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ECONOMICS

DIFFERENT CONCEPTS OF MATRIX CALCULUS

by

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Business School
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DISCUSSION PAPER 11.03

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Let Y be a $p \times q$ matrix whose elements y_{ij} s are differentiable functions of the elements x_{rs} s of a $m \times n$ matrix X. We write $Y = Y(X)$ and say Y is a **matrix function** of X. Given such a set up we have $mnpq$ partial derivatives we can consider:

$$\frac{\partial y_{ij}}{\partial x_{rs}} \qquad \begin{array}{l} i = 1,\dots,m \\ j = 1,\dots,m \\ r = 1,\dots,p \\ s = 1,\dots,q. \end{array}$$

The question is how to arrange these derivatives. Different arrangements give rise to different concepts of derivatives in matrix calculus.

Concept 1

The derivative of the $p \times q$ matrix Y with respect to the $m \times n$ matrix X is the $pq \times mn$ matrix.

$$\frac{\partial Y}{\partial X} = \frac{\partial \operatorname{vec} Y}{\partial \operatorname{vec} X'} = \begin{pmatrix} \frac{\partial y_{11}}{\partial x_{11}} & \cdots & \frac{\partial y_{11}}{\partial x_{m1}} & \cdots & \frac{\partial y_{11}}{\partial x_{1n}} & \cdots & \frac{\partial y_{11}}{\partial x_{mn}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial y_{p1}}{\partial x_{11}} & \cdots & \frac{\partial y_{p1}}{\partial x_{m1}} & \cdots & \frac{\partial y_{p1}}{\partial x_{1n}} & \cdots & \frac{\partial y_{p1}}{\partial x_{mn}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial y_{1q}}{\partial x_{11}} & \cdots & \frac{\partial y_{1q}}{\partial x_{m1}} & \cdots & \frac{\partial y_{1q}}{\partial x_{1n}} & \cdots & \frac{\partial y_{1q}}{\partial x_{mn}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial y_{pq}}{\partial x_{11}} & \cdots & \frac{\partial y_{pq}}{\partial x_{m1}} & \cdots & \frac{\partial y_{pq}}{\partial x_{1n}} & \cdots & \frac{\partial y_{pq}}{\partial x_{mn}} \end{pmatrix}.$$

Notice that under this concept the $mnpq$ derivatives are arranged in such a way that a row of $\frac{\partial \operatorname{vec} Y}{\partial \operatorname{vec} X'}$, gives the derivatives of a particular element of Y with respect to each element of X and a column gives the derivatives of all the elements of Y with respect to a particular element of X. Notice also in talking about the derivatives of y_{ij} we have to specify exactly where the ith row is located in this matrix. Likewise when talking of the derivatives of all the elements of Y with respect to particular element x_{rs} of X again we have to specify exactly where the sth column is located in this matrix.

This concept of a matrix derivative is strongly advocated by Magnus and Neudecker [see for example Magnus and Neudecker (1985) and Magnus (2010)]. The feature they like about it is that $\frac{\partial \operatorname{vec} Y}{\partial \operatorname{vec} X'}$ is a straight forward matrix generalization of the **Jacobian Matrix** for $\mathbf{y} = \mathbf{y}(\mathbf{x})$ where **y** is a $p \times 1$ vector which is a real value differentiable function of a $m \times 1$ vector **x.** This Jacobian matrix is defined as $\partial \mathbf{y} / \partial \mathbf{x}'$.

Concept 2

The derivative of the $p \times q$ matrix Y with respect to the $m \times n$ matrix X is the $mp \times nq$ matrix

$$\frac{\delta Y}{\delta X} = \begin{pmatrix} \frac{\delta Y}{\delta x_{11}} & \cdots & \frac{\delta Y}{\delta x_{1n}} \\ \vdots & & \vdots \\ \frac{\delta Y}{\delta x_{m1}} & \cdots & \frac{\delta Y}{\delta x_{mn}} \end{pmatrix}$$

where $\delta Y / \delta x_{rs}$ is the $p \times q$ matrix given by

$$\frac{\delta Y}{\delta x_{rs}} = \begin{pmatrix} \frac{\partial y_{11}}{\partial x_{rs}} & \cdots & \frac{\partial y_{1q}}{\partial x_{rs}} \\ \vdots & & \vdots \\ \frac{\partial y_{p1}}{\partial x_{rs}} & \cdots & \frac{\partial y_{pq}}{\partial x_{rs}} \end{pmatrix}$$

for $r = 1, \ldots, m, \; s = 1, \ldots, n.$

This concept of a matrix derivative is discussed, for example, in Dwyer and MacPhail (1948), Dwyer (1967), Roger (1980) and Graham (1981).

Concept 3

Suppose y is a scalar but a differentiable function of all the elements of a $m \times n$ matrix X. Then we could conceive of the derivative of y with respect to X as the $m \times n$ matrix consisting of all the partial derivatives of y with respect to the elements of X. Denote this $m \times n$ matrix as

$$\frac{\gamma y}{\gamma X} = \begin{pmatrix} \frac{\partial y}{\partial x_{11}} & \cdots & \frac{\partial y}{\partial x_{1n}} \\ \vdots & & \vdots \\ \frac{\partial y}{\partial x_{m1}} & \cdots & \frac{\partial y}{\partial x_{mn}} \end{pmatrix}.$$

We could then conceive of the derivative of Y with respect to X as the matrix made up of the $\gamma y_{ij} / \gamma X$. Denote this $mp \times qn$ matrix $\gamma y / \gamma X$. This leads to the third concept of the derivative of Y with respect to X.

The derivative of the $p \times q$ matrix Y with respect to the $m \times n$ matrix X is the $mp \times nq$ matrix

$$\frac{\gamma Y}{\gamma X} = \begin{pmatrix} \frac{\gamma y_{11}}{\gamma X} & \cdots & \frac{\gamma y_{1q}}{\gamma X} \\ \vdots & & \vdots \\ \frac{\gamma y_{p1}}{\gamma X} & \cdots & \frac{\gamma y_{pq}}{\gamma X} \end{pmatrix}.$$

This is the concept of a matrix derivative studied in detail by MacRae (1974) and discussed by Dwyer (1967), Roger (1980), Graham (1981) and others.

