

# History-Dependent Risk Attitude\*

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## Abstract

We propose a model of history-dependent risk attitude, allowing a decision maker's risk attitude to be affected by his history of disappointments and elations. The decision maker recursively evaluates compound risks, classifying realizations as disappointing or elating using a threshold rule. We establish equivalence between the model and two cognitive biases: risk attitudes are reinforced by experiences (one is more risk averse after disappointment than after elation) and there is a primacy effect (early outcomes have the greatest impact on risk attitude). In dynamic asset pricing, the model yields volatile, path-dependent prices.

**Keywords:** history-dependent risk attitude, reinforcement effect, primacy effect, dynamic reference dependence

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## **1 Introduction**

Theories of decision making under risk typically assume that risk preferences are stable. Evidence suggests, however, that risk preferences may vary with personal experiences. It has been shown that emotions, which may be caused by exogenous factors or by the outcomes of past choices, play a large role in the decision to bear risk. Moreover, individuals are affected by unrealized outcomes, a phenomenon known in the psychological literature as counterfactual thinking.<sup>1</sup>

Empirical work has found evidence of history-dependent risk aversion in a variety of fields. Pointing to adverse consequences for investment and the possibility of poverty traps, development economists have observed a long-lasting increase in risk aversion after natural disasters (Cameron and Shah, 2010) and, studying the dynamics of farming decisions in an experimental setting, increases of risk aversion after failures (Yesuf and Bluffstone, 2009). Malmendier and Nagel (2011) study how personal experiences of macroeconomic shocks affect financial risk-taking. Controlling for wealth, income, age, and year effects, they find that for up to three decades later, “households with higher experienced stock market returns express a higher willingness to take financial risk, participate more in the stock market, and conditional on participating, invest more of their liquid assets in stocks.” Applied work also demonstrates that changing risk aversion helps explain several economic phenomena. Barberis, Huang and Santos (2001) allow risk aversion to decrease with prior stock market gains (and increase with losses), and show that their model is consistent with the well-documented equity premium and excess volatility puzzles. Gordon and St-Amour (2000) study bull and bear markets, allowing risk attitudes to vary stochastically by introducing a state-dependent CRRA parameter in a discounted utility model. They show that countercyclical risk aversion best explains the cyclical nature of equity prices, suggesting that “future work should address the issue of determining the factors that underline the movements in risk preferences” which they identified.

In this work, we propose a model under which such shifts in risk preferences may arise. Our model of history-dependent risk attitude (HDRA) allows the way that risk unfolds over time to affect attitude towards further risk. We derive predictions for the comparative statics of risk aversion. In particular, our model predicts that one becomes more risk averse after a negative experience than after a positive one, and that sequencing matters: the earlier one is disappointed, the more risk averse one becomes.

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<sup>1</sup>On the effect of emotions, see Knutson and Green (2008) and Kuhnen and Knutson (2011), as well as Section 1.1. Roesse and Olson (1995) offers a comprehensive overview of the counterfactual thinking literature.

To ease exposition, we begin by describing our HDRA model in the simple setting of  $T$ -stage, compound lotteries (the model is later extended to stochastic decision problems, in which the DM may take intermediate actions). A realization of a compound lottery is another compound lottery, which is one stage shorter. In the HDRA model, the DM categorizes each realization of a compound lottery as an elating or disappointing outcome. At each stage, the DM's history is the preceding sequence of elations and disappointments. Each possible history  $h$  corresponds to a preference relation over one-stage lotteries, which comes from an admissible set of preferences. These one-stage preferences are rankable in terms of their risk aversion. For example, an admissible collection could be a class of expected utility preferences with a Bernoulli function  $u(x) = \frac{x^{1-\rho_h}}{1-\rho_h}$ , where the coefficient of relative risk aversion  $\rho_h$  is history dependent.

The key features of the HDRA model are that compound lotteries are evaluated recursively and that the DM's history assignment is *internally consistent*. More formally, starting at the final stage of the compound lottery and proceeding backwards, each one-stage lottery is replaced with its appropriate, history-dependent certainty equivalent. At each step of this recursive process, the DM is evaluating only one-stage lotteries, the outcomes of which are certainty equivalents of continuation lotteries. To determine which outcomes are elating and disappointing, the DM uses a threshold rule that assigns a number (a *threshold level*) to each one-stage lottery encountered in the recursive process. Internal consistency requires that if a subplottery is considered an *elating* (*disappointing*) outcome of its parent subplottery, then its certainty equivalent should indeed exceed (or fall below) the threshold level corresponding to its parent subplottery. Mathematically, internal consistency imposes a fixed point requirement on the assignment of histories in a multistage setting.

Our model is general, allowing a wide class of preferences to be used for recursively evaluating lotteries, as well as a variety of threshold rules. The one-stage preferences may come from the betweenness class (Dekel, 1986; Chew, 1989), which includes expected utility as a special case. The DM's threshold rule may be either endogenous (preference-based) or exogenous. In the preference-based case, the DM's threshold moves endogenously with his preference; he compares the certainty equivalent of a subplottery to the certainty equivalent of its parent. In the exogenous case, the DM uses a rule that is independent of preferences but is a function of the lottery at hand; for example, an expectation-based rule that compares the certainty equivalent of a subplottery to his expected certainty equivalent. All of the components of the HDRA model – that is, the single-stage preferences, threshold rule, and history assignment – can be elicited from choice behavior.

Besides internal consistency, we do not place any restriction on *how* risk aversion should depend on the history. Nonetheless, we show that the HDRA model predicts two well-documented cognitive biases; and that these biases are sufficient conditions for an HDRA representation to ex-

ist. First, the DM's risk attitudes are *reinforced* by prior experiences: he becomes less risk averse after positive experiences and more risk averse after negative ones. Second, the DM displays a *primacy effect*: his risk attitudes are disproportionately affected by early realizations. In particular, the earlier the DM is disappointed, the more risk averse he becomes. We discuss evidence for these predictions in Section 1.1 below.

We apply our model to study a multi-period asset pricing problem in a representative agent economy with CARA preferences. We show that the model yields volatile, path-dependent prices. Past realizations of dividends affect subsequent prices, even though they are statistically independent of future dividends and there are no income effects. For example, high dividends bring about price increases, while a sequence of only low dividends leads to an equity premium higher than in the standard, history-independent CARA case. Since risk aversion is endogenously affected by dividend realizations, the risk from holding an asset is magnified by expected future variation in the level of risk aversion. Hence the HDRA model introduces a channel of risk that is reflected in the greater volatility of asset prices. This is consistent with the observation of excess volatility in equity prices, dating to Shiller (1981).

The rest of this paper is organized as follows. Section 1.1 surveys evidence for the reinforcement and primacy effects. Section 1.2 discusses the related literature. Section 2 presents a basic version of our model, which is rich enough to qualitatively explain our main results. Section 3 contains our main result, and studies additional implications of the model. Section 4 extends the model to allow for intermediate choices and applies it to an asset-pricing problem. Section 5 provides the general version of our model, shows that our main results extend, and describes how the components of the model can be elicited from choice behavior.

## **1.1 Evidence for the Reinforcement and Primacy Effects**

Our main predictions, the reinforcement and primacy effects, are consistent with a body of evidence on risk-taking behavior. Thaler and Johnson (1990) find that individuals become more risk averse after negative experiences and less risk averse after positive ones. Guiso, Sapienza and Zingales (2011) estimate a marked increase in risk aversion in a sample of Italian investors after the 2008 financial crisis; the certainty equivalent of a risky gamble drops from 4,000 euros to 2,500, an increase in risk aversion which, as the authors show, cannot be due to changes in wealth, consumption habits, or background risk. As discussed earlier, Malmendier and Nagel (2011) also find that macroeconomic shocks lead to a long-lasting increase of risk aversion. Studying initial public offerings (IPOs), Kaustia and Knupfer (2008) identify pairs of "hot and cold" IPOs with close offer dates and follow the future subscription activities of investors whose first IPO subscription was in

one of those two. They find that “twice as many investors participate in a subsequent offering if they first experience a hot offering rather than a cold offering.” Pointing to a primacy effect, they find that the initial outcome has a strong impact on subsequent offerings, and that “by the tenth offering, 65% of investors in the hot IPO group will have subscribed to another IPO, compared to only 39% in the cold IPO group.” Baird and Zelin (2000) study the impact of sequencing of positive and negative news in a company president’s letter. They find a primacy effect, showing that information provided early in the letter has the strongest impact on evaluations of that company’s performance. In general, sequencing biases such as the primacy effect are robust and long-standing experimental phenomena (early literature includes Anderson (1965)); and several empirical studies, including Guiso, Sapienza and Zingales (2004) and Alesina and Fuchs-Schündeln (2007)), argue that early experiences may shape financial or cultural attitudes.<sup>2</sup>

The biological basis of changes in risk aversion has been studied by neuroscientists.<sup>3</sup> As summarized in Knutson and Green (2008) and Kuhnen and Knutson (2011), neuroimaging studies have shown that two parts of the brain, the nucleus accumbens and the anterior insula, play a large role in risky decisions. The nucleus accumbens processes information on rewards, and is associated with positive emotions and excitement; while the anterior insula processes information about losses, and is associated with negative emotions and anxiety. Controlling for wealth and information, activation of the nucleus accumbens (anterior insula) is associated with bearing greater (lesser) risk in investment decisions. Moreover, Kuhnen and Knutson (2011) note:

Specifically in the context of feedback about decisions under risk [...] activation in the nucleus accumbens increases when we learn that the outcome of a past choice was better than expected (Delgado et al. (2000), Pessiglione et al. (2006)). Activation in the anterior insula increases when the outcome is worse than expected (Seymour et al (2004), Pessiglione et al (2006)), and when actions not chosen have larger payoffs than the chosen one.

In a neuroimaging study with 90 sequential investment decisions by subjects, these feedback effects are shown to influence subsequent risk-taking behavior (Kuhnen and Knutson, 2011).

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<sup>2</sup>Another well-known sequencing bias is the recency effect, according to which more recent experiences have the strongest effect. A recency effect on risk attitude is opposite to the prediction of our model.

<sup>3</sup>At the endocrine level, researchers have also noted that increases in testosterone levels are associated with increased risk-taking behavior (see for example Sapienza, Zingales and Maestripieri (2009) and references within). Moreover, testosterone levels may rise with winning and fall with losing: in particular, it has been shown that both fans watching, and players participating in, sporting events exhibit an increase in testosterone when their team wins (Bernhardt et. al. (1998) and Oliveira et. al. (2009), respectively).

## 1.2 Relations to the literature

In many theories of choice over temporal lotteries, risk aversion could depend on the passage of time, wealth effects or habit formation in consumption; see Kreps and Porteus (1978), Chew and Epstein (1989), Segal (1990), Campbell and Cochrane (1999) and Rozen (2010), among others. We study how risk attitudes are affected by the past, independently of such effects. In the HDRA model, risk attitudes depend on “what might have been.” Such counterfactual thinking means that our model relaxes consequentialism (Machina, 1989; Hanany and Klibanoff, 2007), an assumption that is maintained by the papers above. Counterfactual thinking may lead to a preference for dominated options in some situations; this observation is further explored in Section 3.3.

Our form of history-dependence is conceptually distinct from models where current and future beliefs affect current utility (that is, dependence of utility on “what might be” in the future). This literature includes, for example, Caplin and Leahy (2001) and Köszegi and Rabin (2009). Caplin and Leahy (2001) propose a two-period model where the prize space of a lottery is enriched to contain psychological states, and there is an (unspecified) mapping from physical lotteries to mental states. Depending on how the second-period mental state is specified to depend on the first, Caplin and Leahy’s model could explain various types of risk-taking behaviors in the first period. While discussing the possibility of second-period disappointment, they do not address the question of history-dependence in choices. We conjecture that with additional periods and an appropriate specification of the mapping between mental states, one could replicate the predictions of our model. Köszegi and Rabin (2009) propose a utility function over  $T$ -period risky consumption streams. In their model, period utility is the sum of current consumption utility and the expectation of a gain-loss utility function, over all percentiles, of consumption utility at that percentile under the ex-post belief minus consumption utility at that percentile under the ex-ante belief. Beliefs are determined by an equilibrium notion, leading to multiplicity of possible beliefs. This bears resemblance to the multiplicity of internally consistent history assignments in our model (see Section 3.5 on how different assignments correspond to different attitudes to compound risks). Köszegi and Rabin (2009) do not address the question of history-dependence: given an ex-ante belief over consumption, utility is not affected by prior history (how that belief was formed). While they point out that it would be realistic for comparisons to past beliefs to matter beyond one lag, they suggest one way to potentially model Thaler and Johnson (1990)’s result in their framework: “by assuming that a person receives money, and *in the same period* makes decisions on how to spend the money – with her old expectations still determining current preferences” (Köszegi and Rabin, 2009, Footnote 6). We conjecture that with additional historical differences in beliefs and an appropriate choice of functional forms (and relaxing additivity), one could replicate our predictions.

## 2 A basic model of history-dependent risk attitude

We now present a basic model of history-dependent risk attitude in the setting of three-stage lotteries. Our general model, which we preview at the end of this section, appears in Section 5.

Consider an interval of prizes  $X = [w, b] \subset \mathbb{R}$ , where  $w$  is the worst prize and  $b$  is the best prize. Let  $\mathcal{L}(X)$ , or simply  $\mathcal{L}^1$ , denote the set of all simple, one-stage lotteries over  $X$  (i.e., lotteries that give positive probability for a finite number of outcomes in  $X$ ). The set  $\mathcal{L}^2 = \mathcal{L}(\mathcal{L}^1)$  is the set of two-stage lotteries – that is, simple lotteries whose outcomes are themselves one-stage lotteries. In this section, the domain of choice of the DM is  $\mathcal{L}^3 = \mathcal{L}(\mathcal{L}^2)$ , the set of simple three-stage compound lotteries. Before proceeding, we collect some useful notations and definitions. Elements in  $\mathcal{L}^t$  are denoted by capital letters, e.g.,  $P^t, Q^t$ , or  $R^t$ . For simplicity, we often use lowercase letters to denote one-stage lotteries (e.g.,  $p, q$ , or  $r$ ). For any  $x \in X$ , we write  $\delta_x^t$  for the degenerate  $t$ -stage lottery which gives the prize  $x$  with probability one. Similarly, for any  $p \in \mathcal{L}^1$ ,  $\delta_p^t$  denotes the  $t + 1$ -stage lottery in which the first  $t - 1$  stages are degenerate and the lottery  $p$  is received in period  $t$ . For  $t = 1, 2$ ,  $P^t$  is a *sublottery* of  $P^{t+1}$  if it is in the support of  $P^{t+1}$ ; and a one-stage lottery  $P^1$  is a sublottery of  $P^3$  if it is in the support of some  $P^2$  which is a sublottery of  $P^3$ .<sup>4</sup> We use  $\langle \alpha_1, P_1^t; \dots; \alpha_n, P_n^t \rangle$  to denote the  $(t + 1)$ -stage lottery which gives the sublottery  $P_j^t$  with probability  $\alpha_j$ . A one-stage lottery is simply written as  $\langle \alpha_1, x_1; \dots; \alpha_n, x_n \rangle$ . The last two notations presume the outcomes are all distinct.

An example of a three-stage lottery  $P^3$  is visualized in Figure 1(a), where  $p, q, r$ , and  $s$  are one-stage lotteries. In the first stage of  $P^3$ , there is an equal chance that the DM faces the sublottery giving  $s$  with probability one (the right branch) or faces an additional stage of risk before learning which lottery governs his winnings (the left branch). Under the two-stage sublottery  $P^2$  in the left branch, DM receives each of the lotteries  $p$  and  $r$  with probability  $\frac{1}{4}$ , and receives lottery  $q$  with probability  $\frac{1}{2}$ . To summarize,  $P^3 = \langle .5, P_1^2; .5, P_2^2 \rangle$ ,  $P_1^2 = \langle .25, p; .5, q; .25, r \rangle$  and  $P_2^2 = \langle 1, s \rangle = \delta_s$ .

The DM classifies each sublottery as either an *elating* or a *disappointing* outcome of the sublottery from which it emanates. Formally, the initial history – i.e. prior to any resolution of risk – is empty (0). If a sublottery is degenerate – i.e., it leads to some sublottery with probability one – then the DM is not exposed to risk at that stage and his history is unchanged. If a sublottery is nondegenerate, each sublottery in its support may be an elating ( $e$ ) or a disappointing ( $d$ ) outcome. The DM's history at any sublottery is the preceding sequence of  $e$ 's and  $d$ 's. The set of all possible histories is thus given by  $H = \{0, e, d, ee, ed, de, dd\}$ . For each  $P^3$ , the *history assignment*  $a(\cdot | P^3)$

<sup>4</sup>Note that within a lottery  $P^3$ , the same one-stage lottery could appear in the support of different two-stage sublotteries of  $P^3$ . Keeping this possibility in mind, but in order to economize on notation, throughout the paper we implicitly identify a particular sublottery by the sequence of sublotteries leading to it.

