Kreps & Scheinkman with product differentiation: an expository note

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Abstract

Kreps and Scheinkman's (1983) celebrated result is that in a two-stage model of a market with homogeneous products in which firms noncooperatively pick capacities in the first stage and set prices in the second stage, the equilibrium outcome is that of a one-shot Cournot game. This note derives capacity reaction functions for the first stage and extends the Kreps and Scheinkman result to the case of differentiated products.

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*I am grateful to Norbert Schulz and Xavier Wauthy for comments on an earlier version of these notes. Responsibility for errors is my own.

Contents

1. Introduction

Kreps and Scheinkman (1983) develop an extension of the Cournot and Bertrand duopoly models in which (1983, p. 327)

> *Capacities* are set in the first stage by the two producers. Demand is then determined by Bertrand-like price competition, and production takes place at zero cost, subject to capacity constraints generated by the first-stage decisions.

Equilibrium in this two-stage game has firms selecting capacities in the first stage that are just sufficient to produce the Cournot equilibrium outputs, and producing those outputs in the second period.

The Kreps and Scheinkman model is frequently cited as providing a justification for use of Cournot model, which is characterized as being differentially unsatisfactory compared with the Bertrand model. For example (Maggi, 1996, p. 240):[1]

> The Cournot model of quantity competition has been the subject of considerable criticism in the theory of industrial organization. If taken literally, the Cournot model assumes that firms dump their production on the market and that an auctioneer determines the price that clears the market. In most industries there is nothing that resembles an auctioneer, and firms use prices as a strategic variable.

Such criticisms may be put forward in part to motivate interest in analyses of the Kreps and Scheinkman type. This does not make them compelling. All models are abstractions from reality. The mechanism by which price is determined is also unspecified in the standard model of a perfectly competitive market, a model that is not set aside on that account.

Nor are such criticisms needed to justify interest in the Kreps and Scheinkman model: it is a seminal example of analysis of rivalry in an imperfectly competitive market in which outcomes depend critically on the sequence in which decisions may be taken, and the way in which earlier decisions condition the payoffs associated with later decisions.

[1] Friedman (1982, p. 505) compares the Cournot model, the Bertrand model, and a model of price-setting oligopoly with product differentiation, and concludes that the latter is to be preferred on the grounds that it relies on more satisfactory assumptions without being computationally more complex.

Kreps and Scheinkman assume that the product is homogeneous. One implication of this assumption is that they must include in the model an assumed rationing rule that determines the quantity demanded of a higher-price firm if the capacity of a lower-price firm does not allow it to supply the entire quantity demanded at the lower price. Their results depend on the particular form of rationing rule that is used, as they suggest (Kreps and Scheinkman, 1983, p. 328) and as Davidson and Deneckere (1986) show formally.

If one extends the Kreps and Scheinkman model to differentiated products, the quantity demanded of each firm in the second stage of the game is well defined for all price pairs. This avoids the need to have a rationing rule as part of the model.

In these lecture notes, I show that the Kreps and Scheinkman result holds when the model is extended to the case of differentiated products.

There is a sense in which this result is intuitive. One would not expect firms to hold excess capacity in equilibrium. If there is no excess capacity in the second stage, then when firms maximize profit in the first stage, they are maximizing a payoff function that has then same demand and cost structures as in the corresponding one-shot Cournot game, the difference being that in the first stage of the two-stage game, firms select capacities rather than outputs. This leads to the result that with product differentiation, firms' capacity reaction functions in the neighborhood of equilibrium are functionally identical to the Cournot quantity reaction functions. It is thus to be expected that the equilibrium of the two-stage game should reproduce the Cournot outcome.

This result is obtained by Yin and Ng (1997; 2000) (see also Schulz (1999, 2000), who do not present the capacity reaction functions.

The analysis reported here is tedious. It has in common with many spatial models of imperfectly competitive markets that a complete treatment requires working through many cases which never occur in equilibrium, and which one knows from the beginning, or at least strongly suspects, will not occur in equilibrium. It turns out nonetheless to be necessary to work these cases out in order to demonstrate that they do not occur in equilibrium, and to verify that the most plausible suspect is in fact an equilibrium.

Section 2 presents the model of demand for differentiated products, due to Bowley (1924), that is used in these notes. Section 3 gives the cost function.

For comparison and background, Sections 4.1 and 4.2 give abbreviated treatments of the Bertrand and Cournot duopoly models with product differentiation.

As is standard in two-stage models, the analysis begins in the second stage and works backward. Section 5 introduces the possible segments of a firm's second-stage price reaction function, and Section 6 works out the three possible shapes of the price reaction functions.

Section 7 states the results, in the form of four Lemmas and a Theorem. Lemma 1 gives the relationship between the capacity level chosen in the first stage and the shape of the firm's price reaction function in the second period. Lemma 2 gives the relationship between the capacity levels chosen in the first period and the nature of the price reaction functions in the neighborhood of equilibrium. Lemma 3, the proof of which is entirely mechanical, gives equilibrium prices, quantities, and payoffs for the alternative second-stage equilibria. Lemma 4 gives the properties of the (first-stage) capacity reaction functions. The Theorem is that the result of the Kreps and Scheinkman model holds when products are differentiated.

2. Demand

For the Bowley (1924) linear product differentiation model the representative consumer utility function is

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}b(q_1^2 + 2\theta q_1 q_2 + q_2^2) + m, \tag{2.1}$$

where $0 \leq \theta \leq 1$.

A Lagrangian function to describe the constrained optimization problem is

$$\begin{aligned} \mathcal{L}_1 = \\ U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}b(q_1^2 + 2\theta q_1 q_2 + q_2^2) + m \\ + \lambda_1 (Y - m - p_1 q_1 - p_2 q_2), \end{aligned} \tag{2.2}$$

where Y is income, m all other goods, and $p_m = 1$ the price of all other goods.