From a theoretical point of view Parring (1992) argues that all three concepts are permissible as operators depending on which matrix or vector space we are operating in and how this space is normed.

CASE WHERE Y IS A SCALAR

Suppose Y is a scalar, y say. This case is common in statistics and econometrics. Then concept 2 and concept 3 are the same and concept 1 is the transpose of the vec of either concept. That is for y a scalar and X a $m \times n$ matrix

$$\frac{\delta y}{\delta X} = \frac{\gamma y}{\gamma X} \text{ and } \frac{\partial y}{\partial X} = \left(\text{vec} \frac{\delta y}{\delta X} \right)'. \tag{1}$$

<u>Examples where Y is a scalar</u>

1. Suppose y is the determinant of a non-singular matrix. That is $y = |X|$ where X is a non-singular matrix.

   Then

   $$\frac{\partial y}{\partial X} = |X| \left[\text{vec} \left(X^{-1} \right)' \right]'.$$

From Eq.(1) it follows immediately that

$$\frac{\delta y}{\delta X} = \frac{\gamma y}{\gamma X} = |X|\left(X^{-1}\right)'.$$

2. Consider $y = |Y|$ where $Y = X'AX$ is non-singular.

Then

$$\frac{\delta y}{\delta X} = |Y|\left(A X Y^{-1\prime} + A' X Y^{-1\prime}\right).$$

It follows from Eq. (1) that

$$\frac{\partial y}{\partial X} = |Y|\left\{\left[\left(Y^{-1} \otimes A\right) + \left(Y^{-1} \otimes A'\right)\right] \operatorname{vec} X\right\}'$$

$$= |Y|\left(\operatorname{vec} X\right)'\left[\left(Y^{-1\prime} \otimes A'\right) + \left(Y^{-1\prime} \otimes A\right)\right].$$

3. Consider $y = |Z|$ where $Z = X B X'$.

Then

$$\frac{\partial y}{\partial X} = |Z|\left(\operatorname{vec} X\right)'\left[\left(B \otimes Z^{-1\prime}\right) + \left(B' \otimes Z^{-1}\right)\right].$$

It follows from Eq.(1) that

$$\frac{\delta y}{\delta X} = \frac{\gamma y}{\gamma X} = |Z|\left(Z^{-1} X B + Z^{-1\prime} X B'\right).$$

4. Let $y = \operatorname{tr} AX$.

Then

$$\frac{\delta y}{\delta X} = A'.$$

It follows from Eq.(1) that

$$\frac{\partial y}{\partial X} = \left(\operatorname{vec} A'\right)'.$$

5. Let $y = \operatorname{tr} X'AX$.

Then

$$\frac{\partial y}{\partial X} = \left(\operatorname{vec}\left(A'X + AX\right)\right)'.$$

It follows from Eq.(1) that

$$\frac{\delta y}{\delta X} = \frac{\gamma y}{\gamma X} = A'X + AX.$$

6. Let $y = \mathrm{tr}XAX'B$.

Then

$$\frac{\delta y}{\delta X} = \frac{\gamma y}{\gamma X} = B'XA' + BXA .$$

It follows from Eq.(1) that

$$\frac{\partial y}{\partial X} = \left(\mathrm{vec}\left(B'XA' + BXA\right)\right)' .$$

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These examples suffice to show that it is a trivial matter moving between the different concepts of matrix derivatives when Y is a scalar. In the next section we derive transformation principles that allow us to move freely between the three different concepts of matrix derivatives in more complicated cases. These principles can be regarded as a generalisation of the work done by Dwyer and Macphail (1948) and by Graham (1980).

MATHEMATICAL PREREQUISITES

1. Kronecker Products

Let A = $\{a_{ij}\}$ be a m x n matrix and B be a p x q matrix. The Kronecker product of A and B, denoted by $A \otimes B$ is the mp x nq matrix given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} a^{1\prime} \\ \vdots \\ a^{m\prime} \end{pmatrix} = \left(a_1 \quad \ldots \quad a_n\right).$$

Then

$$A \otimes B = \begin{pmatrix} a^{1\prime} \otimes B \\ \vdots \\ a^{m\prime} \otimes B \end{pmatrix} = \left(a_1 \otimes B \quad \ldots \quad a_n \otimes B\right).$$

Moreover if x is a r x 1 vector then

$$x' \otimes A = \begin{pmatrix} x' \otimes a^{1'} \\ \vdots \\ x' \otimes a^{m'} \end{pmatrix}$$

so the ith row of $x' \otimes A$ is $x' \otimes a^{i'}$ for i = 1,…,m.

Similarly

$$x \otimes B = x \otimes b_1 \ldots x \otimes b_q$$

so the jth column of $x \otimes B$ is $x \otimes b_j$ for j = 1,…,q.

Locating the ith row and the jth column of $A \otimes B$

The ith row

If i is between 1 and p $\qquad a^{1'} \otimes b^{i'}$

If i is between p+1 and 2p $\qquad a^{2'} \otimes b^{i'}$

$\vdots$

If i is between (m-1)p and pm $\qquad a^{m'} \otimes b^{i'}$.

Write

$$i = (c-1)p + \bar{i}$$

where c is between 1 and m, $\bar{i}$ is between 1 and p. Then ith row of $A \otimes B$ is

$$a^{c'} \otimes b^{\bar{i}'}$$

eg. Let A be 2 x 3, B be 4 x 5 and suppose I want the 7th row of $A \otimes B$. Write

$$7 = (2-1)4 + 3.$$

So c = 2, $\bar{i} = 3$ and

$$(A \otimes B)_{7\bullet} = a^{2'} \otimes b^{3'}$$

Consider the n x n identity matrix I_n and write $I_n = \begin{pmatrix} e_1^n & \cdots & e_n^n \end{pmatrix}$. The ith column e_i^n acts as a selection matrix.

$$\text{i.e. } a^{c'} = e_c^{m\prime} A, \; b^{\bar{i}'} = e_{\bar{i}}^{p\prime} B.$$

So

$$(A \otimes B)_{i\bullet} = (e_c^{m\prime} \otimes e_{\bar{i}}^{p\prime})(A \otimes B).$$

The j^{th} column

Write

$$j = (d-1)q + \bar{j}$$

with d between 1and n and $\bar{j}$ between 1 and q.

