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FUZZY CORES AND FUZZY BALANCEDNESS

By Gerwald van Gulick, Henk Norde

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TILBURG ◆ UNIVERSITY

Fuzzy cores and fuzzy balancedness

Gerwald van Gulick[†], Henk Norde[†,*]

Abstract

We study the relation between the fuzzy core and balancedness for fuzzy games. For regular games, this relation has been studied by Bondareva (1963) and Shapley (1967).

First, we gain insight in this relation when we analyse situations where the fuzzy game is continuous. Our main result shows that any fuzzy game has a non-empty core if and only if it satisfies all (fuzzy) balanced inequalities.

We also consider deposit games to illustrate the use of the main result.

JEL classification: C71

Keywords: Cooperative fuzzy games, fuzzy balancedness, fuzzy core

1 Introduction

In a standard cooperative game, if players choose to combine efforts, then they form a coalition. If a player is in a coalition, he is assumed to put in his maximal effort. However, in certain settings it makes sense to consider partial cooperation of players. For instance, if the participation of a player is dependent only on money or time, it makes sense to consider situations where different levels of participation of a player are considered. Consider for example deposit games as analysed in Van Gulick, Borm, De Waegenaere, and Hendrickx (2010). There players have an endowment that they can use to (jointly) deposit at a bank. It is possible for players to commit only a part of their endowment to a coalition. Fuzzy games are used to study the situations where the endowment (or in general the participation level) of each player is assumed to be infinitely divisible. We think this assumption is reasonable when we consider the amount of capital each player has available, such as endowments in deposit games.

Cooperative fuzzy games were introduced by Aubin (1974) (French) and Aubin (1981) (English). For an overview of work on fuzzy games, we refer to Branzei, Dimitrov, and Tijs (2005). The literature mainly focusses on extending situations corresponding to cooperative games to fuzzy settings. We also provide such an extension in this paper: fuzzy deposit games. We use this extension

[†]CentER and Department of Econometrics and Operations Research, Tilburg University.

[*]Corresponding author: P.O. Box 90153, 5000 LE Tilburg, The Netherlands, E-mail: H.Norde@uvt.nl.

to illustrate our main result: generalising the result by Bondareva (1963) and Shapley (1967) that the core of a cooperative game is non-empty if and only if the game is balanced.

The natural extension of the core of a game is the fuzzy core of the fuzzy game: no fuzzy coalition has any incentive to split off from the grand coalition. The fuzzy core is defined by Tijs, Branzei, Ishihara, and Muto (2004) and is studied for convex fuzzy games by Branzei, Dimitrov, and Tijs (2003).[1] It is not so clear what the natural extension of balancedness is in a fuzzy setting. This is the first question we address in this paper, and in section 2 in particular. We find that restricting fuzzy games to be continuous helps us find the appropriate form of the balanced inequalities that need to be satisfied for these games to have a non-empty core. Furthermore, we find that this approach using continuous fuzzy games provides us with insight about the nature of fuzzy games and fuzzy balancedness.

The second question we address provides an explicit link between the fuzzy core and these balanced inequalities. The main result of this work is a small extension of simultaneous work by Azrieli and Lehrer (2007), who need these results in their study of market games. Because we use a different proof, we are able to omit one condition that Azrieli and Lehrer (2007) impose: that the fuzzy game is assumed to be bounded.

This paper is structured as follows. We first introduce some notation in section 2. We proceed in section 3, where we focus on continuous fuzzy games. We show all continuous fuzzy games with a non-empty core satisfy all balanced inequalities as introduced for fuzzy games in section 2, and vice versa. In section 4, we use the insights of continuous fuzzy games, and extend the result to all fuzzy games with a non-empty core. We apply the results of this paper in section 5, where we consider an extension of the term dependent deposit games introduced in Van Gulick, Borm, De Waegenaere, and Hendrickx (2010). By making the endowment of each coalition depend on the level of participation of the players, we construct term dependent fuzzy deposit games. Using the relation between balanced inequalities and the fuzzy core, we show that the core of these games is always non-empty. We conclude in section 6.

2 Notation

In this section, we briefly introduce some notation used throughout this paper. We denote the set of all players $N = \{1, 2, \dots, n\}$ and $S \subseteq N$ is refered to as a coalition. A regular cooperative game is a function $w : 2^N \to \mathbb{R}$, with the convention that $w(\emptyset) = 0$.

Definition 1 *The core of a game w is defined by*

$$Core(w) = \left\{ x \in \mathbb{R}^N \,\middle|\, x_S \geq w(S) \text{ for all } S \subseteq N, \sum_{i \in N} x_i = w(N) \right\}.$$

[1] An extension of the core called the fuzzy core has also been discussed by Butnariu (1980), but the definition differs from the one used in this paper.

Here $x_S = \sum_{i \in S} x_i$.

In a fuzzy game, we express fuzzy coalitions as a combination of the participation levels of all players, *i.e.* $s \in [0,1]^N$ is a fuzzy coalition. A fuzzy game is denoted (N, v) where $v : [0,1]^N \to \mathbb{R}$, again with the convention that $v(0) = 0$. We define $e^S \in [0,1]^N$ for all $S \subseteq N$ by $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. For all $S \subseteq N$ we interpret e^S as the fuzzy coalition corresponding to coalition S in a regular cooperative game. For a fuzzy singleton coalition $i \in N$ we denote $e^i = e^{\{i\}}$, and for the fuzzy grand coalition we denote e^N. The fuzzy core is defined similar to the regular core.

Definition 2 *The fuzzy core of a fuzzy game* (N, v) *is defined by*

$$FCore(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, x_s \geq v(s) \text{ for all } s \in [0,1]^N, \sum_{i \in N} x_i = v(e^N) \right\}.$$

Here $x_s = \sum_{i \in N} s_i x_i$.

In this paper, we analyse the relation between the fuzzy core and balanced inequalities. We first define a balanced collection.

Definition 3 *A finite collection of fuzzy coalitions* $B \subset [0,1]^N$ *is* balanced *if for every* $s \in B$ *there exists a weight* $\lambda_s > 0$ *such that*

$$\sum_{s \in B} \lambda_s s_i = 1, \tag{1}$$

for all $i \in N$.

Observe that this is a generalisation of the notion of balanced collections used for regular cooperative games (see, *i.e.* Shapley (1967)). Now we define a balanced inequality.

Definition 4 *The balanced inequality corresponding to a balanced collection* B *with weights* $\lambda = (\lambda_s)_{s \in B}$ *and a fuzzy game* (N, v) *is*

$$\sum_{s \in B} \lambda_s v(s) \leq v(e^N).$$

The expression $\sum_{s \in B} \lambda_s v(s)$ *is denoted as* $L(B, \lambda, v)$.

We also define the concept of continuity for fuzzy games.

Definition 5 *A fuzzy game* (N, v) *is* continuous *if* $v : [0,1]^N \to \mathbb{R}$ *is a continuous function.*

3 Continuous fuzzy games

This section focusses on continuous fuzzy games, and the relation in these games between non-emptiness of the fuzzy core and balanced inequalities. We introduce limited fuzzy games, which are restricted to coalitions for which $s \in \mathbb{Q}^N$. These games can be seen as a restriction of the fuzzy game to a finite number of coalitions.

