy}(u+\vartheta+\xi, h+p+r) \end{array}\right\} d\vartheta\, d\xi \geqq 0, \tag{4.5}$$

where $\Delta := v - u > 0$ and $\epsilon := w - u > 0$, which is clearly satisfied if condition C8 holds. For later reference we introduce

$$\mathscr{U}_{22} := \{U \in \mathscr{U} \mid \text{ conditions C1 to C8 are fulfilled}\}. \tag{4.6}$$

Furthermore, we will say that situation $\mathbf{s}^*$ results from situation $\mathbf{s}^\circ$ by means of a T_{22}-*transformation* if $\mathbf{s}^*$ can be obtained from $\mathbf{s}^\circ$ by means of an ability increment and/or an income increment and/or a favourable permutation and/or a progressive income transfer and/or a progressive ability transfer and/or a favourable composite income permutation and/or a favourable composite ability permutation and/or a favourable composite permutation. Then, we have:

Remark 4.2 Social welfare does not decrease as the result of a T_{22}-transformation if and only if $U \in \mathscr{U}_{22}$.

Proof. It follows from the discussion above that $U \in \mathscr{U}_{21}$ is a sufficient condition for social welfare not to decrease as the result of a T_{22}-transformation. In order to prove that it is also necessary, we argue a contrario and show that, if any of the conditions C1 to C8 is not fulfilled, then there exists two situations $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$ such that $\mathbf{s}^*$ is obtained from $\mathbf{s}^\circ$ by means of a T_{22}-transformation but $W_U(\mathbf{s}^*) < W_U(\mathbf{s}^\circ)$. Suppose that condition C8 is violated, in which case there exists $y \in \mathscr{D}$ and $h \in \{1, 2, \ldots, H-2\}$ such that $U_{yy}(y, h+2) - 2\,U_{yy}(y, h+1) + U_{yy}(y, h) > 0$. Consider then the situations $\mathbf{s}^\circ$ and $\mathbf{s}^*$ depicted in Figure 4.3, and choose $\Delta := v - u > 0$ and $\epsilon := w - u > 0$ arbitrarily small, $k = h + 1$ and $q = 1$. We get

$$\lim_{\Delta,\epsilon\to 0}\frac{\Delta W_U/\Delta}{\epsilon} = \frac{1}{n}\left[U_{yy}(u, h+2) - 2\,U_{yy}(u, h+1) + U_{yy}(u, h)\right] < 0, \tag{4.7}$$

and we conclude that social welfare has decreased as the result of a favourable composite permutation and, as a consequence as the result of a T_{22}-transformation. We would arrive at the same result when any of the conditions C1 to C7 does not hold by choosing appropriately the situations $\mathbf{s}^\circ$ and $\mathbf{s}^*$. $\square$

Suppose that the utilitarian ethical observer subscribes to the value judgements embedded in conditions C1 to C8. The next result identifies a simple procedure in order to decide when one situation is better than another according to utilitarian unanimity restricted to the class $\mathscr{U}_{22}$.

Proposition 4.2 *Consider two situations* $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$. *Statements* (a) *and* (b) *below are equivalent:*

(a) $W_U(\mathbf{s}^*) \geqq W_U(\mathbf{s}^\circ),\ \forall\, U \in \mathscr{U}_{22}$.

(b1) $\sum_{r=1}^{h} F^*(\overline{v}, r) \leqq \sum_{r=1}^{h} F^\circ(\overline{v}, r),\ \forall\, h = 1, 2, \ldots, H$; *and*

(b2) $\int_{\underline{v}}^{y} F^*(\xi, H)\, d\xi \leqq \int_{\underline{v}}^{y} F^\circ(\xi, H)\, d\xi,\ \forall\, y \in \mathscr{D}$; *and*

(b3) $\sum_{r=1}^{h}\int_{\underline{v}}^{y} F^*(\xi, r)\, d\xi \leqq \sum_{r=1}^{h}\int_{\underline{v}}^{y} F^\circ(\xi, r)\, d\xi,\ \forall\, y \in \mathscr{D},\ \forall\, h \in \mathscr{H}$.

Statement P4.2b is a second degree stochastic dominance condition for distributions of two attributes, one cardinal and the other ordinal. Here again, we would like to stress the crucial role played by the marginal distributions of income and ability. In particular, second degree stochastic dominance for ability (condition P4.2b1) is necessary for social welfare to increase, which underlines the importance of demographic changes for the assessment of living standards in the society. Similarly, considering all types combined, the distribution of income in situation $\mathbf{s}^*$ must dominate to the second order the distribution of income in situation $\mathbf{s}^\circ$ in order that $\mathbf{s}^*$ is ranked above $\mathbf{s}^\circ$ by the unanimity utilitarian rule over the class $\mathscr{U}_{22}$ (condition P4.2b2). Condition P4.2b3 requires that the sums of the integrals of the joint distribution function corresponding to situation $\mathbf{s}^*$ are nowhere above those corresponding to situation $\mathbf{s}^\circ$, for every income level y, where the summation is performed up to ability h, and that this holds for all $h = 1, 2, \dots, H$. The meaning of this condition is perhaps more transparent once we realise that

$$\int_{\underline{v}}^{y} F(\xi, h)\, d\xi = f(\overline{v}, h)\, P(y \,|\, h), \;\; \forall\; y \in \mathscr{D}, \;\; \forall\; h \in \mathscr{H}, \tag{4.8}$$

where $P(y \,|\, h)$ is the poverty gap for the population of households of type h when the poverty line is set to y, namely

$$P(y \,|\, h) := \int_{\underline{v}}^{y} (y - \xi)\, f(\xi \,|\, h)\, d\xi, \;\; \forall\; y \in \mathscr{D}, \;\; \forall\; h \in \mathscr{H}, \tag{4.9}$$

and $f(y \,|\, h)$ is the conditional – upon ability – density function of income. Then, condition P4.2b3 amounts to saying that the average poverty gap for the population consisting of the h more able households is no greater in situation $\mathbf{s}^*$ than in situation $\mathbf{s}^\circ$, whatever the poverty line y and whatever the ability level h.

