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Immunizing conic quadratic optimization problems against implementation errors

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Abstract

We show that the robust counterpart of a convex quadratic constraint with ellipsoidal implementation error is equivalent to a system of conic quadratic constraints. To prove this result we first derive a sharper result for the S-lemma in case the two matrices involved can be simultaneously diagonalized. This extension of the S-lemma may also be useful for other purposes. We extend the result to the case in which the uncertainty region is the intersection of two convex quadratic inequalities. The robust counterpart for this case is also equivalent to a system of conic quadratic constraints. Results for convex conic quadratic constraints with implementation error are also given. We conclude with showing how the theory developed can be applied in robust linear optimization with jointly uncertain parameters and implementation errors, in sequential robust quadratic programming, in Taguchi's robust approach, and in the adjustable robust counterpart.

Keywords: Conic Quadratic Program, hidden convexity, implementation error, robust optimization, simultaneous diagonalizability, S-lemma

JEL Classification: C61

1 Introduction

Robust Optimization (RO) has become an important field in the last decade. For a comprehensive treatment of RO we refer to [3]. The number of applications of RO has increased rapidly in recent years. Moreover, the RO methodology has recently been implemented into a commercial mathematical modelling and optimization system [1]. The goal of RO is to immunize an optimization problem against uncertain parameters in the problem. Such uncertain parameters may arise as a result of estimation errors in the parameter values, or due to implementation errors. Therefore, a so-called *uncertainty region* for the uncertain parameters is defined, and then it is

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required that the constraints should hold for all parameter values in this uncertain region. For several optimization problems, and for several choices of the uncertainty region, the so-called *Robust Counterpart (RC)* can be formulated as a tractable optimization problem. For example the robust counterpart for a linear programming problem with polyhedral or ellipsoidal uncertainty regions can be reformulated as a linear programming and conic quadratic programming problem, respectively.

In this paper we extend the RO results in the following way. First, we deal with convex quadratic constraints with ellipsoidal implementation error. From the literature we know that the corresponding robust counterpart can be written as a linear matrix inequality (LMI). We show, however, that the RC can also be cast as a system of conic quadratic constraints, which is in practice much more tractable than an LMI. To prove this result we first extend the well-known S-lemma in case the two matrices of the quadratic forms are simultaneously diagonalizable (SD). We show that in this case the LMI in the S-lemma can be replaced by a much simpler condition. Second, we deal with conic quadratic constraints with ellipsoidal implementation errors. We show that the corresponding RC reduces to a system of 'nearly conic quadratic constraints'.

In practice many (conic) quadratic problems generate optimal solutions which suffer from implementation errors. This is especially the case in engineering and medical applications. For example, many cancer treatment problems associated with Intensity Modulated Radiation Therapy (IMRT) are modeled as optimization problems with linear constraints and an objective that is quadratic in the beam intensities. These intensities cannot exactly be realized in practice, and hence we have a quadratic optimization problem with implementation error. See e.g. [7]. The same holds for high-dose rate brachytherapy. Again, the problem contains linear constraints and an objective that is quadratic in the so-called dwell times. These dwell times cannot exactly be realized in practice, since the dwell times have to be multiples of say 0.1 seconds. For more details see [10]. There are also many examples of optimization problems that are not conic quadratic, but can be reformulated as such. In the paper [8] a large number of applications are mentioned that can be modelled as conic quadratic problems, many of which are proned to implementation errors. Relevant examples of this type are the logarithmic Chebychev approximation and quadratic/linear fractional problems. Design centering is another important engineering problem: Given is a set of constraints (design specifications), whose solutions may be affected by implementation errors, one would like to find a feasible point that is in the 'center' of the feasible region. More precisely, the problem is to find the maximal inscribed ellipsoid of the feasible region. The problem of design centering for convex quadratic inequalities is also treated on page 418 in [5], where the problem is reformulated as a semi-definite problem (SDP).

In this paper we also show how our results can be used in the following four classes of applications:

1. generating a tractable robust counterpart for robust linear programming with both ellipsoidal uncertainty in the parameters and the implementation error;

2. treating Taguchi robust optimization approach;

3. processing general uncertain convex nonlinear constraints with uncertainty by a sequence of robust quadratic programs in which each iteration solves a convex quadratic optimization problem under ellipsoidal implementation error;

4. extending the linear decision rules in multi-stage problems (see e.g. [3]) by adding pure quadratic terms.

The results in this paper are based on the use of hidden convexity for quadratic problems. Earlier studies of hidden convexity in seemingly nonconvex quadratic problems include [14], [2], [12], [18], [16], and [19]. In particular, problems with a nonconvex quadratic objective function and one or two constraints where studied in [19]. It was shown there that such problems, under suitable assumptions, can be cast as convex SDPs. The assumptions are, among others, the existence of Slater condition for both the primal SDP relaxation of the quadratic problem and its dual. The latter typically means that one of the quadratic forms is definite. In this paper the main goal is to avoid getting an SDP, and this is achieved under a common diagonalizability condition. Instead of an SDP we get a very simple quadratic problem equivalent to the original nonconvex problem. No Slater condition is required to obtain this result. The combination of the results in [19] and those here, then give a more complete picture of when nonconvex quadratic problems with one or two constraints are in fact equivalent to certain explicit convex problems.

The paper is organized as follows. In Section 2 we extend the S-lemma in case of simultaneous diagonalizability. In Section 3 we treat both the convex quadratic case with ellipsoidal uncertainty, and the conic quadratic case. In Section 4 we show that the results can be generalized to Globalized Robust Optimization methodology (see e.g. [3]). In Section 5 we describe four possible classes of applications. We conclude the paper with several conclusions and subjects for further research in Section 6.

2 The case of simultaneously diagonalizable quadratic forms

2.1 Simultaneous diagonalizability

In this paper the concept of simultaneous diagonalizability (SD) appears to play an important role. Therefore, we first treat this important concept, and summarize some well-known results from the literature.

Definition 1 *Real symmetric matrices A and B are called simultaneously diagonalizable (SD) if there exists a nonsingular matrix S such that both S^TAS and S^TBS are diagonal.*

This property plays an important role in the generalized eigenvalue problem. See [6] for more details on this subject. The following theorem, proved in [17], gives a sufficient condition for simultaneous diagonalizability.

Theorem 2 *Let A and B be two real symmetric matrices. Let $Q_A = \{x \mid x^TAx = 0\}$ and $Q_B = \{x \mid x^TBx = 0\}$. If*

$$Q_A \cap Q_B = \{0\} \tag{1}$$

then A and B can be simultaneously diagonalized.

Note that if one of the matrices A and B is definite, then condition (1) holds and these two matrices are SD. In the literature different methods for simultaneously diagonalizing two matrices are given. We refer again to [6] for such methods. Since in this paper we are considering the case in which one of the matrices is positive definite, we briefly describe a method for this case.

Let B be positive definite, next compute the Cholesky factorization $B = GG^T$, and then $C = G^{-1}AG^{-T}$. (Observe that computing G^{-1} is relatively easy since G is a triangular matrix.) Next use the symmetric QR algorithm to compute the Schur decomposition $Q^TCQ =$

diag $(\alpha_1, \cdots, \alpha_n)$, and finally set $S = G^{-T}Q$. It is easily verified that $S^T BS = I$ and $S^T AS =$ diag $(\alpha_1, \cdots, \alpha_n)$, i.e. A and B are both diagonalized by S. See also [9] for an efficient implementation of such a method.

2.2 Nonconvex quadratic problems with one or two constraints

We consider the following nonconvex quadratic optimization problem:

$$(P) \quad \begin{cases} \min_z & \frac{1}{2} z^T D z + e^T z \\ \text{s.t.} & \frac{1}{2} z^T A z + b^T z + c \leq 0, \end{cases}$$

where $D, A \in \mathbb{R}^{n \times n}$ are symmetric but not necessarily definite or semidefinite, $z, b, e \in \mathbb{R}^n$, and $c \in \mathbb{R}$. We assume that A and D can be simultaneously diagonalized by a nonsingular S:

$$S^T AS = \text{diag}\,(\alpha_1, \cdots, \alpha_n)$$

and

$$S^T DS = \text{diag}\,(\delta_1, \cdots, \delta_n).$$

Using the one-to-one change of variables $z = Sx$ and setting $\beta = S^T b$, $\epsilon = S^T e$, we can rewrite problem (P) as follows:

$$(P_0) \quad \begin{cases} \min_x & \sum_i \left(\frac{1}{2}\delta_i x_i^2 + \epsilon_i x_i\right) \\ \text{s.t.} & \sum_i \left(\frac{1}{2}\alpha_i x_i^2 + \beta_i x_i\right) + c \leq 0. \end{cases}$$

First we observe the following: if for some $i \in \{1, ..., n\}$ it holds that $\delta_i \leq 0$ and $\alpha_i \leq 0$, with at least one strict inequality, then

- if $\delta_i < 0$, then the minimal value of (P_0), and thus of (P), is $-\infty$.
- if $\alpha_i < 0$, then first constraint in (P_0), and thus the constraint in (P), is redundant.