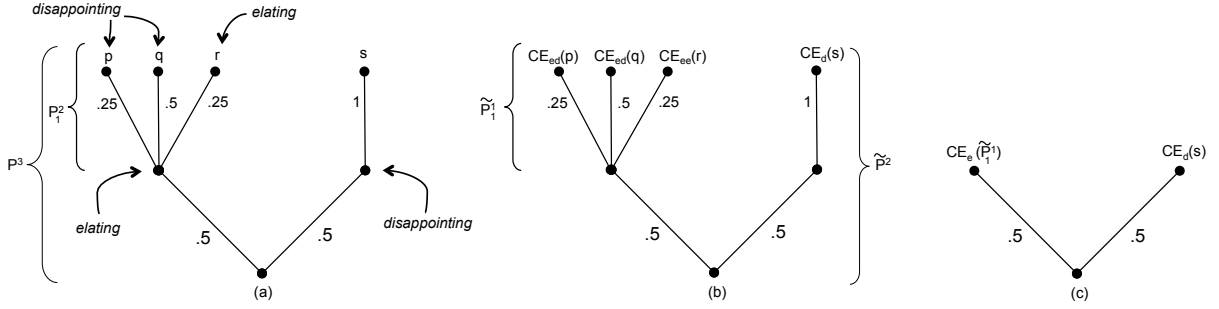


Figure 1: The three-stage lottery  $P^3$ , and a history assignment, are seen in (a). The two-stage lottery  $\tilde{P}^2$  is shown in (b), where each terminal one-stage lottery in  $P^3$  is replaced with its certainty equivalent given the history assignment. The one-stage lottery  $\tilde{P}^1$  is shown in (c), where each terminal one-stage lottery in  $\tilde{P}^2$  is replaced with its certainty equivalent given the history assignment.

denotes the history  $h \in H$  of each sublottery of  $P^3$ . We initialize  $a(P^3|P^3) = 0$ . Within a lottery  $P^3$ , the history assignment is thus *sequential*: if  $P^{t+1}$  is nondegenerate and  $P^t$  is in its support, then  $a(P^t|P^3) \in \{a(P^{t+1}|P^3)\} \times \{e, d\}$ , while if  $P^{t+1}$  is degenerate then  $a(P^t|P^3) = a(P^{t+1}|P^3)$ .

The DM's evaluation of a sublottery is determined by the sequence of elating or disappointing outcomes leading to it. Each history  $h \in H$  corresponds to an expected utility function  $V_h = \mathbb{E}(u_h)$  over one-stage lotteries, with strictly increasing Bernoulli function  $u_h$ . Let  $\mathcal{V} = \{V_h\}_{h \in H}$ . The history assignment  $a$  determines which  $V_h$  is applied. To study how risk attitudes are shaped by prior experiences, we require the utility functions after each history to be (strictly) rankable in terms of their risk aversion, according to the following comparative measure:

**Definition 1.** We say that  $V_h$  is more risk averse than  $V_{h'}$ , denoted  $V_h >_{RA} V_{h'}$ , if for any  $x \in X$  and any nondegenerate  $p \in \mathcal{L}^1$ ,  $V_h(p) \geq V_h(\delta_x)$  implies that  $V_{h'}(p) > V_{h'}(\delta_x)$ .

One example of  $\mathcal{V}$  is a collection of expected CRRA utilities with history-dependent coefficient of relative risk aversion,  $\mathcal{V} = \{\mathbb{E}(x^{1-\rho_h}|\cdot)\}_{h \in H}$ . Another example is a collection of expected CARA utilities,  $\mathcal{V} = \{\mathbb{E}(1 - e^{-\lambda_h x}|\cdot)\}_{h \in H}$ . In these two cases, history-dependent risk aversion is captured by a single parameter.

The DM uses the collection  $\mathcal{V}$  and his history assignment  $a$  to recursively calculate the value of each three-stage lottery. Formally, for  $p \in \mathcal{L}^1$  and  $h \in H$ , let  $CE_h(p)$  be the certainty equivalent of  $p$  calculated using  $V_h$ ; that is,  $V_h(\delta_{CE_h(p)}) = V_h(p)$ . Given  $P^3 \in \mathcal{L}^3$ , the DM first replaces each terminal one-stage lottery with its certainty equivalent given the history assignment. This results in a two-stage lottery  $\tilde{P}^2$ . Note that each two-stage sublottery of  $P^3$  is transformed into a terminal one-stage lottery (over certainty equivalents) in  $\tilde{P}^2$ . The DM then replaces each such one-stage lottery in  $\tilde{P}^2$  with its certainty equivalent, given the history assignment of the two-stage



sublottery in  $P^3$  from which it arose. This procedure reduces  $\tilde{P}^2$  into a one-stage lottery  $\tilde{P}^1$ , which the DM evaluates using  $V_0$ . Figure 1 illustrates this procedure for the previous example where  $P^3 = \langle .5, P_1^2; .5, P_2^2 \rangle$ ,  $P_1^2 = \langle .25, p; .5, q; .25, r \rangle$ , and  $P_2^2 = \delta_s$ . Suppose the DM considers the left branch  $P_1^2$  elating ( $e$ ), the right branch  $P_2^2$  disappointing ( $d$ ), and, within  $P_1^2$ , considers  $r$  elating ( $ee$ ) but  $p$  and  $q$  disappointing ( $ed$ ). Letting  $\tilde{P}_1^1 = \langle .25, CE_{ed}(p); .5, CE_{ed}(q); .25, CE_{ee}(r) \rangle$  and  $\tilde{P}_2^1 = \langle 1, CE_d(s) \rangle$ , the DM first reduces  $P^3$  to  $\tilde{P}^2 = \langle .5, \tilde{P}_1^1; .5, \tilde{P}_2^1 \rangle$ , as seen in Figure 1(b); and then reduces  $\tilde{P}^2$  to the one-stage lottery  $\tilde{P}^1 = \langle .5, CE_e(\tilde{P}_1^1); .5, CE_d(\tilde{P}_2^1) \rangle$ , as seen in Figure 1(c). Finally, the value of  $P^3$  is given by  $V_0(\tilde{P}^1)$  (and its certainty equivalent is  $CE_0(\tilde{P}^1)$ ).

Our model of history-dependent risk attitude requires the history assignments the DM uses to be *internally consistent*. Roughly speaking, if a sublottery is considered elating (disappointing), then its certainty equivalent should indeed be weakly larger than (strictly smaller than) the certainty equivalent of the sublottery from which it emanates. In other words, the certainty equivalent of the parent lottery serves as the disappointment-elation threshold.<sup>5</sup> More precisely, denote by  $CE(\cdot; a, \mathcal{V})$  the recursively-calculated certainty equivalent of a sublottery. For  $t = 1, 2$ , a necessary condition for  $P_j^t$  to be an elating outcome of its parent lottery  $P^{t+1} = \langle \alpha_1, P_1^t; \dots; \alpha_n, P_n^t \rangle$ , is that

$$CE(P_j^t; a, \mathcal{V}) \geq CE_{a(P^{t+1})} \left( \langle \alpha_1, CE(P_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(P_n^t; a, \mathcal{V}) \rangle \right).$$

That is, at the history  $a(P^t)$ , the DM prefers the sublottery  $P_j^t$  to the compound lottery  $P^t$ . Similarly, for  $P_j^t$  to be a disappointing outcome of  $P^{t+1}$ , it must be that

$$CE(P_j^t; a, \mathcal{V}) < CE_{a(P^{t+1})} \left( \langle \alpha_1, CE(P_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(P_n^t; a, \mathcal{V}) \rangle \right).$$

Summarizing the discussion so far, the HDRA model is defined as follows in this simple setting.

**Definition 2 (History-dependent risk attitude, HDRA).** *An HDRA representation over three-stage lotteries consists of a collection  $\mathcal{V} := \{V_h\}_{h \in H}$  of expected utility functions over one-stage lotteries (rankable in terms of risk aversion) and a history assignment  $a$ , such that for each  $P^3 \in \mathcal{L}^3$ , the value of  $P^3$  is calculated recursively and the history assignment of each sublottery is internally consistent. We describe a DM with an HDRA representation by the pair  $(\mathcal{V}, a)$ .*

<sup>5</sup>We assume a DM considers an outcome of a nondegenerate lottery elating if its certainty equivalent is at least as large as the certainty equivalent of the parent lottery. Alternatively, we could assume an outcome is disappointing if its certainty equivalent is at least as small as the certainty equivalent of the parent lottery, or even introduce a third assignment, neutral ( $n$ ), which treats the case of equality differently than elation or disappointment. This would not affect our results. How equality is treated may change the value of a lottery where such equality occurs; but in either case, equality is possible only in a measure zero set of lotteries. Generically, a nonempty history consists of a sequence of strict elations and disappointments.

It is easy to see that the HDRA model is ordinal in nature: the ranking over three-stage lotteries induced by the HDRA model is invariant to increasing, potentially different transformations of each of the members in the collection  $\mathcal{V}$ . The reason is that the HDRA model takes into account only the certainty equivalents of sublotteries after each history  $h$ .

In the HDRA model, the DM's risk attitudes depend on the prior sequence of disappointments and elations, but not on the "intensity" of those experiences. That is, the DM is affected only by his general impressions of past experiences. This simplification of histories can be viewed as an extension of the notions of elation and disappointment for one-stage lotteries suggested in Gul (1991) or Chew (1989). In those works, a prize  $x$  is an elating outcome of a lottery  $p$  if it is preferred to  $p$  itself, and is a disappointing outcome if  $p$  is preferred to it. In the HDRA model described here, the same notion is applied recursively throughout the compound lottery to classify sublotteries as elating or disappointing. While the classification of a realization is binary, the probabilities and magnitudes of realizations determine the threshold for elation and disappointment, and in general affect the value of the lottery. By permitting risk attitude to depend only on prior elations and disappointments, this specification allows us to study endogenously evolving risk attitudes under a parsimonious departure from history independence.<sup>6</sup> Behaviorally, such a classification of histories may also describe a cognitive limitation on the part of the DM. The DM may find it easier to recall whether he was disappointed or elated, than whether he was very disappointed or slightly elated. Keeping track of the "exact" intensity of disappointment and elation for each realization – which is itself a compound lottery – may be difficult, leading the DM to classify his impressions into discrete categories: sequences of elations and disappointments.

Returning to our previous example, we illustrate the internal consistency requirement. Recall that  $\tilde{P}_1 = \langle .25, CE_{ed}(p); .5, CE_{ed}(q); .25, CE_{ee}(r) \rangle$  is the recursively constructed sublottery lottery where the one-stage lotteries  $p, q, r$  are replaced with their history-dependent certainty equivalents. To verify that it is internally consistent for  $r$  to be elating and for  $p$  and  $q$  to be disappointing when  $P_1^1$  is elating, one must check that

$$CE_{ed}(p), CE_{ed}(q) < CE_e(\tilde{P}_1) \leq CE_{ee}(r).$$

Finally, to verify that  $P_1^1$  is indeed elating and  $P_2^2$  is disappointing, one must check that

$$CE_d(s) < CE_0(.5, CE_e(\tilde{P}_1); .5, CE_d(s)) \leq CE_e(\tilde{P}_1).$$

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<sup>6</sup>This means, in particular, that a sure realization is treated (discontinuously) differently than a topologically close sublottery with elating and disappointing outcomes. While there is experimental evidence for various types of discontinuities due to "certainty effects" (see, for example, Andreoni and Sprenger (2009, Section 3)), this feature arises here from the modeling simplifications made.

Observe that internal consistency imposes a fixed point requirement that takes into account the entire history assignment for  $P^3$ . Even if  $p, q$ , and  $r$  have an internally consistent history assignment within  $P_1^2$ , this history assignment must also lead to a certainty equivalent for  $P_1^2$  which will be elating relative to  $CE_d(s)$ . In the next section, we explore the strong implications internal consistency has for risk attitudes.

We conclude this section by previewing how the components of this basic model will be extended in Section 5. (The extension allowing the DM to take intermediate actions is given in Section 4). First, the number of periods can be any finite number  $T$ . Second, while we keep requiring that members of  $\mathcal{V}$  are rankable in term of risk aversion, they need not be expected utility preferences. The collection  $\mathcal{V}$  may consist of members from the *betweenness* class of preferences (Dekel (1986), Chew (1989), which includes popular non-expected utility preferences such as Gul's (1991) model of disappointment aversion. Lastly, the threshold rule, according to which the DM determines which realizations are elating or disappointing, can also be extended. The basic model in this section uses the DM's preference at the parent lottery to determine the threshold for elation or disappointment. We refer to this threshold rule as *endogenous*. In the more general model, we allow the threshold rule, denoted  $\tau$ , to be *exogenous*. For example, in an expectation-based threshold rule, where  $\tau(\cdot|P^{t+1}) = \mathbb{E}(\cdot)$ , internal consistency requires that if the DM considers  $P_j^t$  to be elating in  $P^{t+1} = \langle \alpha_1, P_1^t; \dots; \alpha_n, P_n^t \rangle$ , then it must be that  $CE(P_j^t; a, \mathcal{V}) \geq \mathbb{E}(\langle \alpha_1, CE(P_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(P_n^t; a, \mathcal{V}) \rangle)$ . Similarly, if the DM considers  $P_j^t$  disappointing in  $P^{t+1}$ , it must be that  $CE(P_j^t; a, \mathcal{V}) < \mathbb{E}(\langle \alpha_1, CE(P_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(P_n^t; a, \mathcal{V}) \rangle)$ .

### 3 Properties of history-dependent risk attitude

In this section, we study implications of the HDRA model. First, we show that the existence of an HDRA representation implies regularity properties on  $\mathcal{V}$  that are related to well-known cognitive biases; and that in turn, these properties imply the existence of HDRA. (The extension of this result for the more general setting appears in Section 5.) We then discuss several phenomena that arise.

#### 3.1 The reinforcement and primacy effects

Experimental evidence suggests that an individual's risk attitudes depend on how prior risk resolved. The literature suggests that people's risk attitudes are reinforced by prior experiences: they become less risk averse after positive experiences and more risk averse after negative ones. This effect is captured in the following definition.

**Definition 3.**  $\mathcal{V} = \{V_h\}_{h \in H}$  displays the reinforcement effect if  $V_{hd} >_{RA} V_{he}$  for  $h \in \{0, e, d\}$ .

A body of evidence also suggests that individuals are affected by the position of items in a sequence. One well-documented cognitive bias is the *primacy effect*, according to which early observations have a strong effect on later judgments. In our setting, the order in which elations and disappointments occur affect the DM's risk attitude. The reinforcement effect suggests that after an initial elation, a disappointment increases the DM's risk aversion; and that after an initial disappointment, an elation reduces the DM's risk aversion. The primacy effect further suggests that the shift in attitude from the initial realization has a lasting and disproportionate effect. This means that future elation or disappointment can mitigate, but not overpower, the first impression, as in the following definition.

**Definition 4.**  $\mathcal{V} = \{V_h\}_{h \in H}$  displays the primacy effect if  $V_{de} >_{RA} V_{ed}$ .

The reinforcement effect and primacy effects imply that, comparing any two histories of the same length, the DM is more risk averse the *earlier he is disappointed*. That is, the reinforcement and primacy effects correspond to the lexicographic orderings  $V_d >_{RA} V_e$  and  $V_{dd} >_{RA} V_{de} >_{RA} V_{ed} >_{RA} V_{ee}$ . The following result links these cognitive biases to necessary and sufficient conditions for the existence of an HDRA representation.<sup>7</sup>

**Theorem 1 (Necessary and sufficient conditions for HDRA).** *An HDRA representation  $(\mathcal{V}, a)$  exists – that is, there exists an internally consistent assignment  $a$  – if and only if  $\mathcal{V}$  displays the reinforcement and primacy effects.*

As seen in Definition 2, the HDRA model takes as given a collection of expected utility preferences  $\mathcal{V}$  which are ranked in terms of risk aversion, but does not specify *how* they are ranked. Theorem 1 shows that the internal consistency requirement places strong restrictions on how risk aversion evolves with elations and disappointments. We sketch below the main steps of the proof; formally, the result is a special case of Theorem 4 (in Section 5), which is proved in Appendix A.

**Sketch of proof.** We first describe an algorithm for finding an internally consistent assignment when  $\mathcal{V}$  displays the reinforcement and primacy effects. Consider first a two-stage lottery  $P = \langle \alpha_1, p_1; \dots; \alpha_n, p_n \rangle$ . Assume  $p_1$  has the highest  $CE_e(\cdot)$  among all  $p_i$ 's,  $p_n$  has the lowest  $CE_d(\cdot)$  among  $p_2, \dots, p_n$ , and  $CE_e(p_i) > CE_e(p_{i+1})$  for  $i = 1, \dots, n-2$ . Initially set  $p_1$  to be elating

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<sup>7</sup>To have the most concise statement of our results, we ruled out the standard case of history independence by requiring the  $V_h$ 's to be ranked in terms of risk aversion. However, for the standard model, the existence of an internally consistent assignment is trivial: any assignment is consistent, because history does not affect valuations.

and  $p_j$  to be disappointing for  $j > 1$ . If this assignment is internally consistent, we are done. If not, consider  $p_2$ . If  $CE_d(p_2) \geq CE_0(\langle \alpha_1, CE_e(p_1); \alpha_2, CE_d(p_2); \dots; \alpha_n, CE_d(p_n) \rangle)$ , switch  $p_2$  to an elation and replace  $CE_d(p_2)$  with  $CE_e(p_2)$  above. One key observation is that the threshold of any lottery increases by less than an increase in one of its prizes. Hence we know  $CE_e(p_2) \geq CE_0(\langle \alpha_1, CE_e(p_1); \alpha_2, CE_e(p_2); \dots; \alpha_n, CE_d(p_n) \rangle)$ . If this new assignment is internally consistent, we are done; otherwise, repeat this procedure on  $p_3$ , and so on and so forth. The second key observation is that by the ordering of the  $p_i$ 's, the assignment of previous  $p_i$ 's remains intact. If we reach  $p_n$ , the resulting assignment can be shown to be internally consistent. This procedure extends to a three-stage lottery  $\langle \alpha_1, P_1^2; \dots; \alpha_n, P_n^2 \rangle$ , if  $CE_e(\tilde{P}_i^1) > CE_d(\tilde{P}_i^1)$  for any assignment within a nondegenerate  $P_i^2$ , where  $\tilde{P}_i^1$  is derived by replacing all terminal lotteries in  $P_i^2$  with their certainty equivalents. But this property follows from the reinforcement and primacy effects.