5. Comparison with the Jenkins-Lambert Criterion

We restrict here attention to the case where ability is considered an ordinal variable, so that only comparisons of utility levels between households of different types make sense. The dominance criterion introduced in Proposition 3.2 (conditions P3.2b1 and P3.2b2) differs from the procedure proposed by Jenkins and Lambert (1993). Fundamentally, the difference between the two criteria originates in the way abilities are taken into account in the assessment of household well-being. As we have seen, condition C1 defines implicitly an ordering of household types by assuming that a household of type h is less able – or equivalently has more needs – than a household of type $h+1$. In place of condition C1, Jenkins and Lambert (1993) proposed the following requirement:

C1* $\quad U(\overline{v}, h) = U(\overline{v}, h+1), \;\; \forall\; h = 1, 2, \dots, H-1.$

This condition has been used subsequently by a number of scholars, including Chambaz and Maurin (1998), Lambert and Ramos (2002), and Fleurbaey, Hagneré, and Trannoy (2003).

While, at first sight, conditions C1 and C1* differ significantly, they are not totally independent as C1* and C3 together imply C1. In conjunction with C3, condition C1* is stronger than condition C1 to the extent that it requires that the utilities derived by the different types are the same for the highest conceivable income. The idea is that, for sufficiently large incomes, the differences in abilities have a negligible impact on household well-being. Assuming that we agree with this view – it is always possible to compensate for differences in needs – it remains an important difficulty relative to the implementation of the dominance criteria, as we will show later on. The dominance criterion of Jenkins and Lambert (1993) is obtained by substituting condition C1* for condition C1 in Proposition 3.2. More precisely, letting

$$\mathscr{U}^*_{12} := \{U \in \mathscr{U} \mid \text{ conditions C1}^*\text{, C2, C3, C4 and C5 are fulfilled}\}, \tag{5.1}$$

their result can be stated as:

Proposition 5.1 *Consider two situations* $\mathbf{s}^\circ, \mathbf{s}^* \in \mathscr{X}$. *The following two statements are equivalent:*

(a) $W_U(\mathbf{s}^*) \geqq W_U(\mathbf{s}^\circ),\ \forall\, U \in \mathscr{U}^*_{12}$.

(b) $\displaystyle\int_{\underline{v}}^{y} F^*(\xi, h)\, d\xi \leqq \int_{\underline{v}}^{y} F^\circ(\xi, h)\, d\xi,\ \forall\, y \in \mathscr{D},\ \forall\, h \in \mathscr{H}$.

The comparison of Propositions 3.2 and 5.1 makes clear that the Jenkins-Lambert criterion is more powerful than the dominance criterion defined by P3.2b. Indeed, condition P3.2b2 is the condition obtained by Jenkins and Lambert (1993) when one requires utilitarian unanimity to hold over the class $\mathscr{U}^*_{12}$. Technically, this is not surprising: since the class $\mathscr{U}^*_{12}$ is a proper subset of the class $\mathscr{U}_{12}$, the quasi-ordering we get by requiring unanimity over $\mathscr{U}_{12}$ is a subrelation of the quasi-ordering obtained starting with $\mathscr{U}^*_{12}$. Given the conditions we have imposed on the household utility function, condition P3.2b2, while necessary, is insufficient to ensure that social welfare unambiguously improves. One must actually add a condition on the marginal distributions of ability: whatever the ability level, the proportion of households whose abilities do not exceed that level must be smaller in situation $\mathbf{s}^*$ than in situation $\mathbf{s}^\circ$. Condition C1* amounts to considering that, for sufficiently high incomes, ability differences are negligible. The immediate question is to decide what is the income value from which we consider that differences in ability no longer matter as far as household well-being is concerned. It is far from clear that we can agree on such a value and it is thus dubious that condition C1* can be unanimously approved. While, from a normative point of view, the difficulty can be circumvented by letting $\overline{v} = +\infty$, for in this case Fleurbaey *et al.* (2003, Remark 5.1) have shown that Proposition 5.1 boils down to Proposition 3.2, the difficulty still remains in practice of choosing the right value of $\overline{v}$.

To see the inherent problem with the Jenkins and Lambert (1993) approach, consider first the simple case where the population consists of identical households. In this case, we are back to the standard unidimensional framework and the interval over which we compare the distribution functions of different situations is defined by the smallest and the largest incomes we observe. Thus, if we have to compare three distributions of income $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, then the pairwise ranking we get is consistent in the sense that the corresponding binary relation is transitive. There is no need to define a common – to all distributions under comparison – support since the rankings obtained by using the supports corresponding to all pairs or by using the common support are identical, which results in an appreciable gain in terms of computations.