Hence, in this paper we assume the following:

Assumption 3 *There does not exist an $i \in \{1, ..., n\}$ such that $\delta_i \leq 0$ and $\alpha_i \leq 0$, with at least one strict inequality.*

By setting $y_i = \frac{1}{2}x_i^2$, problem (P_0) is written equivalently as

$$(P_1) \quad \begin{cases} \min_{x,y} & \delta^T y + \epsilon^T x \\ \text{s.t.} & \alpha^T y + \beta^T x + c \leq 0 \\ & \frac{1}{2}x_i^2 - y_i = 0, \quad \forall i. \end{cases}$$

We consider the following convex relaxation of (P_1):

$$(P_2) \quad \begin{cases} \min & \delta^T y + \epsilon^T x \\ \text{s.t.} & \alpha^T y + \beta^T x + c \leq 0 \\ & \frac{1}{2}x_i^2 - y_i \leq 0, \quad \forall i. \end{cases}$$

Let (x^*, y^*) be an optimal solution of (P_2), and define

$$J := \left\{ i \ : \ \frac{1}{2}(x_i^*)^2 < y_i^*, \ i = 1, ..., n \right\}.$$

Then, if $J = \emptyset$ then x^* is an optimal solution of (P_1). The following theorem shows the equivalence of (P_1) and (P_2), i.e. it shows that if there exists an optimal solution to (P_2), then there exists an optimal solution to (P_1). The theorem also shows how to construct such a solution.

Theorem 4 *Consider the nonconvex problem (P) and its equivalent problem (P_1). Let assumption 3 hold. If (x^*, y^*) is an optimal solution of the convex quadratic problem (P_2), then $(\bar{x}, \bar{y})$ is an optimal solution of problem (P_1), where*

$$\bar{x}_i = \begin{cases} x_i^* & \text{if} \quad (i \notin J) \vee [(i \in J) \wedge (\alpha_i = 0) \wedge (\beta_i \neq 0)] \\ \sqrt{2y_i^*} & \text{if} \quad (i \in J) \wedge (\alpha_i = \beta_i = \epsilon_i = 0) \\ \frac{1}{\alpha_i}\left(-\beta_i \pm \sqrt{\beta_i^2 + 2\alpha_i \theta_i^*}\right) & \text{if} \quad (i \in J) \wedge (\alpha_i \neq 0), \end{cases}$$

and

$$\bar{y}_i = \frac{1}{2}\bar{x}_i^2$$

where

$$\theta_i^* = \alpha_i y_i^* + \beta_i x_i^*.$$

The optimal solution of problem (P) is then $z = S\bar{x}$.

Proof: For $i \notin J$ we have $\bar{x}_i = x_i^*$ and $\bar{y}_i = y_i^*$. For $i \in J$ we distinguish three cases:

- **Case I:** $\alpha_i = \beta_i = \epsilon_i = 0$. For this case it is easy to see that $\bar{y}_i = y_i^*$ and $\bar{x}_i = \sqrt{2y_i^*}$ is optimal to (P_1).

- **Case II:** $\alpha_i = \beta_i = 0,\ \epsilon_i \neq 0$. For this case we prove that $i \notin J$, and we set $\bar{x}_i = x_i^*$ and $\bar{y}_i = y_i^*$. To prove this, we choose

$$\tilde{x}_i = \begin{cases} +\sqrt{2y_i^*} & \text{if} \quad \epsilon_i < 0 \\ -\sqrt{2y_i^*} & \text{if} \quad \epsilon_i > 0, \end{cases}$$

  and $\tilde{y}_i = y_i^*$. Then, $(\tilde{x}, \tilde{y})$ is a feasible solution, and

$$\epsilon_i \tilde{x}_i = -|\epsilon_i||\tilde{x}_i| < \epsilon_i |x_i^*|,$$

  which implies that $(\tilde{x}, \tilde{y})$ is strictly better than (x^*, y^*), which is a contradiction.

- **Case III:** α_i and β_i are not both zero. Since (x^*, y^*) is optimal for (P_2), it must satisfy the Fritz-John conditions:

$$\exists \mu_0 \geq 0, u \geq 0, \mu_i \geq 0, i = 1, ..., n, \text{ not all zero, such that } \forall i = 1, ..., n : \tag{2}$$

$$\mu_0 \delta_i + u\alpha_i - \mu_i = 0 \tag{3}$$
$$\mu_0 \epsilon_i + u\beta_i + \mu_i x_i = 0 \tag{4}$$

  We proceed to show that $\mu_0 > 0$. Assume $\mu_0 = 0$, then, for $i \in J$, the Fritz-John conditions reduces to $u\alpha_i = 0$ and $u\beta_i = 0$, implying $u = 0$, and then by (3) that $\mu_i = 0, i = 1, ..., n$. This contradicts (2).

Hence, we have $\mu_0 > 0$, and the Fritz-John conditions become the KKT conditions, which are then necessary and sufficient to optimality of (x^*, y^*). The KKT conditions for $i \in J$ are

$$\delta_i + u\alpha_i = 0 \tag{5}$$
$$\epsilon_i + u\beta_i = 0. \tag{6}$$

Now we distinguish two subcases:

- **Case IIIa:** $\alpha_i = 0$. For this case we have by (5) that $\delta_i = 0$ and then $\bar{x}_i = x_i^*$, $\bar{y}_i = \frac{1}{2}\bar{x}_i^2$ is an optimal solution of (P_1).
- **Case IIIb:** $\alpha_i \neq 0$. Set $\theta_i^* = \alpha_i y_i^* + \beta_i x_i^*$ and choose $\bar{x}_i$ and $\bar{y}_i$ as follows

  $$\bar{y}_i = \frac{1}{2}\bar{x}_i^2,$$

  and $\bar{x}_i$ is a solution of the quadratic equation

  $$\frac{1}{2}\alpha_i \bar{x}_i^2 + \beta_i \bar{x}_i = \theta_i^*. \tag{7}$$

  This means that

  $$\bar{x}_i = \begin{cases} x_i^* & \text{if} \quad \alpha_i = 0 \\ \frac{1}{\alpha_i}\left(-\beta_i \pm \sqrt{\Delta_i^*}\right) & \text{if} \quad \alpha_i \neq 0, \end{cases}$$

  provided of course that $\Delta_i^* = \beta_i^2 + 2\alpha_i\theta_i^*$ is nonnegative. Indeed

  $$\begin{aligned} \Delta_i^* &= \beta_i^2 + 2\alpha_i\theta_i^* \\ &= \beta_i^2 + 2\alpha_i(\alpha_i y_i^* + \beta_i x_i^*) \\ &= (\beta_i + \alpha_i x_i^*)^2 + 2\alpha_i^2\left(y_i^* - \frac{1}{2}(x_i^*)^2\right) \\ &> 0, \end{aligned}$$

  where the last inequality follows since $y_i^* > \frac{1}{2}(x_i^*)^2$. We have shown that the vectors $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^n$ chosen as:

  $$\bar{x}_i = \begin{cases} x_i^* & \text{if} \quad i \notin J \\ \frac{1}{\alpha_i}\left(-\beta_i \pm \sqrt{\Delta_i^*}\right) & \text{if} \quad i \in J, \end{cases}$$

  and

  $$\bar{y}_i = \frac{1}{2}\bar{x}_i^2,$$

  are a feasible pair for problem (P_0).

  We now show that $(\bar{x}, \bar{y})$ achieves the same optimal value as (x^*, y^*). For $i \notin J$ we have $\bar{x}_i = x_i^*$ and $\bar{y}_i = y_i^*$, so

  $$\delta_i y_i^* + \epsilon_i x_i^* = \delta_i \bar{y}_i + \epsilon_i \bar{x}_i. \tag{8}$$

  For $i \in J$, we have since $\mu_i = 0$, and using (5) and (6):

  $$\begin{aligned} \delta_i \bar{y}_i + \epsilon_i \bar{x}_i &= (-u\alpha_i)\bar{y}_i + (-u\beta_i)\bar{x}_i \\ &= -u(\alpha_i \bar{y}_i + \beta_i \bar{x}_i) \\ &= -u(\frac{1}{2}\alpha_i \bar{x}_i^2 + \beta_i \bar{x}_i) \\ &= -u\theta_i^*, \end{aligned}$$

where the last equality follows from (7). On the other hand, again by (5) and (6), we have $\delta_i y_i^* + \epsilon_i x_i^* = -u^*\theta_i^*$. Hence, this verifies (8) for $i \in J$. This shows that $(\bar{x}, \bar{y})$ achieves the same optimal value as (x^*, y^*). Hence, $(\bar{x}, \bar{y})$ is an optimal solution for (P_1) and hence for (P_0).

$\square$

The final result is that even a nonconvex quadratic problem (P) can be solved by solving a convex quadratic optimization problem (P_2). We illustrate this by the following simple example.

Example. We consider the following nonconvex quadratic optimization problem:

$$\begin{aligned} \min \;& -\frac{1}{2}z_1^2 - \frac{1}{2}z_2^2 - z_2 \\ \text{s.t.} \;& z_1^2 + \frac{1}{2}z_2^2 + z_2 \leq 1. \end{aligned}$$

This problem is already in diagonal form, and hence the corresponding problem (P_2) is:

$$\begin{aligned} \min \;& -y_1 - y_2 - x_2 \\ \text{s.t.} \;& 2y_1 + y_2 + x_2 \leq 1 \\ & \frac{1}{2}x_1^2 - y_1 \leq 0 \\ & \frac{1}{2}x_2^2 - y_2 \leq 0. \end{aligned}$$

It can easily be checked that the optimal objective value of this problem is -1, and the optimal solution $x_1^* = 0, x_2^* = 0, y_1^* = 0, y_2^* = 1$, and the KKT mupliers $u = 1, \mu_1 = 1, \mu_2 = 0$. This solution clearly does not satisfy $y_i^* = \frac{1}{2}(x_i^*)^2$. However, such a solution is given by $\bar{x}_1 = 0, \bar{x}_2 = -1 \pm \sqrt{3}, \bar{y}_1 = 0, \bar{y}_2 = 2 \mp \sqrt{3}$, with objective value -1. $\square$

Problem (P_2) has a simple dual problem.