The Kuhn-Tucker conditions are

$$a - b(q_1 + \theta q_2) - \lambda p_1 \leq 0 \qquad q_1 [a - b(q_1 + \theta q_2) - \lambda p_1] = 0 \qquad q_1 \geq 0 \tag{2.3}$$

$$a - b(\theta q_1 + q_2) - \lambda p_2 \leq 0 \qquad q_2 [a - b(\theta q_1 + q_2) - \lambda p_2] = 0 \qquad q_2 \geq 0 \tag{2.4}$$

$$1 - \lambda_1 \leq 0 \qquad m(1 - \lambda_1) = 0 \qquad m \geq 0 \tag{2.5}$$

$$Y - m - p_1 q_1 - p_2 q_2 \geq 0 \qquad \lambda_1 (Y - m - p_1 q_1 - p_2 q_2) = 0 \qquad \lambda_1 \geq 0. \tag{2.6}$$

Assume Y is sufficiently large so that $m > 0$. Then (2.5) implies that

$$\lambda_1 = 1 \tag{2.7}$$

and (2.6) implies

$$m = Y - (p_1 q_1 + p_2 q_2). \tag{2.8}$$

Substitute (2.7) in (2.3) and (2.4) to obtain

$$a - b(q_1 + \theta q_2) - p_1 \leq 0 \qquad q_1 \left[a - b(q_1 + \theta q_2) - p_1\right] = 0 \qquad q_1 \geq 0 \tag{2.9}$$

$$a - b(\theta q_1 + q_2) - p_2 \leq 0 \qquad q_2 \left[a - b(\theta q_1 + q_2) - p_2\right] = 0 \qquad q_2 \geq 0. \tag{2.10}$$

Case 1: $q_1 > 0$, $q_2 > 0$.
Then (2.9) and (2.10) imply that the inverse demand curves are

$$p_1 = a - b(q_1 + \theta q_2) \tag{2.11}$$

$$p_2 = a - b(\theta q_1 + q_2). \tag{2.12}$$

The equations of the inverse demand curves can be inverted to obtain the equations of the demand curves when consumption of both varieties is positive:,

$$q_1 = \frac{a - p_1 - \theta(a - p_2)}{b(1 - \theta^2)} \tag{2.13}$$

$$q_2 = \frac{a - p_2 - \theta(a - p_1)}{b(1 - \theta^2)} \tag{2.14}$$

Case 2: $q_1 > 0$, $q_2 = 0$.
Then (2.9) implies that the inverse demand curve for variety 1 is

$$p_1 = a - b q_1, \tag{2.15}$$

with corresponding demand curve

$$q_1 = \frac{a - p_1}{b}. \tag{2.16}$$

(2.10) implies

$$p_2 \geq a - b\theta q_1. \tag{2.17}$$

Case 3: $q_1 = 0$, $q_2 > 0$.

(2.10) implies

$$p_2 = a - b\theta q_2. \tag{2.18}$$

with corresponding demand curve

$$q_2 = \frac{a - p_2}{b}. \tag{2.19}$$

(2.9) implies that

$$p_1 \geq a - b\theta q_2. \tag{2.20}$$

3. Cost

Let

k_i = firm i's capacity

ρ = long-run cost per unit of capacity

The cost of capacity is fixed; once the firm gets to the second period, its cost function is

$$C(q_i; k_i) = cq_i + \rho k_i,\ q_i \leq k_i. \tag{3.1}$$

The units in which capacity is measured are normalized so that one unit of capacity allows production of one unit of output.

4. Benchmark cases

4.1. Bertrand duopoly

If firms compete in prices, marginal cost is x, and the quantity demanded of both firms is positive, firm 1's profit function is

$$\pi_1 = (p_1 - x)\frac{(1-\theta)(a-x) - (p_1 - x) + \theta(p_2 - x)}{b(1-\theta^2)} \tag{4.1}$$

The first-order condition is

$$2(p_1 - x) - \theta(p_2 - x) = (1-\theta)(a-x) \tag{4.2}$$

and symmetric Bertrand equilibrium prices are

$$p_B(x) = x + \frac{1-\theta}{2-\theta}(a-x) \tag{4.3}$$

Substitute in the equation of the demand function to obtain Bertrand equilibrium quantities demanded: so that

$$q_B(x) = \frac{1}{(1+\theta)(2-\theta)}\frac{a-x}{b}. \tag{4.4}$$

The Bertrand equilibrium payoff with marginal cost x is

$$\pi_B(x) = \frac{(p_B(x)-x)^2}{b(1-\theta^2)} = \frac{1}{1+\theta}\frac{1-\theta}{(2-\theta)^2}\frac{(a-x)^2}{b} \tag{4.5}$$

4.2. Cournot duopoly

If firms compete in quantities and marginal cost is x, firm 1's profit function is

$$\pi_1 = [a - x - b(q_1 + \theta q_2)]q_1 \tag{4.6}$$

The first-order condition is

$$2q_1 + \theta q_2 = \frac{a-x}{b}, \tag{4.7}$$

leading to symmetric equilibrium outputs

$$q_C(x) = \frac{1}{2+\theta}\frac{a-x}{b}. \tag{4.8}$$

The corresponding Cournot equilibrium prices are

$$p_C(x) = x + \frac{a-x}{2+\theta} \tag{4.9}$$

Cournot equilibrium profit per firm is

$$\pi_C(x) = \frac{1}{(2+\theta)^2}\frac{(a-x)^2}{b} \tag{4.10}$$

5. Segments of the price reaction function

Firms first choose capacities, then set prices, then produce the quantities demanded at those prices.

Here we consider the nature of firm 1's price reaction function, taking capacity as given. There are at most four segments of the price reaction function; the actual number of segments may be two, three, or four, depending on the capacity level chosen in the first stage and on the parameters of the model.