This property is not satisfied in the case of the Jenkins-Lambert dominance criterion. Actually, the choice of the upper bound of the income interval over which the distribution functions are compared affects the result of the comparisons. To highlight the difficulty, consider a society comprising three households $S = \{1, 2, 3\}$ and the three following situations:

$$\mathbf{s}^1 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \\ 2 & 2 \end{bmatrix};\quad \mathbf{s}^2 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 3 & 2 \end{bmatrix};\quad \mathbf{s}^3 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 5 & 2 \end{bmatrix}.$$

The joint densities corresponding to the situations $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^3$, which we denote respectively by $f^1(y,h)$, $f^2(y,h)$ and $f^3(y,h)$ are defined in Table 5.1. We also indicate the marginal densities of income $f_1^1(y)$, $f_1^2(y)$ and $f_1^3(y)$, and the marginal densities of ability $f_2^1(h)$, $f_2^2(h)$ and $f_2^3(h)$. One can verify that a progressive income transfer is needed to convert situation $\mathbf{s}^3$ into situation $\mathbf{s}^2$, and Proposition 3.2 indicates that $\mathbf{s}^2$ is considered a better situation than $\mathbf{s}^3$ by all utilitarian ethical observers who subscribe to the value judgements embedded in the class $\mathscr{U}_{12}$. On the other hand, it does not seem possible to obtain situation $\mathbf{s}^1$ from situation

Table 5.1: Density functions of situations $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^3$

		Income					
	$f^1(y,h)$	1	2	3	4	5	$f_2^1(h)$
Ability	1	0	0	2/3	0	0	2/3
	2	0	1/3	0	0	0	1/3
	$f_1^1(y)$	0	1/3	2/3	0	0	

		Income					
	$f^2(y,h)$	1	2	3	4	5	$f_2^2(h)$
Ability	1	0	1/3	0	0	0	1/3
	2	0	0	2/3	0	0	2/3
	$f_1^2(y)$	0	1/3	2/3	0	0	

		Income					
	$f^3(y,h)$	1	2	3	4	5	$f_2^3(h)$
Ability	1	0	1/3	0	0	0	1/3
	2	1/3	0	0	0	1/3	2/3
	$f_1^3(y)$	1/3	1/3	0	0	1/3	

$\mathbf{s}^2$ or from situation $\mathbf{s}^3$ – or *vice versa* – by means of ability increments, income increments, favourable permutations, progressive income transfers or income favourable composite permutations. Actually, the pairs $\{\mathbf{s}^1,\mathbf{s}^2\}$ and $\{\mathbf{s}^1,\mathbf{s}^3\}$ violate condition P3.2b, and we conclude from Proposition 3.2 that the utilitarian unanimity over the class $\mathscr{U}_{12}$ cannot provide conclusive rankings for these two pairs of situations. Applying now the test proposed by Jenkins and Lambert (1993) to the pairs of situations $\{\mathbf{s}^1,\mathbf{s}^2\}$, $\{\mathbf{s}^2,\mathbf{s}^3\}$, and $\{\mathbf{s}^1,\mathbf{s}^3\}$, with

(5.2) $$\overline{v} = \max\{x_i^1, x_i^2\},\ \overline{v} = \max\{x_i^2, x_i^3\} \text{ and } \overline{v} = \max\{x_i^1, x_i^3\},$$

successively, we get

(5.3) $$\mathbf{s}^1 \geq_{\mathscr{U}_{12}^*} \mathbf{s}^2,\ \mathbf{s}^2 \geq_{\mathscr{U}_{12}^*} \mathbf{s}^3,\ \mathbf{s}^1 \text{ and } \mathbf{s}^3 \text{ are non-comparable},$$

where $\geq_{\mathscr{U}_{12}^*}$ is the utilitarian quasi-ordering when unanimity is imposed over the class $\mathscr{U}_{12}^*$. Thus, contrary to what happens in the unidimensional case, the ranking obtained by means of pairwise comparisons is not transitive.

Assuming that we choose as the upper bound $\overline{v}$ the largest income in the two distributions under comparison, a direct implication is that it is possible to manipulate the ranking of the situations by introducing a new situation. In the preceding example, if the agenda consists only of situations $\mathbf{s}^1$ and $\mathbf{s}^2$, then we conclude that $\mathbf{s}^1 \geq_{\mathscr{U}_{12}^*} \mathbf{s}^2$. Let us now introduce situation $\mathbf{s}^3$ which is dominated by situation $\mathbf{s}^2$ when we proceed by means of pairwise comparisons. One intuitively expects that taking into account this new situation will not call into question the ranking of situations $\mathbf{s}^1$ and $\mathbf{s}^2$. Actually, the introduction of situation $\mathbf{s}^3$ into the agenda results in a modification of the ranking of situations $\mathbf{s}^1$ and $\mathbf{s}^2$ that become non-comparable.

By the same token, this example demonstrates that the result of the comparisons depends crucially on the upper bound of the income support used for implementing the Jenkins-Lambert dominance criterion. In the present case, we note that

$$\int_{\underline{v}}^{y} \Big[F^1(\xi,1) - F^2(\xi,1)\Big]\, d\xi \; \begin{cases} \leqq 0, & \forall\; y \leqq 4, \\ > 0, & \forall\; y > 4. \end{cases} \tag{5.4}$$

The marginal distributions of income being identical in situations $\mathbf{s}^1$ and $\mathbf{s}^2$, it follows from Proposition 5.1 that $\mathbf{s}^1$ dominates $\mathbf{s}^2$ as long as $\overline{v} \leqq 4$, while no conclusive verdicts emerges if $\overline{v} > 4$. Obviously, the problem vanishes if we choose an upper bound value sufficiently high: greater than 4 in our example and possibly greater than Bill Gates' fortune in empirical work. Beyond the computational costs that this may cause, there is no guarantee that in a near future the wealth of Bill Gates will not be exceeded, which might challenge all results obtained up to that time.