Then

$$(A \otimes B)_{\bullet j} = a_d \otimes b_{\bar{j}} = (A \otimes B)(e_d^n \otimes e_{\bar{j}}^q) .$$

2. Generalized Vecs and Rvecs

Let A be a m x n matrix and write

$$A = \begin{pmatrix} a^{1\prime} \\ \vdots \\ a^{m\prime} \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} .$$

Then

$$\text{vec}A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \ \text{rvec}A = \begin{pmatrix} a^{1\prime} & \cdots & a^{m\prime} \end{pmatrix}.$$

Let A be a m x np matrix and write

$$A = \begin{pmatrix} \underset{mxn}{A_1} & \cdots & \underset{mxn}{A_p} \end{pmatrix}.$$

Then

$$\text{vec}_n A = \begin{pmatrix} A_1 \\ \vdots \\ A_p \end{pmatrix}.$$

Similarly if B is np x q and write

$$B = \begin{pmatrix} \underset{pxq}{B_1} \\ \vdots \\ \underset{pxq}{B_n} \end{pmatrix}.$$

Then

$$\text{rvec}_p B = \begin{pmatrix} B_1 & \cdots & B_n \end{pmatrix}.$$

Relationships

i) If A is m x np then

$$(\text{vec}_n A)' = \text{rvec}_n A'$$

ii) A generalized vec can always be undone by taking an appropriate generalized rvec and vice versa. For example, if A is m x n and $\text{vec}_j A$ and $\text{rvec}_i A$ exist then

$$\text{rvec}_m(\text{vec}_j A) = A$$
$$\text{vec}_n(\text{rvec}_i A) = A.$$

iii) Suppose a and b are vectors, b being p x 1. Then

$$\text{vec}_p(a' \otimes b') = ab'$$
$$\text{rvec}_p(a \otimes b) = ba'.$$

3. Elementary Matrices

The elementary matrix E_{ij}^{mn} is a m x n zero-one matrix whose elements are all zero except in the $(i,j)^{th}$ position which is 1. i.e.

$$E_{ij}^{mn} = e_i^m e_j^{n\prime}.$$

Recall for A and B m x n and p x q matrices respectively

$$(A \otimes B)_{\bar{i}\bullet} = a^{c\prime} \otimes b^{\bar{i}\prime}.$$

Hence,

$$\begin{aligned} \text{vec}_q(A \otimes B)_{\bar{i}\bullet} &= a^c b^{\bar{i}\prime} = A' e_c^m e_{\bar{i}}^p B \\ &= A' E_{c\bar{i}}^{mp} B \end{aligned} \quad (2)$$

Similarly,

$$\text{rvec}_p(A \otimes B)_{\bullet \bar{j}} = B E_{\bar{j}d}^{qn} A'. \quad (3)$$

4. Commutation Matrix K_{mn}

If A is a m x n matrix then K_{mn} is the mn x mn zero-one matrix defined by

$$K_{mn}\text{vec}A = \text{vec}A'$$

Results about K_{mn}

i) $$K_{mn} = \begin{pmatrix} E_{11}^{nm} & \cdots & E_{n1}^{nm} \\ \vdots & & \vdots \\ E_{1m}^{nm} & \cdots & E_{nm}^{nm} \end{pmatrix}$$

ii) If A is m x n, B is p x q then

$$K_{pm}(A\otimes B)=\begin{pmatrix} a^{1\prime}\otimes b^{1\prime} \\ \vdots \\ a^{m\prime}\otimes b^{1\prime} \\ \vdots \\ a^{1\prime}\otimes b^{p\prime} \\ \vdots \\ a^{m\prime}\otimes b^{p\prime} \end{pmatrix}=\left(B\otimes A\right)K_{qn}$$

and $\left(B\otimes A\right)K_{qn}=\left(B\otimes a_1 \ldots B\otimes a_n\right)$.

iii) i^{th} row of $K_{pm}(A\otimes B)$

By a similar analysis to that of above.

$$[K_{pm}(A\otimes B)]_{i\bullet}=a^{\bar{i}\prime}\otimes b^{c\prime}$$

for $i=(c-1)m+\bar{i}$ and

$$vec_q[K_{pm}(A\otimes B)]_{i\bullet}=A'E^{mp}_{\bar{i}c}B$$

iv) The jth column of $\left(B\otimes A\right)K_{qn}$

By a similar analysis to that of above

$$\left(\left(B\otimes A\right)K_{qn}\right)_{\bullet j}=b_{\bar{j}}\otimes a_d$$

where $j=\left(d-1\right)_q+\bar{j}$ and

$$rvec_m\left(\left(B\otimes A\right)K_{qn}\right)_{\bullet j}=A\,E^{nq}_{d\bar{j}}B'.$$

v) If X is a $m\times n$ matrix then

$$vec\left(X\otimes I_G\right)=\left(I_m\otimes vec_m K_{mG}\right)vecX.$$

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5. The Matrix U_{mn}

U_{mn} is the $m^2\times n^2$ matrix given by

$$U_{mn}=\begin{pmatrix} E^{mn}_{11} & \cdots & E^{mn}_{1n} \\ \vdots & & \vdots \\ E^{mn}_{m1} & \cdots & E^{mn}_{mn} \end{pmatrix}.$$

Let A, B, C, D be $r\times m$, $s\times m$, $n\times u$ and $n\times v$ matrices respectively. Then

$$\left(A\otimes B\right)U_{mn}\left(C\otimes D\right)=\left(vecB\,A'\right)\left(rvecC'D\right).$$

RELATIONSHIPS BETWEEN THE DIFFERENT CONCEPTS

We can use our generalized vec and rvec operators to spell out the relationships that exist between our three concepts of matrix derivatives. We consider two concepts in turn.