It is apparent that this approach needs continuous fuzzy games in order to be viable. Only by assuming continuity for the fuzzy game v, it is true that the values of v for fuzzy coalitions in $\mathbb{Q}^N$ fully determine the values of v on the full domain $[0,1]^N$. We specifically use that all balanced inequalities of limited fuzzy games provide information about the value of the coalitions in $\mathbb{Q}^N$, and hence, by continuity, also for all coalitions. The latter is not true in general for non-continuous fuzzy games, and therefore we take a different approach in the next section. This section however highlights this interesting technique described above that is more insightful than the general approach. Another contribution of this section, is that it sheds light on the definition of balanced inequalities, as in Definition 4. In particular, one might wonder why in fuzzy games balanced inequalities are expressed as finite sums, rather than as integrals.

First, let us give a formal definition of a limited fuzzy game.

Definition 6 *Let $k \in \mathbb{N}$. Define $S_k = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}^N$. A* k-limited fuzzy game *with set of players N is a function $v : S_k \to \mathbb{R}$ where $v(0) = 0$. We have that $\mathbb{R}^{S_k}$ is the collection of all k-limited fuzzy games.*

Rather than considering infinitely many coalitions, the number of coalitions in a k-limited fuzzy game is finite. We refer to S_k as a grid. Observe that a 1-limited fuzzy game is in fact a standard cooperative game with coalitions $S \subseteq N$. We also introduce terminology if the k-limited fuzzy games are restrictions of a fuzzy game.

Definition 7 *Let (N, v) be a fuzzy game and let $k \in \mathbb{N}$. Its k-limited restriction is the k-limited fuzzy game $v_k : S_k \to \mathbb{R}$ defined by $v_k(s) = v(s)$ for all $s \in S_k$.*

This k-limited fuzzy game can be seen as the fuzzy game (N, v), but restricted on a grid which depends on k. For a k-limited fuzzy game, the natural extension of the fuzzy core is simply to restrict the core conditions to fuzzy coalitions in S_k.

Definition 8 *Let $k \in \mathbb{N}$ and $v : S_k \to \mathbb{R}$ a k-limited fuzzy game. The k-core of v is defined by*

$$FCore^k(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, x_s \geq v(s) \text{ for all } s \in S_k, \sum_{i \in N} x_i = v(e^N) \right\}.$$

Note that $FCore^1(v) = Core(v)$ for 1-limited fuzzy games v (which are of course standard cooperative games). It is also straightforward how to extend the notion of additivity to a k-limited fuzzy game.

Definition 9 *Let $k \in \mathbb{N}$. A k-limited fuzzy game v is* additive *if there exists an $a \in \mathbb{R}^N$ such that for all $s \in S_k$ it holds that $v(s) = \sum_{i \in N} s_i a_i$.*

Note that all (fuzzy) additive k-limited games have a non-empty (fuzzy) k-core, and that a is the single element in that (fuzzy) k-core.

Our aim is to show that there exist balancedness conditions that, when satisfied, ensure that the k-core of the k-limited fuzzy game is non-empty. We show moreover that there are (linear) conditions that are satisfied by all k-limited fuzzy games with a non-empty k-core. First, we show that the class of all k-limited fuzzy games with a non-empty k-core is a closed convex cone. The following lemma is a first step in that direction.

Lemma 10 *Let $k \in \mathbb{N}$. Every convergent sequence of k-limited fuzzy games with a non-empty k-core converges to a k-limited fuzzy game with a non-empty k-core.*

Proof: Let $v_1, v_2, \ldots$ be a convergent sequence of k-limited fuzzy games with a non-empty k-core, and let $x^1, x^2, \ldots$ be core elements for these games. For all $s \in S_k$ we define

$$v(s) = \lim_{n \to \infty} v_n(s).$$

Let $i \in N$. As $v(e^i) = \lim_{n \to \infty} v_n(e^i)$, the sequence $v_1(e^i), v_2(e^i), \ldots$ is bounded, so there is a number d_i such that $v_n(e^i) \geq d_i$ for all $n \in \mathbb{N}$. Similarly, because $v(e^N) = \lim_{n \to \infty} v_n(e^N)$, the sequence $v_1(e^N), v_2(e^N), \ldots$ is bounded, so there is a number u such that $v_n(e^N) \leq u$ for all $n \in \mathbb{N}$. So,

$$d_i \leq v_n(e^i) \leq x_i^n = v_n(e^N) - \sum_{j \in N \setminus \{i\}} x_j^n \leq u - \sum_{j \in N \setminus \{i\}} d_j,$$

for every $n \in \mathbb{N}$.

So $x^1, x^2, \ldots$ is bounded. According to Bolzano-Weierstrass, there is a converging subsequence $x^{m_1}, x^{m_2}, \ldots$ of core elements in $\mathbb{R}^N$ for some of these k-limited fuzzy games. Define $x = \lim_{n \to \infty} x^{m_n}$, then

$$x_s = \lim_{n \to \infty} x_s^{m_n} \geq \lim_{n \to \infty} v_{m_n}(s) = v(s),$$

for all $s \in S_k$, and

$$x_{e^N} = \lim_{n \to \infty} x_{e^N}^{m_n} = \lim_{n \to \infty} v_{m_n}(e^N) = v(e^N),$$

so x is a core element of v. □

The next step is to show that the class of all k-limited fuzzy games with a non-empty k-core is a convex cone.

Theorem 11 *Let $k \in \mathbb{N}$. Let v, w be k-limited fuzzy games with a non-empty k-core and $\alpha > 0$. Then $v + w$ and $\alpha \cdot v$ are k-limited fuzzy games with a non-empty k-core. So, all k-limited fuzzy games with a non-empty k-core form a convex cone.*

Proof: Let x^v, x^w core allocations of the games v and w. Define $x^{v+w} = x^v + x^w$. Then for all $s \in S_k$

$$x_s^{v+w} = x_s^v + x_s^w \geq v(s) + w(s) = (v+w)(s),$$

and moreover

$$x_{e^N}^{v+w} = x_{e^N}^v + x_{e^N}^w = v(e^N) + w(e^N) = (v+w)(e^N).$$

Hence $v + w$ has a non-empty k-core. Also for all $s \in S_k$ we have

$$\alpha \cdot x_s^v \geq \alpha \cdot v(s) = (\alpha \cdot v)(s),$$

and

$$\alpha \cdot x_{e^N}^v = \alpha \cdot v(e^N) = (\alpha \cdot v)(e^N).$$

So $\alpha \cdot v$ has a non-empty k-core. □

Lemma 10 implies that this convex cone is in fact closed.

Next, we define a class of linear forms and inequalities that is more general than the class of balanced inequalities. We show that the closed convex cone of all k-limited fuzzy games with a non-empty k-core can be defined using these inequalities. Later, we refine this class of inequalities.

Definition 12 *Let $k \in \mathbb{N}$. A linear k-form is a linear function $L : \mathbb{R}^{S_k} \to \mathbb{R}$. This linear function is fully determined by its coefficients $(\gamma_s)_{s \in S_k}$,* i.e.

$$L(v) = \sum_{s \in S_k} \gamma_s \cdot v(s),$$

for all $v \in \mathbb{R}^{S_k}$. A linear k-form is prebalanced *if it vanishes for additive functions v, and the corresponding inequality*

$$L(v) \leq 0$$

is called a prebalanced inequality*.*

Note that every balanced inequality is also a prebalanced inequality. A link with k-limited fuzzy games with a non-empty k-core is provided below.