Theorem 5 *Assume there exists a strictly feasible solution to (P_2). Then, the objective values of (P_2) and the following dual problem are equal:*

$$(D_2) \quad \begin{cases} max_{v\in\mathbb{R}} \; -\sum_i \frac{(v\beta_i+\epsilon_i)^2}{2(\delta_i+v\alpha_i)} + cv \\ \quad s.t. \qquad \delta_i + v\alpha_i \geq 0, \quad \forall i \\ \qquad\qquad\qquad v \geq 0. \end{cases}$$

Proof: We compute the Lagrange dual of (P_2):

$$\begin{aligned} L(x, y, u, v) &= \delta^T y + \epsilon^T x + \sum_i u_i(\frac{1}{2}x_i^2 - y_i) + v(\alpha^T y + \beta^T x + c) \\ &= \sum_i y_i(\delta_i - u_i + v\alpha_i) + \sum_i x_i(\epsilon_i + \frac{1}{2}u_i x_i + v\beta_i) + vc. \end{aligned}$$

It can easily be verified that the dual objective is

$$g(v, w) = \begin{cases} -\sum_i \frac{(v\beta_i+\epsilon_i)^2}{2u_i} + vc & \text{if} \quad \delta_i - u_i + v\alpha_i = 0, \ \forall i \\ -\infty & \text{elsewhere.} \end{cases}$$

Hence, we obtain that (P_2) is equivalent to the following problem:

$$\begin{cases} \max - \sum_i \frac{(v\beta_i+\epsilon_i)^2}{2u_i} + vc \\ \delta_i - u_i + v\alpha_i = 0, \quad \forall i \\ u \geq 0, v \geq 0. \end{cases}$$

By eliminating u_i, we obtain problem (D_2). □

Problem (D_2) is a very simple problem: only one variable and the objective is concave. Note that problem (D_2) can be cast as a conic quadratic one, which is done in Section 3 to obtain the RC for quadratic constraints affected by ellipsoidal implementation error.

Theorem 4 can be extended to the case that there are two quadratic constraints. We consider the following optimization problem:

$$(\bar{P}) \quad \begin{cases} \min & \frac{1}{2}z^T Dz + e^T z \\ \text{s.t.} & \frac{1}{2}z^T Az + b^T z + c \leq 0 \\ & \frac{1}{2}z^T Gz + h^T z + k \leq 0, \end{cases}$$

where $D, A, G \in \mathbb{R}^{n\times n}$ are symmetric, $z, b, e, h \in \mathbb{R}^n$, and $c, k \in \mathbb{R}$. We assume that A, D and G can be simultaneously diagonalized: $\exists$ nonsingular S such that

$$S^T AS = \text{diag}\,(\alpha_1, \cdots, \alpha_n),\ \ S^T DS = \text{diag}\,(\delta_1, \cdots, \delta_n),\ \ S^T GS = \text{diag}\,(\eta_1, \cdots, \eta_n).$$

Important cases that satisfy this simultaneous diagonalizability are the following cases:

1. The second constraint is linear (i.e. G is the zero matrix) and matrices D and A are SD. Note that an optimization problem with a nonconvex quadratic objective function and one conic quadratic constraint
$$\|Ax - b\| \leq c^T x - d$$
can be reformulated to such a problem with one nonconvex quadratic constraint and one linear constraint:
$$\begin{cases} \|Ax - b\|^2 - (c^T x - d)^2 \leq 0 \\ c^T x - d \geq 0. \end{cases} \tag{9}$$

2. The two constraints originate from a single two-sided quadratic constraint; in this case $G = -A$.

3. Matrix G is the identity matrix, and matrices A and D commute. This case may happen in robust optimization when using ball uncertainty regions.

4. The three matrices A, D and G commute.

Using the change of variables $z = Sx$ and change of parameters $\beta = S^T b$, $\epsilon = S^T e$, $\theta = S^T h$, we can rewrite problem $(\bar{P})$ as follows:

$$(\bar{P}_0) \quad \begin{cases} \min & \sum_i \left(\frac{1}{2}\delta_i x_i^2 + \epsilon_i x_i\right) \\ \text{s.t.} & \sum_i \left(\alpha_i x_i^2 + \beta_i x_i\right) + c \leq 0 \\ & \sum_i \left(\eta_i x_i^2 + \theta_i x_i\right) + k \leq 0 \end{cases}$$

We will assume the following (cf. Assumption 3):

Assumption 6 *There does not exists an $i \in \{1, ..., n\}$ such that $\delta_i \leq 0$, $\alpha_i \leq 0$, and $\eta_i \leq 0$, with at least one strict inequality.*

By setting $y_i = \frac{1}{2}x_i^2$ problem (P_0) is written equivalently as

$$(\bar{P}_1) \quad \begin{cases} \min & \delta^T y + \epsilon^T x \\ \text{s.t.} & \alpha^T y + \beta^T x + c \leq 0 \\ & \eta^T y + \theta^T x + k \leq 0 \\ & \frac{1}{2}x_i^2 - y_i = 0, \quad \forall i. \end{cases}$$

We consider the following convex relaxation of $(\bar{P}_1)$:

$$(\bar{P}_2) \quad \begin{cases} \min & \delta^T y + \epsilon^T x \\ \text{s.t.} & \alpha^T y + \beta^T x + c \leq 0 \\ & \eta^T y + \theta^T x + k \leq 0 \\ & \frac{1}{2}x_i^2 - y_i \leq 0, \quad \forall i. \end{cases} \tag{10}$$

For the case of two constraints we also need the following assumption:

Assumption 7 *Let x^* be an optimal solution of $(\bar{P}_2)$, and $u^* \geq 0$, $\lambda^* > 0$, and $\mu_i^* \geq 0$ be the corresponding KKT multipliers of the constraints of $(\bar{P}_2)$. Then $\lambda^* = 0$ or $u^* = 0$. Henceforth, we assume $\lambda^* = 0$.*

For the case of quadratic objective and conic quadratic constraint mentioned earlier (see (9)) it can easily be seen that Assumption 7 holds if

$$\begin{pmatrix} b \\ d \end{pmatrix} \notin R\begin{pmatrix} A \\ c^T \end{pmatrix}.$$

Let (x^*, y^*) be an optimal solution of (P_2), and define, as in the case of one constraint,

$$J := \left\{ i \ : \ \frac{1}{2}(x_i^*)^2 < y_i^*, \ i = 1, ..., n \right\}.$$

Then, if $J = \emptyset$ then x^* is an optimal solution of (P_1). The following theorem shows the equivalence of $(\bar{P}_1)$ and $(\bar{P}_2)$, i.e. it shows that if there exists an optimal solution to $(\bar{P}_2)$, then there exists an optimal solution to $(\bar{P}_1)$. The theorem also shows how to construct such a solution.

Theorem 8 *Consider the nonconvex problem $(\bar{P})$ and its equivalent problem $(\bar{P}_1)$. Let assumption 6 and 7 hold. If (x^*, y^*) is an optimal solution of the convex quadratic problem $(\bar{P}_2)$, then $(\bar{x}, \bar{y})$ is an optimal solution of problem $(\bar{P}_1)$, where*

$$\bar{x}_i = \begin{cases} x_i^* & \text{if} \quad (i \notin J) \vee [(i \in J) \wedge (\alpha_i = \beta_i = \delta_i = 0) \wedge (\epsilon_i \neq 0)] \\ \sqrt{2y_1^*} & \text{if} \quad (i \in J) \wedge (\alpha_i = \beta_i = \delta_i = \epsilon_i = 0) \wedge (\theta_i \leq 0) \\ -\sqrt{2y_1^*} & \text{if} \quad (i \in J) \wedge (\alpha_i = \beta_i = \delta_i = \epsilon_i = 0) \wedge (\theta_i > 0) \\ \max(\tilde{x}_i^+, \tilde{x}_i^-) & \text{if} \quad (i \in J) \wedge (\alpha_i = \beta_i = 0) \wedge (\delta_i \neq 0) \wedge (\frac{\eta_i \epsilon_i}{\delta_i} \geq \theta_i) \\ \min(\tilde{x}_i^+, \tilde{x}_i^-) & \text{if} \quad (i \in J) \wedge (\alpha_i = \beta_i = 0) \wedge (\delta_i \neq 0) \wedge (\frac{\eta_i \epsilon_i}{\delta_i} < \theta_i) \\ \max(\bar{x}_i^+, \bar{x}_i^-) & \text{if} \quad (i \in J) \wedge (\alpha_i \neq 0) \wedge (\frac{\eta_i \beta_i}{\alpha_i} \geq \theta_i) \\ \min(\bar{x}_i^+, \bar{x}_i^-) & \text{if} \quad (i \in J) \wedge (\alpha_i \neq 0) \wedge (\frac{\eta_i \beta_i}{\alpha_i} < \theta_i) \end{cases}$$

and

$$\bar{y}_i = \frac{1}{2}\bar{x}_i^2,$$

where

$$\begin{aligned}
\tilde{x}_i^+ &= \frac{1}{\delta_i}\left(-\epsilon_i + \sqrt{2\delta_i\sigma_i^*}\right),\\
\tilde{x}_i^- &= \frac{1}{\delta_i}\left(-\epsilon_i - \sqrt{2\delta_i\sigma_i^*}\right),\\
\bar{x}_i^+ &= \frac{1}{\alpha_i}\left(-\beta_i + \sqrt{\beta_i^2 + 2\alpha_i\theta_i^*}\right),\\
\bar{x}_i^- &= \frac{1}{\alpha_i}\left(-\beta_i - \sqrt{\beta_i^2 + 2\alpha_i\theta_i^*}\right),
\end{aligned}$$

and

$$\begin{aligned}
\sigma_i^* &= \delta_i y_i^* + \epsilon_i x_i^*,\\
\theta_i^* &= \alpha_i y_i^* + \beta_i x_i^*.
\end{aligned}$$