Concept 1 and Concept 2

The submatrices in $\delta Y / \delta X$ are

$$\frac{\delta Y}{\delta x_{rs}} = \begin{pmatrix} \frac{\partial y_{11}}{\partial x_{rs}} & \cdots & \frac{\partial y_{1q}}{\partial x_{rs}} \\ \vdots & & \vdots \\ \frac{\partial y_{p1}}{\partial x_{rs}} & \cdots & \frac{\partial y_{pq}}{\partial x_{rs}} \end{pmatrix}$$

for $r = 1, \ldots, m$ and $s = 1, \ldots, n$. In forming the submatrix $\delta Y / \delta x_{rs}$ we need the partial derivatives of the elements of Y with respect to x_{rs}. When we turn to concept 1 we note that these partial derivatives all appear in a column of $\partial Y / \partial X$. Just as we did in locating a column of a Kronecker product we have to specify exactly where this column is located in the matrix $\partial Y / \partial X$. If s is 1 then the partial derivatives appear in the rth column, if s is 2 then they appear in the $m + r$ th column, if s is 3 in the $2m + r$ th column and so on until s is n in which case the partial derivatives appear in the $(n-1)m + r$ th column. To cater for all these possibilities we say x_{rs} appears in the ℓ th column of $\partial Y / \partial X$ where

$$\ell = (s-1)m + r$$

and $s = 1, \ldots, n$. The partial derivatives we seek appear in that column as the column vector

$$\begin{pmatrix} \frac{\partial y_{11}}{\partial x_{rs}} \\ \vdots \\ \frac{\partial y_{p1}}{\partial x_{rs}} \\ \vdots \\ \frac{\partial y_{1q}}{\partial x_{rs}} \\ \vdots \\ \frac{\partial y_{pq}}{\partial x_{rs}} \end{pmatrix}.$$

If we take the rvec_p of this vector we get $\delta Y / \delta x_{rs}$ so

$$\delta Y / \delta x_{rs} = \text{rvec}_p \left(\frac{\partial Y}{\partial X} \right)_{\bullet \ell} \tag{7}$$

where $\ell = (s-1)m + r,$ for $s = 1, \ldots, n$ and $r = 1, \ldots, m.$

Now this generalized rvec can be undone by taking the vec so

$$\left(\frac{\partial Y}{\partial X} \right)_{\bullet \ell} = \text{vec} \left(\frac{\delta Y}{\delta x_{rs}} \right) \tag{8}$$

If we are given $\partial Y / \partial X$ and we can identify the ℓth column of this matrix then Eq.(7) allows us to move from concept 1 to concept 2. If, however, we have in hand $\delta Y / \delta X$ we can identify the submatrix $\delta Y / \delta x_{rs}$ and Eq.(8) will then allow us to move from concept 2 to concept 1.

Concept 1 and Concept 3

The submatrices in $\gamma Y / \gamma X$ are

$$\frac{\gamma y_{ij}}{\gamma X} = \begin{pmatrix} \frac{\partial y_{ij}}{\partial x_{11}} & \cdots & \frac{\partial y_{ij}}{\partial x_{1n}} \\ \vdots & & \vdots \\ \frac{\partial y_{ij}}{\partial x_{m1}} & \cdots & \frac{\partial y_{ij}}{\partial x_{mn}} \end{pmatrix}$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, q.$ In forming the submatrix $\gamma y_{ij} / \gamma X$ we need the partial derivative of y_{ij} with respect to the elements of X. When we examine $\partial Y / \partial X$ we see that these derivatives appear in a row of $\partial Y / \partial X.$

Again we have to specify exactly where this row is located in the matrix $\partial Y / \partial X.$ If j is 1 then the partial derivatives appear in the ith row, if $j = 2$ then they appear in the $p + i$th row, if $j = 3$ then in the $2p + i$th row and so on until $j = q$ in which case the partial derivative appear in $(q-1)p + i$th row. To cater for all possibilities we say the partial derivatives appear in the tth row of $\partial Y / \partial X$ where

$$t = (j-1)p + i$$

and $j = 1, \ldots, q.$ In this row they appear as the row vector

$$\left(\frac{\partial y_{ij}}{\partial x_{11}} \cdots \frac{\partial y_{ij}}{\partial x_{m1}} \cdots \frac{\partial y_{ij}}{\partial x_{1n}} \cdots \frac{\partial y_{ij}}{\partial x_{mn}} \right).$$

If we take the vec_m of this vector we obtain the matrix

$$\begin{pmatrix} \frac{\partial y_{ij}}{\partial x_{11}} & \cdots & \frac{\partial y_{ij}}{\partial x_{m1}} \\ \vdots & & \vdots \\ \frac{\partial y_{ij}}{\partial x_{1n}} & \cdots & \frac{\partial y_{ij}}{\partial x_{mn}} \end{pmatrix}$$

which is $\left(\gamma y_{ij} / \gamma X\right)'$. So we have

$$\frac{\gamma y_{ij}}{\gamma X} = \left(\text{vec}_m \left(\frac{\partial Y}{\partial X} \right)_{t\bullet} \right)' \tag{9}$$

where $t = (j-1)p + i,$ for $j = 1, \ldots, q$ and $i = 1, \ldots, p.$

As

$$\text{vec}_m \left(\frac{\partial Y}{\partial X} \right)_{t\bullet} = \left(\frac{\gamma y_{ij}}{\gamma X} \right)'$$

and this generalized vec can be undone by taking the rvec we have

$$\left(\frac{\partial Y}{\partial X} \right)_{t\bullet} = \text{rvec} \left(\frac{\gamma y_{ij}}{\gamma X} \right)'. \tag{10}$$

If we have in hand $\partial Y / \partial X$ and if we can identify the t th row of this matrix the Eq.(9) allows us to move from concept 1 to concept 3. If, however, we have obtained $\gamma Y / \gamma X$ so we can identify the submatrix $\gamma y_{ij} / \gamma X$ of this matrix then Eq.(10) allows us to move from concept 3 to concept 1.

Concept 2 and Concept 3

Returning to concept 3, the submatrices of $\gamma Y / \gamma X$ are

$$\frac{\gamma y_{ij}}{\gamma X} = \begin{pmatrix} \frac{\partial y_{ij}}{\partial x_{11}} & \cdots & \frac{\partial y_{ij}}{\partial x_{1n}} \\ \vdots & & \vdots \\ \frac{\partial y_{ij}}{\partial x_{m1}} & \cdots & \frac{\partial y_{ij}}{\partial x_{mn}} \end{pmatrix}$$

and the partial derivative $\frac{\partial y_{ij}}{\partial x_{rs}}$ is given by the (r, s)th element of this submatrix. That is

$$\frac{\partial y_{ij}}{\partial x_{rs}} = \left(\frac{\gamma y_{ij}}{\gamma X} \right)_{rs}.$$