Next, to show necessity of the reinforcement effect, consider a nondegenerate one-stage lottery  $p$ . Suppose, by contradiction, that  $V_e >_{RA} V_d$ , which means that  $CE_e(p) < CE_d(p)$ . Pick any  $x \in (CE_e(p), CE_d(p))$  and consider  $P^3 = \langle \alpha, \delta_p; 1 - \alpha, \delta_x^2 \rangle$ . Since  $CE_e(p) < x$ ,  $\delta_p$  cannot be an elating outcome in  $P^3$ . Similarly, since  $CE_d(p) \geq x$ ,  $\delta_p$  cannot be a disappointing outcome in  $P^3$ . Therefore,  $P^3$  would have no internally consistent assignment. To prove that  $V_{ed} >_{RA} V_{ee}$  we may use a lottery of the form  $\langle \alpha, \langle \beta, p; 1 - \beta, \delta_x \rangle; 1 - \alpha, \delta_w^2 \rangle$ ; similarly for  $V_{dd} >_{RA} V_{de}$ .

Finally, we sketch necessity of the primacy effect. The reinforcement effect implies that  $CE_d(\langle .5, b; .5, w \rangle) < CE_e(\langle .5, b; .5, w \rangle)$ ; pick any lottery  $p$  whose support lies in this interval and consider  $P^2 = \langle \alpha, \delta_b; 1 - 2\alpha, p; \alpha, \delta_w \rangle$ . Suppose  $P^2$  is an elating subplottery of some  $P^3$ . To determine whether  $p$  is elating or disappointing in  $P^2$ , we must compare the values of  $CE_h(p)$  and  $CE_e(\langle \alpha, b; 1 - 2\alpha, CE_h(p); \alpha, w \rangle)$ , where  $h$  is either  $ee$  or  $ed$ . By properties of expected utility, this reduces to comparing  $CE_h(p)$  with  $CE_e(\langle .5, b; .5, w \rangle)$ . Since  $CE_h(p)$  is in between the elation and disappointment values of  $\langle .5, b; .5, w \rangle$ ,  $p$  must be disappointing if  $P^2$  is elating, and elating if  $P^2$  is disappointing. Suppose, by contradiction, that  $V_{ed} >_{RA} V_{de}$ . If  $P^2$  is recursively reduced to  $\tilde{P}^1$ , then for small  $\alpha$ ,  $CE_e(\tilde{P}^1) \simeq CE_{ed}(p) < CE_{de}(p) \simeq CE_d(\tilde{P}^1)$ . But then for any  $x \in (CE_e(\tilde{P}^1), CE_d(\tilde{P}^1))$ , the three-stage lottery  $\langle \beta, P^2, 1 - \beta, \delta_x^2 \rangle$  has no internally consistent history assignment. ■

### 3.2 Preferring to lose late rather than early: the “second serve” effect

A recent New York Times article<sup>8</sup> documents a widespread phenomenon in professional tennis: to avoid a double fault after missing the first serve, many players employ a “more timid, perceptibly slower” second serve that is likely to get the ball in play but leaves them vulnerable in the subse-

<sup>8</sup>See “Benefit of Hitting Second Serve Like the First,” August 29, 2010, available for download at <http://www.nytimes.com/2010/08/30/sports/tennis/30serving.html?pagewanted=all>.

quent rally. In the article, Daniel Kahneman attributes this to the fact that “people prefer losing late to losing early.” Kahneman says that “a game in which you have a 20 percent chance to get to the second stage and an 80 percent chance to win the prize at that stage... is less attractive than a game in which the percentages are reversed.” Such a preference was first noted in Ronen (1973).

To study this idea formally within the HDRA model, let us take  $\alpha \in (.5, 1)$  and any two prizes  $H > L$ . How does the two-stage lottery  $P_{ll}^2 = \langle \alpha, \langle 1 - \alpha, \delta_H; \alpha, \delta_L \rangle; 1 - \alpha, \delta_L^2 \rangle$ , where the DM has a good chance of delaying losing, compare with  $P_{le}^2 = \langle 1 - \alpha, \langle \alpha, \delta_H; 1 - \alpha, \delta_L \rangle; \alpha, \delta_L^2 \rangle$ , where the DM is likely to lose earlier? (For simplicity we need only consider two stages here, but to embed this into  $\mathcal{L}^3$  we may either raise the power on each  $\delta$  by one, or consider each of  $P_{ll}^2$  and  $P_{le}^2$  as a sublottery evaluated under a history  $h \in \{0, e, d\}$ ). The standard expected utility model predicts indifference over  $P_{ll}^2$  and  $P_{le}^2$ , because the distribution over final outcomes is the same. To examine the predictions of the HDRA model, let  $u_h$  denote the Bernoulli utility corresponding to expected utility function  $V_h$ . Note that in both lotteries, reaching the final stage is elating under the HDRA model, since  $H > L$ . Then, the HDRA value of  $P_{ll}^2$  is higher than the value of  $P_{le}^2$  starting from history  $h$  if and only if

$$\begin{aligned} \alpha u_h \left( u_{he}^{-1} \left( (1 - \alpha) u_{he}(H) + \alpha u_{he}(L) \right) \right) + (1 - \alpha) u_h(L) > \\ (1 - \alpha) u_h \left( u_{he}^{-1} \left( \alpha u_{he}(H) + (1 - \alpha) u_{he}(L) \right) \right) + \alpha u_h(L). \end{aligned} \quad (1)$$

**Proposition 1.** *The DM prefers losing late to losing early (that is, Equation (1) holds for all  $H > L$ ,  $\alpha \in (.5, 1)$  and  $h \in \{0, e, d\}$ ) if and only if  $V_{he} <_{RA} V_h$ .*

**Proof.** See Appendix A, where we show that Equation (1) is equivalent to  $u_h$  being a concave transformation of  $u_{he}$ . ■

### 3.3 Nonmonotonic behavior: thrill of winning and pain of losing

A DM with an HDRA representation may violate first-order stochastic dominance for certain compound lotteries. For example, if  $\alpha$  is very high, the lottery  $\langle \alpha, \delta_p; 1 - \alpha, \delta_w^2 \rangle$  may be preferred to  $\langle \alpha, \delta_p; 1 - \alpha, \delta_b^2 \rangle$ ; in the former,  $p$  is evaluated as an elation, while in the latter, it is evaluated as a disappointment. Because the prizes  $w$  and  $b$  are received with very low probability, the “thrill of winning” the lottery  $p$  may outweigh the “pain of losing” the lottery  $p$ . This arises from the reinforcement effect on compound risks. While monotonicity with respect to compound first-order stochastic dominance may be normatively appealing, the appeal of such monotonicity is rooted in the assumption of consequentialism (that “what might have been” does not matter). As Mark Machina points out, once consequentialism is relaxed, as is explicitly done in this paper, violations

of monotonicity may naturally occur.<sup>9</sup> In our model, violations of monotonicity arise only on particular compound risks, in situations where the utility gain or loss from a change in risk attitude outweighs the benefit of a prize itself. The idea that winning is enjoyable and losing is painful may also translate to nonmonotonic behavior in more general settings. For example, Lee and Malmendier (2011) show that forty-two percent of auctions for board games end at a price which is higher than the simultaneously available buy-it-now price.

### 3.4 Statistically reversing risk attitudes

Another implication of the reinforcement effect is *statistically reversing risk attitudes*: disappointment is more likely after elation, and vice versa. Greater risk aversion means that each one-stage lottery has a lower certainty equivalent. Using our notions of disappointment and elation, for any nondegenerate  $p \in \mathcal{L}^1$ , and  $h, h'$  such that  $V_h >_{\text{RA}} V_{h'}$ , whenever a prize  $x$  is (i) disappointing in  $p$  under  $V_h$ , then it is disappointing in  $p$  under  $V_{h'}$ , and (ii) elating in  $p$  under  $V_{h'}$ , then it is elating in  $p$  under  $V_h$ . This feature implies that a DM who has been elated is not only less risk averse than if he had been disappointed, but also has a higher elation threshold. In other words, the reinforcement effect implies that after a disappointment, the DM is more risk averse and “settles for less”; whereas after an elation, the DM is less risk averse and “raises the bar.” Therefore, the probability of elation in any sublottery increases if that sublottery is disappointing instead of elating.<sup>10</sup>

### 3.5 Optimism and pessimism

In the HDRA model, it is possible for two individuals to have the same collection of utilities  $\mathcal{V}$  but to disagree on what outcomes are elating and disappointing. For example, consider the two-stage lottery  $\langle \alpha, p; 1 - \alpha, \delta_x \rangle$  and suppose that  $CE_e(p) > x > CE_d(p)$ . In this case, it would be internally consistent for the lottery  $p$  to be either elating or disappointing. The value of a lottery depends on the DM’s history assignment  $a$ , which in turn is revealed by his choice behavior (see Section 5). In this section, we define notions of optimism and pessimism that are behaviorally related to the DM’s risk attitude over compound lotteries.

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<sup>9</sup>As discussed in Mas-Colell, Whinston and Green (1995), Machina offers the example of a DM who would rather take a trip to Venice than watch a movie about Venice, and would rather watch the movie than stay home. Due to the disappointment he would feel watching the movie in the event of not winning the trip itself, Machina points out that the DM might prefer a lottery over the trip and staying home, to a lottery over the trip and watching the movie.

<sup>10</sup>The psychological literature, in particular Parducci (1995) and Smith, Diener and Wedell (1989), provides support for the prediction that elation thresholds increase (decrease) after positive (negative) experiences. Summarizing these works, Schwarz and Strack (1998) observe that “an extreme negative (positive) event increased (decreased) satisfaction with subsequent modest events....Thus, the occasional experience of extreme negative events facilitates the enjoyment of the modest events that make up the bulk of our lives, whereas the occasional experience of extreme positive events reduces this enjoyment.”

**Definition 5.** Given a collection  $\mathcal{V}$ , the DM is an *optimist* (*pessimist*) if for each  $P^3$  he selects the internally consistent history assignment  $a$  that maximizes (minimizes) the HDRA utility of  $P^3$ . If  $\mathcal{V}^A, \mathcal{V}^B$  are ordinally equivalent then  $(\mathcal{V}^A, a^A)$  is more optimistic than  $(\mathcal{V}^B, a^B)$  if  $CE(P^3; a^A, \mathcal{V}^A) \geq CE(P^3; a^B, \mathcal{V}^B)$  for each  $P^3$ , with at least one strict comparison.

When multiple internally consistent assignments are possible, the optimist selects the most favorable interpretation of events and the pessimist selects the least favorable one. To tie Definition 5 to choice behavior, we begin by considering a class of compound lotteries  $\mathcal{L}_u^3$  under which the history assignment is unambiguous. In words,  $\mathcal{L}_u^3$  consists of compound lotteries where in each of the first two stages, either the DM learns he will receive an extreme prize ( $b$  or  $w$ ) or must incur further risk. Formally,

$$\mathcal{L}_u^3 = \left\{ \langle \alpha_1, \langle \alpha_2, p; 1 - \alpha_2, z_2 \rangle; 1 - \alpha_1, z_1 \rangle \mid p \in \mathcal{L}^1, z_i \in \{b, w\} \text{ and } \alpha_i \in [0, 1] \right\}.$$

For example, if  $z_1 = b$  and  $z_2 = w$ , then upon reaching  $p$  the DM would have been disappointed in the first stage and elated in the second.<sup>11</sup>

Consider two HDRA preferences  $\succeq^A$  and  $\succeq^B$  over three-stage lotteries. Extending the standard comparative notion of risk aversion, we say that  $\succeq^A$  is *less risk averse over compound lotteries* than  $\succeq^B$  if for any  $P^3 \in \mathcal{L}^3$  and  $x \in X$ ,  $P^3 \succeq^B \delta_x^3$  implies  $P^3 \succeq^A \delta_x^3$ , with at least one strict comparison.<sup>12</sup>

**Proposition 2.** Let  $\succeq^A$  and  $\succeq^B$  be HDRA preferences over  $\mathcal{L}^3$  arising from  $(\mathcal{V}^A, a^A)$  and  $(\mathcal{V}^B, a^B)$ .

- (i)  $\succeq^A$  and  $\succeq^B$  agree on  $\mathcal{L}_u^3$  if and only if  $\mathcal{V}^A$  and  $\mathcal{V}^B$  are ordinally equivalent.
- (ii)  $\succeq^A$  is more optimistic than  $\succeq^B$  if and only if  $\succeq^A$  and  $\succeq^B$  agree on  $\mathcal{L}_u^3$  and  $\succeq^A$  is less risk averse over compound lotteries than  $\succeq^B$ .

**Proof.** Part (i) follows from applying the model on  $\mathcal{L}_u^3$ ; part (ii) follows from the definition. ■

It follows immediately from part (ii) that  $\succeq^A$  is an optimist (pessimist) if and only if  $\succeq^A$  is less (more) risk averse over compound lotteries than any preference  $\succeq^B$  arising from an HDRA representation  $(\mathcal{V}^B, a^B)$  and which agrees with  $\succeq^A$  on  $\mathcal{L}_u^3$ . Figure 2 illustrates this distinction between optimism and pessimism, depicting for each  $\omega \in (0, 1)$  all possible HDRA values of the lottery  $P^3(\omega) = \langle 1, P^2(\omega) \rangle$ , where  $P^2(\omega) = \langle \frac{1}{3}, \delta_1; \frac{1}{3}, \delta_2; \frac{1}{3}, p(\omega) \rangle$  and  $p(\omega) = \langle \omega, 3; 1 - \omega, 0 \rangle$ , using a CRRA collection  $\mathcal{V}$ . An increase in  $\omega$  is a first-order stochastic improvement of  $p(\omega)$ . While

<sup>11</sup>Whenever  $\alpha_i = 1$ , the DM does not learn anything and the history assignment is unchanged.

<sup>12</sup>For any  $P^3$ , let  $E(P^3)$  be the expected value of the probability distribution over final outcomes induced by  $P^3$ . The DM is risk averse over compound lotteries if for all  $P^3$ , he prefers the sure outcome  $E(P^3)$  to  $P^3$ .



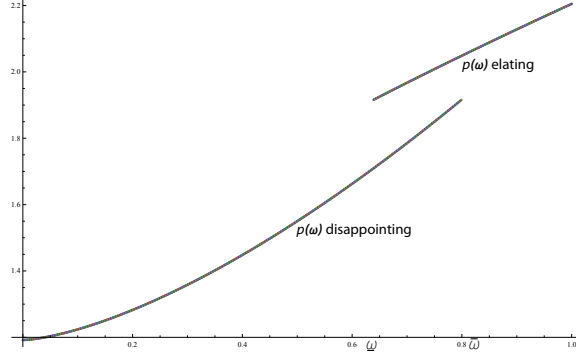


Figure 2: Possible HDRA utilities of  $P^3(\omega)$  are pictured on the vertical axis for each  $\omega \in (0, 1)$  on the horizontal axis, given CRRA utilities  $V_h = \mathbb{E}\left(\frac{x^{1-\rho_h}}{1-\rho_h}; \cdot\right)$ , where  $\rho_e = 0$ ,  $\rho_0 = 1/4$ ,  $\rho_d = 1/2$ . The subplottery  $p(\omega)$  can be viewed as an elation or a disappointment in the range  $[\underline{\omega}, \bar{\omega}]$ .

$p(\omega)$  is unambiguously elating (disappointing) for high (low) values of  $\omega$ , there is an intermediate range  $[\underline{\omega}, \bar{\omega}]$  where  $p(\omega)$  can be viewed either as an elation or as a disappointment. The optimist views  $p(\omega)$  as an elation as soon as possible (for all  $\omega \geq \underline{\omega}$ ), while the pessimist views  $p(\omega)$  as a disappointment for as long as possible (for all  $\omega \leq \bar{\omega}$ ). More generally, a DM may have a cutoff  $\omega^* \in [\underline{\omega}, \bar{\omega}]$  at which  $p(\omega)$  switches from a disappointment to an elation. If one DM is more optimistic than another, then his cutoff  $\omega^*$  must be lower. In the range of  $\omega$ 's where they disagree, there are sure outcomes (those in between the two utilities of  $P^2(\omega)$ ) that the more pessimistic DM prefers to the risky lottery  $P^2(\omega)$ , while the more optimistic DM prefers to face the risk.

## 4 Extension to intermediate actions, and a dynamic asset pricing problem

In this section, we extend our previous results to settings where the DM may take intermediate actions while risk resolves. We then apply the model to a three-period asset pricing problem to examine the impact of history-dependent risk attitude on prices.

### 4.1 HDRA with intermediate actions

In the HDRA model with intermediate actions, the DM categorizes each realization of a dynamic (stochastic) decision problem – which is a choice set of shorter dynamic decision problems – as elating or disappointing. He then recursively evaluates all the alternatives in each choice problem based on the preceding sequence of elations and disappointments.

Formally, for any set  $Z$ , let  $K(Z)$  be the set of finite, nonempty subsets of  $Z$ . A one-stage decision problem is simply a one-stage lottery; we write  $\mathcal{D}^1 = \mathcal{L}^1$ . The set of finite, nonempty

sets of one-stage decision problems is given by  $\mathcal{A}^1 = K(\mathcal{D}^1)$ . The set of two-stage decision problems is thus  $\mathcal{D}^2 = \mathcal{L}(\mathcal{A}^1)$ . The set of finite, nonempty sets of two-stage decision problems is  $\mathcal{A}^2 = K(\mathcal{D}^2)$ . The DM's domain of choice is  $\mathcal{D}^3 = \mathcal{L}(\mathcal{A}^2)$ .