The optimal solution of problem $(\bar{P})$ *is then* $z = S\bar{x}$*.*

Proof: For $i \notin J$ we have $\bar{x}_i = x_i^*$ and $\bar{y}_i = y_i^*$. For $i \in J$ we distinguish the following cases:

- **Case I:** $\alpha_i = \beta_i = 0$ and $\delta_i = \epsilon_i = 0$. For this case it is easy to see that $\bar{y}_i = y_i^*$ and

  $$\bar{x}_i = \begin{cases} +\sqrt{2\bar{y}_i} & \text{if} \quad \theta_i \le 0 \\ -\sqrt{2\bar{y}_i} & \text{if} \quad \theta_i > 0, \end{cases}$$

  is an optimal solution of $(\bar{P}_1)$.

- **Case II:** α_i and β_i are not both zero. It can easily be shown by using Assumption 7 that the Fritz-John conditions reduce to KKT conditions.

- **Case IIa:** $\alpha_i = 0$, $\beta_i \neq 0$. Using the KKT conditions it can easily be shown that $\delta_i = 0$, and hence, by assumption 6, $\eta_i > 0$. Hence, this case is basically the same as Case IIIb.

- **Case IIb:** $\alpha_i \neq 0$. The first part of the proof is the same as the proof for Case IIIb of Theorem 4. We now only have to show that $(\bar{x}, \bar{y})$ also satisfies the second constraint of problem $(\bar{P}_2)$:

  $$\eta^T y + \theta^T x + k \le 0. \tag{11}$$

  First, we prove a needed result for the two roots $\frac{1}{\alpha_i}\left(-\beta_i \pm \sqrt{\beta_i^2 + 2\alpha_i\theta_i^*}\right)$, denoted by $\bar{x}_i^+$ and $\bar{x}_i^-$. We claim that

  $$\min(\bar{x}_i^+, \bar{x}_i^-) \le x_i^* \le \max(\bar{x}_i^+, \bar{x}_i^-). \tag{12}$$

  To prove this we define

  $$\tau(x) = \frac{1}{2}\alpha_i x^2 + \beta_i x - (\alpha_i y_i^* + \beta_i x_i^*).$$

  Let us first assume that $\alpha_i > 0$. Then, it is easy to verify that $\tau(\bar{x}_i^-) = \tau(\bar{x}_i^+) = 0$ and that $\tau(x)$ is convex. Moreover,

  $$\tau(x_i^*) = \alpha_i\left(\frac{1}{2}(x_i^*)^2 - y_i^*\right) < 0.$$

Hence, $\bar{x}_i^- \leq x_i^* \leq \bar{x}_i^+$. Let us now assume that $\alpha_i < 0$. Then, it is easy to verify that $\tau(\bar{x}_i^-) = \tau(\bar{x}_i^+) = 0$ and that $\tau(x)$ is concave. Moreover,

$$\tau(x_i^*) = \alpha_i \left(\frac{1}{2}(x_i^*)^2 - y_i^* \right) > 0.$$

Hence, $\bar{x}_i^- \leq x_i^* \leq \bar{x}_i^+$. This proves (12).

Now we prove that $(\bar{x}, \bar{y})$ also satisfies (10). For $i \notin J$ we have $\bar{x}_i = x_i^*$ and $\bar{y}_i = \frac{1}{2}\bar{x}_i^2$ and hence

$$\eta_i \bar{y}_i + \theta_i \bar{x}_i \leq \eta_i y_i^* + \theta_i x_i^*.$$

We are left to show that $\forall i \in J$, we have that $\bar{x}_i$, and $\bar{y}_i = \frac{1}{2}\bar{x}_i^2$ satisfy

$$\eta_i \bar{y}_i + \theta_i \bar{x}_i < \eta_i y_i^* + \theta_i x_i^*,$$

where $\bar{x}_i$ is one of the two roots $\bar{x}_i^+$ and $\bar{x}_i^-$. Indeed

$$\begin{aligned}
\eta_i \bar{y}_i + \theta_i \bar{x}_i &= \frac{1}{2}\eta_i \bar{x}_i^2 + \theta_i \bar{x}_i \\
&= \eta_i \left(y_i^* + \frac{\beta_i}{\alpha_i} x_i^* - \frac{\beta_i}{\alpha_i} \bar{x}_i \right) + \theta_i \bar{x}_i \\
&= \eta_i y_i^* + \theta_i x_i^* + \eta_i \frac{\beta_i}{\alpha_i}(x_i^* - \bar{x}_i) + \theta_i(\bar{x}_i - x_i^*) \\
&= \eta_i y_i^* + \theta_i x_i^* + \left(\eta_i \frac{\beta_i}{\alpha_i} - \theta_i \right)(x_i^* - \bar{x}_i) \\
&\leq \eta_i y_i^* + \theta_i x_i^*.
\end{aligned}$$

To prove the last inequality it remains to show that by choosing $\bar{x}_i$ one of the two roots x_i^+ and x_i^- we have

$$\left(\eta_i \frac{\beta_i}{\alpha_i} - \theta_i \right)(x_i^* - \bar{x}_i) \leq 0.$$

Using (12) it is easy to verify that the right choice for $\forall i \in J, \alpha_i \neq 0$, is

$$\bar{x}_i = \begin{cases} \max(\bar{x}_i^+, \bar{x}_i^-) & \text{if} \quad \eta_i \frac{\beta_i}{\alpha_i} \geq \theta_i \\ \min(\bar{x}_i^+, \bar{x}_i^-) & \text{if} \quad \eta_i \frac{\beta_i}{\alpha_i} < \theta_i. \end{cases}$$

Finally, similar as in Case IIIb of the proof of Theorem 4 it can be shown that $(\bar{x}, \bar{y})$ achieves the same optimal value as (x^*, y^*). Hence, $(\bar{x}, \bar{y})$ is an optimal solution for $(\bar{P}_1)$ and thus for $(\bar{P})$.

- **Case IIIa:** $\alpha_i = \beta_i = 0, \delta_i \neq 0$. This case is similar as Case IIb, and the proof easily follows from that case by working with the objective function instead of the first constraint.

- **Case IIIb:** $\alpha_i = \beta_i = 0,\ \delta_i = 0,\ \epsilon_i \neq 0$. Then by Assumption 6 we have $\eta_i > 0$, and then we choose $\bar{x}_i = x_i^*$ and $\bar{y}_i = \frac{1}{2}\bar{x}_i^2$. This solution has the same objective value as (x^*, y^*) and also the same value for the first constraint, and is moreover also feasible for the second constraint since $\eta_i \bar{y}_i < \eta_i y_i^*$. Hence this solution is also optimal for $(\bar{P}_1)$.

□

Assumption 7 is necessary for the validity of Theorem 8, which is demonstrated by the following example.

Example. We consider the following nonconvex quadratic optimization problem with two constraints:

$$\begin{aligned}
\min \;& -\frac{1}{2}z_1^2 - \frac{1}{2}z_2^2 - 2z_2 \\
\text{s.t. } & z_1^2 + \frac{1}{2}z_2^2 + z_2 \le 2 \\
& z_1 + z_2 \le -1.
\end{aligned}$$

This problem is already in diagonal form, and hence the corresponding problem $(\bar{P}_2)$ is:

$$\begin{aligned}
\min \;& -y_1 - y_2 - 2x_2 \\
\text{s.t. } & 2y_1 + y_2 + x_2 \le 2 \\
& x_1 + x_2 \le -1 \\
& \frac{1}{2}x_1^2 - y_1 \le 0 \\
& \frac{1}{2}x_2^2 - y_2 \le 0.
\end{aligned}$$

It can easily be checked that the unique optimal objective value of this problem is $-\frac{3}{2}$, and the optimal solution $x_1^* = -1, x_2^* = 0, y_1^* = \frac{1}{2}, y_2^* = 1$, and the KKT mupliers $u = 1, \lambda = 1, \mu_1 = 1, \mu_2 = 0$. This solution clearly does not satisfy $y_i^* = \frac{1}{2}(x_i^*)^2$, hence $J = \{2\}$. However, such a solution is given by $\bar{x}_1 = -1, \bar{x}_2 = -1-\sqrt{3}, \bar{y}_1 = \frac{1}{2}, \bar{y}_2 = 2+\sqrt{3}$, with objective value $-\frac{1}{2}+\sqrt{3}$. The objective values are not equal, which is caused by the fact that both u and λ are strictly positive, and hence assumption 7 is not satisfied. □

The dual problem of $(\bar{P}_2)$ is a 2-variable convex problem, given in the following theorem.