It follows that

$$\frac{\delta Y}{\delta x_{rs}} = \begin{pmatrix} \left(\frac{\gamma y_{11}}{\gamma X}\right)_{rs} & \cdots & \left(\frac{\gamma y_{1q}}{\gamma X}\right)_{rs} \\ \vdots & & \vdots \\ \left(\frac{\gamma y_{p1}}{\gamma X}\right)_{rs} & \cdots & \left(\frac{\gamma y_{pq}}{\gamma X}\right)_{rs} \end{pmatrix}. \tag{11}$$

Starting now with concept 2, the submatrices of $\delta Y / \delta X$ are

$$\frac{\delta Y}{\delta x_{rs}} = \begin{pmatrix} \frac{\partial y_{11}}{\partial x_{rs}} & \cdots & \frac{\partial y_{1q}}{\partial x_{rs}} \\ \vdots & & \vdots \\ \frac{\partial y_{p1}}{\partial x_{rs}} & \cdots & \frac{\partial y_{pq}}{\partial x_{rs}} \end{pmatrix}$$

and the partial derivative $\partial y_{ij} / \partial x_{rs}$ is the (i, j)th element of this submatrix. That is

$$\frac{\partial y_{ij}}{\partial x_{rs}} = \left(\frac{\delta Y}{\delta x_{rs}}\right)_{ij}.$$

It follows that

$$\frac{\gamma y_{ij}}{\gamma X} = \begin{pmatrix} \left(\frac{\delta Y}{\partial x_{11}}\right)_{ij} & \cdots & \left(\frac{\delta Y}{\delta x_{1n}}\right)_{ij} \\ \vdots & & \vdots \\ \left(\frac{\delta Y}{\delta x_{m1}}\right)_{ij} & \cdots & \left(\frac{\delta Y}{\delta x_{mn}}\right)_{ij} \end{pmatrix}. \tag{12}$$

If we have in hand $\gamma y / \gamma X$ then Eq.(11) allows us to build up the submatrices we need for $\delta Y / \delta X$. If however, we have a result for $\delta Y / \delta X$ then Eq.(12) allows us to obtain the submatrices we need for $\gamma Y / \gamma X$.

Tranformation Principles One

Several matrix calculus results when we use concept 1 involve Kronecker products whereas the equivalent results, using concepts 2 and 3 involve elementary matrices. In this section we see that this is no coincidence.

We have just seen that

$$\frac{\delta Y}{\delta x_{rs}} = \text{rvec}_p \left(\frac{\partial Y}{\partial X} \right)_{\bullet \ell} \qquad (13)$$

where $\ell = (s-1)m + r$ and that

$$\frac{\gamma y_{ij}}{\gamma X} = \left(\text{vec}_m \left(\frac{\partial Y}{\partial X} \right)_{t\bullet} \right)' \qquad (14)$$

where $t = (j-1)p + i$. Suppose now that $\partial Y / \partial X = A \otimes B$ where A is a $q \times n$ matrix and B is a $p \times m$ matrix.

Then from Eq.(3) we have

$$\text{rvec}_p (A \otimes B)_{\bullet \ell} = B E^{mn}_{rs} A',$$

so using Eq.(13) we have that

$$\frac{\delta Y}{\delta x_{rs}} = B E^{mn}_{rs} A'.$$

From Eq.(2) we have

$$\text{vec}_m (A \otimes B)_{t\bullet} = A' E^{qp}_{ji} B$$

so from Eq.(14)

$$\frac{\gamma y_{ij}}{\gamma X} = \left(A' E^{qp}_{ji} B \right)' = B' E^{pq}_{ji} A.$$

This leads us to our first transformation principle.

The First Transformation Principle

Let A be a $q \times n$ matrix and B be a $p \times m$ matrix. Whenever

$$\frac{\partial Y}{\partial X} = A \otimes B$$

regardless of whether A and B are matrix functions of X or not

$$\frac{\delta Y}{\delta x_{rs}} = B E^{mn}_{rs} A'$$

and

$$\frac{\gamma y_{ij}}{\gamma X} = B' E^{pq}_{ij} A$$

and the converse statements are true also.

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For this case

$$\frac{\delta Y}{\delta X}=\begin{pmatrix} B E_{11}^{mn} A' & \cdots & B E_{1n}^{mn} A' \\ \vdots & & \vdots \\ B E_{m1}^{mn} A' & \cdots & B E_{mn}^{mn} A' \end{pmatrix}=\left(I_m \otimes B\right) U_{mn}\left(I_n \otimes A'\right),$$

where U_{mn} is the $m^2 \times n^2$ matrix, given by

$$U_{mn}=\begin{pmatrix} E_{11}^{mn} & \ldots & E_{1n}^{mn} \\ \vdots & & \vdots \\ E_{m1}^{mn} & \ldots & E_{mn}^{mn} \end{pmatrix}.$$

From Eq.(6)

$$\left(A \otimes B\right) U_{mn}\left(C \otimes D\right)=\left(\operatorname{vec} BA'\right)\left(\operatorname{rvec} C'D\right),$$

so

$$\frac{\delta Y}{\delta X}=\left(\operatorname{vec} B\right)\left(\operatorname{rvec} A'\right).$$

In terms of concept 3 for this case

$$\frac{\gamma Y}{\gamma X}=\begin{pmatrix} B' E_{11}^{pq} A & \ldots & B' E_{1q}^{pq} A \\ \vdots & & \vdots \\ B' E_{p1}^{pq} A & \ldots & B' E_{pq}^{pq} A \end{pmatrix}=\left(I_p \otimes B'\right) U_{pq}\left(I_q \otimes A\right)=\left(\operatorname{vec} B'\right)\left(\operatorname{rvec} A\right).$$

In terms of the entire matrices we can express the First Transformation Principle by saying that the following statements are equivalent:

$$\frac{\partial Y}{\partial X}=A \otimes B$$
$$\frac{\delta Y}{\delta X}=\left(\operatorname{vec} B\right)\left(\operatorname{rvec} A'\right)$$
$$\frac{\gamma Y}{\gamma X}=\left(\operatorname{vec} B'\right)\left(\operatorname{rvec} A\right).$$

Examples of the Use of the First Transformation Principle

1. $Y = A \times B$ for A $p \times m$ and B $n \times q$.