Once k_1 has been chosen, firm 1's profit function for

$$q_1 = \frac{(1-\theta)a - p_1 + \theta p_2}{b(1-\theta^2)} \leq k_1 \tag{5.1}$$

is

$$\pi_1 = (p_1 - c)\frac{(1-\theta)a - p_1 + \theta p_2}{b(1-\theta^2)} - \rho k_1. \tag{5.2}$$

Call the first-order condition to maximize (5.2) when the capacity constraint (5.1) is not binding *branch one* of firm 1's price reaction function; this is the Bertrand reaction function with marginal cost equal to c, with equation

$$2(p_1 - c) - \theta(p_2 - c) = (1-\theta)(a-c) \tag{5.3}$$

If the capacity constraint is binding, firm 1's output equals capacity; the equation of the binding capacity constraint may be variously written

$$q_1 = \frac{(1-\theta)a - p_1 + \theta p_2}{b(1-\theta^2)} = k_1, \tag{5.4}$$

or

$$p_2 = \frac{p_1 - (1-\theta)a + b(1-\theta^2)k_1}{\theta} \tag{5.5}$$

or

$$p_1 = \theta p_2 + (1-\theta)a - b(1-\theta^2)k_1. \tag{5.6}$$

Call this *branch two* of firm 1's price reaction function

On branch two of its reaction function, firm 1's profit function is

$$\pi_1 = (p_1 - c - \rho)k_1. \tag{5.7}$$

If p_2 rises sufficiently, the quantity demanded of firm 2 goes to zero. At that point, the quantity demanded of firm 1 is less than $\min(k_1, q_{m(c)})$, where $q_{m(c)}$ is the output of a single-variety monopolist with marginal cost c per unit and the nonnegativity constraint $q_2 \geq 0$ becomes binding, and it is the $q_2 = 0$ equation,

$$-\theta p_1 + p_2 = (1-\theta)a \tag{5.8}$$

(from (2.14)) that is the equation of firm 1's price reaction function. Call this segment of the price reaction function *branch four* of firm 1's reaction function.

In the *KSPD* model, there is no *branch three* of the price reaction function. Maggi (1996) develops an extension of the Kreps and Scheinkman model in which firms may expand capacity after demand is realized, at a differentially higher unit cost, if it is profitable to do so. In the Maggi model, branch three is the segment of a firm's price reaction function when it maximizes profit, expanding capacity beyond the level chosen in stage one. Branch three does not appear in the *KSPD* model.

Finally, if p_2 rises sufficiently, the quantity demanded of firm 1 equals $\min(k_1, q_{m(c)})$, at which point firm 1's price reaction function becomes vertical: firm 2's price is so high that firm 1 can sell as close to monopoly output as its capacity level permits, without creating a positive demand for variety 2.

6. Price reaction functions

Four configurations are possible for firm 1's price reaction function, depending on its capacity level and on the parameters of the model.

(b2,b5): for very low capacity levels (as specified in Lemma 1), firm 1 is capacity constrained until p_2 reaches such a high level that the quantity demanded of firm 2 is zero even when firm 1 sells all it can produce, given its capacity; see Figure 6.1.

For larger capacity levels, but not exceeding a limit specified in Lemma 1, firm 1's price reaction function begins with the unconstrained, branch one segment, then moves on to branch two and branch five. See Figure 6.2.

For still higher values of k_1, but not exceed a level specified in Lemma 1, firm 1's price reaction function has all four segments. See Figure 6.3.

For very high capacity levels, firm one is not capacity constrained, and sets price along its branch one, until p_2 rises so high that firm 1 is able to set the unconstrained monopoly price and sell the unconstrained monopoly output. See Figure 6.4.