Theorem 9 *Assume that assumptions 6 and 7 are satisfied. Moreover, assume that there exists a strictly feasible solution to $(\bar{P}_2)$. Then, the objective values of $(\bar{P}_2)$ and the following dual problem are equal:*

$$(\bar{D}_2) \quad \begin{cases} max_{v_1,v_2\in\mathbb{R}} \; -\sum_i \frac{(v_1\beta_i+v_2\theta_i+\epsilon_i)^2}{2(\delta_i+v_1\alpha_i+v_2\eta_i)} + cv \\ s.t. \quad \delta_i + v_1\alpha_i + v_2\eta_i \ge 0, \quad \forall i \\ \qquad v_1, v_2 \ge 0. \end{cases}$$

Proof: Similar as the proof of Theorem 5. □

To conclude, the (probably nonconvex) quadratic problem $(\bar{P})$ satisfying the assumptions of Theorem 8 can be solved by solving a convex quadratic optimization problem $(\bar{P}_2)$, or by its dual $(\bar{D}_2)$.

2.3 Refining the S-lemma

We start with the inhomogeneous version of the fundamental S-lemma. For an excellent overview on the S-lemma we refer to [13].

Lemma 10 *Let A, D be symmetric matrices of the same size, and let the quadratic form $z^TAz + 2b^Tz + c$ be strictly positive at some point. Then the implication*

$$z^TAz + 2b^Tz + c \ge 0 \Rightarrow z^TDz + 2e^Tz + f \ge 0$$

holds true if and only if

$$\exists \lambda \geq 0 : \left(\begin{array}{c|c} D - \lambda A & e - \lambda b \\ \hline e^T - \lambda b^T & f - \lambda c \end{array} \right) \succeq 0. \tag{13}$$

□

Before we state a sharpened version of this lemma in case the matrices A and D are SD, we show that the Slater conditions in Lemma 5 and Lemma 10 are equivalent.

Lemma 11 *Suppose that the nonsingular matrix S diagonalizes A, i.e. $S^T AS = diag(\alpha_1, ..., \alpha_n)$. Then, the condition*

$$\exists z : z^T Az + 2b^T z + c > 0 \tag{14}$$

is equivalent to

$$\exists x, y : \begin{cases} \alpha^T y + 2\beta^T x + c > 0 \\ x_i^2 - y_i < 0, \end{cases} \tag{15}$$

in which $\beta = S^T b$.

Proof: First we show that if (14) holds then (15) holds. Suppose that $\bar{z}$ satisfies $\bar{z}^T A\bar{z} + 2\bar{b}^T z + c > 0$. We substitute $\bar{z} = S\bar{x}$ into (14) and obtain

$$\sigma := \sum_i \alpha_i \bar{x}_i^2 + 2\beta^T \bar{x} + c > 0. \tag{16}$$

We define $\bar{y}_i = \bar{x}_i^2 + \frac{\sigma}{2\|\alpha\|_1}$. Evidently we have $\bar{y}_i > \bar{x}_i^2$. Moreover, using this expression for $\bar{y}$ gives

$$\begin{aligned} \sum_i \alpha_i \bar{y}_i + 2\beta^T \bar{x} + c &= \sum_i \alpha_i \bar{x}_i^2 + \sum_i \alpha_i \frac{\sigma}{2\|\alpha\|_1} + 2\beta^T \bar{x} + c \\ &\geq \sum_i \alpha_i \bar{x}_i^2 - \frac{1}{2}\sigma + 2\beta^T \bar{x} + c \\ &> 0, \end{aligned}$$

where the last inequality follows from (16). Hence, $\bar{x}$ and $\bar{y}$ satisfy (15).

Now we prove that if (15) holds then (14) holds. First of all, we may assume that all eigenvalues of A are nonpositive. This is true, since if there exists a positive eigenvalue of A then the corresponding eigenvector will satisfy (14). It is well-known that a congruence transformation preserves the signs of the original matrix, and hence we have that $\alpha_i \leq 0, \forall i$. Suppose now that $(\bar{x}, \bar{y})$ satisfies (15), i.e.

$$\begin{cases} \alpha^T \bar{y} + 2\beta^T \bar{x} + c > 0 \\ \bar{x}_i^2 - \bar{y}_i < 0. \end{cases}$$

We can write

$$\bar{y}_i = \bar{x}_i^2 + \theta_i,$$

where $\theta_i > 0$. Hence, we have

$$\sum_i \alpha_i \bar{x}_i^2 + \sum_i \alpha_i \theta_i + 2\beta^T \bar{x} + c > 0.$$

By substituting $\bar{z} = S\bar{x}$ we obtain

$$\bar{z}^T A\bar{z} + 2b^T\bar{z} + c > -\sum_i \alpha_i\theta_i \geq 0.$$

Hence, $\bar{z}$ satisfies (15). $\square$

In case that the matrices A and D are SD we can sharpen the S-lemma, i.e., the LMI can be replaced by a simple convex constraint.

Lemma 12 *Let A, D be symmetric matrices of the same size and SD into* $\operatorname{diag}(\alpha_1, ..., \alpha_n)$ *and* $\operatorname{diag}(\delta_1, ..., \delta_n)$*, respectively. Let the quadratic form $z^T Az + 2b^T z + c$ be strictly positive at some point. Then the implication*

$$z^T Az + 2b^T z + c \geq 0 \Rightarrow z^T Dz + 2e^T z + f \geq 0$$

holds true if and only if there exist w and $v \in \mathbb{R}$ such that

$$\begin{cases} -\sum_i \frac{(v\beta_i - \epsilon_i)^2}{\delta_i - v\alpha_i} - cv + f \geq 0 \\ \delta_i - v\alpha_i \geq 0, \quad \forall i \\ v \geq 0, \end{cases}$$

in which $\beta = S^T b$ and $\epsilon = S^T e$.

Proof: The implication

$$z^T Az + 2b^T z + c \geq 0 \Rightarrow z^T Dz + 2e^T z + f \geq 0$$

holds true if and only if the optimal value of the following optimization problem is nonnegative:

$$\begin{aligned} \min \quad & \frac{1}{2} z^T Dz + e^T z + \frac{1}{2} f \\ \text{s.t.} \quad & \frac{1}{2} z^T Az + b^T z + \frac{1}{2} c \geq 0. \end{aligned}$$

Using Theorem 4 we immediately obtain the result of the theorem.

Another proof is by starting from the inhomogeneous S-lemma (Lemma 10). Since a congruence transformation preserves all signs of the eigenvalues we have that the LMI (13) is equivalent to:

$$\left(\begin{array}{c|c} S & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{c|c} D - \lambda A & e - \lambda b \\ \hline e^T - \lambda b^T & f - \lambda c \end{array}\right) \left(\begin{array}{c|c} S & 0 \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} \operatorname{diag}(\delta_1 - \lambda\alpha_1, ..., \delta_n - \lambda\alpha_n) & \epsilon - \lambda\beta \\ \hline (\epsilon - \lambda\beta)^T & f - \lambda c \end{array}\right) \succeq 0. \quad (17)$$

The theorem follows easily by using the Schur complement for the most right matrix in (17). $\square$

In the next section we show that this result enables us to prove that the robust counterpart of a quadratic constraint with ellipsoidal implementation error is equivalent to a system of conic quadratic constraints instead of LMIs. The above lemma, however, may also be used in other cases where the S-lemma is used and the matrices are SD, to get a system of conic quadratic constraints instead of LMIs. Note that, although both conic quadratic programming and SDP can be solved in polynomial time by interior point methods, in practice SDPs are much more difficult to solve. Another advantage of our analysis is that it provides an explicit way to extract the solution for the original optimization problem (P) contrary to the S-lemma. Moreover, our analysis leads to the generalization of an optimization problem with two quadratic constraints (problem $(\bar{P})$) and even much more general problems, which will be treated in a forthcoming paper. Finally, our analysis sharpens the above lemma in the sense that the Slater condition may be dropped, which is stated in the following lemma.

Lemma 13 *Let A, D be symmetric matrices of the same size and SD by the matrix S into* $\operatorname{diag}(\alpha_1, ..., \alpha_n)$ *and* $\operatorname{diag}(\delta_1, ..., \delta_n)$*, respectively. Then the implication*

$$z^T Az + 2b^T z + c \geq 0 \Rightarrow z^T Dz + 2e^T z + f \geq 0$$

holds true if and only if there exist $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ such that

$$\begin{cases} \delta^T y + 2\epsilon^T x + f \geq 0 \\ \alpha^T y + 2\beta^T x + c \geq 0 \\ x_i^2 - y_i \leq 0, \quad \forall i, \end{cases}$$

in which $\beta = S^T b$ and $\epsilon = S^T e$.

Proof: The proof is essentially the same as the proof of Theorem 4. □

Finally, we note that by using Theorem 9 the S-lemma can also be generalized to the case of three quadratic forms.