Lemma 13 *Let $k \in \mathbb{N}$ and let L be a linear k-form such that $L(v) \leq 0$ for all k-limited fuzzy games with a non-empty k-core. Then L is necessarily prebalanced.*

Proof: Let a be an additive game. Then $-a$ is an additive game as well. Hence a and $-a$ both have a non-empty k-core. So, $L(a) \leq 0$ and $-L(a) = L(-a) \leq 0$. Hence $L(a) = 0$. □

In fact, the relation between k-limited fuzzy games with a non-empty k-core and these prebalanced inequalities is more specific, as shown in the next lemma.

Lemma 14 *Let $k \in \mathbb{N}$. The set of k-limited fuzzy games with a non-empty k-core can be defined by a system of prebalanced inequalities.*

Proof: A closed convex cone is the intersection of the closed half-spaces that contain it. Half-spaces are defined by a linear inequality, and for the set of k-limited fuzzy games with a non-empty k-core these are necessarily prebalanced by Lemma 13. □

We are able to be more particular about the form of the prebalanced inequalities in Lemma 14.

Theorem 15 *Let $k \in \mathbb{N}$. A k-limited fuzzy game has a non-empty k-core if, and only if, it satisfies all balanced inequalities corresponding to a balanced collection $B \subseteq S_k$.*

Proof: The defining inequalities of a k-limited fuzzy game has a non-empty k-core are prebalanced by Lemma 13, hence they are linear. Let

$$L(v) = \sum_{s \in S_k} \gamma_s \cdot v(s) \leq 0,$$

by such a prebalanced inequality. Consider a k-limited fuzzy game v with a non-empty k-core. Then by decreasing the value of any coalition $s \in S_k$ where $s \neq e^N$, the game still has a non-empty k-core. So this coalition must have a non-negative coefficient γ. By increasing the value of coalition e^N the game remains to have a non-empty k-core. So γ_{e^N} is non-positive. If $\gamma_{e^N} = 0$, then $\gamma_s = 0$ for all $s \in S_k$, because a linear k-form vanishes for additive functions. Hence alternatively, if $\gamma_{e^N} < 0$, the prebalanced inequality can be written as

$$\sum_{s \in S_k \backslash \{e^N\}} \gamma'_s \cdot v(s) \leq v(e^N),$$

where $\gamma'_s \geq 0$ is obtained from dividing γ_s by $-\gamma_{e^N}$ for all $s \in S_k \backslash \{e^N\}$. Because a linear k-form vanishes for additive functions, this prebalanced inequality corresponds to a balanced inequality if we define

$$B = \{s \in S_k \backslash \{e^N\} | \gamma'_s > 0\}.$$

It remains to show that all k-limited fuzzy games with a non-empty k-core satisfy all balanced inequalities. Take such a game v, with core element x. Then we make an additive game x' with value x_i for all $i \in N$. Let $B \subseteq S_k$. By definition of a balanced inequality we have

$$\sum_{s \in B} \lambda_s x'(s) = x'(e^N),$$

where $\lambda_s > 0$ for all $s \in B$. From the fact that for all $s \in S_k$ we have $x'(s) \geq v(s)$ and $x'(e^N) = v(e^N)$ it follows that

$$\sum_{s \in B} \lambda_s v(s) \leq v(e^N),$$

where $\lambda_s > 0$ for all $s \in B$, which completes the proof. $\square$

We are now able to present a partial result of the relationship between balanced inequalities and the fuzzy core.

Theorem 16 *A continuous fuzzy game (N, v) has a non-empty core if, and only if, every k-limited fuzzy game of v has a non-empty k-core.*

Proof: Let (N, v) be a continuous fuzzy game, and let x be a core element of (N, v). Then for all $k \in \mathbb{N}$ and $s \in S_k$ we have

$$x_s \geq v(s) = v_k(s),$$

and also

$$x_{e^N} = v(e^N) = v_k(e^N),$$

so x is also a core element for every k-limited fuzzy game of v.

Let $k \in \mathbb{N}$ and $x^k \in FCore^k(v_k)$. Then for each $i \in N$ we have

$$x_i^k \geq v_k(e^i) = v(e^i),$$

and also

$$x_i^k = v_k(e^N) - \sum_{j \in N \setminus \{i\}} x_j^k \leq v(e^N) - \sum_{j \in N \setminus \{i\}} v(e^j).$$

So $x^1, x^2, \ldots$ is bounded. According to Bolzano-Weierstrass there exists at least one subsequence that converges. Let $x^{k_1}, x^{k_2}, \ldots$ be such a subsequence and define $x = \lim_{n \to \infty} x^{k_n}$. We now show that $x \in FCore(v)$.

Let $s \in [0,1]^N$. Define $s_i^{k_n} = \frac{\lfloor k_n \cdot s_i \rfloor}{k_n}$ for all $n \in \mathbb{N}$ and all $i \in N$. It is clear that $s^{k_n} \in S_{k_n}$. We have that $|s_i - s_i^{k_n}| \leq \frac{1}{k_n}$ for all $i \in N$. Therefore, we find that

$$\lim_{n \to \infty} s^{k_n} = s.$$

Then

$$\begin{aligned} x_s &= \sum_{i \in N} x_i \cdot s_i = \sum_{i \in N} \lim_{n \to \infty} (x_i^{k_n} \cdot s_i^{k_n}) \\ &= \lim_{n \to \infty} \sum_{i \in N} x_i^{k_n} \cdot s_i^{k_n} = \lim_{n \to \infty} x_{s^{k_n}}^{k_n} \\ &\geq \lim_{n \to \infty} v_{k_n}(s^{k_n}) = \lim_{n \to \infty} v(s^{k_n}) \\ &= v(s), \end{aligned}$$

where the final equality follows from continuity of the fuzzy game v. Obviously

$$x_{e^N} = \lim_{n \to \infty} x_{e^N}^{k_n} = \lim_{n \to \infty} v_{k_n}(e^N) = \lim_{n \to \infty} v(e^N) = v(e^N),$$

so $x \in FCore(v)$. $\square$

As a corollary of Theorem 15, Theorem 16 and continuity in general, the main result of this section follows.

Theorem 17 *A continuous fuzzy game has a non-empty core if, and only if, it satisfies all balanced inequalities.*

This result is generalised in the next section.

4 Main result

In this section, we generalise the result of section 3 to the class of all fuzzy games with a non-empty core. We start this section with a visual representation of the method we use to derive this result. In Figure 1 the graph of a fuzzy game (N, v) is depicted, where $N = \{1, 2\}$ and $v(s) = (s_1 + 2s_2)^2$. This fuzzy game (N, v) satisfies all balancedness conditions. For this fuzzy game we find a function f that provides an upper bound for the value of each coalition, such that all balanced inequalities hold. The intuition is that $f(s)$ also gives information about $v(e^N - s)$, because s and $e^N - s$ together form a balanced collection. The graph of f is depicted in Figure 2. We show that there exists a linear function g that separates f and v^2. The plane corresponding to g is graphically represented in Figure 3. Because the graph of this function includes $(e^N, v(e^N))$, it gives rise to an element of the fuzzy core. In Figure 4 we have highlighted the graphs of the functions v, f and g on the diagonal from e^1 to e^2. The core element of (N, v) that g gives rise to can be readily seen in Figure 4, since for any player $i \in N$ we only need to look at the value of g in e^i. In this case, the function g gives rise to $x = (3, 6)$, and it can be verified that indeed $x \in FCore(v)$.