The set of possible histories  $H$  and collection of expected utility preferences  $\mathcal{V} = \{V_h\}_{h \in H}$  are the same as before, with the understanding that histories now refer to choice nodes. For each  $P^3 \in \mathcal{D}^3$ , the history assignment  $a(\cdot|P^3)$  maps each choice set in  $P^3$  (an element of  $\mathcal{A}^1 \cup \mathcal{A}^2$ ) to a history in  $H$  that describes the preceding sequence of elations and disappointments. We initialize  $a(P^3|P^3) = 0$ . The DM recursively evaluates decision problems in the following way. Given a three-stage decision problem  $P^3$  and a history assignment  $a(\cdot|P^3)$  over decision problems, the DM first replaces each terminal one-stage decision problem  $A^1 \in \mathcal{A}^1$  with  $\max_{p \in A^1} CE_{a(A^1|P^3)}(p)$ . This results in a two-stage decision problem  $\tilde{P}^2$ . Note that each original choice set  $A^2 \in \mathcal{A}^2$  of two-stage decision problems is transformed in  $\tilde{P}^2$  into a terminal choice set  $\tilde{A}^1$  consisting of one-stage lotteries over certainty equivalents. The DM then replaces each such terminal choice set in  $\tilde{P}^2$  with  $\max_{p \in \tilde{A}^1} CE_{a(A^2|P^3)}(p)$ . This reduces  $\tilde{P}^2$  into a one-stage lottery  $\tilde{P}^1$ , which he evaluates using  $V_0$ .

In summary, the history assignment of a choice set determines the one-stage utility  $V_h$  used to evaluate each (recursively reduced) one-stage decision problem inside it. The value of a choice set is the maximal value of those (recursively reduced) one-stage decision problems. As before, the history assignment of choice sets must be internally consistent. That is, for  $A_j^t$  to be an elating (disappointing) outcome of  $P^{t+1} = \langle \alpha_1, A_1^t; \dots; \alpha_n, A_n^t \rangle$ , where  $P^{t+1} \in A^{t+1}$ , then we must have  $CE(A_j^t; a, \mathcal{V}) \geq (<) CE_{a(A^{t+1})}(\langle \alpha_1, CE(A_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(A_n^t; a, \mathcal{V}) \rangle)$ . The definition of HDRA is almost the same as before.

**Definition 6 (HDRA with intermediate actions).** *An HDRA representation over three-stage decision problems consists of a collection  $\mathcal{V} := \{V_h\}_{h \in H}$  of expected utility functions over one-stage lotteries (rankable in terms of risk aversion) and a history assignment  $a$  such that for each  $P^3 \in \mathcal{D}^3$ , the value of  $P^3$  is calculated recursively and the history assignment of each choice set is internally consistent.*

Observe that the DM is “sophisticated” under HDRA with intermediate actions. From any future choice set, the DM anticipates selecting the best continuation decision problem. That choice leads to an internally consistent history assignment of that choice set. When reaching a choice set, the single-stage utility he uses to evaluate the choices therein is the one he anticipated using, and his choice is precisely his anticipated choice. Internal consistency is thus a stronger requirement than before, because it takes optimal choices into account. However, our previous result extends.

**Theorem 2 (Extension to intermediate actions).** *An HDRA representation with intermediate actions  $(\mathcal{V}, a)$  exists if and only if  $\mathcal{V}$  displays the reinforcement and primacy effects.*

**Proof.** See Appendix A. ■

## 4.2 A three-period asset pricing problem

We now apply the HDRA model with intermediate actions to a three period asset-pricing problem in a representative-agent economy. We show that the model yields predictable, path-dependent prices that exhibit excess volatility arising from actual and anticipated changes in risk aversion.

In each period  $t = 1, 2, 3$ , there are two assets traded, one safe and one risky. At the end of the period, the risky asset yields  $\tilde{y}$ , which is equally likely to be *High* ( $H$ ) or *Low* ( $L$ ). The second is a risk-free asset returning  $R = 1 + r$ , where  $r$  is the risk-free rate of return. Asset returns are in the form of a perishable consumption good that cannot be stored; it must be consumed in the current period. Each agent is endowed with one share of the risky asset in each period. The realization of the risky asset in period  $t$  is denoted  $y_t$ . In the beginning of each period  $t > 1$ , after a sequence of realizations  $(y_1, \dots, y_{t-1})$ , each agent can trade in the market, at price  $P(y_1, \dots, y_{t-1})$  for the risky asset, with the risk-free asset being the numeraire. At  $t = 1$ , there are no previous realizations and the price is simply denoted  $P$ . At the end of each period, the DM learns the realization of  $\tilde{y}$  and consumes the perishable return. The payoff at each terminal node of the three-stage decision problem is the sum of per-period consumptions.<sup>13</sup> The decision problem of the representative agent in each period  $t$  is to determine the share  $\alpha(y_1, \dots, y_{t-1})$  of property rights to retain on his unit of risky asset given  $(y_1, \dots, y_{t-1})$ . The agent is purchasing additional shares when  $\alpha(y_1, \dots, y_{t-1}) > 1$ , and is short-selling when  $\alpha(y_1, \dots, y_{t-1}) < 0$ .

The representative agent has HDRA preferences with underlying CARA utilities; that is, the Bernoulli function after history  $h$  is  $u_h(x) = 1 - e^{-\lambda_h x}$ . The CARA specification means that our results will not arise from wealth effects. For this section, we use the following simple parametrization of the agent's coefficients of absolute risk aversion. Consider  $a, b$  satisfying  $0 < a < 1 < b$  and  $\lambda_0 > 0$ . In the first period, elation scales down the agent's risk aversion by  $a^2$ , while disappointment scales it up by  $b^2$ . In the second period, elation scales down the agent's current risk aversion by  $a$ , while disappointment scales it up by  $b$ . In summary,  $\lambda_e = a^2 \lambda_0$ ,  $\lambda_d = b^2 \lambda_0$ ,  $\lambda_{ee} = a^3 \lambda_0$ ,  $\lambda_{ed} = a^2 b \lambda_0$ ,  $\lambda_{de} = b^2 a \lambda_0$ , and  $\lambda_{dd} = b^3 \lambda_0$ . This parametrization satisfies the reinforcement and primacy effects, and has the feature that  $\lambda_h \in (\lambda_{he}, \lambda_{hd})$  for all  $h$ .

<sup>13</sup>Alternatively, one could let the terminal payoff be some function of the consumption vector, in which case the agent is evaluating lotteries over terminal utility instead of total consumption.

As can be seen from our analysis below, if the agent's risk aversion is independent of history and fixed at  $\lambda_0$  at every stage, then the asset price is constant over time. By contrast, in the HDRA model, prices depend on past realizations of the asset, even though past and future realizations are statistically independent. The following result formalizes the predictions of the HDRA model.

**Theorem 3.** *Under the HDRA model given the parametrization above, the price moves in response to past realizations as follows:*

- (i)  $P(H, H) > P(H, L) > P(L, H) > P(L, L)$  at  $t = 3$ .
- (ii)  $P(H) > P(L)$  at  $t = 2$ .
- (iii) Price increases after each *High* realization:  $P(H, H) > P(H) > P(0)$  and  $P(L, H) > P(L)$ .
- (iv) The prices  $P(0)$ ,  $P(L)$ , and  $P(L, L)$  are all below the (constant) price under history independent risk aversion  $\lambda_0$ .

**Proof.** A partial proof is provided in the text below. For omitted details, see Appendix B. ■

Theorem 3 is illustrated in Figure 3, which depicts the simulated price path for the specification  $\lambda_0 = .005$ ,  $b = 1.2$ ,  $a = .8$ ,  $H = 20$  and  $L = 0$ .<sup>14</sup> In that case, the price also decreases after each *Low* realization:  $P(0) > P(L) > P(L, L)$  and  $P(H) > P(H, L)$ .<sup>15</sup> In general, this need not be true. Notice that the DM faces one fewer stage of risk each time there is a realization of the asset. Nonetheless, price is constant with history-independent CARA preferences. With history-dependent risk aversion, a compound risk may become even riskier due to the fact that the continuation certainty equivalents fluctuate with risk aversion. That is, expected future risk aversion movements introduce an additional source of risk, causing price volatility. The price after each history is a convex combination of  $H$  and  $L$ , with weights that depend on the product of current risk aversion and the spread between the future certainty equivalents (as seen in Table 1). Depending on how much risk aversion fluctuates, there may be an upward trend in prices, simply from having fewer stages of risk left. As will be seen in our analysis below, elation (*High* realizations) reinforces that trend, because the agent is both less risk averse and faces fewer stages of risk. However, there is tension between these two forces after disappointment (*Low* realizations), because the agent is more

<sup>14</sup>Estimates of CARA coefficients in the literature are highly variable, ranging from .00088 (Cohen and Einav (2007)) to .0085 to .14 (see Saha, Shumway and Talpaz (1994) for a summary of estimates). Using  $\lambda_0 = .005$ ,  $b = 1.2$  and  $a = .8$  yields CARA coefficients between .0025 and .00864.

<sup>15</sup>Barberis, Huang and Santos (2001) propose and calibrate a model where investors have linear loss aversion preferences and derive gain-loss utility only over fluctuations in financial wealth. In our model, introducing consumption shocks would induce shifts in risk aversion. They assume that the amount of loss aversion decreases with a statistic that depends on past stock prices. In their calibration, this leads to price increases (decreases) after good (bad) dividends and high volatility.

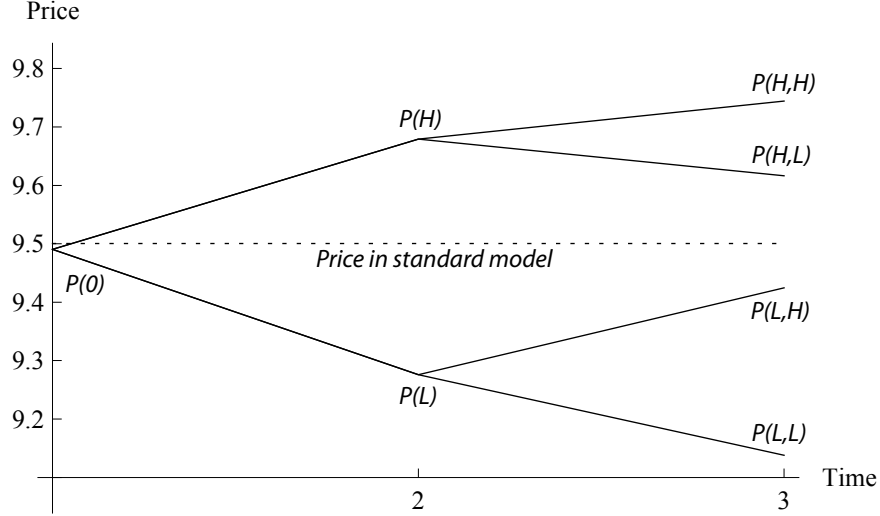


Figure 3: The predicted HDRA price path when  $\lambda_0 = .005$ ,  $a = .8$ ,  $b = 1.2$ ,  $H = 20$  and  $L = 0$ .

risk averse even though he faces a shorter horizon. One can find parameter values where the upward trend dominates, and *Low* realizations yield a (quantitatively very small) price increase – while maintaining the rankings in Theorem 3. Intuitively, this occurs when disappointment has a very weak effect on risk aversion ( $b \approx 1$ ) but elation has a strong effect, because then expected variability in future risk aversion (hence expected variability in utility) prior to a realization may overwhelm the small increase in risk aversion after a *Low* realization occurs.

We now discuss the proof of Theorem 3. We begin by illustrating how the agent uses the HDRA model to rebalance his portfolio. To do this, we first solve the agent’s optimization problem under the recursive application of one-stage preferences using an arbitrary history assignment, and later find the internally consistent one. Since in equilibrium the representative agent must hold his initial share of the asset, the first-order conditions from portfolio optimization pin down prices given a history assignment.

#### *The agent’s optimization problem given a history assignment*

We solve the agent’s problem by backward induction, denoting by  $h(y_1, \dots, y_{t-1})$  the agent’s history assignment after the sequence of realizations  $(y_1, \dots, y_{t-1})$  and later solving for the right assignment. After the realizations  $(y_1, y_2)$  but before learning  $y_3$ , the agent solves the problem:

$$\max_{\alpha(y_1, y_2)} \mathbb{E}_{\tilde{y}_3} \left[ u_{h(y_1, y_2)} \left( c(y_1) + c(y_2|y_1) + \alpha(y_1, y_2)\tilde{y}_3 + (1 - \alpha(y_1, y_2))P(y_1, y_2)R \right) \right],$$

where  $c(y_1) = \alpha y_1 + (1 - \alpha)PR$  denotes the realized consumption in period 1 and  $c(y_2|y_1) =$

$\alpha(y_1)y_t + (1 - \alpha(y_1))P(y_1)R$  denotes the realized consumption in period 2. Using the CARA form, the first-order condition given  $y_1$  and  $y_2$  simplifies to

$$P(y_1, y_2)R = \frac{1}{1 + \exp(\lambda_{h(y_1, y_2)} \alpha(y_1, y_2)(H - L))} H + \frac{\exp(\lambda_{h(y_1, y_2)} \alpha(y_1, y_2)(H - L))}{1 + \exp(\lambda_{h(y_1, y_2)} \alpha(y_1, y_2)(H - L))} L. \quad (2)$$

Let  $\alpha^*(y_1, y_2)$  be the optimal choice and let  $c^*(\tilde{y}_3 | y_1, y_2) = \alpha^*(y_1, y_2)\tilde{y}_3 + (1 - \alpha^*(y_1, y_2))P(y_1, y_2)R$  denote the optimal consumption plan for  $t = 3$  given  $y_1$  and  $y_2$ .

We now introduce some useful notation. Given a one-dimensional random variable  $\tilde{x}$  and a function  $f$  of that random variable, we let  $\Gamma_{\tilde{x}}(\lambda, f(\tilde{x}))$  denote the certainty equivalent of  $f(\tilde{x})$  given CARA preferences with coefficient  $\lambda$ . That is,

$$\Gamma_{\tilde{x}}(\lambda, f(\tilde{x})) = -\frac{1}{\lambda} \ln \mathbb{E}_{\tilde{x}} \left[ \exp(-\lambda f(\tilde{x})) \right].$$

Using this notation and the CARA functional form, the certainty equivalent  $CE(y_1, y_2)$  of the choice problem in  $t = 3$  after  $(y_1, y_2)$  is then:

$$CE(y_1, y_2) = c(y_1) + c(y_2 | y_1) + \Gamma_{\tilde{y}_3} \left( \lambda_{h(y_1, y_2)}, c^*(\tilde{y}_3 | y_1, y_2) \right),$$

which is the sum of period-1 consumption, period-2 consumption, and the certainty equivalent of period-3 optimal consumption given risk aversion  $\lambda_{h(y_1, y_2)}$ .

Proceeding backwards, after observing  $y_1$  but before learning  $y_2$ , the agent solves the problem<sup>16</sup>

$$\max_{\alpha(y_1)} \mathbb{E}_{\tilde{y}_2} \left[ u_{h(y_1)} \left( CE(y_1, \tilde{y}_2) \right) \right],$$

where  $CE(y_1, y_2)$ , defined above, is a function of  $\alpha(y_1)$  through  $c(y_2 | y_1)$ . Using the CARA form, the first-order condition at  $t = 2$  simplifies to

$$P(y_1)R = \frac{1}{1 + \exp(\lambda_{h(y_1)} (CE(y_1, H) - CE(y_1, L)))} H + \frac{\exp(\lambda_{h(y_1)} (CE(y_1, H) - CE(y_1, L)))}{1 + \exp(\lambda_{h(y_1)} (CE(y_1, H) - CE(y_1, L)))} L. \quad (3)$$

Let  $\alpha^*(y_1)$  be the optimal choice and denote by  $c^*(\tilde{y}_2 | y_1) = \alpha^*(y_1)\tilde{y}_2 + (1 - \alpha^*(y_1))P(y_1)R$  the agent's optimal consumption plan for  $t = 2$  given  $y_1$ . The certainty equivalent  $CE(y_1)$  of the choice

<sup>16</sup>Note that the  $t = 1, 2$  problems fix future histories regardless of the choice of  $\alpha$  and  $\alpha(y_1)$ . Because period-3 prices will be such that the agent holds the asset at *some* positive level of risk aversion given  $(y_1, y_2)$ , we know that  $P(y_1, y_2) < \frac{1}{R} \frac{H+L}{2}$ . In period 3, the agent is thus willing to hold some amount of the asset at *any* level of risk aversion. This means that in periods  $t = 1, 2$ , even if the agent were hypothetically not to hold the asset, he would still be exposed to risk at  $t = 1, 2$  due to the influence on period-3 prices; hence the agent's history is as specified.

problem in  $t = 2$  after the realization  $y_1$  is then:

$$CE(y_1) = c(y_1) + \Gamma_{\tilde{y}_2} \left( \lambda_{h(y_1)}, c^*(\tilde{y}_2|y_1) + \Gamma_{\tilde{y}_3} \left( \lambda_{h(y_1, \tilde{y}_2)}, c^*(\tilde{y}_3|y_1, \tilde{y}_2) \right) \right).$$

The random variable  $c^*(\tilde{y}_2|y_1) + \Gamma_{\tilde{y}_3} \left( \lambda_{h(y_1, \tilde{y}_2)}, c^*(\tilde{y}_3|y_1, \tilde{y}_2) \right)$ , which is a function of  $\tilde{y}_2$ , is the sum of period-2 optimal consumption and the certainty equivalent of period-3 optimal consumption given risk aversion  $\lambda_{h(y_1, \tilde{y}_2)}$ . The term  $CE(y_1)$  is simply the certainty equivalent of this random variable given risk aversion  $\lambda_{h(y_1)}$ .