3 (Conic) quadratic constraint with implementation errors

3.1 Quadratic constraint

We start with a convex quadratic constraint that is affected by ellipsoidal implementation error:

$$(x + a)^T D(x + a) + 2e^T (x + a) \leq f \quad \forall\, a : a^T Aa \leq \rho^2, \tag{18}$$

in which $a \in \mathbb{R}^n$ is the additive implementation error, $x, e \in \mathbb{R}^n$, $A, D \in \mathbb{R}^{n\times n}$, and $\rho, f \in \mathbb{R}$. We assume that A is positive definite. By setting

$$d(x) = Dx + e$$

and

$$\gamma(x) = f - (x^T Dx + 2e^T x)$$

we get that (18) is equivalent to

$$a^T Da + 2a^T d \leq \gamma \quad \forall\, a : a^T Aa \leq \rho^2. \tag{19}$$

Note that $d = d(x)$ and $\gamma = \gamma(x)$ only depend on x and not on a. Moreover, $d(x)$ is linear in x, and $\gamma(x)$ is concave in x if $D \succeq 0$.

We first note that by using the S-lemma, we can rewrite (19) as

$$\exists \lambda \geq 0 : \left(\begin{array}{c|c} \lambda A - D & -d(x) \\ \hline -d(x)^T & \gamma(x) - \lambda\rho^2 \end{array}\right) \succeq 0, \tag{20}$$

or by substituting the expressions for $d(x)$ and $\gamma(x)$:

$$\exists \lambda \geq 0 : \left(\begin{array}{c|c} \lambda A - D & -Dx - e \\ \hline -Dx - e & f - x^T Dx - 2e^T x - \lambda \rho^2 \end{array} \right) \succeq 0. \tag{21}$$

This is equivalent to

$$\exists \lambda \geq 0 : \left(\begin{array}{c|c|c} f + e^T D^{-1} e - \lambda \rho^2 & 0 & -(x + D^{-1} e)^T \\ \hline 0 & \lambda A & I \\ \hline -(x + D^{-1} e) & I & D^{-1} \end{array} \right) \succeq 0, \tag{22}$$

which can easily be verifed by checking that the Schur complement of D^{-1} in (22) is exactly the left hand side of the LMI in (21). However, although theoretically speaking (22) is tractable, we would like to avoid LMIs, since LMIs are practically speaking intractable. In the remainder of this section we give an equivalent conic quadratic formulation.

Since A is positive definite, if follows that A and D can be simultaneously diagonalized by a nonsingular matrix S. This means

$$S^T DS = \operatorname{diag}(\delta_1, ..., \delta_n) \quad \text{and} \quad S^T AS = \operatorname{diag}(\alpha_1, \ldots, \alpha_n).$$

Using our improved S-lemma, Lemma 12, we obtain the following result.

Lemma 14 *Assume that there exists $\bar{a}$ such that $\bar{a}^T A \bar{a} < \rho^2$. Then (18) holds if and only if there exist $v \in \mathbb{R}$ and $w \in \mathbb{R}^n$ such that*

$$(RQ) \quad \begin{cases} \sum_i w_i + \rho^2 v + x^T Dx + 2e^T x \leq f \\ \left[S^T(Dx + e)\right]_i^2 + (w_i - v\alpha_i + \delta_i)^2 \leq (w_i + v\alpha_i - \delta_i)^2, \quad \forall i \\ v\alpha_i - \delta_i \geq 0, \quad \forall i \\ v \geq 0. \end{cases}$$

Proof: (19) is equivalent to

$$a^T Aa \leq \rho^2 \Longrightarrow a^T Da + 2d^T a \leq \gamma. \tag{23}$$

Since A and D are SD by a nonsingular matrix S, we can apply Lemma 12, which yields that (23) is equivalent to the following: there exists $v \in \mathbb{R}$ such that

$$\begin{cases} \sum_i \frac{(S^T d)_i^2}{v\alpha_i - \delta_i} + \rho^2 v \leq \gamma \\ v\alpha_i - \delta_i \geq 0, \quad \forall i \\ v \geq 0. \end{cases}$$

This system of constraints can be reformulated as a system of conic quadratic constraints:

$$\begin{cases} \sum_i w_i + \rho^2 v \leq \gamma \\ (S^T d)_i^2 + (w_i - v\alpha_i + \delta_i)^2 \leq (w_i + v\alpha_i - \delta_i)^2, \quad \forall i \\ v\alpha_i - \delta_i \geq 0, \quad \forall i \\ v \geq 0. \end{cases}$$

Substituting $d = d(x)$ and $\gamma = \gamma(x)$ we immediately obtain the result in the theorem. □

The lemma states that the robust counterpart of a convex quadratic inequality with ellipsoidal implementation error can be written as a system of conic quadratic constraints. Note that in the case of a convex quadratic inequality, i.e., D is positive semi-definite, then also the first constraint of (RQ) is convex. Note, however, that the lemma still holds in the case of a nonconvex quadratic inequality, i.e., D is not positive semi-definite. It is interesting to observe that, although in that case the first constraint of (RQ) is of course nonconvex, the implementation error does not introduce extra nonconvexities.

Finally, we note that Lemma 14 can also be generalized to the case of two ellipsoidal constraints by using Theorem 9.

3.2 Conic quadratic constraint

We consider a conic quadratic constraint with implementation error

$$\left.\begin{array}{c} \|Q(x+a)-q\|^2 \leq [p^T(x+a)-r]^2 \\ p^T(x+a) \geq r \end{array}\right\} \ \forall\, a : a^T A a \leq \rho^2, \tag{24}$$

in which $a \in \mathbb{R}^n$ is the additive implementation error, $x, p, q \in \mathbb{R}^n$, $A, Q \in \mathbb{R}^{n\times n}$, and $\rho, r \in \mathbb{R}$. We assume that A is positive definite. The second constraint is equivalent to:

$$p^T x - r + \min_{a^T A a \leq \rho^2} p^T a \geq 0 \Longleftrightarrow p^T x - r - \rho(p^T A^{-1} p)^{1/2} \geq 0, \tag{25}$$

which is a linear constraint.

The first constraint of (24) can be written as

$$\|Qx-q\|^2 + a^T Q^T Q a + 2(Qx-q)^T Q a \leq (p^T x - r)^2 + a^T p p^T a + 2(p^T x - r) p^T a$$

Let us define

$$\begin{array}{l} D = Q^T Q - p p^T \\ d = Q^T(Qx-q) - (p^T x - r) p \\ \gamma = (p^T x - r)^2 - \|Qx - q\|^2. \end{array}$$

Note that x itself is feasible, i.e. $p^T x - r \geq 0$, and $\|Qx-q\|^2 \leq (p^T x - r)^2$ so $\gamma \geq 0$. Moreover, D is symmetric but not necessarily definite or even nonsingular. Then the first constraint of (24) is equivalent to

$$a^T D a + 2 d^T a \leq \gamma \quad \forall\, a : a^T A a \leq \rho^2. \tag{26}$$

First, we observe that in this case it is not possible to rewrite (26) as an LMI, as we have done for the quadratic case. Since A is positive definite, if follows that A and D can be simultaneously diagonalized by a nonsingular matrix S. This means

$$S^T D T = \mathrm{diag}\,(\delta_1, ..., \delta_n) \quad \text{and} \quad S^T A S = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_n).$$

Using our improved S-lemma, Lemma 12, we obtain the following result.

Lemma 15 *Assume that there exists $\bar{a}$ such that $\bar{a}^T A \bar{a} < \rho^2$. Then the first constraint of (24) holds if and only if there exist $v \in \mathbb{R}$ and $w \in \mathbb{R}^n$ such*

$$(RCQ) \quad \left\{ \begin{array}{l} \sum_i w_i + \rho^2 v + \|Qx-q\|^2 \leq (p^T x - r)^2 \\ \left[S^T Q^T Q x - S^T Q^T q - (p^T x - r) S^T p\right]_i^2 + (w_i - v\alpha_i + \delta_i)^2 \leq (w_i + v\alpha_i - \delta_i)^2, \quad \forall i \\ p^T x - r - \rho(p^T A^{-1} p)^{1/2} \geq 0 \\ v\alpha_i - \delta_i \geq 0, \quad \forall i \\ v \geq 0. \end{array}\right.$$

Proof: Since (26) is the same as (19) for the quadratic case (only γ and d differ), we can again use the proof of Lemma 14. Finally, we obtain that the first constraint of (24) is equivalent to

$$\begin{cases} \sum_i w_i + \rho^2 v \leq \gamma \\ (S^T d)_i^2 + (w_i - v\alpha_i + \delta_i)^2 \leq (w_i + v\alpha_i - \delta_i)^2, \quad \forall i \\ v\alpha_i - \delta_i \geq 0, \quad \forall i \\ v \geq 0. \end{cases}$$

The result follows by substitution of $d = d(x)$ and $\gamma = \gamma(x)$, and by adding constraint (25). $\square$

Note that the second set of constraints of problem (RCQ) is conic quadratic. The first constraint of problem (RCQ), however, is not conic quadratic, and even not convex. However, since the feasible set of (24) is convex, and (RCQ) is equivalent to (24), we also know that the level sets for the first constraint in (RCQ) are convex. Hence, each KKT point for (RCQ) is a global optimizer.