   Then it is know that

$$\frac{\partial AXB}{\partial X}=B' \otimes A.$$

   It follows that

$$\frac{\delta AXB}{\delta x_{rs}}=A E_{rs}^{mn} B$$

and

$$\frac{\gamma\left(AXB\right)_{ij}}{\gamma X} = A' E_{ij}^{pq} B'$$

Moreover

$$\frac{\delta AXB}{\delta X} = \left(\text{vec}\, A\right)\left(\text{rvec}\, B\right)$$

$$\frac{\gamma AXB}{\gamma X} = \left(\text{vec}\, A'\right)\left(\text{rvec}\, B'\right).$$

2. $Y = XAX$ where X is a $n \times n$ symmetric matrix.

Then it is know that

$$\frac{\delta XAX}{\delta x_{rs}} = E_{rs}^{nn} AX + XAE_{rs}^{nn}.$$

It follows that

$$\frac{\gamma\left(XAX\right)_{ij}}{\gamma X} = E_{ij}^{nn} XA' + A'XE_{ij}^{nn}$$

and that

$$\frac{\partial XAX}{\partial X} = \left(X'A \otimes I_n\right) + \left(I_n \otimes A'X\right).$$

Moreover

$$\frac{\delta XAX}{\delta X} = \left(\text{vec}\, I_n\right)\left(\text{rvec}\, A'X\right) + \left(\text{vec}\, A'X\right)\left(\text{rvec}\, I_n\right)$$

$$\frac{\gamma Y}{\gamma X} = \left(\text{vec}\, I_n\right)\left(\text{rvec}\, X'A\right) + \left(\text{vec}\, X'A\right)\left(\text{rvec}\, I_n\right).$$

3. $Y = X \otimes I_G$ where X is a $m \times n$ matrix.

We have seen that $\text{vec}\left(X \otimes I_G\right) = \left(I_n \otimes \text{vec}_m K_{mG}\right) \text{vec}\, X$ so

$$\frac{\partial\left(X \otimes I_G\right)}{\partial X} = I_n \otimes \text{vec}_m K_{mG}.$$

It follows that

$$\frac{\delta\left(X \otimes I_G\right)}{\delta x_{rs}} = \left(\text{vec}_m K_{Gm}\right) E_{rs}^{mn}$$

and

$$\frac{\gamma\left(X \otimes I_G\right)}{\gamma X} = \left(\text{vec}_m K_{Gm}\right)' E_{ij}^{kn} \quad \text{where} \quad k = G^2 n.$$

Moreover

$$\frac{\delta\left(X\otimes I_G\right)}{\delta X}=\text{vec}\left(\text{vec}_m K_{mG}\right)\left(\text{rvec}\, I_n\right)=\left(\text{vec}\, I_{mG}\right)\left(\text{rvec}\, I_n\right)$$
$$\frac{\gamma\left(X\otimes I_G\right)}{\gamma X}=\text{vec}\left(\text{vec}_m K_{mG}\right)'\left(\text{rvec}\, I_n\right)=\left(\text{vec}\, I_{mG}\right)\left(\text{rvec}\, I_n\right).$$

4. $Y=AX^{-1}B$ where A is $p\times n$ and B is $n\times q$. Then it is known that

$$\frac{\gamma\left(AX^{-1}B\right)_{ij}}{\gamma X}=-X^{-1\prime}A'E_{ij}^{pq}B'X^{-1\prime}.$$

It follows straight away that

$$\frac{\delta AX^{-1}B}{\delta x_{rs}}=-AX^{-1}E_{rs}^{nn}X^{-1}B,$$

and that

$$\frac{\partial AX^{-1}B}{\partial X}=-B'X^{-1\prime}\otimes AX^{-1}.$$

Moreover

$$\frac{\delta AX^{-1}B}{\delta X}=-\left(\text{vec}\, AX^{-1}\right)\left(\text{rvec}\, X^{-1}B\right)$$

and

$$\frac{\gamma AX^{-1}B}{\gamma X}=-\left(\text{vec}\, X^{-1\prime}A'\right)\left(\text{rvec}\, B'X^{-1\prime}\right).$$

5. $Y=AXBXC$ where X is $m\times n$, A is $p\times m$, B is $n\times m$ and C is $n\times q$.

Then it is well known that

$$\frac{\delta AXBXC}{\delta x_{rs}}=A\,E_{rs}^{mn}BXC+AXBE_{rs}^{mn}C.$$

It follows that

$$\frac{\gamma\left(AXBXC\right)_{ij}}{\gamma X}=A'E_{ij}^{pq}C'X'B'+B'X'A'E_{ij}^{pq}C'$$

and

$$\frac{\partial AXBXC}{\partial X}=\left(C'X'B'\otimes A\right)+\left(C'\otimes AXB\right).$$

Moreover

$$\frac{\delta AXBXC}{\delta X} = (\operatorname{vec} A)(\operatorname{rvec} BXC) + (\operatorname{vec} AXB)(\operatorname{rvec} C).$$

and

$$\frac{\gamma AXBXC}{\gamma X} = (\operatorname{vec} A')(\operatorname{rvec} C'X'B') + (\operatorname{vec} B'X'A')(\operatorname{rvec} C').$$

As I hope these examples make clear this transformation principle ensure is a very easy matter to move from a result involving one of the concepts of matrix derivatives to the corresponding results for the other two concepts. Although this principle covers a lot of cases, it does not cover them all. Several matrix calculus results for concept 1 involve multiplying a Kronecker product by a commutation matrix. The following transformation principal covers this case.

Transformation Principle Two

Suppose then that

$$\frac{\partial Y}{\partial X} = K_{qp}(C \otimes D) = (D \otimes C) K_{mn}$$

where C is a $p \times n$ matrix and D is a $q \times m$ matrix. Forming $\partial Y / \partial x_{rs}$ from this matrix requires that we first obtain the ℓ th column of this matrix where $\ell = (s-1)m + r$ and we take the rvec_p of this column. From Eq.(5) we get

$$\frac{\delta Y}{\delta x_{rs}} = C E_{sr}^{nm} D'$$

In forming $\gamma y_{ij} / \gamma X$ from $\partial Y / \partial X$ we first have to obtain the t th row of this matrix, for $t = (j-1)p + i$ and then we take the vec_m of this row. The required matrix $\gamma y_{ij} / \gamma X$ is the transpose of the matrix thus obtained. From Eq.(4) we get

$$\frac{\gamma y_{ij}}{\gamma X} = \left(C' E_{ij}^{pq} D\right)' = D' E_{ji}^{qp} C.$$

This leads us to our second transformation principle.