Should the agent choose to hold an initial share  $\alpha$  of the risky asset, the value of the decision problem the agent faces at  $t = 1$  is given by  $\mathbb{E}_{\tilde{y}_1} [u_0(CE(\tilde{y}_1))]$ , where  $CE(y_1)$  depends on  $\alpha$  through  $c(y_1)$ . The agent then chooses the optimal initial share  $\alpha^*$ :

$$\max_{\alpha} \mathbb{E}_{\tilde{y}_1} \left[ u_0 \left( CE(\tilde{y}_1) \right) \right].$$

Using the CARA form, and letting  $P$  be the initial price of the risky asset in period  $t = 1$  prior to any realizations, the first-order condition at  $t = 1$  simplifies to

$$PR = \frac{1}{1 + \exp(\lambda_0(CE(H) - CE(L)))} H + \frac{\exp(\lambda_0(CE(H) - CE(L)))}{1 + \exp(\lambda_0(CE(H) - CE(L)))} L. \quad (4)$$

In view of (2), (3), and (4) and the fact that  $\exp(\cdot) \geq 0$ , we conclude that all the  $P(y_1, \dots, y_{t-1})R$ 's are a convex combination of  $H$  and  $L$ , where the weights depend on the agent's risk aversion after the realizations  $y_1, \dots, y_{t-1}$ . To determine those levels of risk aversion, we use the equilibrium condition that the representative agent must optimally hold his per-period endowment of one share of risky asset after any realization and given his history assignment. That is, we plug  $\alpha^*(y_1, y_2) = \alpha^*(y_1) = \alpha^* = 1$  into the formulas for  $CE(y_1, y_2)$  and  $CE(y_1)$  and look for an internally consistent history assignment in order to deduce the equilibrium prices.

#### *Verifying internal consistency of the history assignment*

We now check that it is internally consistent for the agent to consider each *High* realization elating and each *Low* realization disappointing (Lemma 6 in the Appendix shows that this is also the unique internally consistent history assignment). We proceed recursively, fixing a realization  $y_1 \in \{L, H\}$ . Being elated by  $y_2 = H$  and disappointed by  $y_2 = L$  is internally consistent if the resulting

certainty equivalents for the  $t = 3$  choice problems satisfy

$$CE(y_1, H) \geq CE_{h(y_1)} \left( \left\langle \frac{1}{2}, CE(y_1, H); \frac{1}{2}, CE(y_1, L) \right\rangle \right) > CE(y_1, L).$$

But this holds if and only if  $CE(y_1, H) > CE(y_1, L)$ , which in turns holds if and only if

$$H + \Gamma_{\tilde{y}_3}(\lambda_{h(y_1)e}, \tilde{y}_3) > L + \Gamma_{\tilde{y}_3}(\lambda_{h(y_1)d}, \tilde{y}_3). \quad (5)$$

As  $\lambda$  increases, risk aversion increases and the certainty equivalent  $\mu_{\tilde{y}_3}(\lambda, \tilde{y}_3)$  decreases. Hence the reinforcement effect, or  $\lambda_{h(y_1)e} < \lambda_{h(y_1)d}$ , implies that Equation (5) must hold.

Proceeding backwards, observe that being elated by  $y_1 = H$  and disappointed by  $y_2 = L$  is internally consistent if, similarly to our previous calculation, the resulting certainty equivalents for the  $t = 2$  choice problems satisfy  $CE(H) > CE(L)$ . In turn, this holds if and only if

$$H + \Gamma_{\tilde{y}_2} \left( \lambda_e, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(H,\tilde{y}_2)}, \tilde{y}_3) \right) > L + \Gamma_{\tilde{y}_2} \left( \lambda_d, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(L,\tilde{y}_2)}, \tilde{y}_3) \right), \quad (6)$$

where the history assignments above are  $h(H, H) = ee$ ,  $h(H, L) = ed$ ,  $h(L, H) = de$ , and  $h(L, L) = dd$ . Using this, it is easy to compare the certainty equivalent  $\Gamma_{\tilde{y}_3}(\lambda_{h(y_1,\tilde{y}_2)}, \tilde{y}_3)$  on each side of Equation (6). By the primacy and reinforcement effects,  $\lambda_{ee} < \lambda_{ed} < \lambda_{de} < \lambda_{dd}$ . Thus, given any realization of  $\tilde{y}_2$ , the random variable  $\tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(y_1,\tilde{y}_2)}, \tilde{y}_3)$  takes a larger value when  $y_1 = H$  than when  $y_1 = L$ . Moreover, due to the reinforcement effect, the random variable is also evaluated using a less risk-averse coefficient when  $y_1 = H$  than when  $y_1 = L$ . Hence Equation (6) must also hold, and our proposed history assignment is internally consistent.

### *Implications for prices*

Finally, we apply the results above to study prices. It is immediately clear from (2)-(4) that the asset price would be constant over time in the standard setting, where risk aversion is history independent. If risk aversion is fixed at  $\lambda_0$ , then the price  $\bar{P}$  in the standard model would simply be

$$\bar{P} = \frac{1}{R} \left( \frac{1}{1 + \exp(\lambda_0(H-L))} H + \frac{\exp(\lambda_0(H-L))}{1 + \exp(\lambda_0(H-L))} L \right).$$

In the HDRA model, however, the asset price depends on  $y_1$  and  $y_2$ , not only through current risk aversion, but also through the impact on future certainty equivalents. Setting  $\alpha^*(y_1, y_2) = \alpha^*(y_1) = \alpha^* = 1$  in the first-order conditions (2)-(4) and using the history assignment in the result, the asset price  $P(y_1, \dots, y_t)$  takes the form  $\frac{1}{R} \left( \frac{1}{1 + \exp(f(y_1, \dots, y_t))} H + \frac{\exp(f(y_1, \dots, y_t))}{1 + \exp(f(y_1, \dots, y_t))} L \right)$ , where:



Realizations	Weight $f(y_1, \dots, y_t)$
0	$\lambda_0(H - L + \Gamma_{\tilde{y}_2}(\lambda_e, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(H, \tilde{y}_2}), \tilde{y}_3)) - \Gamma_{\tilde{y}_2}(\lambda_d, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(L, \tilde{y}_2)}, \tilde{y}_3)))$
$y_1$	$\lambda_{h(y_1)}(H - L + \Gamma_{\tilde{y}_3}(\lambda_{h(y_1)e}, \tilde{y}_3) - \Gamma_{\tilde{y}_3}(\lambda_{h(y_1)d}, \tilde{y}_3))$
$y_1, y_2$	$\lambda_{h(y_1, y_2)}(H - L)$

Table 1: The weighting function  $f(y_1, \dots, y_t)$  for prices.

In all these cases, an increase in  $f(y_1, \dots, y_t)$  decreases the weight on  $H$  and thus decreases the price of the asset. Since  $H > L$ , the ranking  $P(H, H) > P(H, L) > P(L, H) > P(L, L)$  in Theorem 3(i) follows immediately from the reinforcement and primacy effects. The ranking  $P(H) > P(L)$  in Theorem 3(ii) is proved in Lemma 7 in the Appendix. To see why some argument is required, notice that  $\lambda_e < \lambda_d$  is not sufficient to show that  $P(H) > P(L)$  unless we know something more about how the difference in the certainty equivalents of period-three consumption,  $\Gamma_{\tilde{y}_3}(\lambda_{h(y_1)e}, \tilde{y}_3) - \Gamma_{\tilde{y}_3}(\lambda_{h(y_1)d}, \tilde{y}_3)$ , compares after  $y_1 = H$  versus  $y_1 = L$ .

The rankings  $P(H, H) > P(H)$  and  $P(L, H) > P(L)$  in Theorem 3(iii) are easily seen using the table above. First, the parameterization implies  $\lambda_{ee} < \lambda_e$ , and  $\lambda_{de} < \lambda_d$ . The proof is complete by recalling that the term  $\Gamma_{\tilde{y}_3}(\lambda_{h(y_1)e}, \tilde{y}_3) - \Gamma_{\tilde{y}_3}(\lambda_{h(y_1)d}, \tilde{y}_3)$  is positive, as shown in our argument for internal consistency. An additional step of proof, given in Lemma 8 of the Appendix, is needed to show that  $P(H) > P(0)$ . Similarly, Theorem 3(iv) follows from  $\lambda_{dd} > \lambda_d > \lambda_0$  combined with the fact that  $\lambda_h$  always multiplies a term strictly larger than  $H - L$ . Hence the prices  $P(0)$ ,  $P(L)$ , and  $P(L, L)$  all fall below the price  $\bar{P}$  in the standard model with constant risk aversion  $\lambda_0$ .

## 5 Generalization

In the basic version of the HDRA model studied in the previous sections, we confined our attention to (i) three time periods, (ii) expected utility preferences in each stage, and (iii) a rule for classifying disappointing and elating outcomes based on the DM's endogenously changing preferences. In this section, we present a model of history-dependent risk attitude that incorporates an arbitrary number of time periods, a more general class of preferences in each stage, and a more general threshold rule for determining disappointments and elations. We then extend our results on the evolution of risk attitudes to this generalized setting, and describe how the primitives of the model may be elicited from choice behavior.

## **$T$ -stage lotteries**

To simplify exposition, we present our results for  $T$ -stage lotteries; the extension to  $T$ -stage decision problems is immediate. Keeping the set of prizes  $X = [w, b]$  and the set of simple one-stage lotteries  $\mathcal{L}^1$  as before, the set of  $T$ -stage lotteries,  $\mathcal{L}^T$ , is defined by the inductive relation  $\mathcal{L}^T = \mathcal{L}(\mathcal{L}^{T-1})$  for  $T \geq 2$ . A typical element  $P^T$  of  $\mathcal{L}^T$  has the form  $P^T = \langle \alpha_1, P_1^{T-1}; \dots; \alpha_m, P_m^{T-1} \rangle$ , where each  $P_j^{T-1} \in \mathcal{L}^{T-1}$  is a  $(T-1)$ -stage lottery. If  $P_j^{T-1}$  is the outcome of  $P^T$ , then all remaining uncertainty is resolved according to  $P_j^{T-1}$ . The degenerate lottery  $\delta_x^T \in \mathcal{L}^T$  gives the lottery  $\delta_x^{T-1}$  with probability one (i.e.,  $x$  is received with probability one after  $T$  stages). We also use  $\delta_{P^t}^{T-t}$  for the lottery where the  $t$ -stage sublottery  $P^t$  is received after  $T-t$  degenerate stages. As before, a  $t$ -stage lottery  $P^t$  is a *sublottery* of  $P^T$  if there is a sequence  $P^{t+1}, P^{t+2}, \dots, P^T$  such that for every  $t' \in \{t, \dots, T-1\}$ ,  $P^{t'} \in \text{supp } P^{t'+1}$ .

For any  $T$ -stage lottery  $P^T$ , the initial history is empty – that is,  $a(P^T | P^T) = 0$ . If a sublottery is degenerate – i.e., it leads to some sublottery with probability one – then the DM is not exposed to risk at that stage and his history is unchanged. If a sublottery is nondegenerate, each sublottery in its support may be an elating ( $e$ ) or disappointing ( $d$ ) outcome. The DM's history  $a(P^t | P^T)$  at any sublottery  $P^t$  is the preceding sequence of  $e$ 's and  $d$ 's. The set of all possible histories is thus

$$H = \{0\} \cup \bigcup_{t=1}^{T-1} \{e, d\}^t.$$

Each history  $h \in H$  corresponds to a utility function  $V_h : \mathcal{L}^1 \rightarrow \mathbb{R}$  over one-stage lotteries.

## **Betweenness preferences**

The DM may recursively apply continuous and monotone single-stage preferences  $\succeq_1$  satisfying the *betweenness* property: for all  $p, q \in \mathcal{L}^1$  and  $\alpha \in [0, 1]$ ,  $p \succeq_1 q$  implies  $p \succeq_1 \alpha p + (1 - \alpha)q \succeq_1 q$ , where  $\alpha p + (1 - \alpha)q$  is the one-stage lottery which is the convex combination of  $p$  and  $q$ . Betweenness is a weakened form of the vNM-independence axiom. It implies neutrality towards randomization among equally-good lotteries. Chew (1989) and Dekel (1986) show that a preference relation  $\succeq$  satisfies continuity, monotonicity, and betweenness if and only if it has a utility representation  $V$  such that each  $V(p)$  is defined implicitly as the unique  $v \in [0, 1]$  solving

$$\sum_x p(x) u(x, v) = v, \tag{7}$$

where  $u : X \times [0, 1] \rightarrow [0, 1]$  is continuous in both arguments, strictly increasing in the first argument, and satisfies  $u(w, v) = 0$  and  $u(b, v) = 1$  for all  $v \in [0, 1]$ . The function  $u(x, v)$  can be interpreted as the value of the prize  $x$  relative to a reference utility level  $v$ . This class contains expected utility preferences, where  $u(x, v) \equiv u(x)$ . Another well-known model in this class is Gul (1991)'s theory of disappointment aversion,<sup>17</sup> where the value  $V(p; \beta, u)$  of a lottery  $p$  is the unique  $v$  solving

$$v = \frac{\sum_{\{x|u(x) \geq v\}} p(x)u(x) + (1 + \beta) \sum_{\{x|u(x) < v\}} p(x)u(x)}{1 + \beta \sum_{\{x|u(x) < v\}} p(x)}. \quad (8)$$

That is, lotteries are evaluated by calculating their “expected utility,” except that disappointing outcomes (those that are worse than the lottery) get a uniformly greater (or smaller) weight depending on  $\beta \in (-1, \infty)$ .

As before, we require the utility functions after each history to be rankable in terms of risk aversion. An example of an admissible class  $\mathcal{V}$  with non-expected utility preferences is a collection of disappointment aversion preferences with history-dependent  $\beta$  and history-independent utility function over prizes; i.e.,  $\mathcal{V} = \{V(\cdot; \beta_h, u)\}_{h \in H}$ , where  $V(\cdot; \beta_h, u)$  is given by (8). Indeed, Gul (1991, Proposition 5) shows that the DM becomes increasingly risk averse as  $\beta$  increases, holding fixed the utility function over prizes.

### Threshold rules

Our model of history-dependent risk attitude requires the history assignment the DM uses to be internally consistent. In the basic version of the model, we confined our attention to a preference-based rule where a sublottery may be considered elating (disappointing) if its recursively-calculated certainty equivalent is greater (smaller) than that of its parent lottery. Denote by  $CE(\cdot; a, \mathcal{V})$  the certainty equivalent of a sublottery (of a  $T$ -stage compound lottery). Generalizing our earlier notion, the DM may determine if  $P_j^t$  is an elating or disappointing outcome of its parent lottery  $P^{t+1} = \langle \alpha_1, P_1^t; \dots; \alpha_n, P_n^t \rangle$  by examining whether  $CE(P_j^t; a, \mathcal{V})$  falls above or below a *threshold level* which is a function of  $\langle \alpha_1, CE(P_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(P_n^t; a, \mathcal{V}) \rangle$ . We consider two types of threshold-generating rules different DM's may use: an exogenous rule (independent of preference) and an endogenous, preference based specification.

*Exogenous threshold.* Consider any function in the betweenness class (as in (7)) and let the threshold rule  $\tau : \mathcal{L}^1 \rightarrow \mathbb{R}$  be its inverse (certainty equivalent) function. Internal consistency re-

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<sup>17</sup>Gul's model was first intended to explain the Allais paradox. In a dynamic setting, it proved useful to address, for example, the equity premium puzzle (Ang, Bekaert and Liu, 2005) and a statistically significant negative correlation between volatility and private investment (Aizenman and Marion, 1999).

quires that if the DM considers  $P_j^t$  to be elating in  $P^{t+1} = \langle \alpha_1, P_1^t; \dots; \alpha_n, P_n^t \rangle$ , then it must be that  $CE(P_j^t; a, \mathcal{V}) \geq \tau(\langle \alpha_1, CE(P_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(P_n^t; a, \mathcal{V}) \rangle)$ . If  $P_j^t$  is disappointing in  $P^{t+1}$ , then  $CE(P_j^t; a, \mathcal{V}) < \tau(\langle \alpha_1, CE(P_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(P_n^t; a, \mathcal{V}) \rangle)$ .

Note that the function  $\tau$  is independent of preferences, even though it takes as an input lotteries that have been generated from the DM's preference. One example is an expectation-based threshold rule where  $\tau(\cdot) = \mathbb{E}(\cdot)$ , in which case the DM compares the certainty equivalent of a sublottery to his expected certainty equivalent.

The other threshold rule studied is our earlier preference-based specification.

*Endogenous, preference-based threshold.* The DM uses his preference at a sublottery to determine which realizations are elating or disappointing. The threshold rule  $\tau$  is a collection of preference-based thresholds where  $\tau(\cdot | P^{t+1}) : \mathcal{L}^1 \rightarrow \mathbb{R}$  is the certainty equivalent under the DM's preference  $V_{a(P^{t+1})}(\cdot)$  at  $P^{t+1}$ ; under our earlier notation,  $\tau(\cdot | P^{t+1}) = CE_h(\cdot)$  where  $h = a(P^{t+1})$ . Internal consistency requires that if the DM considers  $P_j^t$  elating in  $P^{t+1} = \langle \alpha_1, P_1^t; \dots; \alpha_n, P_n^t \rangle$ , then  $CE(P_j^t; a, \mathcal{V}) \geq CE_{a(P^{t+1})}(\langle \alpha_1, CE(P_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(P_n^t; a, \mathcal{V}) \rangle)$ . If  $P_j^t$  is disappointing in  $P^{t+1}$ , then it must be that  $CE(P_j^t; a, \mathcal{V}) < CE_{a(P^{t+1})}(\langle \alpha_1, CE(P_1^t; a, \mathcal{V}); \dots; \alpha_n, CE(P_n^t; a, \mathcal{V}) \rangle)$ .