Finally, we note that Lemma 15 can also be generalized to the case of two ellipsoidal constraints by using Theorem 9.

4 Globalized Robust Optimization

In this section we study the Globalized Robust Counterpart (GRC). For the generalized robust counterpart we define two uncertainty regions:

$$U_1 = \{a : a^T A a \leq \rho_1^2\}$$

and

$$U_2 = \{a : a^T A a \leq \rho_2^2\},$$

such that $\rho_1 < \rho_2$, i.e., $U_1 \subset U_2$, and in which $A \succ 0$. For $a \in U_1$ we enforce that (19) should hold, and for $a \in U_2$ we allow some violation $\theta(\mathrm{dist}(a, U_1))$, where $\theta(t)$ is a nondecreasing function for $t \geq 0$, with $\theta(0) = 0$, and $\theta(t) = 0$, for $t < 0$, and

$$\mathrm{dist}(a, U_1) = \min_{a' \in U_1} \|a' - a\|_A = \min_{a' \in U_1} \sqrt{(a' - a)^T A (a' - a)}$$

is the distance of a to U_1 measured in the A-norm. Let us now focus on the quadratic case. Instead of (19) we now get:

$$a^T D a + 2a^T d \leq \gamma + \theta(\mathrm{dist}(a, U_1)) \quad \forall\, a \in U_2. \tag{27}$$

Observe that if $a \in U_1$ then $\mathrm{dist}(a, U_1) = 0$, and the constraint should hold without violation. Note that this framework is different than the Globalized Robust Optimization approach proposed in [3], where U_1 is a compact convex set, and U_2 is equal to U_1 plus a convex cone. In our approach both regions are ellipsoid. Moreover, $\theta(t) = t$ in [3] , while here we have more freedom in choosing $\theta(t)$.

Constraint (27) can be written as:

$$\max_{a \in U_2} \{a^T D a + 2a^T d - \theta(\mathrm{dist}(a, U_1))\} \leq \gamma. \tag{28}$$

Since

$$\mathrm{dist}(a, U_1) = \min_{a' \in U_1} \|a - a'\|_A = \begin{cases} 0 & \text{if} \quad a \in U_1 \\ \|a\|_A - \rho_1 & \text{otherwise} \end{cases}$$

(28) becomes:

$$\max_{a \in U_2} \{a^T Da + 2a^T d - \theta(\|a\|_A - \rho_1)\} \leq \gamma. \tag{29}$$

Since $\theta(.)$ is nondecreasing, this is equivalent to

$$\max_{\rho_1 \leq t \leq \rho_2} \max_{\|a\|_A \leq t} \{a^T Da + 2a^T d - \theta(t - \rho_1)\} \leq \gamma. \tag{30}$$

Now we focus on the inner problem

$$\max \left\{ a^T Da + 2a^T d \ : \ \|a\|_A \leq t \right\}.$$

This is a quadratic optimization problem with one quadratic constraints, for which the results obtained in the previous section can be applied.

Assuming that A and D can be simultaneously diagonalized we get that (30) is equivalent to

$$\begin{cases} \sum_i w_i + \max_{\rho_1 \leq t \leq \rho_2} \{t^2 v - \theta(t - \rho_1)\} + x^T Dx + 2e^T x \leq f \\ \left[S^T(Dx + e)\right]_i^2 + (w_i - v\alpha_i + \delta_i)^2 \leq (w_i + v\alpha_i - \delta_i)^2, \quad \forall i \\ v\alpha_i - \delta_i \geq 0, \quad \forall i \\ v, w \geq 0. \end{cases}$$

Note that the function $\varphi(v) := \max_{\rho_1 \leq t \leq \rho_2} \{t^2 v - \theta(t - \rho_1)\}$ is convex in v, even if $\theta(.)$ is not convex.

Example 1: θ is linear. Let $\theta(t) = \omega t$. Then we have

$$\max_{\rho_1 \leq t \leq \rho_2} \{t^2 v - \omega(t - \rho_1)\} = \max\{\rho_1^2 v, \rho_2^2 v - \omega(\rho_2 - \rho_1)\}.$$

Finally x is a GRC solution if and only if $x \in \mathbb{R}^n, v \in \mathbb{R}$ and $w \in \mathbb{R}^n$ solve

$$\begin{cases} \sum_i w_i + \max\{\rho_1^2 v, \rho_2^2 v - \omega(\rho_2 - \rho_1)\} + x^T Dx + 2e^T x \leq f \\ \left[S^T(Dx + e)\right]_i^2 + (w_i - v\alpha_i + \delta_i)^2 \leq (w_i + v\alpha_i - \delta_i)^2, \quad \forall i \\ v\alpha_i - \delta_i \geq 0, \quad \forall i \\ v, w \geq 0. \end{cases}$$

Example 2: θ is quadratic. Let us take for example $\theta(t) = \omega t^2$. Then we have

$$\max_{\rho_1 \leq t \leq \rho_2} \{t^2 v - \theta(t - \rho_1)\} = \max_{\rho_1 \leq t \leq \rho_2} \{t^2 v - \omega(t - \rho_1)^2\}.$$

The function $t^2 v - \omega(t - \rho_1)^2$ is maximal for

$$t = \frac{\omega \rho_1}{\omega - v},$$

and the corresponding optimal value is $\rho_1^2 \omega \left(-1 + \frac{\omega}{\omega - v}\right)$. This optimal t is in the interval $[\rho_1, \rho_2]$ if $v \leq \omega \left(1 - \frac{\rho_1}{\rho_2}\right)$. It can easily be verified that

$$\varphi(v) = \begin{cases} \rho_1^2 \omega \left(-1 + \frac{\omega}{\omega - v}\right) & \text{if} \quad v < \omega \left(1 - \frac{\rho_1}{\rho_2}\right) \\ \rho_2^2 v - \omega(\rho_2 - \rho_1)^2 & \text{if} \quad v \geq \omega \left(1 - \frac{\rho_1}{\rho_2}\right), \end{cases}$$

which is convex.

The derivation of the GRC for the conic quadratic case can be done in a similar way.

5 Applications

In this section we describe four important classes of applications of the results obtained in this paper.

Linear optimization with both parameter uncertainty and implementation error. Consider the following linear constraint

$$b^T x \leq c,$$

that is affected by both uncertainty in the parameter b and implementation error:

$$(b + Bp)^T (x + a) \leq c, \quad \forall p : p^T R p \leq \bar{\rho}^2 \quad \forall a : a^T A a \leq \rho^2,$$

in which the matrices A and R are positive definite. First, we calculate the robust counterpart with respect to the parameter uncertainty, and obtain

$$b^T (x + a) + \bar{\rho} \| R^{-1/2} B^T (x + a) \| \leq c, \quad \forall a : a^T A a \leq \rho^2.$$

This is basically (24), a conic quadratic constraint with implementation error. By using Lemma 15, we obtain the robust counterpart:

$$\begin{cases} \sum_i w_i + \rho^2 v + \bar{\rho}^2 \| R^{-1/2} B^T x \|^2 \leq (c - b^T x)^2 \\ \left[\bar{\rho}^2 S^T B R^{-1} B^T x + (c - b^T x) S^T b \right]_i^2 + (w_i - v\alpha_i + \delta_i)^2 \leq (w_i + v\alpha_i - \delta_i)^2, \quad \forall i \\ c - b^T x - \rho (b^T A^{-1} b)^{1/2} \geq 0 \\ v\alpha_i - \delta_i \geq 0, \quad \forall i \\ v \geq 0, \end{cases}$$

in which S is the matrix that simultaneously diagonalizes A and $\bar{\rho}^2 B R^{-1} B^T - bb^T$.

This situation of both parameter uncertainty and implementation error happens often in practice, e.g., in engineering. Linear functions are often estimated via simulation or physical experiments, hence there is (much) uncertainty on the coefficients. Moreover, in many cases we also have implementation errors. For an example we refer to the TV-design problem in [15].

General nonlinear robust optimization. Let us first start with the general nonlinear design centering problem. Suppose we have a nonlinear (convex) constraint with implementation error, and we would like to solve the optimization problem in a robust way. Hence, we consider,

$$f(x + a) \leq 0, \qquad \forall a : a^T A a \leq \rho^2.$$

One possible way to solve this is by using Sequential Robust Quadratic Programming. In each iteration i we solve:

$$q^i (x + a) \leq 0, \quad \forall a : a^T A a \leq \rho^2,$$

where $q^i(x)$ is the quadratic approximation of $f(x)$ in the i-th iteration. This subproblem is exactly the one studied in the previous section. Note that without loss of generality we may assume $A = I$, thereby simplifying the process of simultaneous diagonalizing.

A similar Sequential Robust Quadratic Programming approach may also be used for a more general problem (i.e., not only for implementation error):

$$f(x, a) \leq 0, \qquad \forall a : a^T A a \leq \rho^2,$$

in which a is the uncertain parameter, and $f(x,a)$ is convex in (x,a). In each iteration i we use the quadratic approximation $q^i(x,a)$ of $f(x,a)$ in the current iterate (x,a), i.e. we solve:

$$q^i(x,a) \le 0, \quad \forall a : a^T A a \le \rho^2. \tag{31}$$

Note that $q^i(x,a)$ is quadratic in x and in a. Hence, we may use the methods of the previous section to solve subproblem (31). In each iteration we have to simultaneously diagonalize A and $\nabla^2 q^i(x,a)$, which process can be simplified by assuming without loss of generality that $A = I$. Note that [4] in fact proposes such a sequential method but they use linear approximations. Moreover, they also work with both parameter uncertainty and implementation error. They first linearize the nonlinear function both with respect to x and the uncertain parameters (so there are no cross terms). Then they solve the robust counterpart for this linear model. As described in the previous section, we can solve the robust counterpart for linear models with both implementation and parameter uncertainty, hence we can also handle cross terms.