<u>The Second Transformation Principle</u>

Let C be a $p \times n$ matrix and D be a $q \times m$ matrix. Whenever

$$\frac{\partial Y}{\partial X} = K_{qp}(C \otimes D)$$

regardless of whether C and D are matrix functions of X or not

$$\frac{\delta Y}{\delta x_{rs}} = C\,E_{sr}^{nm}\,D'$$

$$\frac{\gamma y_{ij}}{\gamma X} = D'\,E_{ji}^{qp}\,C$$

and the converse statements are true also.

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For this case

$$\frac{\delta Y}{\delta X} = \begin{pmatrix} C\,E_{11}^{nm}D' & \dots & C\,E_{n1}^{nm}D' \\ \vdots & & \vdots \\ C\,E_{1m}^{nm}D' & \dots & C\,E_{nm}^{nm}D' \end{pmatrix} = \left(I_m \otimes C\right)\begin{pmatrix} E_{11}^{nm} & \dots & E_{n1}^{nm} \\ \vdots & & \vdots \\ E_{1m}^{nm} & \dots & E_{nm}^{nm} \end{pmatrix}\left(I_n \otimes D'\right) = \left(I_m \otimes C\right)K_{mn}\left(I_n \otimes D'\right).$$

In terms of $\gamma Y/\gamma X$ we have

$$\frac{\gamma Y}{\gamma X} = \begin{pmatrix} D'E_{11}^{qp}C & \dots & D'E_{q1}^{qp}C \\ \vdots & & \vdots \\ D'E_{1p}^{qp}C & \dots & D'E_{qp}^{qp}C \end{pmatrix} = \left(I_p \otimes D'\right)K_{pq}\left(I_q \otimes C\right).$$

In terms of the full matrices we can express the Second Transformation Principle as saying that the following statements are equivalent:

$$\frac{\partial Y}{\partial X} = K_{qp}\left(C \otimes D\right)$$

$$\frac{\delta Y}{\delta X} = \left(I_m \otimes C\right)K_{mn}\left(I_n \otimes D'\right)$$

$$\frac{\gamma Y}{\gamma X} = \left(I_p \otimes D'\right)K_{pq}\left(I_q \otimes C\right).$$

As an example of the use of this second transformation principle let $Y = AX'B$ where A is $p \times n$ and B is $m \times q$. Then it is known that

$$\frac{\partial AX'B}{\partial X} = K_{pq}\left(B' \otimes A\right).$$

It follows that

$$\frac{\delta AX'B}{\delta x_{rs}} = B'E_{sr}^{mn}A'$$

and that

$$\frac{\gamma\left(AX'B\right)_{ij}}{\gamma X} = A'E_{ji}^{pq}\,B'.$$

In terms of the entire matrices we

$$\frac{\delta Y}{\delta X} = \left(I_n \otimes B'\right)K_{nm}\left(I_m \otimes A'\right)$$

$$\frac{\gamma Y}{\gamma X} = \left(I_q \otimes A'\right)K_{qp}\left(I_p \otimes B\right).$$

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Principle 2 comes into its own when it is used in conjunction with principle 1. Many matrix derivatives come in two parts: one where principle 1 is applicable and the other where principle 2 is applicable.

For example we often have

$$\frac{\partial Y}{\partial X} = A \otimes B + K_{qp}\left(C \otimes D\right),$$

so we would apply principle 1 to the $A \otimes B$ part and principle 2 to the $K_{qp}\left(C \otimes D\right)$ part.

Examples of the Combined Use of Principles One and Two

1. Let $Y = X'AX$ where X is $m \times n$, A is $m \times m$. Then it is well known that

$$\frac{\partial X'AX}{\partial X} = K_{nn}\left(I_n \otimes X'A'\right) + \left(I_n \otimes X'A\right).$$

It follows that

$$\frac{\delta X'AX}{\delta x_{rs}} = E_{sr}^{nm}AX + X'A\,E_{rs}^{mn}$$

and that

$$\frac{\gamma\left(X'AX\right)_{ij}}{\gamma X} = AXE_{ji}^{nn} + A'XE_{ij}^{nn}.$$

Moreover

$$\frac{\delta X'AX}{\delta X} = K_{mn}\left(I_n \otimes AX\right) + \left(I_m \otimes X'A\right)U_{mn} = K_{mn}\left(I_n \otimes AX\right) + \left(\operatorname{vec} X'A\right)\left(\operatorname{rvec} I_n\right).$$

$$\frac{\gamma X'AX}{\gamma X} = \left(I_n \otimes AX\right)K_{nn} + \left(I_n \otimes A'X\right)U_{nn} = \left(I_n AX\right)K_{nn} + \left(\operatorname{vec} A'X\right)\left(\operatorname{rvec} I_n\right).$$

2. Let $Y = XAX'$ where X is $m \times n$ and A is $n \times n$. Then it is known that

$$\frac{\delta XAX'}{\delta x_{rs}} = XAE_{sr}^{nm} + E_{rs}^{mn} AX'.$$

It follows that

$$\frac{\gamma\left(XAX'\right)_{ij}}{\gamma X} = E_{ji}^{mm} XA + E_{ij}^{mm} XA'$$

and

$$\frac{\partial XAX'}{\partial X} = K_{mm}\left(XA \otimes I_m\right) + \left(XA' \otimes I_m\right).$$

Moreover

$$\frac{\delta XAX'}{\delta X} = \left(I_m \otimes XA\right) K_{mn} + U_{mn}\left(I_n \otimes AX'\right) = \left(I_m \otimes XA\right) K_{mn} + \left(\operatorname{vec} I_m\right)\left(\operatorname{rvec} AX'\right).$$

and

$$\frac{\gamma XAX'}{\gamma X} = K_{mm}\left(I_m \otimes XA'\right) + U_{mm}\left(I_m \otimes XA'\right) = K_{mm}\left(I_m \otimes XA'\right) + \left(\operatorname{vec} I_m\right)\left(\operatorname{rvec} AX'\right).$$

3. Let $Y = AX'BXC$ where A is $p \times n$, B is $m \times m$ and C is $n \times q$. Then it is known that

$$\frac{\gamma\left(AX'BXC\right)_{ij}}{\gamma X} = BXCE_{ji}^{qp} A + B'XA' E_{ij}^{pq} C'.$$