While it is not immediately clear, the only way to guarantee that the ranking of situations by the Jenkins-Lambert criterion is robust to the kind of manipulation described above is actually to impose in addition condition P3.2b1. Suppose indeed that we want to make sure that

$$\int_{\underline{v}}^{y} F^*(\xi,h)\, d\xi \leqq \int_{\underline{v}}^{y} F^\circ(\xi,h)\, d\xi,\; \forall\; y \leqq \overline{v},\; \forall\; h \in \mathscr{H}, \tag{5.5}$$

always implies that

$$\int_{\underline{v}}^{y} F^*(\xi,h)\, d\xi \leqq \int_{\underline{v}}^{y} F^\circ(\xi,h)\, d\xi,\; \forall\; y \leqq \tilde{v},\; \forall\; h \in \mathscr{H}, \tag{5.6}$$

whenever $\tilde{v} \geqq \overline{v} = \max\{x_i^*, x_i^\circ\}$. A necessary and sufficient condition for this to be the case is that

$$F^*(y,h) \leqq F^\circ(y,h),\; \forall\; y \geqq \overline{v},\; \forall\; h \in \mathscr{H}. \tag{5.7}$$

Since by definition there is no income in situations $\mathbf{s}^*$ and $\mathbf{s}^\circ$ greater than $\overline{v}$, the difference $F^*(y,h) - F^\circ(y,h)$ is constant over the interval $[\overline{v}, +\infty)$, and (5.7) reduces to condition P3.2b1. We insist on the fact that all the dominance criteria we have proposed in this paper are transitive and cannot be manipulated by the *strategic* introduction of a non-relevant situation.

6. Concluding Remarks

Building upon the results of Atkinson and Bourguignon (1982) in the case of continuous and cardinal variables, we have proposed four dominance criteria in order to compare income distributions for populations of households with differing needs. These criteria can be considered extensions of the sequential dominance criteria introduced by Atkinson and Bourguignon (1987) for making comparisons of income distributions between populations whose marginal distributions of needs are identical. No restrictions were imposed on the marginal distributions of needs and our criteria can in principle be used for evaluating the impact on the society's standards of living of changing demographic patterns. From this point of view, our results are in line with those of Jenkins and Lambert (1993) and Chambaz and Maurin (1998). However, our criteria differ in the way needs are taken into account in the assessment of social welfare. Beyond this normative difference, our criteria have the advantage – once the choice of the ordered list of abilities has been agreed on – of being immune to manipulation: it is not possible to modify the ranking of the situations under comparison by introducing an irrelevant situation. The second contribution of this article consists in uncovering the elementary transformations of the distributions underpinning the normative conditions imposed on the household utility functions. These transformations, which involve at the same time income and ability, highlight the normative meaning of the restrictions placed on the utility functions – and beyond that the value judgements – that some might consider difficult to understand or even arbitrary.

While the approach taken in the paper is framed in terms of comparisons of income distributions for populations of households who differ in terms of their demographic characteristics, it applies more generally to any situation where one attribute is continuous, cardinal and transferable between the individuals and the other attribute is ordinal and non-transferable. As we have alluded to in Section 2, the application of the technique of stochastic dominance to the comparison of distributions of household's well-being involves strong and specific assumptions about the way the household utility function is derived. The comparison of income distributions where the income recipients differ in health is an instance of a case exempt of such problems. Other potential applications concern the comparisons of income distributions when additional information is available about the environment of the income recipients such as the exposure to risks and crime, or the presence of local public goods. In all these cases, the ethical observer may have some reluctance to attach a cardinal meaning to the values taken by these variables and may prefer to build on an ordinal information. More generally, the approach can be easily adapted to the situation where both attributes are cardinally measurable and continuous as it is the case in the original article of Atkinson and Bourguignon (1982). Then, the partial summations involved in statements P4.1b2, P4.2b1 and P4.2b3 would have to be replaced by integrals and the first and second utility differences in conditions C1, C3, C5, C6, C7 and C9 by the appropriate derivatives. Fundamentally, the difference between our approach and that of Atkinson and Bourguignon (1982) is the recognition that the variables that contribute to the society's welfare are not treated in a symmetric way. Allowing for asymmetries into the analysis opens the route to dominance criteria at variance with the standard ones and may provide in some cases a more appropriate framework (see for instance Muller and Trannoy (2010)).

The approach presented here has a number of limitations, among which an important one is the fact that we focused on bidimensional distributions. Given the asymmetry between the variables taken into account, one might think of introducing *either* more than one cardinal variable *or* more than one ordinal variable. In principle the approach presented here extends naturally to these cases even though one should expect that the number of incomparabilities will increase dramatically. An important motivation of the present analysis was to uncover the transformations of the situations that correspond to different standard properties of the utility function. While we have shown that there exist one-to-one relationships between the elementary transformations of the situations and the properties of the utility functions, a lot still remains to be achieved. In particular, one might be willing to identify the precise way in which these transformations have to be combined in order to generate the dominating situation starting from the dominated one in a finite number of steps. Although there exist attempts in the literature in this direction – for instance Moyes (1996) has results for favourable permutations in the degenerate case where there is only one individual of each type – it must be recognised that we do not yet know a lot. Building on Bourguignon (1989), it is possible to consider a class of utility functions that lies between classes $\mathscr{U}_{11}$ and $\mathscr{U}_{12}$. This is done by requiring that the household utility function is non-decreasing in ability, non-decreasing in income at a declining rate with ability, and concave with respect to income. Gravel and Moyes (2010) have identified the transformations which improve social welfare when situations are evaluated by means of utilitarian unanimity. Finally part of our results continue to hold if one substitutes welfarist unanimity for utilitarian unanimity (see Gravel and Moyes (2011)).