Taguchi. Another important potential class of applications is Taguchi robust optimization. In Taguchi's approach we finally get the following response function by doing experiments:

$$\sum_i b_i x_i + \sum_j c_j z_j + \sum_{i,j} d_{ij} x_i z_j + e$$

in which x_i are the optimization variables and z_j the noise factors. See Chapter 11 of [11] for more details. This function may contain four categories of uncertainty:

1. Simulation / experimental errors, which lead to uncertainty in the coefficients b_i, c_j, and d_{ij};
2. Model errors, which lead to uncertainty in the coefficients b_i, c_j, and d_{ij};
3. Uncertainty in the noise factors z_j;
4. Implementation error in x_i.

Normally speaking Taguchi only looks at uncertainty in the noise factors. We are now able to deal with e.g. noise factors and implementation error or experimental errors and implementation errors. Let us consider the case that there is ellipsoidal uncertainty in the noise factor, and ellipsoidal implementation error. We are then facing the following problem:

$$\min_x \max_{a:a^T A a \le \rho^2} \max_{z:z^T R z \le \bar{\rho}^2} \sum_i b_i (x_i + a_i) + \sum_j c_j z_j + \sum_{i,j} d_{ij} (x_i + a_i) z_j + e,$$

in which A and R are positive definite. Note that this problem can be written as:

$$\min_x \max_{a:a^T A a \le \rho^2} \max_{z:z^T R z \le \bar{\rho}^2} b^T (x+a) + z^T (c + D^T (x+a)) + e,$$

in which the (i,j) element of D is d_{ij}. By first solving the inner most maximization problem, we get a conic quadratic function with implementation error:

$$\min_x \max_{a:a^T A a \le \bar{\rho}^2} b^T (x+a) + \bar{\rho} \| R^{-1/2} (c + D^T (x+a)) \| + e.$$

Using Lemma 15 the robust counterpart for this problem can be stated as the following problem in variables x, v, and w:

$$\begin{cases} \sum_i w_i + \rho^2 v + \bar{\rho}^2 \|R^{-1/2}(c + D^T x)\|^2 \leq (b^T x + e)^2 \\ \left[\bar{\rho}^2 S^T D R^{-1} D^T x + \bar{\rho}^2 S^T D R^{-1} c - (b^T x + e) S^T b\right]_i^2 + (w_i - v\alpha_i + \delta_i)^2 \leq (w_i + v\alpha_i - \delta_i)^2, \quad \forall i \\ b^T x + e + \rho (b^T A^{-1} b)^{1/2} \leq 0 \\ v\alpha_i - \delta_i \geq 0, \quad \forall i \\ v \geq 0, \end{cases}$$

in which S is the matrix that simultaneously diagonalizes A and $\bar{\rho}^2 D R^{-1} D^T - bb^T$.

Adjustable robust counterpart Suppose we have the following constraint, that corresponds to a multistage optimization problem with fixed recourse:

$$(a_0 + a)^T x + b^T y \leq c, \quad \forall a : a^T A a \leq \rho^2, \tag{32}$$

in which a is the uncertain parameter, a_0, b are certain, and x is non-adjustable, y is adjustable. In [3] linear decision rules $y = u + Va$ are introduced to approximate (32), in which case (32) can be reformulated as a conic quadratic problem. It is also shown in [3] that using a full quadratic decision rule $y_i = u_i + v_i^T a + \sum_{j \leq k} w_{ijk} a_j a_k$ leads to an SDP problem. Although in general such full quadratic decision rules leads to better solutions, solving large SDP problems is practically speaking still difficult and time-consuming. We therefore propose partial quadratic decision rule $y = u + Va + W\bar{a}$, in which $\bar{a}_i = a_i^2$. We show next that for such rules (32) can be reformulated as a conic quadratic problem. Assume without loss of generality $A = I$. Then (32) becomes

$$(a_0 + a)^T x + b^T (u + Va + W\bar{a}) \leq c, \quad \forall (a, \bar{a}) : a^T A a \leq \rho^2, \bar{a}_i = a_i^2,$$

or

$$a_0^T x + (W^T b)^T \bar{a} + (x + V^T b)^T a + b^T u \leq c, \quad \forall (a, \bar{a}) : \sum_i \bar{a}_i \leq \rho^2, \bar{a}_i \geq a_i^2.$$

Using Lemma 12 one can verify that this is equivalent to the following set of conic quadratic constraints in the variables x, s, u, v, w and z:

$$\begin{cases} a_0^T x + b^T u + \sum_i z_i + \rho^2 s \leq c \\ (x_i + (V^T b)_i)^2 + (s - (W^T b)_i - z_i)^2 \leq (s - (W^T b)_i + z_i)^2 \quad \forall i \\ s - (W^T b)_i \geq 0, \quad \forall i \\ s \geq 0. \end{cases}$$

6 Concluding remarks

We have shown that quadratic optimization problems with one or two constraints can be reformulated as convex quadratic optimization problems in the case that the two or three matrices involved are SD. This result sharpens the S-lemma. Moreover, we have shown that this result can be used to show that a convex quadratic constraint with ellipsoidal uncertainty error can be reformulated as a set of conic quadratic constraints. Moreover, a conic quadratic constraint with ellipsoidal uncertainty error can be reformulated as a set of 'nearly' conic quadratic constraints. The feasible set of this problem is certainly convex. Besides the many direct applications of (conic) quadratic optimization problems with implementation error, we also described four important classes of indirect applications.

For further research we mention the extension of Theorem 4 to other classes of separable problems. Moreover, an interesting topic for further research is the analysis and numerical testing of the Sequential Robust Quadratic Programming idea and the numerical testing of the proposed partial quadratic decision rule in Section 5.

References

[1] AIMMS. Paragon Decision Technology, http://www.aimms.com/operations-research/mathematical-programming/robust-optimization, 2011.

[2] A. Ben-Tal, M. Teboulle. Hidden convexity in some nonconvex quadratically constrained quadratic programming. *Mathematical Programming*, 72, 51–63, 1995.

[3] A. Ben-Tal, L. El-Ghaoui, A. Nemirovski. *Robust Optimization.* Princeton Series in Applied Mathematics, 2009.

[4] D. Bertsimas, O. Nohadani, K.M. Teo. Robust optimization for unconstrained simulation-based problems. *Operations Research*, 58(1), 161–178, 2010.

[5] S. Boyd, L. Vandenberghe. *Convex Optimization.* Cambridge University Press, Cambridge, 2004.

[6] G.H. Golub, C.F. van Loan. *Matrix Computations* (2nd edition). Johns Hopkins University Press, London, UK. 1989.

[7] M. Lahanas, E. Schreibmann, D. Baltas. Multiobjective inverse planning or intensity modulated radiotherapy with constraint-free gradient-based optimization algorithms. *Physics in Medicine and Biology*, 48(17), 2843–2871, 2003.

[8] M.S. Lobo, L. Vandenberghe, S. Boyd, et al. Applications of second-order cone programming. *Linear Algebra And Its Applications*, 284(1-3), 193–228, 1998.

[9] R.S. Martin, J.H. Heiberger. Reduction of a symmetric eigenproblem $Ax = \lambda Bx$ and related problems to standard form. *Numerische Mathematik*, 11, 99–110, 1968.

[10] N. Milickovic, M. Lahanas, M. Papagiannopoulou, N. Zamboglou, D. Baltas. Multiobjective anatomy-based dose optimization for HDR-brachytherapy with constraint free deterministic algorithms. *Physics in Medicine and Biology*, 47(13), 2263–2280, 2002.

[11] R.H. Myers, D.C. Montgomery. *Response Surface Methodology.* Wiley Series in Probability and Statistics, Second Edition, 2002

[12] Y.E. Nesterov. Semidefinite relaxation and nonconvex quadratic optimization. *Optimization Methods and Software*, 9, 141–160, 1998.

[13] I. Polik, T. Terlaky. A survey of the S-lemma. *SIAM Review*, 49(3), 371–418, 2007.

[14] R.J. Stern, H. Wolkowicz. Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations. *SIAM Journal on Optimization*, 5, 286–313, 1995.

[15] E. Stinstra, D. den Hertog. Robust optimization using computer experiments. *European Journal of Operational Research*, 191(3), 816–837, 2008.

[16] J.F. Sturm, S. Zhang. On cones of nonnegative quadratic functions. *Mathematics of Operations Research*, 28(2), 246–267, 2003.

[17] F. Uhlig. Definite and semidefinite matrices in a real symmetric matrix pencil. *Pacific Journal of Mathematics*, 49 (2), 561–568, 1972.

[18] Y. Ye. Approximating quadratic programming with bound and quadratic constraints. *Mathematical Programming*, 84, 219–226, 1999.

[19] Y. Ye, S. Zhang. New results on quadratic minimization. *SIAM Journal on Optimization*, 14 (1), 245–267, 2003.