It follows using our principles that

$$\frac{\delta AX'BXC}{\delta x_{rs}} = CE_{sr}^{nm} BXC + AX'BE_{rs}^{mn} C$$

and that

$$\frac{\partial AX'BXC}{\partial X} = K_{qp}\left(A \otimes C'X'B'\right) + \left(C' \otimes AX'B\right).$$

In terms of the entire matrices we have

$$\begin{aligned}\frac{\delta AX'BXC}{\delta X} &= \left(I_m \otimes A\right) K_{mn}\left(I_n BXC\right) + \left(I_m \otimes AX'B\right) U_{mn}\left(I_n \otimes C\right) \\ &= \left(I_m \otimes A\right) K_{mn}\left(I_n \otimes BXC\right) + \left(\operatorname{vec} AX'B\right)\left(\operatorname{rvec} C\right).\end{aligned}$$

$$\begin{aligned}\frac{\gamma AX'BXC}{\gamma X} &= \left(I_p \otimes BXC\right) K_{pq}\left(I_q \otimes A'\right) + \left(I_p \otimes B'XA'\right) U_{pq}\left(I_q \otimes C'\right) \\ &= \left(I_p \otimes BXC\right) K_{pq}\left(I_q \otimes A'\right) + \left(\operatorname{vec} B'XA'\right)\left(\operatorname{rvec} C'\right).\end{aligned}$$

4. Let $Y = AXBX'C$ where A is $p\times m$, B is $n\times n$ and C is $m\times q$. Then it is well known that

$$\frac{\partial AXBX'C}{\partial X} = K_{qp}\left(AXB\otimes C'\right)+\left(C'XB'\otimes A\right).$$

Using our principles we obtain

$$\frac{\delta AXBX'C}{\delta x_{rs}} = AXBE_{sr}^{nm}C + AE_{rs}^{mn}BX'C$$

and

$$\frac{\gamma\left(AXBX'C\right)_{ij}}{\gamma X} = CE_{ji}^{qp}AXB + A'E_{ij}^{pq}C'XB'.$$

Moreover, we have

$$\begin{aligned}\frac{\delta AXBX'C}{\delta X} &= \left(I_m\otimes AXB\right)K_{mn}\left(I_n\otimes D\right)+\left(I_m\otimes A\right)U_{mn}\left(I_n\otimes BX'C\right)\\ &= \left(I_m\otimes AXB\right)K_{mn}\left(I_n\otimes D\right)+\left(\operatorname{vec}A\right)\left(\operatorname{rvec}BX'C\right).\end{aligned}$$

$$\begin{aligned}\frac{\gamma AXBX'C}{\gamma X} &= \left(I_p\otimes C\right)K_{pq}\left(I_q\otimes B'X'A'\right)+\left(I_p\otimes A'\right)U_{pq}\left(I_q\otimes C'XB'\right)\\ &= \left(I_m\otimes AXB\right)K_{mn}\left(I_n\otimes D\right)+\left(\operatorname{vec}A'\right)\left(\operatorname{rvec}C'XB'\right).\end{aligned}$$

The following results are not as well known:

5. Let $Y = D'D$ where $D = A + BXC$ with A $p\times q$, B $p\times m$ and C $n\times q$.

Then from Lutkepohl (1996) p.191 we have

$$\frac{\partial D'D}{\partial X} = K_{qq}\left(C'\otimes D'B\right)+C'\otimes D'B.$$

Using our principles we obtain

$$\frac{\partial D'D}{\partial x_{rs}} = C'E_{sr}^{nm}B'D + B'DE_{rs}^{mn}C$$

and

$$\frac{\gamma\left(D'D\right)_{ij}}{\gamma X} = B'DE_{ji}^{qq}C' + B'DE_{ij}^{qq}C'.$$

In terms of the complete matrices we have

$$\frac{\partial D'D}{\partial X} = \left(I_m \otimes C'\right) K_{mn} \left(I_n \otimes B'D\right) + \left(I_m \otimes D'B\right) U_{mn} \left(I_n \otimes C\right)$$
$$= \left(I_m \otimes C'\right) K_{mn} \left(I_n \otimes B'D\right) + \left(\operatorname{vec} D'B\right)\left(\operatorname{rvec} C\right).$$

$$\frac{\gamma D'D}{\gamma X} = \left(I_q \otimes B'D\right) K_{qq} \left(I_q \otimes C'\right) + \left(I_q \otimes B'D\right) U_{qq} \left(I_q \otimes C'\right)$$
$$= \left(I_q \otimes B'D\right) K_{qq} \left(I_q \otimes C'\right) + \left(\operatorname{vec} B'D\right)\left(\operatorname{rvec} C'\right).$$

6. Let $Y = DD'$ where D is as in 5.

Then from Lutkepohl (1996) p.191 again we have

$$\frac{\partial DD'}{\partial X} = K_{pp} \left(DC' \otimes B\right) + \left(DC' \otimes B\right).$$

It follows that

$$\frac{\delta DD'}{\delta x_{rs}} = DC'E_{sr}^{nm}B' + BE_{rs}^{mn}CD'$$

$$\frac{\gamma \left(DD'\right)_{ij}}{\gamma X} = B'E_{ji}^{pp}DC' + B'E_{ij}^{pp}DC'$$

or in terms of complete matrices

$$\frac{\delta DD'}{\delta X} = \left(I_m \otimes DC'\right) K_{mn} \left(I_n \otimes B'\right) + \left(I_m \otimes B\right) U_{mn} \left(I_n \otimes CD\right)$$
$$= \left(I_m \otimes DC'\right) K_{mn} \left(I_n \otimes B'\right) + \left(\operatorname{vec} B\right)\left(\operatorname{rvec} CD'\right)$$

$$\frac{\gamma DD'}{\gamma X} = \left(I_p \otimes B'\right) K_{pp} \left(I_p \otimes DC'\right) + \left(I_p \otimes B'\right) U_{pp} \left(I_p \otimes DC'\right)$$
$$= \left(I_p \otimes B'\right) K_{pp} \left(I_p \otimes DC'\right) + \left(\operatorname{vec} B'\right)\left(\operatorname{rvec} DC'\right).$$

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