Let (N, v) be a fuzzy game that satisfies all balancedness conditions, where N is the set of players $\{1, 2, \ldots, n\}$ and $v : [0, 1]^N \to \mathbb{R}$ its characteristic function. We define the set of values a coalition $s \in [0, 1]^N$ can attain without violating a balancedness condition in which this coalition is involved, given all other values in this balancedness condition remain unchanged, by

$$\mathcal{M}(s) = \left\{ \frac{v(e^N) - \sum_{j=1}^{k} \lambda_j v(s^j)}{\lambda_0} \;\middle|\; \begin{array}{l} \exists k \in \mathbb{N},\, s^1, s^2, \ldots, s^k \in [0, 1]^N, \\ \qquad \lambda_0, \lambda_1, \ldots, \lambda_k > 0 : \lambda_0 s + \sum_{j=1}^{k} \lambda_j s_j = e^N \end{array} \right\}$$

Every balancedness condition that includes coalition s provides a value in $\mathcal{M}(s)$.

Lemma 18 *Let (N, v) be a fuzzy game that satisfies all balancedness conditions. The set $\mathcal{M}(s)$ is non-empty for all $s \in [0, 1]^N$.*

Proof: Let $s \in [0, 1]^N$, $k = \lambda_0 = \lambda_1 = 1$, and $s_1 = e^N - s \in [0, 1]^N$. We find that $v(e^N) - v(e^N - s) \in \mathcal{M}(s)$, so $\mathcal{M}(s)$ is non-empty. □

We define a function $f : [0, 1]^N \to \mathbb{R}$ that is an upper bound for the value each coalition in N can attain without violating any balancedness condition, given

[2] We say that a function g separates functions f and v if for all $s \in [0, 1]^N$ it holds $v(s) \leq g(s) \leq f(s)$.

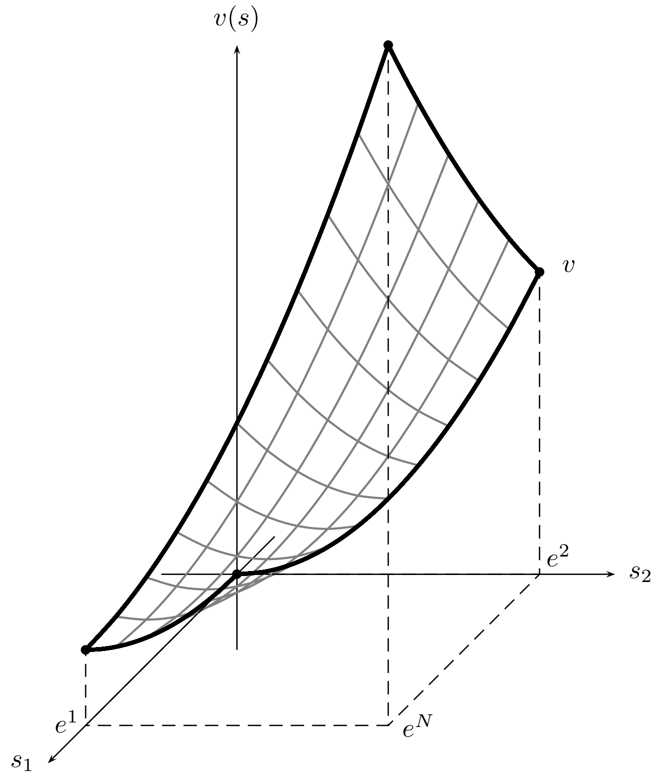

Figure 1: The graph of $v(s) = (s_1 + 2s_2)^2$.

the values of all other coalitions remain the same, by

$$f(s) = \inf \mathcal{M}(s). \tag{2}$$

First, we have to show that this function f is well-defined.

Lemma 19 *Let (N, v) be a fuzzy game that satisfies all balancedness conditions. The function f in equation (2) is well-defined. Furthermore $f(s) \geq v(s)$ for all $s \in [0, 1]^N$.*

Proof: Let $s \in [0, 1]^N$. We show that $\mathcal{M}(s)$ has lower bound $v(s)$. Any non-empty set that has a lower bound has an infimum, thus $f(s)$ is well-defined. Also, this lower bound is then a lower bound for $f(s)$ as well, so $v(s) \leq f(s)$.

For any balanced inequality containing s it holds that

$$\lambda_0 v(s) + \sum_{j=1}^{k} \lambda_j v(s^j) \leq v(e^N),$$

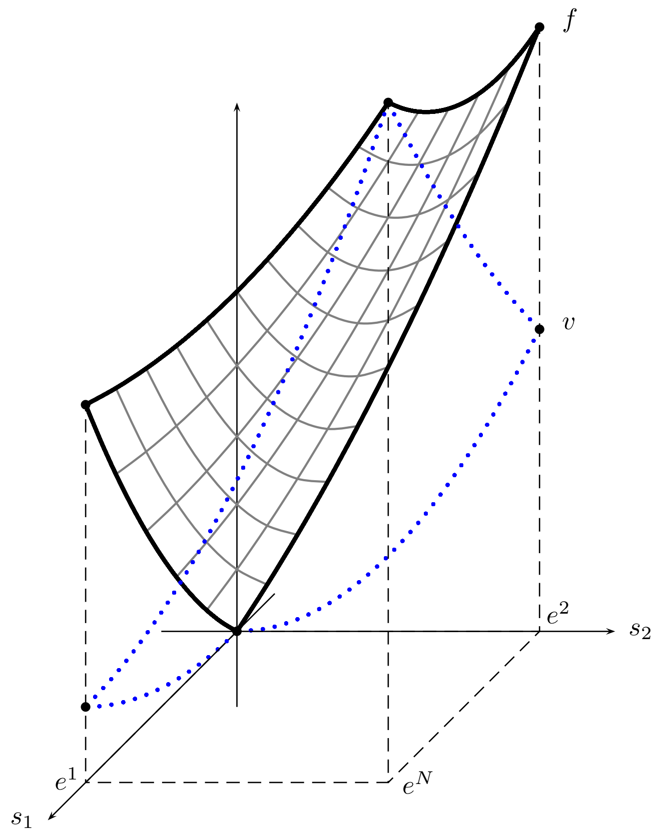

Figure 2: The extension of Figure 1 with the graph of f.

where $k \in \mathbb{N}$, $s^1, s^2, \ldots, s^k \in [0,1]^N$, and $\lambda_0, \lambda_1, \ldots, \lambda_k > 0$. So, we have that

$$v(s) \leq \frac{v(e^N) - \sum_{j=1}^{k} \lambda_j v(s^j)}{\lambda_0},$$

proving there is a lower bound. □

Next, we show that on the diagonal from 0 to e^N, f is linear, and furthermore that the values of f and v are equal in 0 and e^N. Note that consequently for any affine function g that separates f and v, the values on this diagonal are equal to those of f as a result. It is readily observed that if g is linear as well, an efficient allocation for v can be derived from g.