To illustrate the difference between the endogenous and exogenous threshold rules, consider a one-stage lottery that gives the prizes  $\{0, 1, \dots, 1000\}$  with equal probabilities. If the DM is risk averse and uses the (endogenous) preference-based threshold rule, then he may be elated by prizes smaller than 500, where the cutoff for elation is his certainty equivalent for this lottery. By contrast, if he uses the exogenous threshold rule, then only prizes exceeding 500 are elating. That is, exogenous threshold rules separate the classification of disappointment and elation from preferences.

### Generalized model of history-dependent risk attitude

The generalized model of history-dependent risk attitude combines the elements outlined above as follows.

**Definition 7 (History-dependent risk attitude, HDRA).** *An HDRA representation over  $T$ -stage lotteries consists of a collection  $\mathcal{V} := \{V_h\}_{h \in H}$  of utilities over one-stage lotteries from the betweenness class (rankable in terms of risk aversion), a history assignment  $a$ , and an (endogenous*

or exogenous) threshold rule  $\tau$ , such that for each  $P^T \in \mathcal{L}^T$ , the value of  $P^T$  is calculated recursively and the history assignment of each sublottery is internally consistent given the threshold rule  $\tau$ . We identify a DM with an HDRA representation by the triple  $(\mathcal{V}, a, \tau)$  satisfying the above.

To formulate our characterization of HDRA in this setting, we extend Definitions 3 and 4 of the reinforcement and primacy effects. For any  $t$ , let  $d^t$  (or  $e^t$ ) denote  $t$  repetitions of  $d$  (or  $e$ ). The history  $hed^t$ , for instance, corresponds to experiencing one elation and  $t$  successive disappointments after history  $h$ , under the implicit assumption that the resulting history is in  $H$ .

**Definition 8.**  $\mathcal{V} = \{V_h\}_{h \in H}$  displays the reinforcement effect if  $V_{hd} >_{RA} V_{he}$  for all  $h$ .

**Definition 9.**  $\mathcal{V} = \{V_h\}_{h \in H}$  displays the primacy effect if  $V_{hde^t} >_{RA} V_{hed^t}$  for all  $h$  and  $t$ .

The reinforcement and primacy effects together imply strong restrictions on the collection  $\mathcal{V}$ ; these are seen in the following observation. We refer below to the lexicographic order on histories of the same length as the ordering where  $\tilde{h}$  precedes  $h$  if it precedes it alphabetically. Since  $d$  comes before  $e$ , this is interpreted as “the DM is disappointed earlier in  $\tilde{h}$  than in  $h$ .”

**Observation 1.**  $\mathcal{V}$  displays the reinforcement and primacy effects if and only if for  $h, \tilde{h}$  of the same length,  $V_{\tilde{h}} >_{RA} V_h$  if  $\tilde{h}$  precedes  $h$  lexicographically. Moreover, under the additional assumption  $V_{hd} >_{RA} V_h >_{RA} V_{he}$  for all  $h \in H$ ,  $\mathcal{V}$  displays the reinforcement and primacy effects if and only if for any  $h, h', h''$ , we have  $V_{hdh''} >_{RA} V_{heh'}$ .

The content of Observation 1 is visualized Figure 4. The first statement corresponds to the lexicographic ordering across the rows. Under the additional assumption  $V_{hd} >_{RA} V_h >_{RA} V_{he}$ , which says an elation reduces (and a disappointment increases) the DM’s risk aversion *relative to his initial level*, one obtains the vertical lines and consecutive row alignment. Observe that along a realized path, this imposes no restriction on how current risk aversion compares to risk aversion two or more periods ahead *when the continuation path* consists of both elating and disappointing outcomes: e.g., one can have both  $V_h <_{RA} V_{hed}$  or  $V_h >_{RA} V_{hed}$ .

We complete this section by generalizing our main characterization result and showing how one may elicit the primitives  $(\mathcal{V}, a, \tau)$  from choice behavior. We begin with the following theorem, which extends Theorem 1

**Theorem 4 (Necessary and sufficient conditions for HDRA).** Consider  $\mathcal{V}$  and an exogenous or endogenous threshold rule  $\tau$ . An HDRA representation  $(\mathcal{V}, a, \tau)$  exists – that is, there exists an internally consistent assignment  $a$  – if and only if  $\mathcal{V}$  displays the reinforcement and primacy effects.

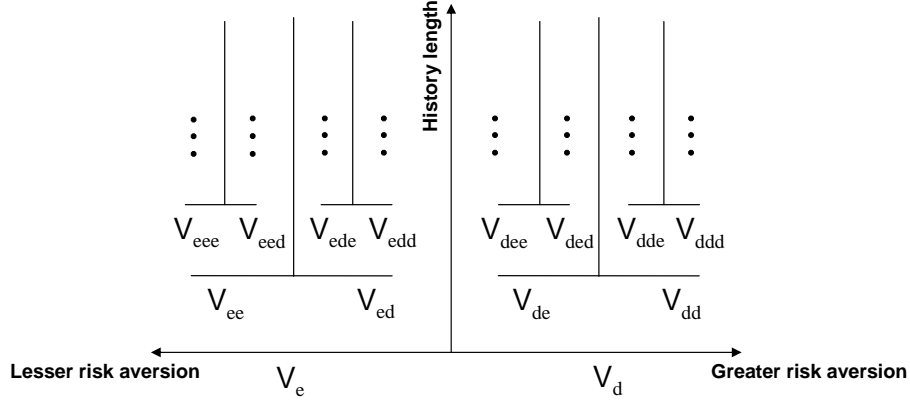


Figure 4: Starting from the bottom, each row depicts the risk aversion rankings  $>_{\text{RA}}$  of the  $V_h$  for histories of length  $t = 1, 2, 3, \dots, T - 1$ . The reinforcement effect and the primacy effect imply the lexicographic ordering in each row. The vertical boundaries and consecutive row alignment would be implied by the additional assumption  $V_{hd} >_{\text{RA}} V_h >_{\text{RA}} V_{he}$  for all  $h \in H$ .

**Proof.** See Appendix A. ■

An immediate implication of Theorem 4 is that the recursively calculated certainty equivalent of any  $t$ -stage sublottery is larger when the sublottery is considered elating, than when it is considered disappointing. In other words, the DM also exhibits less risk aversion over *compound* lotteries (as defined in Section 3.5) after elation than after disappointment. The same feature arises in the DM's choices when intermediate actions are permitted; Theorem 4 readily extends, as before. A second implication of Theorem 4 under an endogenous threshold is *statistically reversing risk attitudes*; this feature does not arise under exogenous threshold rules. As discussed in Section 3.4, when the threshold moves with preference, elation becomes more likely after disappointment because the certainty equivalent of any lottery decreases (and vice-versa). The intensity of reversals in risk attitude may well persist, even under the additional assumption in Observation 1 that  $V_{hd} >_{\text{RA}} V_h >_{\text{RA}} V_{he}$  for all  $h$ . As visualized in Figure 4, this assumption means that after an elation, the DM's greatest possible degree of risk aversion in the future decreases; and conversely, after a disappointment, the DM's lowest possible degree of risk aversion in the future increases. However, the "mood swings" of a DM with an endogenous threshold need not moderate with experience. To see this, suppose for simplicity that the DM's risk aversion in  $\mathcal{V}$  is described by a collection of risk aversion coefficients  $\{\rho_h\}_{h \in H}$ , and note that for any fixed time horizon  $T$ , the parameters *need not* satisfy  $|\rho_{ed} - \rho_e| \geq |\rho_{ede} - \rho_{ed}| \geq |\rho_{eded} - \rho_{ede}| \dots$ .

In what follows, we discuss the question of *elicitation* – that is, how to recover the primitives  $(\mathcal{V}, a, \tau)$  from the choice behavior of a DM who applies the HDRA model. While the utility func-

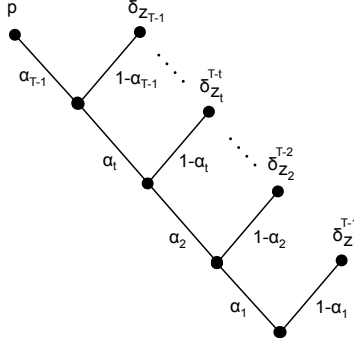


Figure 5: A representative lottery in  $\mathcal{L}_u^T$ , where  $z_i \in \{b, w\}$  for all  $i = 1, \dots, T-1$  and  $p \in \mathcal{L}^1$ .

tions in  $\mathcal{V}$  are used to evaluate certainty equivalents in all  $T$ -stage lotteries, they may be elicited using only choice behavior over a simple subclass of lotteries illustrated in Figure 5. Extending the definition of  $\mathcal{L}_u^3$  from Section 3.5, let

$$\mathcal{L}_u^T = \left\{ \left\langle \alpha_1, \left\langle \alpha_2, \dots \left\langle \alpha_{T-1}, p; 1 - \alpha_{T-1}, \delta_{z_{T-1}}^{T-(T-1)} \right\rangle \dots; 1 - \alpha_2, \delta_{z_2}^{T-2} \right\rangle; 1 - \alpha_1, \delta_{z_1}^{T-1} \right\rangle \right\} \\ \text{such that } p \in \mathcal{L}^1, z_i \in \{b, w\}, \text{ and } \alpha_i \in [0, 1]$$

be the set of lotteries where in each period, either the DM learns he will receive one of the extreme prizes  $b$  or  $w$  for sure, or he must incur further risk (which is ultimately resolved by  $p$  if an extreme prize has not been received). For lotteries in the class  $\mathcal{L}_u^T$  the history assignment is unambiguous. The DM is disappointed by any continuation subplottery received *instead* of the best prize  $b$ , and elated by any continuation subplottery received *instead* of the worst prize  $w$ . To illustrate how the class  $\mathcal{L}_u^T$  allows elicitation of  $\mathcal{V}$ , consider a history  $h = (h_1, \dots, h_t)$  of length  $t \leq T-1$ . Pick any sequence  $\alpha_1, \dots, \alpha_t \in (0, 1)$  and pick  $\alpha_i = 1$  for  $i > t$ . (Note that anytime the continuation probability  $\alpha_i$  is one, the history is unchanged). Construct the sequence  $z_1, \dots, z_t$  such that for every  $i \leq t$ ,  $z_i = b$  if  $h_i = d$ , and  $z_i = w$  if  $h_i = e$ . Finally, define  $\ell_h : \mathcal{L}^1 \rightarrow \mathcal{L}_u^T$  by  $\ell_h(r) \equiv \langle \alpha_1, \langle \alpha_2, \dots \langle \alpha_t, r; 1 - \alpha_t, \delta_{z_t}^{T-t} \rangle \dots; 1 - \alpha_2, \delta_{z_2}^{T-2} \rangle; 1 - \alpha_1, \delta_{z_1}^{T-1} \rangle$  for any  $r \in \mathcal{L}^1$ . It is easy to see that the history assignment of  $r$  must be  $h$ . Moreover, under the HDRA model,  $V_h(p) \geq V_h(q)$  if and only if the DM prefers  $\ell_h(p)$  to  $\ell_h(q)$ . As in our basic model, only the ordinal rankings represented by the collection  $\mathcal{V}$  affect choice behavior.

We may also elicit the DM's (endogenous or exogenous) threshold rule  $\tau$  from his choices. For ease of exposition, we describe the elicitation procedure within a two-stage setting and under the initial history. (More generally, one may embed this procedure in a  $T$ -stage setting, and after any history  $h$ , using lotteries analogous to those in  $\mathcal{L}_u^T$  but where the final one-stage lottery is replaced with a two-stage lottery). Recall that  $\tau$  is a function of one-stage lotteries. To determine how the

prize  $z$  compares to the threshold  $\tau$  of a one-stage lottery  $\langle \alpha_{1,x_1}; \dots; \alpha_{n,x_n} \rangle$ , it suffices to examine the DM's behavior over two-stage lotteries of the form:

$$P^2(p) \equiv \langle \frac{1}{2}, p; \frac{\alpha_1}{2}, \delta_{x_1}; \dots, \frac{\alpha_n}{2}, \delta_{x_n} \rangle, \text{ for } p \in \mathcal{L}^1.$$

Suppose that for some nondegenerate  $\bar{p} \in \mathcal{L}^1$  satisfying  $CE_e(\bar{p}) = z$ , the DM is indifferent between  $P^2(\bar{p})$  and  $P^2(\delta_z)$ . Under the HDRA model, it cannot be that  $\bar{p}$  is disappointing in  $P^2(\bar{p})$ , since  $CE_d(\bar{p}) < CE_e(\bar{p})$  implies the DM strictly prefers  $P^2(\bar{p})$  to  $P^2(CE_d(\bar{p}))$  by monotonicity of his one-stage preferences. Therefore,  $\bar{p}$  is elating in  $P^2(\bar{p})$  and

$$z = CE_e(\bar{p}) \geq \tau(\langle \frac{1}{2}, CE_e(\bar{p}); \frac{\alpha_1}{2}, x_1; \dots, \frac{\alpha_n}{2}, x_n \rangle). \quad (9)$$

The one-stage lottery  $\langle \frac{1}{2}, z; \frac{\alpha_1}{2}, x_1; \dots, \frac{\alpha_n}{2}, x_n \rangle$  is a convex combination of the sure prize  $z$  and  $\langle \alpha_{1,x_1}; \dots; \alpha_{n,x_n} \rangle$ . Because  $\tau$  is the certainty equivalent of a betweenness function, it satisfies betweenness itself. Hence (9) holds if and only if  $z \geq \tau(\langle \alpha_{1,x_1}; \dots; \alpha_{n,x_n} \rangle)$ . Similarly, if for some nondegenerate  $\underline{p} \in \mathcal{L}^1$  satisfying  $CE_d(\underline{p}) = z$ , the DM is indifferent between  $P^2(\underline{p})$  and  $P^2(\delta_z)$ , then  $z < \tau(\langle \alpha_{1,x_1}; \dots; \alpha_{n,x_n} \rangle)$ . We show in Appendix A.1 that there always exists such a  $\bar{p}$  (or  $\underline{p}$ ) whenever  $z \geq (<) \tau(\langle \alpha_{1,x_1}; \dots; \alpha_{n,x_n} \rangle)$ . Thus, the auxiliary relation  $\succeq_\tau$ , defined by  $z \succeq_\tau \langle \alpha_{1,x_1}; \dots; \alpha_{n,x_n} \rangle$  (respectively,  $\langle \alpha_{1,x_1}; \dots; \alpha_{n,x_n} \rangle \succ_\tau z$ ) if there is a nondegenerate  $p \in \mathcal{L}^1$  such that  $P^2(p) \sim P^2(\delta_z)$  and  $CE_e(p) = z$  (respectively,  $CE_d(p) = z$ ), represents  $\tau$ 's comparisons between a lottery and a sure prize. Appendix A.1 shows how to complete this relation using choice over lotteries of the form  $P^2(\cdot)$ , and proves that it represents the threshold  $\tau$ .

Finally, to recover the history assignment of a  $T$ -stage lottery, one needs to iteratively ask the DM what sure outcomes should replace the terminal lotteries to keep him indifferent. Generically, his chosen outcome must be the certainty equivalent of the corresponding subplottery under his history assignment  $a$ .<sup>18</sup>

## 6 Conclusion and directions for further research

We propose a model of history-dependent risk attitude which has tight predictions for how disappointments and elations affect the attitude to future risks. The model permits a wide class of preferences and threshold rules, and is consistent with a body of evidence on risk-taking behavior.

To study endogenous reference dependence under a minimal departure from recursive history

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<sup>18</sup>Since  $H$  is finite, if there is  $P^T$  such that two assignments yield the same value, then there is an open ball around  $P^T$  within which every other lottery has the property that no two assignments yield the same value.



independent preferences, HDRA posits the categorization of each sublottery as either elating or disappointing. The DM’s risk attitudes depend on the prior sequence of disappointments or elations, but not on the “intensity of those experiences.” As seen in the application to asset pricing, this makes the model particularly easy to apply. It is possible to generalize our model so that the more a DM is “surprised” by an outcome, the more his risk aversion shifts away from a baseline level. As shown in a supplementary appendix, the equivalence between the generalized model and the reinforcement and primacy effects remains. Extending the model requires introducing an additional component (a *sensitivity function* capturing dependence on probabilities) and parametrizing risk aversion in the one-stage utility functions using a continuous real variable. By contrast, allowing the size of risk aversion shifts to depend on the magnitude of outcomes would be a more substantial change. Finding the history assignment involves a fixed point problem which would then become quite difficult to solve. Moreover, the testable implications of such a model depend on whether it is possible to identify the extent to which a realization is disappointing or elating, as that designation depends on the extent to which other outcomes are considered elating or disappointing.

Finally, this paper considers a finite-horizon model of decision making. In an infinite-horizon setting, our methods extend to prove necessity of the reinforcement and primacy effects. However, our methods do not immediately extend to ensure the existence of an infinite-horizon internally consistent history assignment. One possible way to embed the finite-horizon HDRA preferences into an infinite-horizon economy is through the use of an overlapping generations model.

## Appendix A: Proofs

**Proofs of Theorems 1 and 4.** We first prove a sequence of four lemmas.

**Lemma 1.** *For any  $h$  and  $t$ , and any  $h'$  with length  $t$ , we have  $V_{hd^t} >_{RA} V_{hh'} >_{RA} V_{he^t}$ .*

**Proof.** For simplicity and without loss of generality, we may assume  $h = 0$  because one can always append the lotteries constructed below to a beginning lottery whose support in each stage consists of a continuation lottery and a prize  $z$ , where  $z \in \{b, w\}$ . In an abuse of notation, if we write a lottery or prize as a outcome when there are more stages left than present in the outcome, we mean that lottery is resolved immediately, with the prize in the terminal stage.