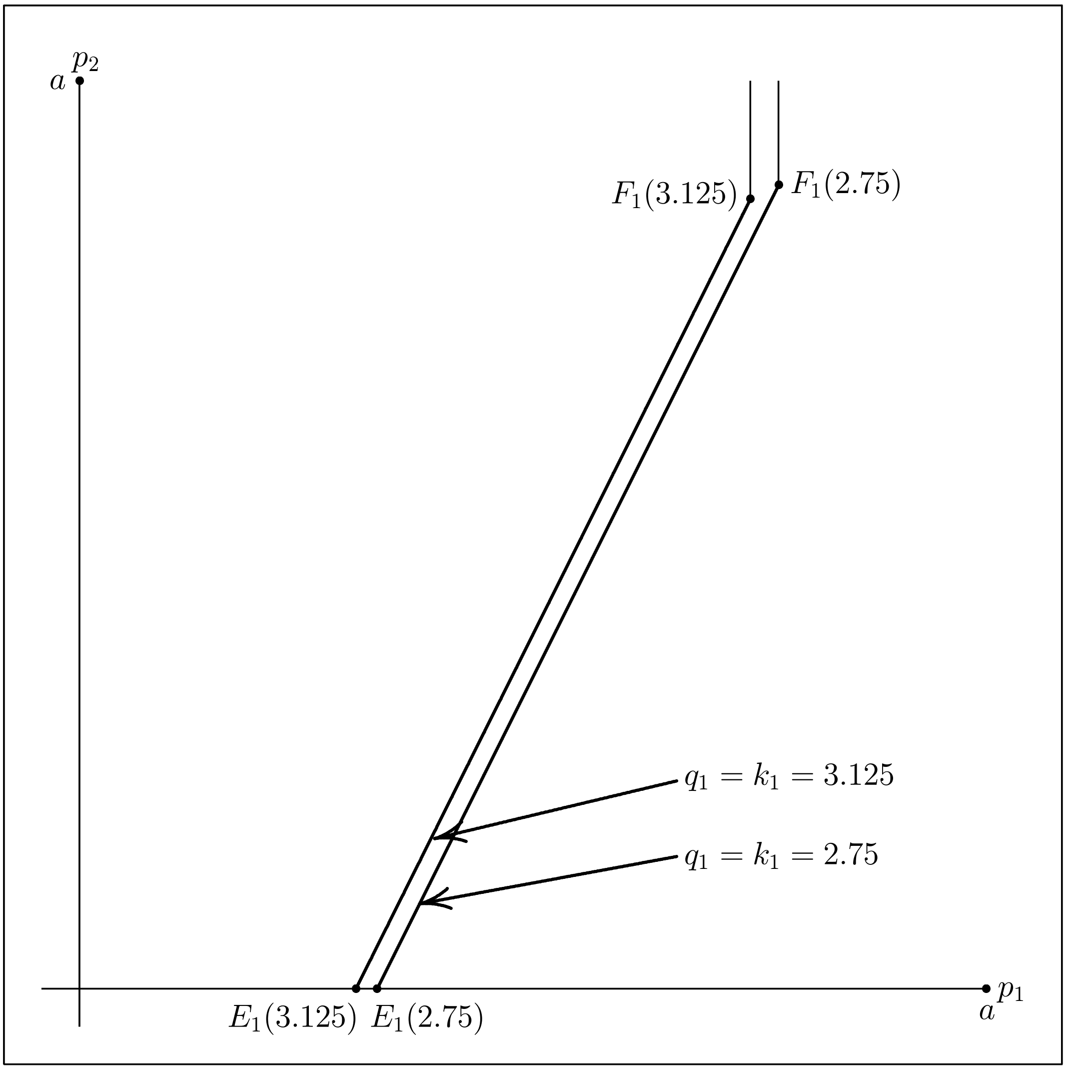

Figure 6.1: Firm 1's (b2,b5) price reaction function, $k_1 \leq k_A$

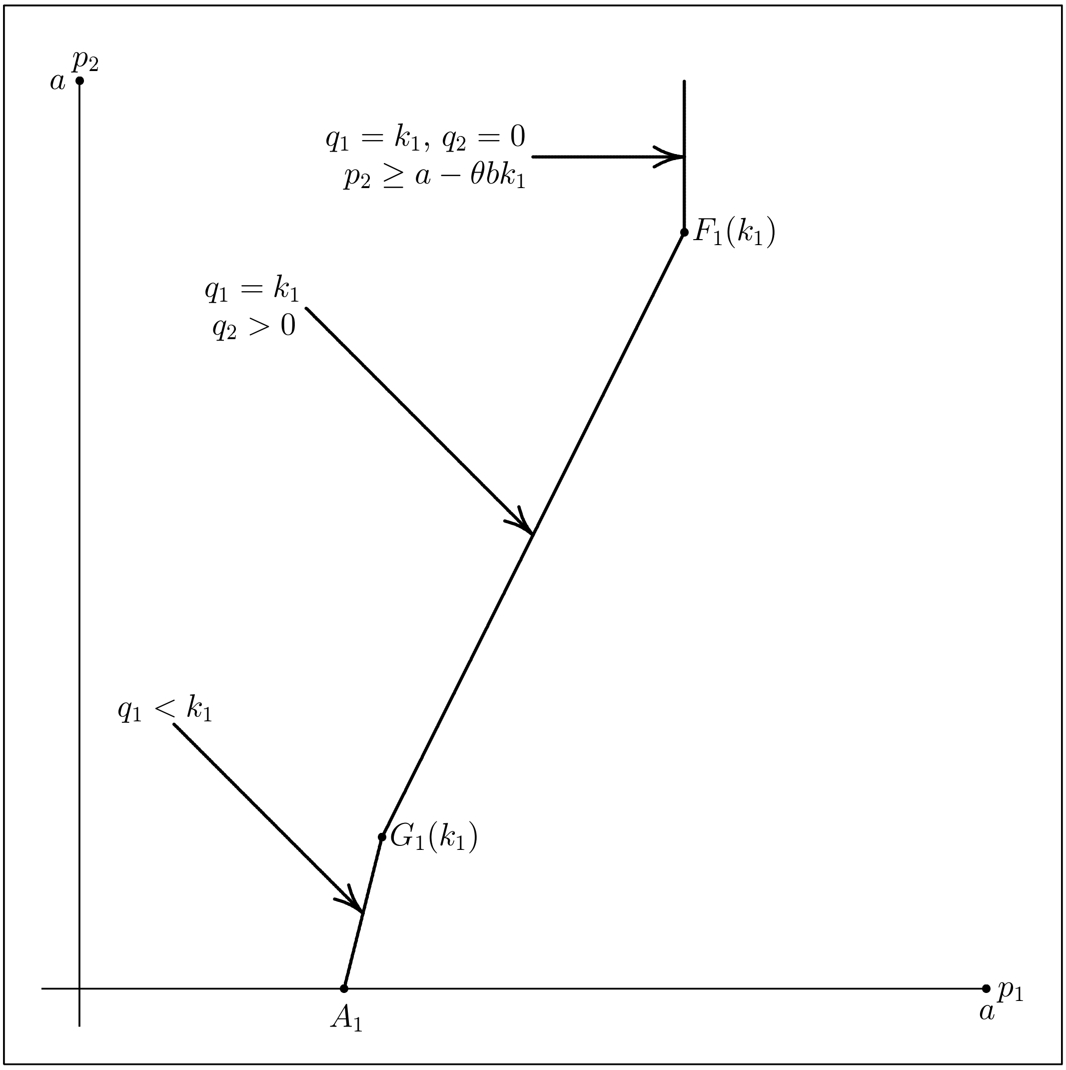

Figure 6.2: Firm 1's (b1,b2,b5) price reaction function, $k_A \leq k_1 \leq q_m$

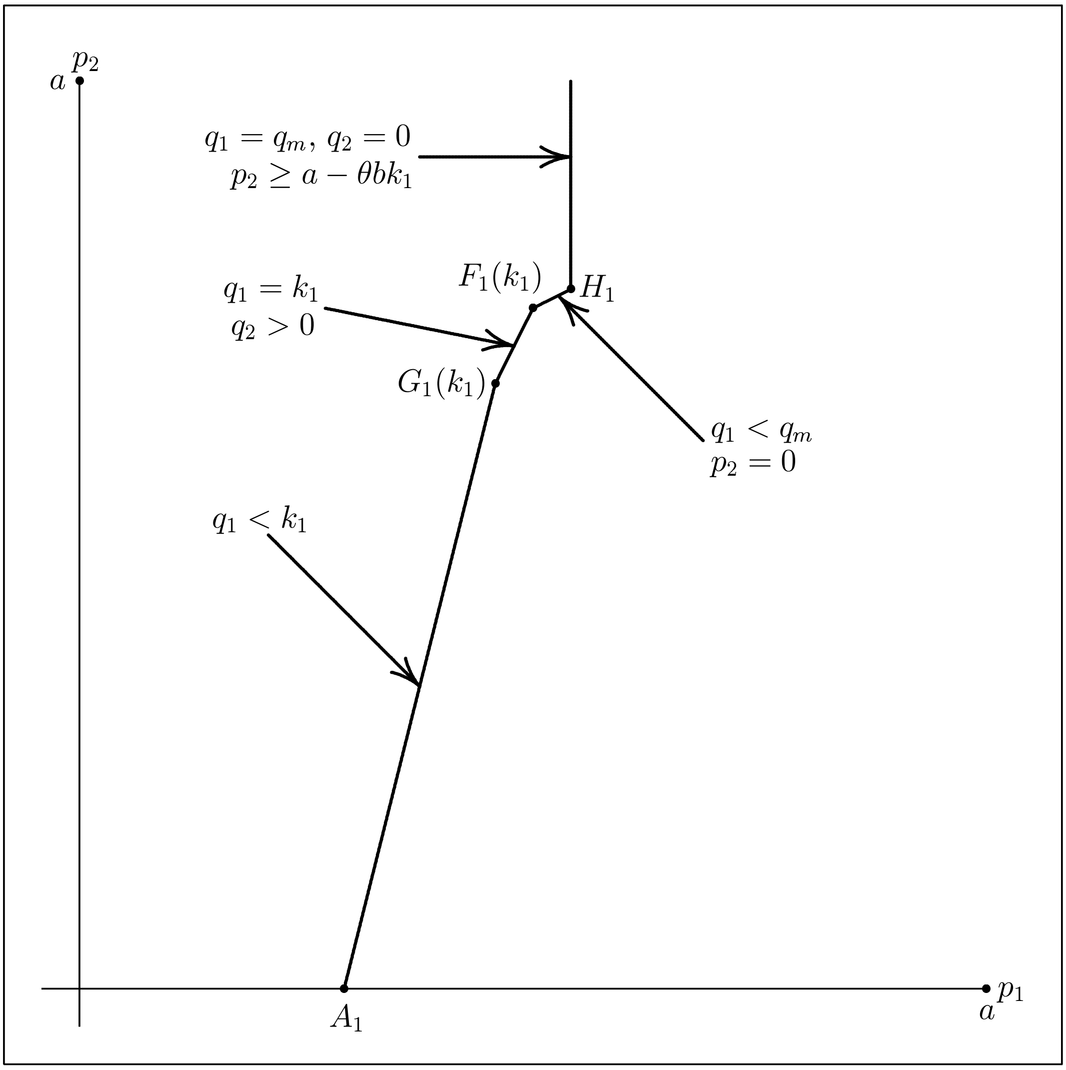

Figure 6.3: Firm 1's (b1,b2,b4,b5) price reaction function, $q_m \leq k_1 \leq k_D$