A. Proofs of the Main Results

Sufficient Conditions. Let $\Delta W_U := W_U(\mathbf{s}^*) - W_U(\mathbf{s}^\circ)$ and $\Delta f(y,h) := f^*(y,h) - f^\circ(y,h)$, for all $y \in \mathscr{D}$ and all $h \in \mathscr{H}$. The variation in social welfare when we go from situation $\mathbf{s}^\circ$ to situation $\mathbf{s}^*$ is given by:

$$\Delta W_U = \sum_{h=1}^{H} \left\{ \int_{\underline{v}}^{\overline{v}} U(y,h) \Delta f(y,h)\, dy \right\}. \tag{A.1}$$

Case 1: $U \in \mathscr{U}_{11}$. Integrating by parts the term within braces in (A.1) and letting $\Delta F(y,h) := F^*(y,h) - F^\circ(y,h)$, we get:

$$\Delta W_U = \sum_{h=1}^{H} \left\{ U(\overline{v},h) \int_{\underline{v}}^{\overline{v}} \Delta f(y,h)\, dy - \int_{\underline{v}}^{\overline{v}} U_y(y,h) \left[\int_{\underline{v}}^{y} \Delta f(\xi,h)\, d\xi \right] dy \right\}. \tag{A.2}$$

We refer the reader to Fishburn and Vickson (1978, pp. 72–73) for a justification of this way of proceeding in the case of non-continuous distribution functions. Rearranging (A.2) and applying Abel's decomposition rule, we get

$$\begin{aligned} \Delta W_U = & - \sum_{h=1}^{H-1} [U(\overline{v},h+1) - U(\overline{v},h)] \Delta F(\overline{v},h) + U(\overline{v},H) \Delta F(\overline{v},H) \\ & + \int_{\underline{v}}^{\overline{v}} \left\{ \sum_{h=1}^{H-1} \Big[U_y(y,h+1) - U_y(y,h) \Big] \Delta F(y,h) - U_y(y,H) \Delta F(y,H) \right\} dy. \end{aligned} \tag{A.3}$$

Since by definition $\Delta F(\overline{v},H) = 0$, condition P3.1b guarantees that $\Delta W_U \geqq 0$, for all $U \in \mathscr{U}_{11}$.

Case 2: $U \in \mathscr{U}_{12}$. Integrating by parts the second integral in the expression within braces in (A.2), we obtain

$$\begin{aligned} \Delta W_U = & \sum_{h=1}^{H} U(\overline{v},h) \int_{\underline{v}}^{\overline{v}} \Delta f(y,h)\, dy - \sum_{h=1}^{H} U_y(\overline{v},h) \int_{\underline{v}}^{\overline{v}} \left[\int_{\underline{v}}^{y} \Delta f(\xi,h)\, d\xi \right] dy \\ & + \int_{\underline{v}}^{\overline{v}} \left\{ \sum_{h=1}^{H} U_{yy}(y,h) \int_{\underline{v}}^{y} \left[\int_{\underline{v}}^{\xi} \Delta f(\vartheta,h)\, d\vartheta \right] d\xi \right\} dy. \end{aligned} \tag{A.4}$$

Applying Abel's decomposition rule to every sum in (A.4), we get

$$\begin{aligned} \Delta W_U = & - \sum_{h=1}^{H-1} \Big[U(\overline{v},h+1) - U(\overline{v},h) \Big] \Delta F(\overline{v},h) + U(\overline{v},H) \Delta F(\overline{v},H) \\ & + \sum_{h=1}^{H-1} \Big[U_y(\overline{v},h+1) - U_y(\overline{v},h) \Big] \int_{\underline{v}}^{\overline{v}} \Delta F(y,h) dy - U_y(\overline{v},H) \int_{\underline{v}}^{\overline{v}} \Delta F(y,H) dy \\ & - \int_{\underline{v}}^{\overline{v}} \left\{ \sum_{h=1}^{H-1} \Big[U_{yy}(y,h+1) - U_{yy}(y,h) \Big] \int_{\underline{v}}^{y} \Delta F(\xi,h)\, d\xi - U_{yy}(y,H) \int_{\underline{v}}^{y} \Delta F(\xi,H)\, d\xi \right\} dy, \end{aligned} \tag{A.5}$$

and conditions P3.2b1 and P3.2b2 ensure that $\Delta W_U \geqq 0$, for all $U \in \mathscr{U}_{12}$.

Case 3: $U \in \mathscr{U}_{21}$. Applying Abel's decomposition one more time to each sum in (A.3), we obtain

$$
\begin{aligned}
\Delta W_U = & \sum_{h=1}^{H-2} \Big[U(\overline{v}, h+2) - 2\, U(\overline{v}, h+1) + U(\overline{v}, h) \Big] \sum_{r=1}^{h} \Delta F(\overline{v}, r) \\
& - \Big[U(\overline{v}, H) - U(\overline{v}, H-1) \Big] \sum_{r=1}^{H-1} \Delta F(\overline{v}, r) \\
& + U(\overline{v}, H) \Delta F(\overline{v}, H) \\
& - \int_{\underline{v}}^{\overline{v}} \Bigg\{ \sum_{h=1}^{H-2} \Big[U_y(y, h+2) - 2\, U_y(y, h+1) + U_y(y, h) \Big] \sum_{r=1}^{h} \Delta F(y, r) \\
& - \Big[U_y(y, H) - U_y(y, H-1) \Big] \sum_{r=1}^{H-1} \Delta F(y, r) \\
& + U_y(y, H) \Delta F(y, H) \Bigg\} dy.
\end{aligned}
\tag{A.6}
$$

Conditions P4.1b1 and P4.1b2 ensure that $\Delta W_U \geqq 0$, for all $U \in \mathscr{U}_{21}$.