We proceed by induction. For  $t = 1$ , this is the reinforcement effect. Suppose by contradiction that for any non degenerate  $p$ ,  $CE_e(p) < CE_d(p)$ . Fix  $p$  and let  $x$  be a number in between. Then  $\langle \alpha, p; 1 - \alpha, \delta_x \rangle$  has no internally consistent decomposition.

Now assume the claim holds for all  $s \leq t - 1$ , and suppose by contradiction that  $V_{d^t}$  is not the most risk averse. Then there is  $h'$  with length  $t$  such that  $V_{h'} >_{RA} V_{d^t}$ . It must be that  $h' = eh''$

where  $h''$  has length  $t - 1$ , otherwise there is a contradiction to the inductive step using  $h = d$ . By the inductive step,  $V_{eh''}$  is less risk averse than  $V_{ed^{t-1}}$ , so  $V_{ed^{t-1}} >_{RA} V_{d^t}$ . For any nondegenerate  $p$ , this means  $CE_{ed^{t-1}}(p) < CE_{d^t}(p)$ . Iteratively define the lottery  $P^{t-1}$  by  $P^2 = \langle \alpha, p; 1 - \alpha, b \rangle$ , and for each  $3 \leq s \leq t - 1$ ,  $P^s = \langle \alpha, P^{s-1}; 1 - \alpha, b \rangle$ . Finally, let  $P^t = \langle \beta, P^{t-1}; 1 - \beta, x \rangle$ , where  $x \in (CE_{ed^{t-1}}(p), CE_{d^t}(p))$ . Note that the assignment of  $p$  must be  $d^{t-1}$  within  $P^{t-1}$  and that for  $\alpha$  close to one, the value of  $P^{t-1}$  is either close to  $CE_{ed^{t-1}}(p)$  (if  $P^{t-1}$  is an elation) or close to  $CE_{d^t}(p)$  (if  $P^{t-1}$  is a disappointment). But then for  $\alpha$  close enough to 1, there is no consistent decomposition given the choice of  $x$ . Hence  $V_{d^t}$  is most risk averse. Analogously, to show that  $V_{e^t}$  is least risk averse, assuming it is not true implies  $CE_{dh''}(p) > CE_{e^t}(p)$ , and a similar construction with  $w$  instead of  $b$  in  $P^t$ ,  $x \in (CE_{e^t}(p), CE_{dh''}(p))$  and  $\alpha$  close to one, yields a contradiction. ■

**Lemma 2.** For any  $h$  and  $t$ , we have  $V_{hd^t} >_{RA} V_{hed^t}$ .

**Proof.** We use the same simplifications as in the previous lemma (without loss of generality). Also abusing notation, we write  $\tau_h$  to denote either the exogenous threshold rule (where  $\tau_h(\cdot) = \tau(\cdot)$  for all  $h$ ) or the endogenous threshold rule (where  $\tau_h(\cdot) = CE_h(\cdot)$  for all  $h$ ). We prove the result by induction. For  $t = 1$ , suppose by contradiction that  $V_{ed} >_{RA} V_{de}$ . Consider a nondegenerate lottery  $p$  with  $w \notin \text{supp } p$ . For any  $\beta \in (0, 1)$ , let  $P^3 = \langle \beta, P, 1 - \beta, x \rangle$ , where

$$P = \langle \varepsilon(1 - \alpha), w; \varepsilon\alpha, p; 1 - 2\varepsilon, q \rangle$$

and  $\varepsilon, \alpha, q, x$  are chosen as follows. We want  $q$  to necessarily be elating (disappointing) if  $P$  is disappointing (elating). Consider the conditions

$$\begin{aligned} CE_{dd}(q) &> \tau_d(\langle \alpha, CE_{de}(p); 1 - \alpha, w \rangle), \\ CE_{ee}(q) &< \min\{\tau_e(\langle \alpha, CE_{ee}(p); 1 - \alpha, w \rangle), CE_{ed}(p)\}. \end{aligned}$$

By Lemma 1,  $\tau_d(\langle \alpha, CE_{de}(p); 1 - \alpha, w \rangle) < \tau_e(\langle \alpha, CE_{ee}(p); 1 - \alpha, w \rangle)$  for each choice of  $\alpha, \beta$ . This is because the lottery on the RHS first-order stochastically dominates that on the LHS, and moreover is evaluated using a less risk-averse threshold. By monotonicity of  $\tau_h$  in  $\alpha$ , choose  $\alpha$  such that  $\tau_e(\langle \alpha, CE_{ee}(p); 1 - \alpha, w \rangle) < CE_{ed}(p)$ . Choose any non degenerate  $q$  where

$$\text{supp } q \subseteq \tau_d(\langle \alpha, CE_{de}(p); 1 - \alpha, \tau_e(\langle \alpha, CE_{ee}(p); 1 - \alpha, w \rangle)).$$

Using betweenness, this condition on the support of  $q$  implies that it must be elating (disappointing) when  $P$  is disappointing (elating). For  $\varepsilon$  sufficiently small, the value of  $P$  is either close to  $CE_{ed}(q)$

(when  $P$  is elating) or close to  $CE_{de}(q)$  (when  $P$  is disappointing). Pick  $x \in (CE_{ed}(q), CE_{de}(q))$  and notice that  $P^3$  has no consistent decomposition.

Assume the lemma is true for  $s \leq t-1$ . We prove it for  $s = t$  by first proving  $V_{de^t} >_{RA} V_{ed^{t-1}e}$ . Suppose  $V_{ed^{t-1}e} >_{RA} V_{de^t}$  by contradiction. Define, for any  $p^1, \dots, p^{t-1} \in \mathcal{L}^1$ , and  $s \in \{2, \dots, t\}$ ,

$$\begin{aligned} a_s(\alpha) &:= \tau_{de^{t-s}}(\alpha, CE_{de^{t-s}e}(p^{s-1}); 1 - \alpha w), \\ b_s(\alpha) &:= \tau_{ed^{t-s}}(\alpha, CE_{ed^{t-s}d}(p^{s-1}); 1 - \alpha, w). \end{aligned}$$

Pick  $\underline{y}, \bar{y}$  such that  $w < \underline{y} < \bar{y} < b$  and ensure  $\alpha$  is sufficiently small that  $\tau_{ed^{t-1}}(\alpha, \bar{y}; 1 - \alpha, w) < \underline{y}$ . Take a nondegenerate  $p^1 \in \mathcal{L}^1$  where  $\text{supp } p^1 \subseteq [\underline{y}, \bar{y}]$  and  $p^1(\bar{y}), p^1(\underline{y}) > 0$ . Let  $a(\alpha) := a_1(\alpha)$ . Now construct a sequence  $p^2, \dots, p^{t-1}$  where  $\text{supp } p^s = \text{supp } p^1$  and  $a_s(\alpha) = a$  for each  $s$ , as follows. To construct  $p^2$ , compare  $\tau_{de^{t-3}}(\alpha, CE_{de^{t-3}d}(p^1); 1 - \alpha, w)$  with  $\tau_{de^{t-2}}(\alpha, CE_{de^{t-2}d}(p^1); 1 - \alpha, w)$ . If the latter (reps., former) is the smaller of the two, construct  $p^2$  by a first-order reduction (reps., improvement) in  $p^1$  by mixing with  $\delta_{\underline{y}}$  (reps.,  $\delta_{\bar{y}}$ ) using the appropriate weight, which exists because  $\tau_h$  satisfies betweenness. Similarly construct the rest of the sequence. By the inductive step,  $a_s(\alpha) < b_s(\alpha)$  for each  $s \in \{2, \dots, t\}$  and the sequence of  $p^s$  above,  $a_s(\alpha) = a(\alpha)$ . Therefore,

$$\bigcap_{s=2}^t (a_s(\alpha), b_s(\alpha)) = (a(\alpha), \min_{s \in \{2, \dots, t\}} b_s(\alpha)) \neq \emptyset.$$

Let  $q$  be a nondegenerate lottery with  $\text{supp } q \subseteq (a(\alpha), \min_{s \in \{2, \dots, t\}} b_s(\alpha))$ . Construct the lottery  $Q = Q^2 = \langle 1 - \varepsilon, q; \varepsilon, w \rangle$  and for each  $s \in \{3, \dots, t+1\}$ , define

$$Q^s = \langle \varepsilon(1 - \alpha), w; \varepsilon\alpha, p^{s-2}; 1 - \varepsilon, Q^{s-1} \rangle.$$

Finally, define  $Q^{t+2} = \langle \gamma, Q^{t+1}; 1 - \gamma, x \rangle$  where  $x \in (CE_{ed^{t-1}e}(q), CE_{de^t}(q))$ . Using Lemma 1 and the choice of  $q$ 's support in the interval above, each branch  $Q^s$  (for  $2 \leq s \leq t$ ) is disappointing (elating) if  $Q^{t+1}$  is elating (disappointing). For  $\varepsilon$  sufficiently small, the value of  $Q^{t+1}$  is either very close to  $CE_{ed^{t-1}e}(q)$  when it is elating or  $CE_{de^t}(q)$  when it is disappointing. But by the choice of  $x$ , there is no consistent decomposition.

To complete the proof, now assume by contradiction that  $V_{ed^t} >_{RA} V_{de^t}$ . Recall  $a_s, b_s$  from above and the sequence  $p^1, \dots, p^{t-1}$ , constructed so that  $a_s(\alpha) = a(\alpha)$  for every  $s \in \{2, \dots, t\}$ . Now define, for any  $p^0$ ,

$$\begin{aligned} a_1(\alpha) &:= \tau_{de^{t-1}}(\alpha, CE_{de^{t-1}e}(p^0); 1 - \alpha, w), \\ b_1(\alpha) &:= \tau_{ed^{t-1}}(\alpha, CE_{ed^{t-1}e}(p^0); 1 - \alpha, w). \end{aligned}$$

Notice that  $a_1 < b_1$  by the claim we just proved and also the inductive hypothesis applied to  $\tau_h$  ( $t_{d^{e^{t-1}}}$  is weakly more risk averse than  $t_{e^{d^{t-1}}}$ ). Construct  $p^0$  with the same support as  $p^1$  such that  $a_1(\alpha) = a$ . By the choice of  $\alpha$ , notice that  $b_1 < CE_{e^{d^{t-1}d}}(p^0)$ . Let  $\tilde{q}$  be a nondegenerate lottery with  $\text{supp } \tilde{q} \subseteq (a(\alpha), \min_{s \in \{1, \dots, t\}} b_s(\alpha))$ . For each  $s \in \{2, \dots, t+1\}$ , define

$$\tilde{Q}^s = \langle \varepsilon(1 - \alpha), w; \varepsilon\alpha, p^{s-2}; 1 - \varepsilon, \tilde{Q}^{s-1} \rangle.$$

Finally, define  $\tilde{Q}^{t+2} = \langle \gamma, \tilde{Q}^{t+1}; 1 - \gamma, x \rangle$  where  $x \in (CE_{e^{d^{t-1}e}}(\tilde{q}), CE_{d^{e^t}}(\tilde{q}))$ . For  $\varepsilon$  sufficiently small, the certainty equivalent of  $\tilde{Q}^{t+1}$  is either very close to  $CE_{e^{d^{t-1}e}}(\tilde{q})$  when it is elating or  $CE_{d^{e^t}}(\tilde{q})$  when it is disappointing. But then  $\tilde{Q}^{t+1}$  has no internally consistent assignment. ■

For any  $V : \mathcal{L}^1 \rightarrow \mathbb{R}$ , define, for any  $p \in \mathcal{L}^1$ ,  $e(p) := \{x \in \text{supp } p \mid V(\delta_x) > V(p)\}$  and  $d(p) := \{x \in \text{supp } p \mid V(\delta_x) < V(p)\}$ .

**Lemma 3.** Consider any lottery  $p = \langle p(x_1), x_1; \dots; p(x_j), x_j; \dots; p(x_m), x_m \rangle$  and another lottery  $p' = \langle p(x_1), x'_1; \dots; p(x_j), x_j; \dots; p(x_m), x_m \rangle$  which differs by one prize. For  $V$  in the betweenness class: (1) if  $x_1 \notin d(p)$  and  $x'_1 > x_1$  then  $x'_1 \in e(p')$ ; and (2) if  $x_1 \notin e(p)$  and  $x'_1 < x_1$  then  $x'_1 \in d(p')$ .

**Proof.** We prove statement (1), since the proof of (2) is analogous. If  $x_1 \notin d(p)$  then  $\delta_{x_1} \succeq p$ , where  $\succeq$  represents  $V$ . Note that  $p$  can be written as the convex combination of the lotteries  $\delta_{x_1}$  (with weight  $p(x_1)$ ) and  $p_{-1} = \langle \frac{p(x_2)}{1-p(x_1)}, x_2; \dots; \frac{p(x_m)}{1-p(x_1)}, x_m \rangle$  (with weight  $1 - p(x_1)$ ). By betweenness, this implies that  $\delta_{x_1} \succeq p_{-1}$ . Since  $x'_1 > x_1$ , monotonicity implies  $\delta_{x'_1} \succ \delta_{x_1} \succeq p_{-1}$ , and thus that  $\delta_{x'_1} \succ p_{-1}$ . But then again by betweenness and the fact that  $p(x_1) \in (0, 1)$ ,  $x'_1$  must be strictly preferred to the convex combination of  $\delta_{x'_1}$  (with weight  $p(x_1)$ ) and  $p_{-1}$  (with weight  $1 - p(x_1)$ ). But that convex combination is  $p'$ , meaning that  $x'_1 \in e(p')$ . ■

**Lemma 4.** Suppose that for any nondegenerate  $p \in \mathcal{L}^1$ ,  $CE_e(p) > CE_d(p)$ . Then for any nondegenerate  $P \in \mathcal{L}^2$ , a consistent history assignment (using only strict elation and disappointment for nondegenerate lotteries in its support) exists.

**Proof.** Consider  $P = \langle \alpha_1, p_1; \dots; \alpha_m, p_m \rangle$ . Suppose for simplicity that all  $p_i$  are nondegenerate (if  $p_i = \delta_x$  is degenerate, then  $CE_e(p_i) = CE_d(p_i)$ , so the algorithm can be run on the nondegenerate sublotteries, with the degenerate ones labeled ex-post according to internal consistency). Without loss of generality, suppose that the indexing in  $P$  is such that  $p_1 \in \arg \max_{i=1, \dots, m} CE_e(p_i)$ ,  $p_m \in \arg \min_{i=2, \dots, m} CE_d(p_i)$ , and  $CE_e(p_2) \geq CE_e(p_3) \geq \dots \geq CE_e(p_{m-1})$ . A consistent decomposition is constructed by the following algorithm (consistency means that all  $p_i$  set as elations (disappointments) have  $CE_{e(d)}(p_i)$  weakly larger (strictly smaller) than the certainty equivalent of  $P$  calculated by folding back using this assignment). Set  $a^1(p_1) = e$  and  $a^1(p_j) = d$  for all  $i > 1$ .