Lemma 20 *Let (N, v) be a fuzzy game that satisfies all balancedness conditions. Let f be the function defined in equation (2). For all $0 \leq \alpha \leq 1$ it holds that $f(\alpha e^N) = \alpha v(e^N)$.*

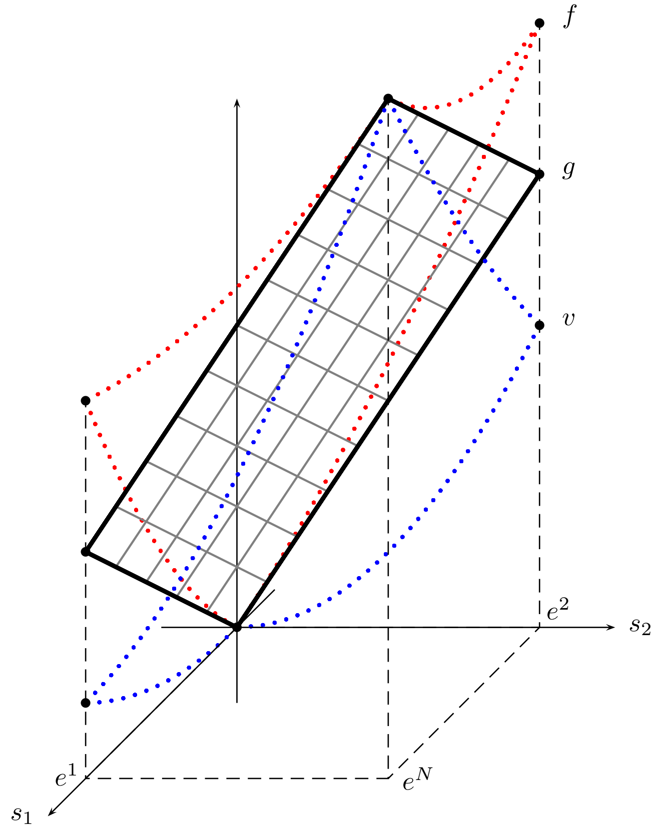

Figure 3: The extension of Figure 2 with the graph of g.

Proof: Let $0 \leq \alpha \leq 1$. Let $k = \lambda_0 = 1$, $\lambda_1 = 1 - \alpha$, $s = \alpha e^N$ and $s^1 = e^N$. This corresponds to a balancedness condition, since obviously

$$\lambda_0 s + \lambda_1 s_1 = \alpha e^N + (1-\alpha)e^N = e^N.$$

Hence,

$$\frac{v(e^N) - (1-\alpha)v(e^N)}{1} = \alpha v(e^N) \in \mathcal{M}(\alpha e^N).$$

So $f(\alpha e^N) = \inf \mathcal{M}(\alpha e^N) \leq \alpha v(e^N)$.

If $\alpha = 1$, it follows from the above inequality combined with Lemma 19 that $v(e^N) \leq f(e^N) \leq v(e^N)$, hence $f(e^N) = v(e^N)$. We now consider the case when $\alpha < 1$. Let $k \in \mathbb{N}$, $\lambda_0, \ldots, \lambda_k > 0$ and $s^1, \ldots, s^k \in [0,1]^N$, with

$$\lambda_0 \alpha e^N + \sum_{j=1}^{k} \lambda_j s^j = e^N. \tag{3}$$

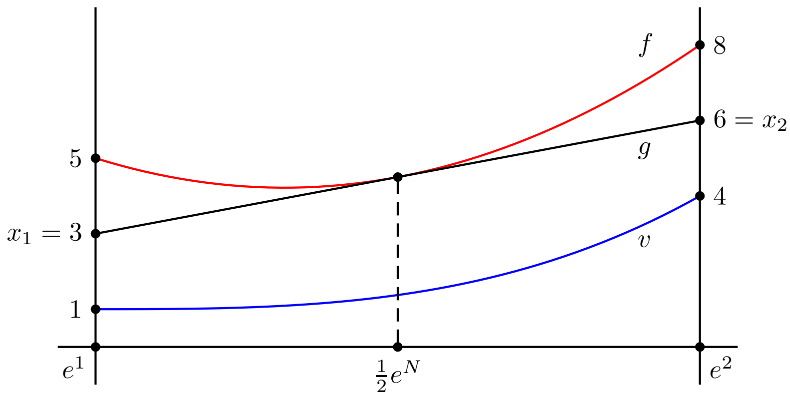

Figure 4: Crossection of Figure 3 along the diagonal from e^1 to e^2, including core element $x = (3, 6)$.

Then

$$\frac{v(e^N) - \sum_{j=1}^{k} \lambda_j v(s^j)}{\lambda_0} \in \mathcal{M}(\alpha e^N).$$

We can rewrite equation (3) to

$$\sum_{j=1}^{k} \lambda_j s^j = (1 - \lambda_0 \alpha) e^N,$$

so $s^1, \ldots, s^k \in [0,1]^N$ needs to be balanced. Therefore,

$$\sum_{j=1}^{k} \lambda_j v(s^j) \leq (1 - \lambda_0 \alpha) v(e^N),$$

and thus

$$\frac{v(e^N) - \sum_{j=1}^{k} \lambda_j v(s^j)}{\lambda_0} \geq \frac{v(e^N) - (1 - \lambda_0 \alpha) v(e^N)}{\lambda_0} = \frac{\lambda_0 \alpha v(e^N)}{\lambda_0} = \alpha v(e^N).$$

So every element of $\mathcal{M}(e^N)$ is bounded from below by $\alpha v(e^N)$. It follows that $f(\alpha e^N) \geq \alpha v(e^N)$. Hence $f(\alpha e^N) = \alpha v(e^N) = \alpha f(e^N)$. □

It follows that $f(0) = 0 \cdot v(e^N) = v(0)$.

We now proceed to show that the function f is convex. It follows that there exists a hyperplane 'below' the graph of f.

Lemma 21 *Let (N, v) be a fuzzy game that satisfies all balancedness conditions. The function f defined in equation (2) is convex.*

Proof: Let $s, t \in [0,1]^N$ and $\alpha \in (0,1)$. We show that

$$f(\alpha s + (1-\alpha)t) \leq \alpha f(s) + (1-\alpha) f(t).$$

For this, we show that any two balanced inequalities, one containing s and the other containing t, give rise to a balanced inequality containing $\alpha s + (1-\alpha)t$. We find an upper bound for these inequalities, hence the infimum of all balanced inequalities containing $\alpha s + (1-\alpha)t$ must have a smaller value than this upper bound.

Let $\varepsilon > 0$. There exist numbers $k, \ell \in \mathbb{N}$, weights $\lambda_0, \ldots, \lambda_k, \mu_0, \ldots, \mu_\ell > 0$ and coalitions $s^1, \ldots, s^k, t^1, \ldots, t^\ell \in [0,1]^N$ such that

$$\frac{v(e^N) - \sum_{j=1}^k \lambda_j v(s^j)}{\lambda_0} < f(s) + \varepsilon,$$

and also

$$\frac{v(e^N) - \sum_{j=1}^\ell \mu_j v(t^j)}{\mu_0} < f(t) + \varepsilon.$$

Using the above equations, we find that

$$\begin{aligned} &\alpha \frac{v(e^N) - \sum_{j=1}^k \lambda_j v(s^j)}{\lambda_0} + (1-\alpha) \frac{v(e^N) - \sum_{j=1}^\ell \mu_j v(t^j)}{\mu_0} \\ &< \alpha f(s) + (1-\alpha) f(t) + \varepsilon. \end{aligned} \tag{4}$$

Define $u = \alpha s + (1-\alpha)t$. We now show that

$$\alpha \frac{v(e^N) - \sum_{j=1}^k \lambda_j v(s^j)}{\lambda_0} + (1-\alpha) \frac{v(e^N) - \sum_{j=1}^\ell \mu_j v(t^j)}{\mu_0} \in \mathcal{M}(u)$$

by combining the two underlying balanced inequalities to one containing u. Let $m \in \mathbb{N}$, $\eta_0, \eta_1, \ldots, \eta_m \in [0,1]^N$ and $u_1, u_2, \ldots, u_m$ be defined by

$$\begin{aligned} m &= k + \ell, \\ \eta_0 &= \frac{\lambda_0 \mu_0}{\alpha \mu_0 + (1-\alpha)\lambda_0}, \\ \eta_i &= \frac{\alpha \mu_0}{\alpha \mu_0 + (1-\alpha)\lambda_0} \lambda_i, && \text{for all } i \in \{1, \ldots, k\}, \\ \eta_i &= \frac{(1-\alpha)\lambda_0}{\alpha \mu_0 + (1-\alpha)\lambda_0} \mu_{i-k}, && \text{for all } i \in \{k+1, \ldots, k+\ell\}, \\ u^i &= s^i, && \text{for all } i \in \{1, \ldots, k\}, \\ u^i &= t^{i-k}, && \text{for all } i \in \{k+1, \ldots, k+\ell\}. \end{aligned}$$