Case 4: $U \in \mathscr{U}_{22}$. Finally, application of Abel's decomposition to each sum in (A.5) gives

$$
\begin{aligned}
\Delta W_U = & \sum_{h=1}^{H-2} \Big[U(\overline{v}, h+2) - 2\, U(\overline{v}, h+1) \big) + U(\overline{v}, h) \Big] \sum_{r=1}^{h} \Delta F(\overline{v}, r) \\
& - \Big[U(\overline{v}, H) - U(\overline{v}, H-1) \Big] \sum_{r=1}^{H-1} \Delta F(\overline{v}, r) \\
& + U(\overline{v}, H) \Delta F(\overline{v}, H) \\
& - \sum_{h=1}^{H-2} \Big[U_y(\overline{v}, h+2) - 2\, U_y(\overline{v}, h+1) + U_y(\overline{v}, h) \Big] \sum_{r=1}^{h} \int_{\underline{v}}^{\overline{v}} \Delta F(y, r)\, dy \\
& + \Big[U_y(\overline{v}, H) - U_y(\overline{v}, H-1) \Big] \sum_{r=1}^{H-1} \int_{\underline{v}}^{\overline{v}} \Delta F(y, r)\, dy \\
& - U_y(\overline{v}, H) \int_{\underline{v}}^{\overline{v}} \Delta F(y, H) dy \\
& + \int_{\underline{v}}^{\overline{v}} \Bigg\{ \sum_{h=1}^{H-2} \Big[U_{yy}(y, h+2) - 2\, U_{yy}(y, h+1) + U_{yy}(y, h) \Big] \sum_{r=1}^{h} \int_{\underline{v}}^{y} \Delta F(\xi, r)\, d\xi \\
& - \Big[U_{yy}(y, H) - U_{yy}(y, H-1) \Big] \sum_{r=1}^{H-1} \int_{\underline{v}}^{y} \Delta F(\xi, r)\, d\xi \\
& + U_{yy}(y, H) \int_{\underline{v}}^{y} \Delta F(\xi, H) d\xi \Bigg\} dy,
\end{aligned}
\tag{A.7}
$$

and conditions P4.2b1, P4.2b2 and P4.2b3 guarantee that $\Delta W_U \geqq 0$, for all $U \in \mathscr{U}_{22}$.

Necessary Conditions. We consider successively Proposition 3.1 to 4.2 and we show that, if condition (b) is not satisfied, then it is possible to find a utility function belonging to the relevant class such that $\Delta W_U := W_U(\mathbf{s}^*) - W_U(\mathbf{s}^\circ) < 0$.

Case 1: $U \in \mathscr{U}_{11}$. Suppose that condition P3.1b is not verified and let (y^*, h^*) be the smallest – in the lexicographic sense – couple (y, h) $(h \neq H)$ such that $F^*(y, h) > F^\circ(y, h)$. Consider then the utility function

$$\phi(y) := \begin{cases} 0 & \text{if } \underline{v} \leqq y < y^*, \\ \vartheta & \text{if } y^* \leqq y \leqq \overline{v}, \end{cases} \tag{A.8}$$

and let ϕ^* be a differentiable approximation of ϕ with positive first derivatives (see Fishburn and Vickson (1978, p. 75) for details). Choose the utility function U such that $U(y, h) := \phi^*(y)$, for all $y \in \mathscr{D}$ and all $h = 1, 2, \ldots, h^* - 1$, and $U(y, h) := \vartheta$, for all $y \in \mathscr{D}$ and all $h = h^*, h^* + 1, \ldots, H$. By construction $U \in \mathscr{U}_{11}$, but we can check that $\Delta W_U < 0$. Hence condition P3.1a is not verified.

Case 2: $U \in \mathscr{U}_{12}$. Suppose to begin with that condition P3.2b1 is not verified and let h^* be the smallest index $h \in \{1, 2, \ldots, H - 1\}$ such that $F^*(\overline{v}, h) > F^\circ(\overline{v}, h)$. Choose U such that $U(y, h) := 0$, for all $y \in \mathscr{D}$ and all $h = 1, 2, \ldots, h^* - 1$, and $U(y, h) := \vartheta > 0$, for all $y \in \mathscr{D}$ and all $h = h^*, h^* + 1, \ldots, H$. Even though $U \in \mathscr{U}_{21}$, one can check that $\Delta W_U < 0$. Suppose next that condition P3.2b2 does not hold and denote by (y^*, h^*) the smallest – in the lexicographic sense – couple (y, h) such that

$$\int_{\underline{v}}^{y} F^*(\xi, h)\, d\xi > \int_{\underline{v}}^{y} F^\circ(\xi, h)\, d\xi. \tag{A.9}$$

Consider next the piecewise linear, non-decreasing and concave function

$$\psi(y) := \begin{cases} y - y^* & \text{if } \underline{v} \leqq y < y^*, \\ 0 & \text{if } y^* \leqq y \leqq \overline{v}, \end{cases} \tag{A.10}$$

and let ψ^* be a differentiable approximation of ψ with non-negative first derivatives and non-positive second derivatives (see Fishburn and Vickson (1978, p. 76) for details). Choose the utility function U such that $U(y, h) := \psi^*(y)$, for all $y \in \mathscr{D}$ and all $h = 1, 2, \ldots, h^* - 1$, and $U(y, h) := 0$, for all $y \in \mathscr{D}$ and all $h = h^*, h^* + 1, \ldots, H$. By definition, $U \in \mathscr{U}_{12}$, but $\Delta W_U < 0$. Hence condition P3.2a is violated.

Case 3: $U \in \mathscr{U}_{21}$. Suppose first that condition P4.1b1 is not fulfilled and denote by y^* the smallest y such that $F^*(y, H) > F^\circ(y, H)$. Consider the function φ defined by

$$\varphi(y) := \begin{cases} 0 & \text{if } \underline{v} \leqq y < y^*, \\ H\vartheta & \text{if } y^* \leqq y \leqq \overline{v}, \end{cases} \tag{A.11}$$

and let φ^* be a differentiable approximation of φ. Consider then the utility function U defined by $U(y, h) := \varphi(y)$, for all $y \in \mathscr{D}$ and all $h \in \mathscr{H}$. While by definition $U \in \mathscr{U}_{21}$, one easily checks that $\Delta W_U < 0$. Hence condition P4.1a does not hold. Suppose next that condition P4.1b2 is not satisfied and let (y^*, h^*) indicate the smallest – in the lexicographic sense – couple (y, h) such that