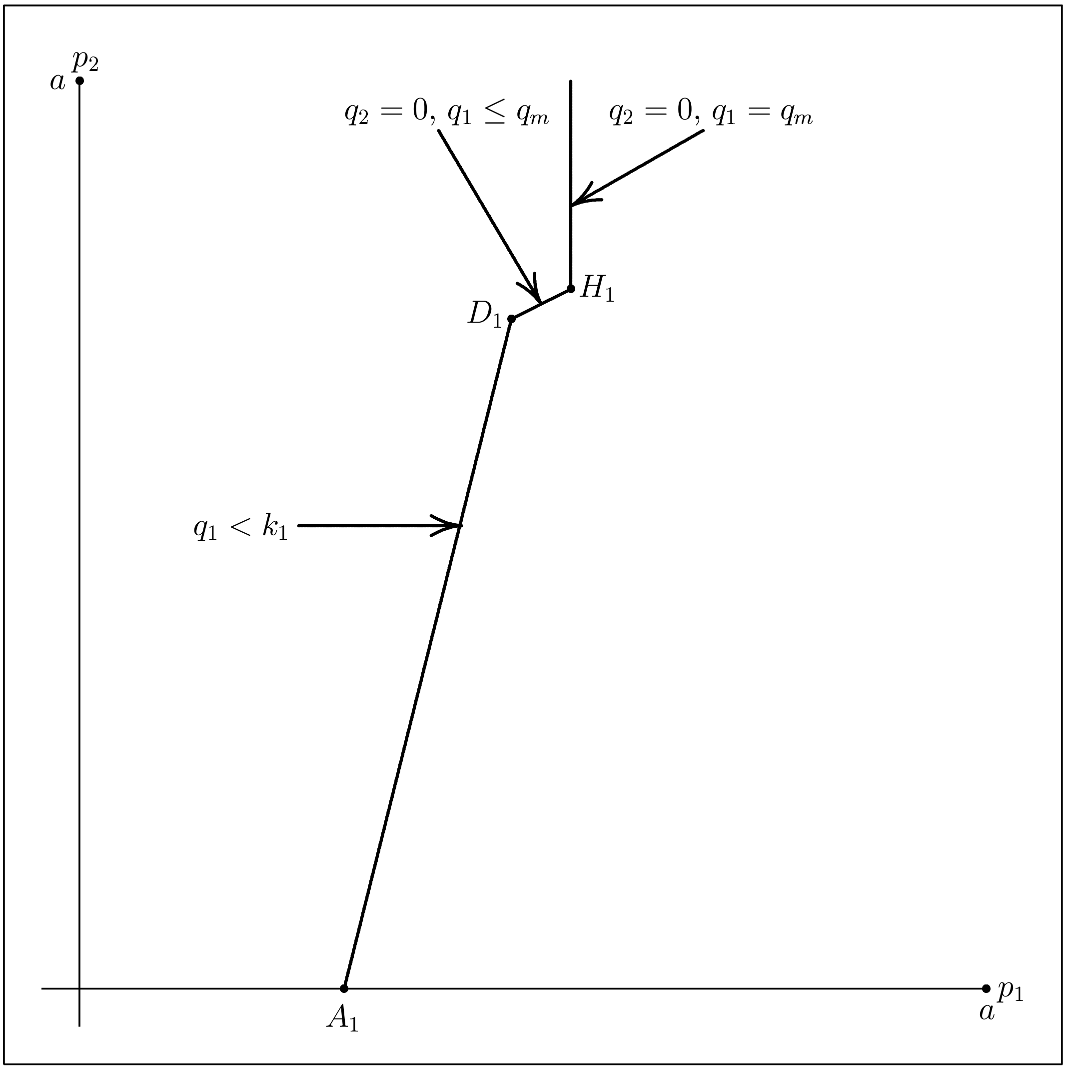

Figure 6.4: Firm 1's (b1,b4,b5) price reaction function, $k_1 \geq k_D$

7. Results

7.1. Lemma 1

Lemma 1: Let

$$k_A \equiv \frac{(1-\theta)(a-c)-\theta c}{2(1-\theta^2)b}, \tag{7.1}$$

$$k_D \equiv \frac{1}{2-\theta^2}\frac{a-c}{b}, \tag{7.2}$$

and assume (here and in what follows) that θ is not too large,

$$\theta \leq 1 - \frac{c}{a} = \frac{a-c}{a}, \tag{7.3}$$

so that $k_A > 0$.

Then

(a) the relation between first-stage capacity k_i and the configuration of the second-stage price reaction function is

$k_i \leq k_A$	$\Rightarrow$	firm i's reaction function is of the form (b2,b5)
$k_A \leq k_i \leq q_{m(c)}$	$\Rightarrow$	firm i's reaction function is of the form (b1,b2,b5)
$q_{m(c)} \leq k_i \leq k_D$	$\Rightarrow$	firm i's reaction function is of the form (b1,b2,b4,b5)
$k_D \leq k_i \leq \frac{a-c-\rho}{b}$	$\Rightarrow$	firm i's reaction function is of the form (b1,b4,b5)

(b) in the second stage, there are 16 possible combinations of price reaction functions of the two firms, one combination for each of the 16 regions shown Figure 7.1.

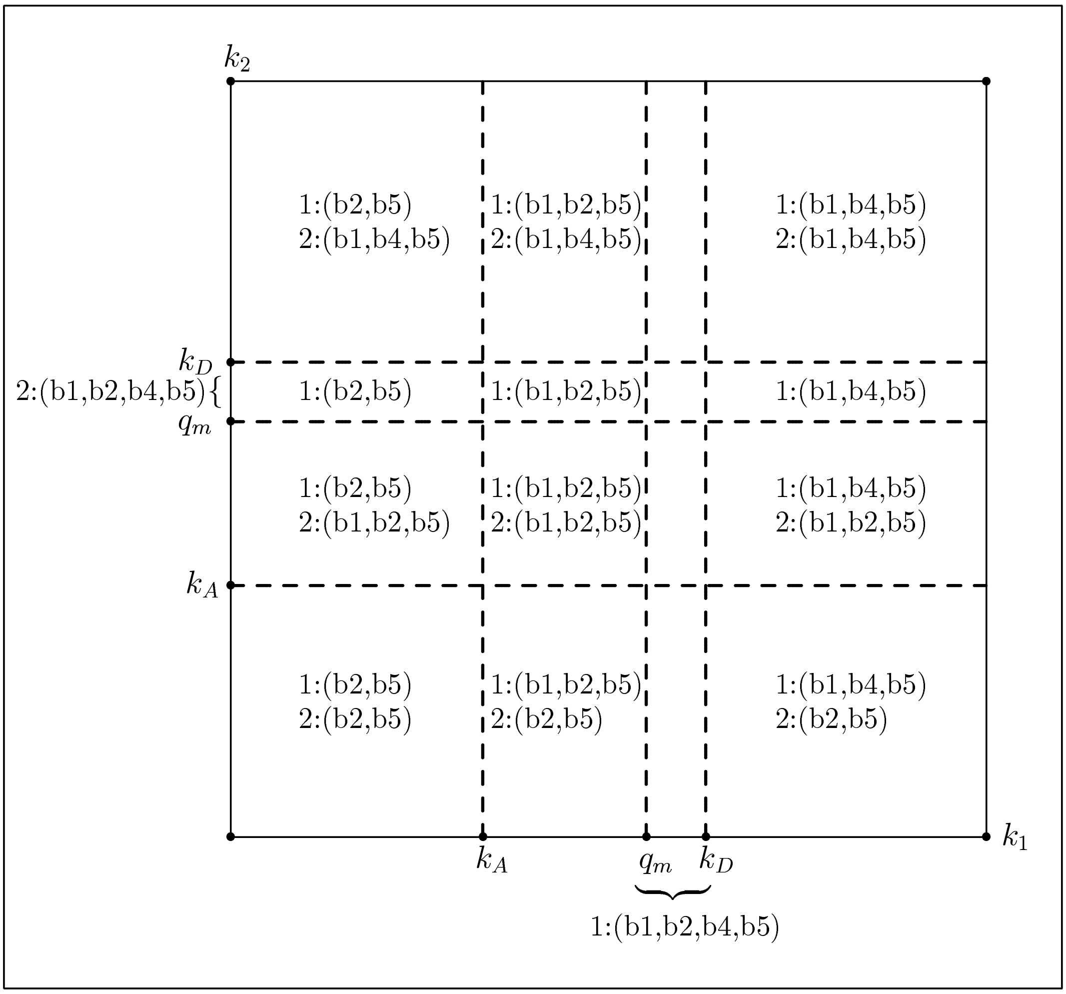

Figure 7.1: Price reaction function configurations, capacity space

7.2. Lemma 2

Lemma 2: Let

$$k^*_{B(c)} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} \tag{7.4}$$

denote the capacity that is just sufficient to allow a firm to produce the Bertrand equilibrium output with marginal cost c and let

$$k^{br}_{B(c)}(k_j) = \frac{1}{2-\theta^2}\left(\frac{a-c}{b} - \theta k_j\right) \tag{7.5}$$

be the capacity level that just allows firm i to produce its Bertrand best-response output when both firms have marginal cost c and firm j produces output level k_j.