We find that

$$
\begin{aligned}
&\eta_0(\alpha s+(1-\alpha)t)+\sum_{j=1}^{m}\eta_j u^j \\
=&\frac{\lambda_0\mu_0}{\alpha\mu_0+(1-\alpha)\lambda_0}(\alpha s+(1-\alpha)t)+\sum_{j=1}^{k}\frac{\alpha\mu_0}{\alpha\mu_0+(1-\alpha)\lambda_0}\lambda_j s^j \\
&\quad+\sum_{j=k+1}^{k+\ell}\frac{(1-\alpha)\lambda_0}{\alpha\mu_0+(1-\alpha)\lambda_0}\mu_{j-k}t^{j-k} \\
&=\frac{\alpha\mu_0}{\alpha\mu_0+(1-\alpha)\lambda_0}\left(\lambda_0 s+\sum_{j=1}^{k}\lambda_j s^j\right)+\frac{(1-\alpha)\lambda_0}{\alpha\mu_0+(1-\alpha)\lambda_0}\left(\mu_0 t+\sum_{j=1}^{\ell}\mu_j t^j\right) \\
&=\frac{\alpha\mu_0}{\alpha\mu_0+(1-\alpha)\lambda_0}e^N+\frac{(1-\alpha)\lambda_0}{\alpha\mu_0+(1-\alpha)\lambda_0}e^N \\
&=e^N,
\end{aligned}
$$

so we conclude that

$$\frac{v(e^N)-\sum_{j=1}^{m}\eta_j v(u^j)}{\eta_0}\in\mathcal{M}(u).$$

Furthermore, we see that for this value it holds that

$$
\begin{aligned}
&\frac{v(e^N)-\sum_{j=1}^{m}\eta_j v(u^j)}{\eta_0} \\
&=\frac{v(e^N)-\sum_{j=1}^{k}\frac{\alpha\mu_0}{\alpha\mu_0+(1-\alpha)\lambda_0}\lambda_j v(s^j)-\sum_{j=k+1}^{k+\ell}\frac{(1-\alpha)\lambda_0}{\alpha\mu_0+(1-\alpha)\lambda_0}\mu_{j-k}v(t^{j-k})}{\frac{\lambda_0\mu_0}{\alpha\mu_0+(1-\alpha)\lambda_0}} \\
&=\frac{(\alpha\mu_0+(1-\alpha)\lambda_0)v(e^N)-\alpha\mu_0\sum_{j=1}^{k}\lambda_j v(s^j)-(1-\alpha)\lambda_0\sum_{j=1}^{\ell}\mu_j v(t^j)}{\lambda_0\mu_0} \\
&=\frac{\alpha\mu_0 v(e^N)-\alpha\mu_0\sum_{j=1}^{k}\lambda_j v(s^j)}{\lambda_0\mu_0}+\frac{(1-\alpha)\lambda_0 v(e^N)-(1-\alpha)\lambda_0\sum_{j=1}^{\ell}\mu_j v(t^j)}{\lambda_0\mu_0} \\
&=\alpha\frac{v(e^N)-\sum_{j=1}^{k}\lambda_j v(s^j)}{\lambda_0}+(1-\alpha)\frac{v(e^N)-\sum_{j=1}^{\ell}\mu_j v(t^j)}{\mu_0}.
\end{aligned}
$$

Because $f(\alpha s+(1-\alpha)t)=f(u)=\inf\mathcal{M}(u)$, we find that

$$f(\alpha s+(1-\alpha)t)\leq\alpha\frac{v(e^N)-\sum_{j=1}^{k}\lambda_j s^j}{\lambda_0}+(1-\alpha)\frac{v(e^N)-\sum_{j=1}^{\ell}\mu_j t^j}{\mu_0}.$$

So, according to equation (4) we have

$$f(\alpha s+(1-\alpha)t)<\alpha f(s)+(1-\alpha)f(t)+\varepsilon.$$

This inequality holds for all $\varepsilon > 0$, so we find

$$f(\alpha s + (1-\alpha)t) \leq \alpha f(s) + (1-\alpha) f(t),$$

which completes the proof that f is a convex function. □

Because f is convex, there exists a hyperplane with the graph of f on one side. We later show that this hyperplane can be chosen such that the graph of v is on the other side. We need the next theorem to show this.

Theorem 22 (Supporting Hyperplane Theorem) (Rockafellar (1970), page 100) *Let C be a convex set, and let D be a non-empty convex subset of C (for instance, a subset consisting of a single point). In order that there exists a non-trivial hyperplane to C containing D, it is necessary and sufficient that D be disjoint from the relative interior of C.*

In this case, we can take the set $C = \{(s, x) | x \in \mathbb{R}, s \in [0,1]^N : f(s) \leq x\}$. This set is obviously convex, because the function f is convex. Then $D = \{(\frac{1}{2}e^N, f(\frac{1}{2}e^N))\}$ is obviously on the boundary of this set, so it is a non-empty convex subset of C, and disjoint from the relative interior of C. So, there exists a hyperplane that contains $D = \{(\frac{1}{2}e^N, f(\frac{1}{2}e^N))\}$, and C is contained entirely in one of the two half-spaces of the hyperplane. We consider this hyperplane as the graph of the affine function $g : [0,1]^N \to \mathbb{R}$. Because the function f is convex, we know that the graph of the function f is in the upper half-space.

Corollary 23 *Let (N, v) be a fuzzy game that satisfies all balancedness conditions. Let f be the function defined in equation* (2)*, and let g be the hyperplane to C containing D. It holds that $g(s) \leq f(s)$ for all $s \in [0,1]^N$.*

Furthermore, we can now show that the function g is linear.

Lemma 24 *Let (N, v) be a fuzzy game that satisfies all balancedness conditions. Let f be the function defined in equation* (2)*. Let g be the affine function for which the graph is the hyperplane to C containing D. The function g is linear.*

Proof: Since g is an affine function, and $g(s) \leq f(s)$ for all $s \in [0,1]^N$, it holds that

$$\alpha g(e^N) + (1-\alpha) g(0) = g(\alpha e^N) \leq f(\alpha e^N) = \alpha f(e^N),$$

for $0 \leq \alpha \leq 1$. By choosing $\alpha = 0$, we see that $g(0) \leq 0$. By choosing $\alpha = \frac{1}{2}$, equality holds by our definition of g and thus we see that $g(e^N) + g(0) = f(e^N)$. Because we know $g(e^N) \leq f(e^N)$, it follows that $g(0) \geq 0$. Thus $g(0) = 0$, so g is linear. □

Note that it follows that

$$g(e^N) = 2g(\frac{1}{2}e^N) = 2f(\frac{1}{2}e^N) = f(e^N) = v(e^N).$$

We now show that while f is in one of the half-spaces of g, v is in the other half-space. This result might seem counterintuitive at first, because f is an upper bound for v, and one can imagine the values of v being equal to this upper bound, in which case v cannot be in the other half-space. We are able to show that g in fact does separate f and v. The intuition behind this result is that for any coalition t we have that $v(t)+v(e^N-t) \leq v(e^N)$ is a balanced inequality, and thus satisfied by v. Let a be an affine function such that $a(t) = v(t)$ and $a(\frac{1}{2}e^N) = v(\frac{1}{2}e^N)$. It follows that also $a(e^N) = v(e^N)$. Then $v(t) = a(t) \leq f(t)$, and it follows that $v(e^N - t) \leq v(e^N) - v(t) = a(e^N) - a(t) = a(e^N - t)$. Thus $v(e^N - t)$ is on the same side of the hyperplane associated with a as $v(t)$. The key in the next proof is that the function g is similar to the affine function a.