$$\sum_{r=1}^{h} F^*(y, r) > \sum_{r=1}^{h} F^\circ(y, r). \tag{A.12}$$

Consider the function χ defined by

$$\chi(y) := \begin{cases} (h - 1)\,\vartheta & \text{if } \underline{v} \leqq y < y^*, \\ h^*\vartheta & \text{if } y^* \leqq y \leqq \overline{v}, \end{cases} \tag{A.13}$$

and let χ^* be a differentiable approximation de χ. Choose the utility function U such that $U(y,h) := \chi^*(y)$, for all $y \in \mathscr{D}$ and all $h = 1, 2, \ldots, h^* - 1$, and $U(y,h) := h^*\vartheta$, for all $y \in \mathscr{D}$ and all $h = h^*, h^* + 1, \ldots, H$. Again, while $U \in \mathscr{U}_{21}$, it can be shown that $\Delta W_U < 0$, and we conclude that condition P4.1a is not satisfied.

Case 4: $U \in \mathscr{U}_{22}$. The proofs of the necessity of conditions P4.2b1 and P4.2b2 are based on similar arguments as those used for Cases 2 and 3 above and they are not reproduced. Suppose then that condition P4.2b3 is violated and let (y^*, h^*) be the smallest – in the lexicographic sense – couple (y,h) such that

$$\sum_{r=1}^{h} \int_{\underline{v}}^{y} F^*(\xi, r)\, d\xi > \sum_{r=1}^{h} \int_{\underline{v}}^{y} F^\circ(\xi, r)\, d\xi. \tag{A.14}$$

Consider the function κ defined by

$$\kappa(y) := \begin{cases} (h^* - h - 1)[y - y^*] & \text{if } \underline{v} \leqq y < y^*, \\ 0 & \text{if } y^* \leqq y \leqq \overline{v}, \end{cases} \tag{A.15}$$

and let κ^* be an appropriate differentiable approximation of χ. Choose the utility function U defined by $U(y,h) := \kappa^*(y)$, for all $y \in \mathscr{D}$ and all $h = 1, 2, \ldots, h^* - 1$, and $U(y,h) := 0$, for all $y \in \mathscr{D}$ and all $h = h^*, h^* + 1, \ldots, H$. While by definition $U \in \mathscr{U}_{22}$, on can easily check that $\Delta W_U < 0$, which proves that condition P4.2a does not hold.

B. Multidimensional Dominance Criteria for Continuous Variables

Let X_1 and X_2 be two continuous attributes – or variables – taking their values in $D_1 := [\,\underline{u}, \overline{u}\,]$ and $D_2 := [\,\underline{v}, \overline{v}\,]$, respectively. For instance, X_1 is the person's income while X_2 is her life expectancy. We are interested in the distribution of attributes X_1 and X_2 and we indicate by F the corresponding joint (cumulative) distribution function defined by

$$F(u,v) = P\left(X_1 \leqq u \text{ and } X_2 \leqq v\right), \ \forall\ (u,v) \in \mathscr{D} := D_1 \times D_2,$$

with $F(\underline{u}, \underline{v}) = 0$ and $F(\overline{u}, \overline{v}) = 1$. We denote by $\mathscr{F}$ the set of the joint distribution functions that are differentiable and we indicate by $\mathscr{U} := \{U : \mathscr{D} \to \mathbb{R} \mid U \text{ is differentiable}\}$ the set of utility functions. We appeal to the (average) utilitarian social welfare function

$$W_U(F) := \int_{\underline{v}}^{\overline{v}} \int_{\underline{u}}^{\overline{u}} U(u,v)\, f(u,v)\, du\, dv,$$

where f is the joint density function corresponding to F, in order to compare different joint distributions. Consider the following properties of the utility function $U \in \mathscr{U}$:

B.1	$U_1 \geqq 0$
B.2	$U_2 \geqq 0$
B.3	$U_{12} \leqq 0$
B.4	$U_{22} \leqq 0$
B.5	$U_{221} \geqq 0$
B.6	$U_{11} \leqq 0$
B.7	$U_{112} \geqq 0$
B.8	$U_{1122} \leqq 0$

The class of utility functions below is important for the next result:

$$\mathscr{U}^* := \{U \in \mathscr{U} \mid \text{conditions B.1, B.2 and B.3 are fulfilled}\}.$$

Then, we have the following result (Atkinson and Bourguignon (1982)):

Proposition B.1 *Consider two distribution functions $F, G \in \mathscr{F}$. The following two statements are equivalent:*

(a) $W_U(F) \geqq W_U(G),\ \forall\ U \in \mathscr{U}^*$.

(b) $F(u,v) \leqq G(u,v),\ \forall\ (u,v) \in \mathscr{D}$.

Letting now

$$\mathscr{U}^{**} := \{U \in \mathscr{U} \mid \text{conditions B.1 to B.8 are fulfilled}\},$$

we obtain (Atkinson and Bourguignon (1982)):

Proposition B.2 *Consider two distribution functions $F, G \in \mathscr{F}$. Statements* (a) *and* (b) *below are equivalent:*

(a) $W_U(F) \geqq W_U(G),\ \forall\ U \in \mathscr{U}^{**}$.

(b1) $\int_{\underline{u}}^{u} F(s,\overline{v})\,ds \leqq \int_{\underline{u}}^{u} G(s,\overline{v})\,ds,\ \forall\ u \in D_1$;

(b2) $\int_{\underline{v}}^{v} F(\overline{u},t)\,dt \leqq \int_{\underline{v}}^{v} F(\overline{u},t)\,dt,\ \forall\ v \in D_2$; *and*

(b3) $\int_{\underline{u}}^{u} \int_{\underline{v}}^{v} F(s,t)\,ds\,dt \leqq \int_{\underline{u}}^{u} \int_{\underline{v}}^{v} G(s,t)\,ds\,dt,\ \forall\ (u,v) \in \mathscr{D}$.