There are four second-stage equilibrium types:

	Region of capacity space	*Firm 1*	*Firm 2*
$(b1, b1)$	$k_1 \geq k^*_{B(c)}$, $k_{2)} \geq k^*_{B(c)}$	branch one	branch one
$(b1, b2)$	$k_1 \geq k^{br}_{B(c)}(k_2)$, $k_2 \leq k^*_{B(c)}$	branch one	branch two
$(b2, b1)$	$k_1 \leq k^*_{B(c)}$, $k_2 \geq k^{br}_{B(c)}(k_1)$	branch two	branch one
$(b2, b2)$	$k_1 \leq k^{br}_{B(c)}(k_2)$, $k_2 \leq k^{br}_{B(c)}(k_1)$	branch two	branch two

These regions are shown in Figure 7.2.

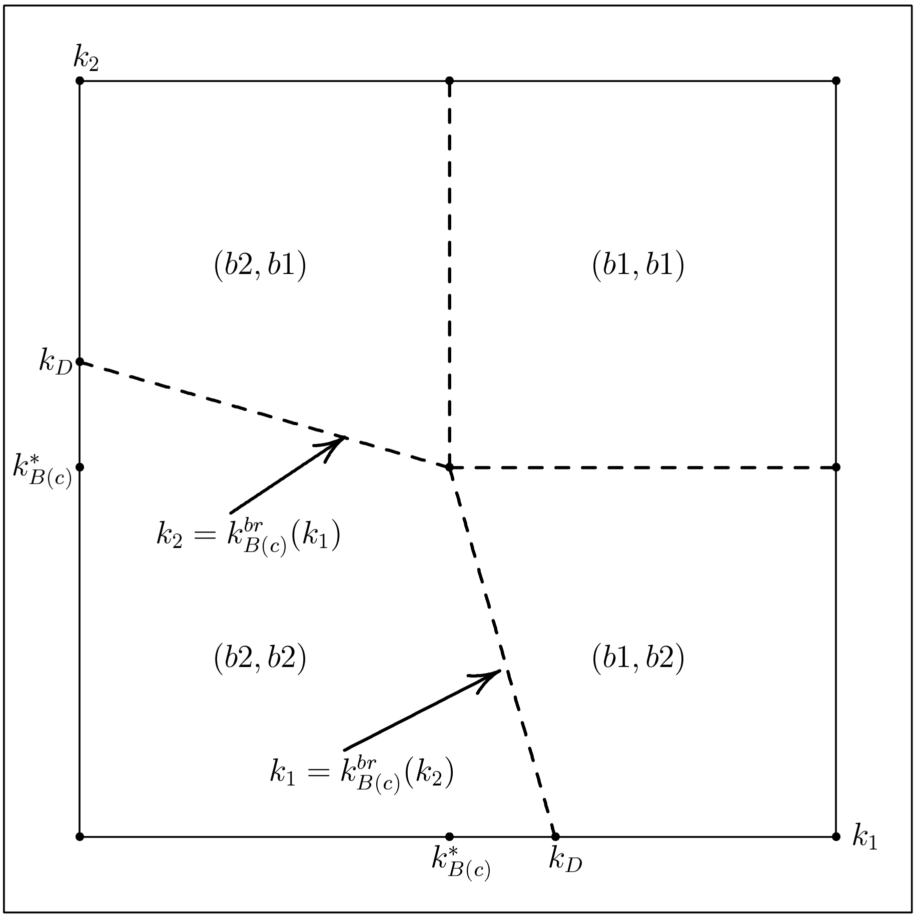

Figure 7.2: Segments of price reaction functions that intersect in equilibrium, capacity space; (bi, bj) indicates that in second-stage equilibrium, firm 1's branch i intersects with firm 2's branch j.

7.3. Lemma 3

Lemma 3: Second-stage equilibrium prices, outputs, and payoffs for the four equilibrium types described in *Lemma 2* are

(a) $(b1, b1)$

Firm 1	*Firm 2*
$p_{1b1b1} = c + \frac{1-\theta}{2-\theta}(a-c)$	$p_{2b1b1} = c + \frac{1-\theta}{2-\theta}(a-c)$
$q_{1b1b1} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b}$	$q_{2b1b1} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b}$
$\pi_{1b1b1} = \frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_1$	$\pi_{2b1b1} = \frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_2$

(b) $(b1, b2)$

Firm 1	*Firm 2*
$p_{1b1b2} = c + \frac{1-\theta^2}{2-\theta^2}(a-c-\theta b k_2)$	$p_{2b1b2} = c + \frac{(1-\theta)(2+\theta)(a-c)-2(1-\theta^2)bk_1}{2-\theta^2}$
$q_{1b1b2} = \frac{1}{2-\theta^2}\frac{a-c-\theta b k_2}{b}$	$q_{2b1b2} = k_2$
$\pi_{1b1b2} = \frac{1-\theta^2}{(2-\theta^2)^2}\frac{(a-c-\theta b k_2)^2}{b} - \rho k_1$	$\pi_{2b1b2} = \left(\frac{(1-\theta)(2+\theta)(a-c)-2(1-\theta^2)bk_2}{2-\theta^2} - \rho\right) k_2$