Lemma 25 *Let (N, v) be a fuzzy game that satisfies all balancedness conditions. Let f be the function defined in equation* (2)*. Let g be the affine function for which the graph is the hyperplane to C containing D. It holds that $g(s) \geq v(s)$ for all $s \in [0,1]^N$.*

Proof: Let $t \in [0,1]^N$ be such that it holds that $g(t) < v(t)$. Let $s = e^N - t$, $k = \lambda_0 = \lambda_1 = 1$ and $s_1 = t$. Then $\lambda_0 s + \lambda_1 s_1 = s + t = e^N - t + t = e^N$, so

$$\frac{v(e^N) - \lambda_1 v(s_1)}{\lambda_0} = v(e^N) - v(t) \in \mathcal{M}(s).$$

Then we find that

$$\begin{aligned} f(s) &= \inf \mathcal{M}(s) \leq v(e^N) - v(t) \\ &< v(e^N) - g(t) = g(e^N) - g(t) = g(e^N - t) \\ &= g(s). \end{aligned}$$

However, because we chose g such that $g(u) \leq f(u)$ for all $u \in [0,1]^N$, we arrive at a contradiction with Corollary 23, and thus a t such that $g(t) < v(t)$ cannot exist. □

We have shown that indeed g is a linear function that separates f and v. Furthermore, we have that $g(e^N) = v(e^N)$. Hence, g gives rise to an efficient allocation in the fuzzy core of v.

Theorem 26 *A fuzzy game (N, v) that satisfies all balancedness conditions has a non-empty fuzzy core.*

Proof: Let (N, v) be a game that satisfies all balancedness conditions. Then there exists an affine g such that $g(e^N) = v(e^N)$, and $g(s) \geq v(s)$ for all $s \in [0,1]^N$. Because g is a linear function, (N, g) is an additive game. Define $x_i = g(e^i)$ for all $i \in N$. We easily verify that

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} g(e^i) = g(e^N) = v(e^N),$$

and also that for all $s \in [0,1]^N$ it holds that

$$\sum_{i=1}^{n} s^i x_i = \sum_{i=1}^{n} s^i g(e^i) = \sum_{i=1}^{n} g(s^i e^i) = g(s) \geq v(s).$$

Hence x is in the fuzzy core of the game (N, v). □

The relationship is even stronger than stated in Theorem 26.

Theorem 27 *A fuzzy game (N, v) has a non-empty fuzzy core if, and only if, it satisfies all balancedness conditions.*

Proof: Theorem 26 already implies one part of this theorem. We only show the remaining part. Let (N, v) be a fuzzy game with a non-empty fuzzy core, and let x be in the fuzzy core of this game. Let $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k > 0$ and $s^1, \dots, s^k \in [0,1]^N$, such that

$$\sum_{j=1}^{k} \lambda_j s^j = e^N.$$

Then it holds that

$$\sum_{j=1}^{k} \lambda_j v(s^j) \leq \sum_{j=1}^{k} \lambda_j \sum_{i=1}^{n} s_i^j x_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{k} \lambda_j s_i^j = \sum_{i=1}^{n} x_i e^i = \sum_{i=1}^{n} x_i = v(e^N),$$

so the balancedness condition holds. □

In the next section we provide an example of how this result can be applied in (an extension of) deposit games.

5 Fuzzy deposit games

In this section, we illustrate the use of Theorem 27 to show that a fuzzy game has a non-empty core. For this, we extend the deposit games from Van Gulick, Borm, De Waegenaere, and Hendrickx (2010). In that paper, two ways to show that the core is non-empty are highlighted. In capital dependent deposit game a core element is constructed, while we show that term dependent deposit games satisfy all balancedness conditions. We were unable to find a core element of term dependent deposit games in general, a task that is made even harder when the fuzzy core is considered, because the fuzzy core is even more restrictive. Therefore, a proper extension of term dependent deposit games to fuzzy games is a candidate to illustrate the usefulness of Theorem 27.

In a deposit situation as analysed in Van Gulick, Borm, De Waegenaere, and Hendrickx (2010), players can combine their endowments, $m(i) \in \mathbb{R}_+$ for $i \in N$, in order to invest in deposits. Recall that N is the set of players. Capital is deposited for a fixed amount of time, called the term of the deposit:

$$T = \{t_1, t_1 + 1, \dots, t_2\}.$$

Then $\mathcal{T}$ is the set of all possible terms of a deposit between 1 and τ. Here $\tau \in \mathbb{N}$ is the final period in which a deposit can be made. The set of all possible deposits is denoted

$$\Delta = \left\{ \delta \in \mathbb{R}^{\tau+1} \,|\, \exists c > 0, T \in \mathcal{T} : \delta = c \cdot h(T) \right\},$$

where we define the function $h : \mathcal{T} \to \mathbb{R}^{\tau+1}$ as a deposit of 1 unit of capital over term $T \in \mathcal{T}$. Note that because these definitions do not depend on $m(i)$ at all, they are all still applicable to the situition where the participation level of the players is allowed to vary, *i.e.* they only invest part of their endowment. We later emphasise where we the participation level comes into play.

The revenue function is given by $P : \Delta \to \mathbb{R}_+$ is the revenue function. A deposit situation is the tuple $(N, \tau, \Delta, P, (m(i))_{i \in N})$. The set of all feasible portfolios for an endowment vector $m \in \mathbb{R}^\tau$, which is still independent of allowing different levels of participation of the players, is given by

$$\mathcal{F}(m) = \left\{ f : \Delta \to \mathbb{N} \cup \{0\} \,\middle|\, \forall t \in \{1, \ldots, \tau\} : \sum_{\delta \in \Delta} f(\delta)\delta_t = m_t + \sum_{\delta \in \Delta} f(\delta) P_t(\delta) \right\}.$$

Recall that we have

$$f + g \in \mathcal{F}(m_1 + m_2) \tag{5}$$

for any two portfolios $f \in \mathcal{F}(m_1)$ and $g \in \mathcal{F}(m_2)$. The total capital at time $\tau + 1$ as a function of a feasible portfolio $f \in \mathcal{F}(m)$ equals

$$\Pi(f) = \sum_{\delta \in \Delta} f(\delta)[P(\delta) - \delta]_{\tau+1},$$

as before. Note that

$$\Pi(f) + \Pi(g) = \Pi(f + g) \tag{6}$$

for any two portfolios $f \in \mathcal{F}(m_1)$ and $g \in \mathcal{F}(m_2)$.