References

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory*, **2**, 244–263.

Atkinson, A. B. and Bourguignon, F. (1982). The comparison of multidimensioned distributions of economic status. *Review of Economic Studies*, **49**, 183–201.

Atkinson, A. B. and Bourguignon, F. (1987). Income distributions and differences in needs. In G. Feiwel, editor, *Arrow and the Foundations of the Theory of Economic Policy*, pages 350–370. Macmillan, New York.

Blackorby, C. and Donaldson, D. (1993). Adult-equivalence scales and the economic implementation of interpersonal comparisons of well-being. *Social Choice and Welfare*, **10**, 335–361.

Blundell, R. and Lewbel, A. (1991). The information content of equivalence scales. *Journal of Econometrics*, **50**, 49–68.

Bourguignon, F. (1989). Family size and social utility: Income distribution dominance criteria. *Journal of Econometrics*, **42**, 67–80.

Buhmann, B., Rainwater, L., Schmaus, G., and Smeeding, T. M. (1988). Equivalence scales, well-being, inequality and poverty: Sensitivity estimates across ten countries using the Luxembourg Income Study (LIS) database. *Review of Income and Wealth*, **34**, 115–142.

Chambaz, C. and Maurin, E. (1998). Atkinson and Bourguignon's dominance criteria: Extended and applied to the measurement of poverty in France. *Review of Income and Wealth*, **44**, 497–513.

d'Aspremont, C. (1985). Axioms for social welfare orderings. In L. Hurwicz, D. Schmeidler, and H. Sonnenschein, editors, *Social Goals ans Social Organization. Essays in Memory of Elisha Pazner*, pages 19–76. Cambridge University Press, Cambridge.

d'Aspremont, C. and Gevers, L. (2002). Social welfare functionals and interpersonal comparability. In K. Arrow, A. K.Sen, and K. Suzumura, editors, *Handbook of Social Choice and Welfare, Volume I*, pages 459–542. Elsevier/North-Holland, Amsterdam/New York/Oxford.

Ebert, U. and Moyes, P. (2003). Equivalence scales reconsidered. *Econometrica*, **71**, 319–343.

Ebert, U. and Moyes, P. (2009). Household decision and equivalence scales. *Journal of Population Economics*, **22**, 1039–1062.

Epstein, L. G. and Tanny, S. (1980). Increasing generalized correlation: A definition and some economic consequences. *Canadian Journal of Economics*, **13**, 16–34.

Figini, P. (1998). Inequality measures, equivalence scales and adjustment for household size and composition. Luxembourg Income Study, Working Paper No. 185.

Fishburn, P. C. and Vickson, R. G. (1978). Theoretical foundations of stochastic dominance. In G. Whitmore and M. Findlay, editors, *Stochastic Dominance*, pages 39–113. Lexington Books, Lexington, MA.

Fleurbaey, M., Hagneré, C., and Trannoy, A. (2003). Welfare comparisons with bounded equivalence scales. *Journal of Economic Theory*, **110**, 309–336.

Fleurbaey, M., Hagneré, C., and Trannoy, A. (2007). Welfare comparisons of income distributions and family size. IDEP Discussion Paper 07-02, CNRS-EHESS-Universités Aix-Marseille II et III.

Foster, J. E. and Shorrocks, A. F. (1988). Poverty orderings and welfare dominance. *Social Choice and Welfare*, **5**, 91–110.

Glewwe, P. (1991). Household equivalence scales and the measurement of inequality: Transfers from the poor to the rich could decrease inequality. *Journal of Public Economics*, **44**, 211–216.

Gravel, N. and Moyes, P. (2010). Ethically robust comparisons of bidimensional distributions with an ordinal attribute. Paper presented at the ParisTech-Journal of Economic Theory Conference on Inequality and Risk, Paris, June 25–26, 2010.

Gravel, N. and Moyes, P. (2011). Utilitarianism and welfarism: Does it make a difference? GREThA, Université Montesquieu Bordeaux IV.

Hardy, G. H., Littlewood, J. E., and Pólya, G. (1952). *Inequalities.* Cambridge University Press, Cambridge, second edition.

Jenkins, S. P. and Lambert, P. J. (1993). Ranking income distributions when needs differ. *Review of Income and Wealth*, **39**, 337–356.

Kolm, S.-C. (1969). The optimal production of social justice. In J. Margolis and H. Guitton, editors, *Public Economics*, pages 145–200. Macmillan, London.

Kolm, S.-C. (1977). Multidimensional egalitarianisms. *Quarterly Journal of Economics*, **91**, 1–13.

Lambert, P. J. and Ramos, X. (2002). Welfare comparisons: Sequential procedures for heterogeneous populations. *Economica*, **69**, 549–562.

Maasoumi, E. (1986). Measurement and decomposition of multi-dimensional inequality. *Econometrica*, **12**, 93–102.

Marshall, A. W. and Olkin, I. (1979). *Inequalities: Theory of Majorization and its Applications*. Academic Press, New York.

Moyes, P. (1996). Sequential rank order dominance implies a finite sequence of favourable permutations. LARE, Université Montesquieu Bordeaux IV.

Muller, C. and Trannoy, A. (2010). Multidimensional inequality comparisons: A compensation perspective. Paper presented at the ParisTech-Journal of Economic Theory Conference on Inequality and Risk, Paris, June 25–26, 2010.

Sen, A. K. (1977). On weights and measures: Informational constraints in social welfare analysis. *Econometrica*, **45**, 1539–1572.

Shorrocks, A. F. (1983). Ranking income distributions. *Economica*, **50**, 3–17.

Shorrocks, A. F. and Foster, J. E. (1987). Transfer sensitive inequality measures. *Review of Economic Studies*, **54**, 485–497.

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