Let  $CE^1$  be the certainty equivalent of  $P$  when it is folded back under  $a^1$ ; if  $CE^1$  is consistent with  $a^1$ , the algorithm and proof are complete. If not, consider  $i = 2$ . If  $CE_d(p_2) \geq CE^1$ , then set  $a^2(p_2) = e$  and  $a^2(p_i) = a^1(p_i)$  for all  $i \neq 2$  (if  $CE_d(p_2) < CE^1$ , let  $a^2(p_i) = a^1(p_i)$  for all  $i$ ). Let  $CE^2$  be the resulting certainty equivalent of  $P$  when it is folded back under  $a^2$ . If  $CE^2$  is consistent with  $a^2$ , the algorithm and proof are complete. If not, move to  $i = 3$ , and so on and so forth, so long as  $i \leq m - 1$ . Notice from Lemma 3 that if  $CE_d(p_i) \geq CE^{i-1}$ , then  $CE_e(p_i) > CE^i$ . Moreover, notice that if  $CE_e(p_i) > CE^i$ , then for any  $j < i$ ,  $CE_e(p_j) \geq CE_e(p_i) > CE^i$ , so previously switched assignments remain strict elations; also, because  $CE^i \geq CE^{i-1}$  for all  $i$ , previous disappointments remain disappointments. If the final step of the algorithm reaches  $i = m - 1$ , notice that  $CE_d(p_m)$  is the lowest disappointment certainty equivalent, therefore the lowest value among  $\{CE_{a^{m-1}(p_j)}(p_j)\}_{j=1,\dots,m}$ . Hence, the final history assignment  $a^{m-1}$  is consistent with  $CE^{m-1}$ . ■

We are now ready to complete the proofs of Theorems 1 and 4. The reinforcement and primacy effects are necessary by Lemmas 1 and 2. By Lemma 4 and the reinforcement effect, an internally consistent (strict) history assignment exists for any nondegenerate  $P \in \mathcal{L}^2$ , using any initial  $V_h$ . By induction, suppose that for any  $(t - 1)$ -stage lottery an internally consistent history assignment exists, using any initial  $V_h$ . Consider a  $t$ -stage nondegenerate lottery  $P^t = \langle \alpha_1, P_1^{t-1}; \dots; \alpha_m, P_m^{t-1} \rangle$ . Notice that the algorithm in Lemma 4 for  $\mathcal{L}^2$  only uses the fact that  $CE_e(p) > CE_d(p)$  for any nondegenerate  $p \in \mathcal{L}^1$ . But the same algorithm can be used to construct an internally consistent history assignment for  $P^t$  if for any  $P^{t-1} \in \mathcal{L}^{t-1}$ ,  $CE_e(P^{t-1}) > CE_d(P^{t-1})$ . While there may be multiple consistent assignments of  $P^{t-1}$  using each of  $V_e$  and  $V_d$ , the primacy effect ensures this strict inequality regardless of the history assignment. Indeed, starting with  $V_e$ , the tree is folded back using higher certainty equivalents sublottery by sublottery, and evaluated using a less risk averse single-stage utility, as compared to starting with the more risk averse  $V_d$ . As in Lemma 4, the history for any degenerate sublottery can be assigned ex-post according to what is consistent; its certainty equivalent is not affected by the assignment of  $e$  or  $d$ . This proves Theorem 4, from which Theorem 1 follows. ■

**Proof of Theorem 2.** The proof of necessity is analogous to that of Theorem 1. The proof of sufficiency is analogous as well, with two additions of note. First, since the reinforcement and strong primacy effects imply the certainty equivalent of each decision problem in a choice set increases when evaluated as an elation, the certainty equivalent of the choice set (the maximum of those values) also increases when viewed as an elation (relative to being viewed as a disappointment). Second, if the certainty equivalent of a choice set is the same when viewed as an elation

and as a disappointment, the best option in both choice sets must be degenerate. Then its history assignment may be made ex-post according to internal consistency. ■

**Proof of Proposition 1.** Rearrange Equation (1) to

$$\frac{u_h\left(u_{he}^{-1}\left((1-\alpha)u_{he}(H)+\alpha u_{he}(L)\right)\right)-u_h(L)}{1-\alpha} > \frac{u_h\left(u_{he}^{-1}\left(\alpha u_{he}(H)+(1-\alpha)u_{he}(L)\right)\right)-u_h(L)}{\alpha}.$$

Observe that we may write  $u_h = f \circ u_{he}$  for some function  $f$ ; hence  $u_{he}^{-1} = u_h^{-1} \circ f$ . Dividing both sides by the positive term  $u_{he}(H) - u_{he}(L)$  then yields

$$\frac{f\left(u_{he}(L)+(1-\alpha)(u_{he}(H)-u_{he}(L))\right)-f(u_{he}(L))}{(1-\alpha)(u_{he}(H)-u_{he}(L))} > \frac{f\left(u_{he}(L)+\alpha(u_{he}(H)-u_{he}(L))\right)-f(u_{he}(L))}{\alpha(u_{he}(H)-u_{he}(L))}.$$

The ratios on each side are slopes of segments joining points on the graph of  $f$ , starting at  $f(u_{he}(L))$ . Note that  $\alpha > .5$  if and only if  $\alpha(u_{he}(H) - u_{he}(L)) > (1 - \alpha)(u_{he}(H) - u_{he}(L))$ . But then the above inequality holding for all  $\alpha \in (.5, 1)$  and  $H > L$  is equivalent to concavity of  $f$ , or equivalently, that  $V_{he} <_{RA} V_h$ . ■

### Appendix A.1: Eliciting the threshold

Define  $\succeq_\tau$  based on the DM's preference  $\succeq$  as follows. For  $x \in X$  and  $q = \langle q_1, x_1; \dots; q_n, x_n \rangle \in \mathcal{L}(X)$ , we say  $\delta_x \succ_\tau q$  (resp.,  $q \succ_\tau \delta_x$ ) if there is  $x' < x$  (resp.,  $x' > x$ ) and  $p \in \mathcal{L}(X)$  nondegenerate such that  $\ell_e(\delta_{x'}) \sim \ell_e(p)$  (resp.,  $\ell_d(\delta_{x'}) \sim \ell_d(p)$ ) and  $\langle .5, \delta_p^{T-1}; .5q_1, \delta_{x_1}^T; \dots; .5q_n, \delta_{x_n}^T \rangle \sim \langle .5, \delta_{x'}^T; .5q_1, \delta_{x_1}^T; \dots; .5q_n, \delta_{x_n}^T \rangle$ . For  $p, q \in \mathcal{L}(X)$ , we say  $p \succ_\tau q$  if there is  $x \in X$  such that  $p \succ_\tau \delta_x \succ_\tau q$ . We say  $p \sim_\tau q$  if  $p \not\succeq_\tau q$  and  $q \not\succeq_\tau p$ .

**Lemma 5.**  $\tau(p) > \tau(q)$  if and only if  $p \succ_\tau q$ .

**Proof.** If  $\tau(p) > \tau(q)$ , then take  $x' > x > x''$  such that  $\tau(p) > \tau(\delta_{x'}) = x' > \tau(\delta_x) = x > \tau(\delta_{x''}) = x'' > \tau(q)$ . By admissibility of  $V_h$ , we can pick  $\alpha', \alpha'' \in (0, 1)$  such that  $r' = \langle \alpha', x'; 1 - \alpha', x'' \rangle$  and  $r'' = \langle \alpha'', x'; 1 - \alpha'', x'' \rangle$  satisfy  $CE_d(r') = CE(\delta_x)$  and  $CE_e(r'') = CE_e(\delta_x)$ . Since  $\tau(q) < x'' < x < x' < \tau(p)$ , we have  $p \succ_\tau \delta_x$  and  $x \succ_\tau \delta_q$ , which means  $p \succ_\tau q$ . Now, suppose  $p \succ_\tau q$ . This means there is  $x$  such that  $p \succ_\tau x \succ_\tau q$ , and so there exist  $x' > x$  such that  $p \succ_\tau \delta_{x'}$  and  $x'' < x$  such that  $\delta_{x''} \succ_\tau q$ . Since  $\langle .5, \delta_r^{T-1}; .5q_1, \delta_{x_1}^T; \dots; .5q_n, \delta_{x_n}^T \rangle \sim \langle .5, \delta_{x'}^T; .5q_1, \delta_{x_1}^T; \dots; .5q_n, \delta_{x_n}^T \rangle$  for some nondegenerate  $r$  such that  $CE_d(r) = CE_d(\delta_{x'})$ , the representation implies that  $\tau(\delta_{x'}) < \tau(\langle .5, \delta_{x'}; .5p_1, \delta_{x_1}; \dots; .5p_n, \delta_{x_n} \rangle)$ , which by betweenness means that  $\tau(\delta_{x'}) < \tau(p)$ . Similarly, from  $x'' \succ_\tau q$ , we have  $\tau(\delta_{x''}) > \tau(q)$ , concluding the proof by transitivity. ■

## Appendix B: Proofs for the asset-pricing application

**Lemma 6.** *The history assignment is unique.*

**Proof.** If it were internally consistent after  $y_1$  for  $H$  to be disappointing and  $L$  to be elating, that would mean  $L + \Gamma_{\tilde{y}_2}(\lambda_{h(y_1)e}, \tilde{y}_2) > H + \Gamma_{\tilde{y}_2}(\lambda_{h(y_1)d}, \tilde{y}_2)$ , or  $H - L < \Gamma_{\tilde{y}_2}(\lambda_{h(y_1)e}, \tilde{y}_2) - \Gamma_{\tilde{y}_2}(\lambda_{h(y_1)d}, \tilde{y}_2)$ . Notice that  $\Gamma_{\tilde{y}_2}(\lambda_{h(y_1)e}, \tilde{y}_2) < \frac{1}{2}H + \frac{1}{2}L$  (the boundary case  $\lambda_{h(y_1)e} = 0$ ) and  $\Gamma_{\tilde{y}_2}(\lambda_{h(y_1)d}, \tilde{y}_2) > L$  (the case  $\lambda_{h(y_1)d} = \infty$ ). Thus,  $H - L < \Gamma_{\tilde{y}_2}(\lambda_{h(y_1)e}, \tilde{y}_2) - \Gamma_{\tilde{y}_2}(\lambda_{h(y_1)d}, \tilde{y}_2) < \frac{1}{2}H - \frac{1}{2}L$ , a contradiction.

Suppose by contradiction that in the first period,  $H$  is disappointing and  $L$  is elating. Internal consistency requires  $L + \Gamma_{\tilde{y}_2}(\lambda_e, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(L, \tilde{y}_2)}, \tilde{y}_3)) > H + \Gamma_{\tilde{y}_2}(\lambda_d, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(H, \tilde{y}_2)}, \tilde{y}_3))$ . But since this is also the certainty equivalent of a (more complex) random variable, we have the bound  $\Gamma_{\tilde{y}_2}(\lambda_d, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(H, \tilde{y}_2)}, \tilde{y}_3)) > L + \Gamma_{\tilde{y}_3}(\lambda_{dd}, \tilde{y}_3) > 2L$ . Similarly, we also have the bound  $\Gamma_{\tilde{y}_2}(\lambda_e, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(L, \tilde{y}_2)}, \tilde{y}_3)) < \frac{1}{2}(H + L + \Gamma_{\tilde{y}_3}(\lambda_{ee}, \tilde{y}_3) + \Gamma_{\tilde{y}_3}(\lambda_{ed}, \tilde{y}_3)) < H + L$ . But then we have a contradiction  $H - L < \Gamma_{\tilde{y}_2}(\lambda_e, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(L, \tilde{y}_2)}, \tilde{y}_3)) - \Gamma_{\tilde{y}_2}(\lambda_d, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(H, \tilde{y}_2)}, \tilde{y}_3)) < H - L$ . ■

**Lemma 7.**  $P(H) > P(L)$ .

**Proof.** Using (3), we know  $P(H) > P(L)$  if and only if the following function is negative:

$$G(\lambda_0, H, L) = \lambda_0(H - L)(a^2 - b^2) + \lambda_0 \left( a^2(f(a^3\lambda_0) - f(a^2b\lambda_0)) - b^2(f(b^2a\lambda_0) - f(b^3\lambda_0)) \right).$$

By pulling out the term  $\exp(-\lambda L)$ , note that  $f(\lambda) = L - \frac{1}{\lambda} \ln \left( \frac{1}{2} \exp(-\lambda(H - L)) + \frac{1}{2} \right)$ . Hence

$$\begin{aligned} G(\lambda_0, H, L) &= \lambda_0(H - L)(a^2 - b^2) + \frac{1}{b} \ln \left( \frac{1}{2} \exp(-\lambda_0 a^2 b (H - L)) + \frac{1}{2} \right) \\ &\quad - \frac{1}{a} \ln \left( \frac{1}{2} \exp(-\lambda_0 a^3 (H - L)) + \frac{1}{2} \right) - \frac{1}{b} \ln \left( \frac{1}{2} \exp(-\lambda_0 b^3 (H - L)) + \frac{1}{2} \right) \\ &\quad + \frac{1}{a} \ln \left( \frac{1}{2} \exp(-\lambda_0 a b^2 (H - L)) + \frac{1}{2} \right). \end{aligned}$$

We want to show that  $\frac{\partial G}{\partial H}(\lambda_0, H, L) < 0$ , implying  $G(\lambda_0, H, L)$  would be maximized at  $H = L$ , where it has value zero. The derivative of  $G$  with respect to  $H$  is given by

$$\frac{\partial G}{\partial H}(\lambda_0, H, L) = \lambda_0 \left( a^2(1 + g(a^3) - g(a^2b)) - b^2(1 - g(b^3) + g(ab^2)) \right),$$

using the definition  $g(x) = \frac{\exp(-\lambda_0 x (H - L))}{\exp(-\lambda_0 x (H - L)) + 1}$ . Note that  $g'(x) = \frac{-\lambda_0 x (H - L) \exp(\lambda_0 x (H - L))}{x(1 + \exp(\lambda_0 x (H - L)))^2}$ . Moreover,  $-\frac{1}{x} < g'(x) < 0$  for all  $x \geq 0$ . Negativity of  $g'(x)$  is clear. To see the left bound, simply observe

that  $\lambda_0 x(H-L) \exp(\lambda_0 x(H-L)) \leq (\exp(\lambda_0 x(H-L)))^2$ . The mean value theorem says that for some  $c_1 \in (a^3, a^2b)$  we have  $g(a^3) - g(a^2b) = a^2(a-b)g'(c_1)$ . Similarly, for some  $c_2 \in (ab^2, b^3)$  we have  $g(ab^2) - g(b^3) = b^2(a-b)g'(c_2)$ . Then,

$$\frac{\partial G}{\partial H}(\lambda_0, H, L) = \lambda_0 \left( a^2 - b^2 + a^4(a-b)g'(c_1) - b^4(a-b)g'(c_2) \right) < \lambda_0 \left( a^2 - b^2 + a^4(a-b) \left( -\frac{1}{ab^2} \right) \right),$$

which equals  $\lambda_0(a-b)(a+b-\frac{a^3}{b^2})$ . This is negative as desired, since  $0 < a < 1 < b$ . ■

**Lemma 8.**  $P(H) > P(0)$ .

**Proof.** Since  $a < 1$ , it suffices to show that  $a^2(\Gamma_{\tilde{y}_3}(a^3\lambda_0, \tilde{y}_3) - \Gamma_{\tilde{y}_3}(a^2b\lambda_0, \tilde{y}_3))$  is smaller than  $\Gamma_{\tilde{y}_2}(a^2\lambda_0, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(H, \tilde{y}_2)}, \tilde{y}_3)) - \Gamma_{\tilde{y}_2}(b^2\lambda_0, \tilde{y}_2 + \Gamma_{\tilde{y}_3}(\lambda_{h(L, \tilde{y}_2)}, \tilde{y}_3))$ . To show this,<sup>19</sup> define  $\hat{\Gamma}(\lambda, m, n) \equiv \Gamma_{\tilde{x}}(\lambda, \tilde{x})$  for the random variable  $\tilde{x}$  which gives each of  $m$  and  $n$  with probability one-half. Also, we define the notation  $\mu = \hat{\Gamma}(a^3\lambda_0, H, L)$  and  $\nu = \hat{\Gamma}(a^2b\lambda_0, H, L)$ . Because  $\hat{\Gamma}(\lambda, m, n)$  is a certainty equivalent, it is increasing in both  $m, n$  and decreasing in  $\lambda$ . Because  $b^2\lambda_0 > ab$ ,  $b^2a > a^3$ , and  $b^3 > a^2b$ , it suffices to show  $\hat{\Gamma}(a^2\lambda_0, H + \mu, L + \nu) - \hat{\Gamma}(ab\lambda_0, H + \mu, L + \nu) > a^2(\mu - \nu)$ , which is stronger. Letting  $\gamma(\lambda, m, n) = -\lambda^2 \frac{\partial \hat{\Gamma}}{\partial \lambda}(\lambda, m, n)$ , we note  $\gamma$  satisfies three properties: (i)  $\frac{\partial \gamma}{\partial \lambda} > 0$ ; (ii) if  $m > n$ ,  $\frac{\partial \gamma}{\partial m} > 0$ ; and (iii)  $\gamma(\lambda, m + c, n + c) = \gamma(\lambda, m, n)$  for all  $c > 0$ . To see this, observe that

$$\gamma(\lambda, m, n) = \frac{-\lambda m \exp(-\lambda m) - \lambda n \exp(-\lambda n)}{\exp(-\lambda m) + \exp(-\lambda n)} - \ln\left(\frac{\exp(-\lambda m) + \exp(-\lambda n)}{2}\right).$$

Property (iii) then follows from simple algebra. Property (i) follows from the fact that

$$\frac{\partial \gamma}{\partial \lambda}(\lambda, m, n) = \frac{\lambda(m-n)^2 \exp(-\lambda(m+n))}{(\exp(-\lambda m) + \exp(-\lambda n))^2} > 0.$$

Using  $p = \exp(-\lambda m)$ ,  $q = \exp(-\lambda n)$  in  $\gamma$ , observe that the derivative of  $\frac{p \ln p + q \ln q}{p+q} - \ln\left(\frac{p+q}{2}\right)$  with respect to  $p$  is  $\frac{q(\ln p - \ln q)}{(p+q)^2} < 0$ , since  $p < q$ . Property (ii) follows by  $\frac{dp}{dm} < 0$ . To complete the proof,

$$\begin{aligned} \hat{\Gamma}(a^2\lambda_0, H + \mu, L + \nu) - \Gamma(ab\lambda_0, H + \mu, L + \nu) &= \int_{a^2\lambda_0}^{ab\lambda_0} -\frac{\partial \hat{\Gamma}}{\partial \lambda}(x, H + \mu, L + \nu) dx \\ \text{(by definition)} &= \int_{a^2\lambda_0}^{ab\lambda_0} \frac{1}{x^2} \gamma(x, H + \mu, L + \nu) dx \end{aligned}$$

<sup>19</sup>We thank Xiaosheng Mu for providing the argument showing this inequality.



$$\begin{aligned}
& \text{(by property (iii) of } \gamma) & = \int_{a^2\lambda_0}^{ab\lambda_0} \frac{1}{x^2} \gamma(x, H + \mu - \nu, L) dx \\
& \text{(by } \mu > \nu, a < 1, \text{ and properties (i)-(ii) of } \gamma) & > \int_{a^2\lambda_0}^{ab\lambda_0} \frac{1}{x^2} \gamma(ax, H, L) dx \\
& \text{(changing variables to } y = ax) & = \int_{a^3\lambda_0}^{a^2b\lambda_0} \frac{1}{\left(\frac{y}{a}\right)^2} \gamma(y, H, L) \frac{1}{a} dy \\
& \text{(by definition)} & = a \int_{a^3\lambda_0}^{a^2b\lambda_0} -\frac{\partial \hat{\Gamma}}{\partial \lambda}(y, H, L) dy \\
& \text{(since } a < 1) & > a^2(\hat{\Gamma}(a^3\lambda_0, H, L) - \hat{\Gamma}(a^2b\lambda_0, H, L)). \quad \blacksquare
\end{aligned}$$

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