In Van Gulick, Borm, De Waegenaere, and Hendrickx (2010), when considering coalitions and investigating the core of the corresponding deposit games, the authors assumed that all players had two options: either they are part of a coalition with their entire endowment, or they were not in the coalition at all. However, since the endowment is capital, and hence divisible, it makes sense to consider fractional players. In that situation, players can decide to join a coalition with only a fraction of the capital available to them. This extension has consequences for the notion of stability as well, since an allocation of the revenue is only stable if no player has an incentive to participate with less than his entire endowment. The endowment for a fuzzy coalition $s \in [0, 1]^N$ is

$$m(s) = \sum_{i \in N} s_i \cdot m(i).$$

The fact that we consider fuzzy coalitions is only reflected in the endowment. The maximal joint revenue of a coalition $s \in [0, 1]^N$ in period $\tau + 1$ is given by

$$v(s) = \sup \left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t(s) \,\middle|\, f \in \mathcal{F}(m(s)) \right\}. \tag{7}$$

This corresponds to the total capital at time $\tau + 1$ minus the capital deposited for a portfolio that is feasible for the endowment of that coalition. A fuzzy deposit game corresponding to a deposit situation $(N, \tau, \Delta, P, (m(i))_{i \in N})$ is v as defined in equation (7); a straightforward extension of the standard deposit game with reinvestment.

Because deposit games do not have a non-empty core in general, since Van Gulick, Borm, De Waegenaere, and Hendrickx (2010) provide a counterexample, the same holds for fuzzy deposit games because the fuzzy core is even more restrictive. Therefore we consider term dependent fuzzy deposit games. A fuzzy deposit game is term dependent if the revenue function is called term dependent, conform Van Gulick, Borm, De Waegenaere, and Hendrickx (2010). Note that the revenue function and its term dependence are defined in deposit situations, not (fuzzy) deposit games.

Definition 28 *A revenue function $P : \Delta \to \mathbb{R}_+^{\tau+1}$ is called* term dependent *if it holds for all $t \in \{1, \ldots, \tau+1\}$, for all deposits $\delta \in \Delta$ and for all $\alpha > 0$, that $P_t(\alpha\delta) = \alpha P_t(\delta)$. If the underlying revenue function is term dependent, also the corresponding fuzzy deposit game is called term dependent.*

We need one more result that holds for any deposit situation, irrespective of the corresponding deposit game being fuzzy or not. The next lemma from Van Gulick, Borm, De Waegenaere, and Hendrickx (2010) states that if we have a feasible portfolio for some endowments, then if we scale these endowments the portfolio that consists of the scaled deposits is feasible for the new problem, and furthermore its revenue is also scaled in the same manner..

Lemma 29 *Let $m \in \mathbb{R}^\tau$, $f \in \mathcal{F}(m)$ and $\lambda > 0$ and define $g(\lambda \cdot \delta) = f(\delta)$ for all $\delta \in \Delta$. Then $g \in \mathcal{F}(\lambda \cdot m)$ and $\Pi(g) = \lambda \cdot \Pi(f)$.*

The next result and its proof are similar to the ones for regular term depedent deposit games, now that we can apply Theorem 27.

Theorem 30 *Every term dependent fuzzy deposit game has a non-empty core.*

Proof: Let (N, v) be a fuzzy deposit game corresponding to a term dependent deposit situation $(N, \tau, \Delta, P, (m(i))_{i \in N})$. We show that (N, v) satisfies all balancedness conditions. Take $k \in \mathbb{N}$, $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$ and $s^1, s^2, \ldots, s^k \in [0,1]^N$ such that $\sum_{j=1}^k \lambda_j s^j = e^N$. Then for every $t \in \{1, \ldots, \tau\}$

$$\begin{aligned} \sum_{j=1}^k \lambda_j m_t(s^j) &= \sum_{j=1}^k \lambda_j \sum_{i \in N} s_i^j m_t(i) = \sum_{i \in N} \sum_{j=1}^k \lambda_j s_i^j m_t(i) \\ &= \sum_{i \in N} m_t(i) \sum_{j=1}^k s_i^j \lambda_j \\ &= \sum_{i \in N} m_t(i) = m_t(e^N). \end{aligned} \tag{8}$$

Consequently,

$$\sum_{j=1}^k \lambda_j v(s^j)$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} \lambda_j \sup\left\{ \Pi(f^j) - \sum_{t=1}^{\tau} m_t(s^j) \,\middle|\, f^j \in \mathcal{F}(m(s^j)) \right\} \\
&= \sup\left\{ \sum_{j=1}^{k} \lambda_j \Pi(f^j) - \sum_{j=1}^{k} \lambda_j \sum_{t=1}^{\tau} m_t(s^j) \,\middle|\, \forall j \leq k : f^j \in \mathcal{F}(m(s^j)) \right\} \\
&= \sup\left\{ \sum_{j=1}^{k} \lambda_j \Pi(f^j) - \sum_{t=1}^{\tau} \sum_{j=1}^{k} \lambda_j m_t(s^j) \,\middle|\, \forall j \leq k : f^j \in \mathcal{F}(m(s^j)) \right\}.
\end{aligned}
$$

We first use equation (8) and then Lemma 29 to the above equation, and obtain

$$
\begin{aligned}
&\sup\left\{ \sum_{j=1}^{k} \lambda_j \Pi(f^j) - \sum_{t=1}^{\tau} m_t(e^N) \,\middle|\, \forall j \leq k : f^j \in \mathcal{F}(m(s^j)) \right\} \\
&= \sup\left\{ \sum_{j=1}^{k} \Pi(f^j) - \sum_{t=1}^{\tau} m_t(e^N) \,\middle|\, \forall j \leq k : f^j \in \mathcal{F}(\lambda_j m(s^j)) \right\}.
\end{aligned}
$$

If we apply equation (6) to the last expression, followed by (5) and (8), we see that this is equal to

$$
\begin{aligned}
&\sup\left\{ \Pi(\sum_{j=1}^{k} f^j) - \sum_{t=1}^{\tau} m_t(e^N) \,\middle|\, \forall j \leq k : f^j \in \mathcal{F}(\lambda_j m(s^j)) \right\} \\
&\leq \sup\left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t(e^N) \,\middle|\, f \in \mathcal{F}\left(\sum_{j=1}^{k} \lambda_j m(s^j) \right) \right\} \\
&= \sup\left\{ \Pi(f) - \sum_{t=1}^{\tau} m_t(e^N) \,\middle|\, f \in \mathcal{F}(m(e^N)) \right\} \\
&= v(e^N).
\end{aligned}
$$

Hence by Theorem 27 the core is non-empty. □

6 Conclusions

In a standard game coalitions between players are based on full cooperation. When a player is in a coalition, he dedicates all his time and resources to that coalition. This model can be extended when we allow the cooperation between players also depends on the level of cooperation of the players. More specifically, if the level of cooperation of the players is infinitely divisible, we consider fuzzy games. In this setting, the natural extension of the core of a game is in that case the fuzzy core of the fuzzy game; no fraction of a player has any incentive to split off from the grand coalition. In general, there are two ways to show that the core of the game is non-empty: either one finds an element of the core, or one shows that all balanced inequalities are satisfied. We extend the latter approach to the fuzzy core.

We show what balanced inequalities are if we consider fuzzy games, and we show that if all balanced inequalities are satisfied, then the fuzzy game has a non-empty fuzzy core. To provide insight in the mechanism used, and in fuzzy games in general, we first consider continuous fuzzy games. In these games, a limit argument holds. We then prove the relationship for all fuzzy games. Finally, we extend the deposit games introduced in Van Gulick, Borm, De Waegenaere, and Hendrickx (2010) to fuzzy deposit games, and show that the balanced inequalities can be used to show that the fuzzy core of these games is non-empty if we assume term dependency.

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