(c) $(b2, b1)$

Firm 1	*Firm 2*
$p_{1b2b1} = c + \frac{(1-\theta)(2+\theta)(a-c)-2(1-\theta^2)bk_1}{2-\theta^2}$	$p_{2b2b1} = c + \frac{1-\theta^2}{2-\theta^2}(a-c-\theta b k_1)$
$q_{1b2b1} = k_1$	$q_{2b2b1} = \frac{1}{2-\theta^2}\frac{a-c-\theta b k_1}{b}$
$\pi_{1b2b1} = \left(\frac{(1-\theta)(2+\theta)(a-c)-2(1-\theta^2)bk_1}{2-\theta^2} - \rho\right) k_1$	$\pi_{2b2b1} = \frac{1-\theta^2}{(2-\theta^2)^2}\frac{(a-c-\theta b k_1)^2}{b} - \rho k_2$

(d) $(b2, b2)$

Firm 1	*Firm 2*
$p_{1b2b2} = a - b(k_1 + \theta k_2)$	$p_{2b2b2} = a - b(\theta k_1 + k_2)$
$q_{1b2b2} = k_1$	$q_{2b2b2} = k_2$
$\pi_{1b2b2} = (a - c - \rho - b(k_1 + \theta k_2))k_1$	$\pi_{2b2b2} = (a - c - \rho - b(\theta k_1 + k_2))k_2$

7.4. Lemma 4

Lemma 4: Let

$$k_S = \frac{1}{\theta}\left\{\frac{a-c-\rho}{b} - 2\sqrt{2\frac{1-\theta^2}{2-\theta^2}\left[\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}\right]}\right\}. \quad (7.6)$$

Equilibrium capacity reaction functions are

Firm j's capacity	*Firm i's best response capacity*
$0 \le k_j \le k_S$	$k_i(k_j) = k^{br}_{C(c+\rho)}(k_2) = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_j\right)$,
$k_S \le k_j \le \frac{a-c-\rho}{b}$	$k_i(k_j) = \frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}$

where $k^{br}_{C(c+\rho)}(k_2)$ is the capacity level that just allows firm i to produce its Cournot best-response output if both firms have unit cost $c+\rho$ and firm j is producing output k_j.

The reaction functions are shown in Figure 7.3.

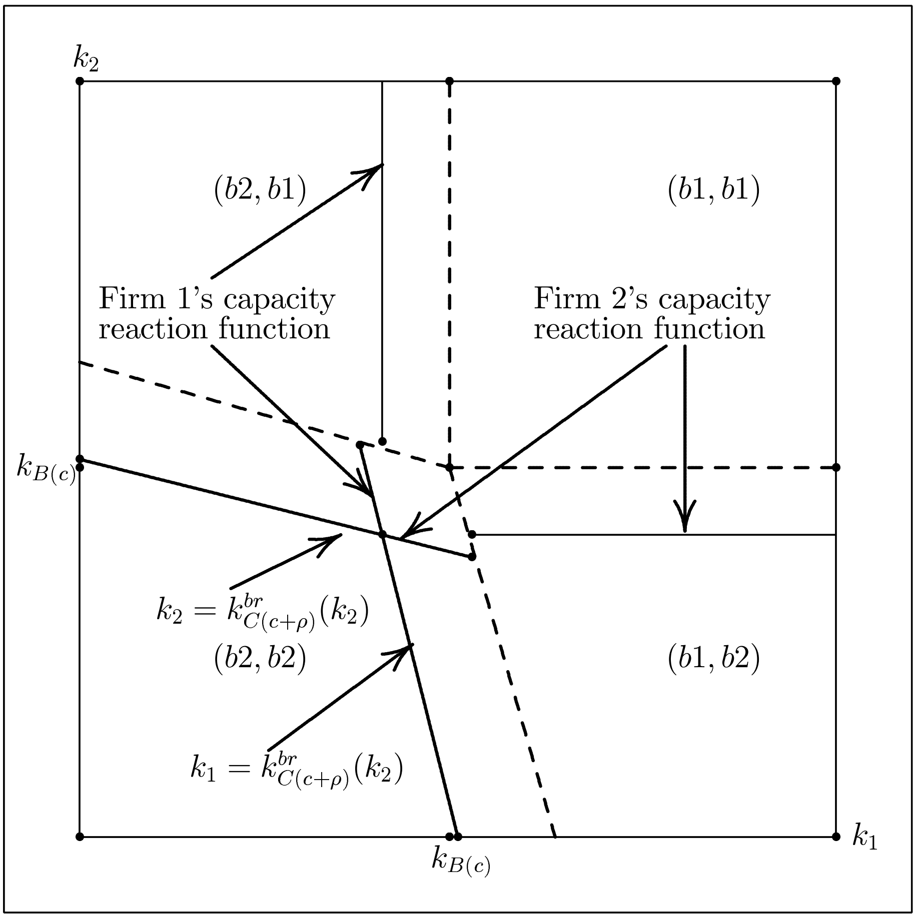

Figure 7.3: Capacity reaction functions, Kreps & Scheinkman model with product differentiation, $a = 12$, $b = c = \rho = 1$, $\theta = 1/2$

7.5. Theorem

Let

$$k_{C(c+\rho)} = \frac{1}{2+\theta}\frac{a-c-\rho}{b} \tag{7.7}$$

denote the minimum capacity that permits a firm to produce the Cournot equilibrium output of the one-shot game when both firms have marginal cost $c+\rho$.

Theorem: In the unique noncooperative equilibrium of the Kreps and Scheinkman model with product differentiation, firms select capacities $k_i = k_{C(c+\rho)}$ in the first stage and set the Cournot equilibrium prices in the second stage.

Proof: this follows from the facts that the segment of the capacity reaction function that is functionally identical to the Cournot quantity reaction function rise above $k^*_{B(c)}$, the capacity level that permits a firm to produce the Bertrand equilibrium output when marginal cost is c,

$$k_S > k^*_{B(c)}$$

and that Bertrand equilibrium output with marginal cost c is greater than Cournot equilibrium output with marginal cost $c+\rho$:

$$k^*_{B(c)} > k_{C(c+\rho)}.$$

The reaction functions are shown in Figure 7.3. Figure 7.4 shows the price reaction functions for the continuation game when the noncooperative equilibrium capacity levels are chosen in the first stage.

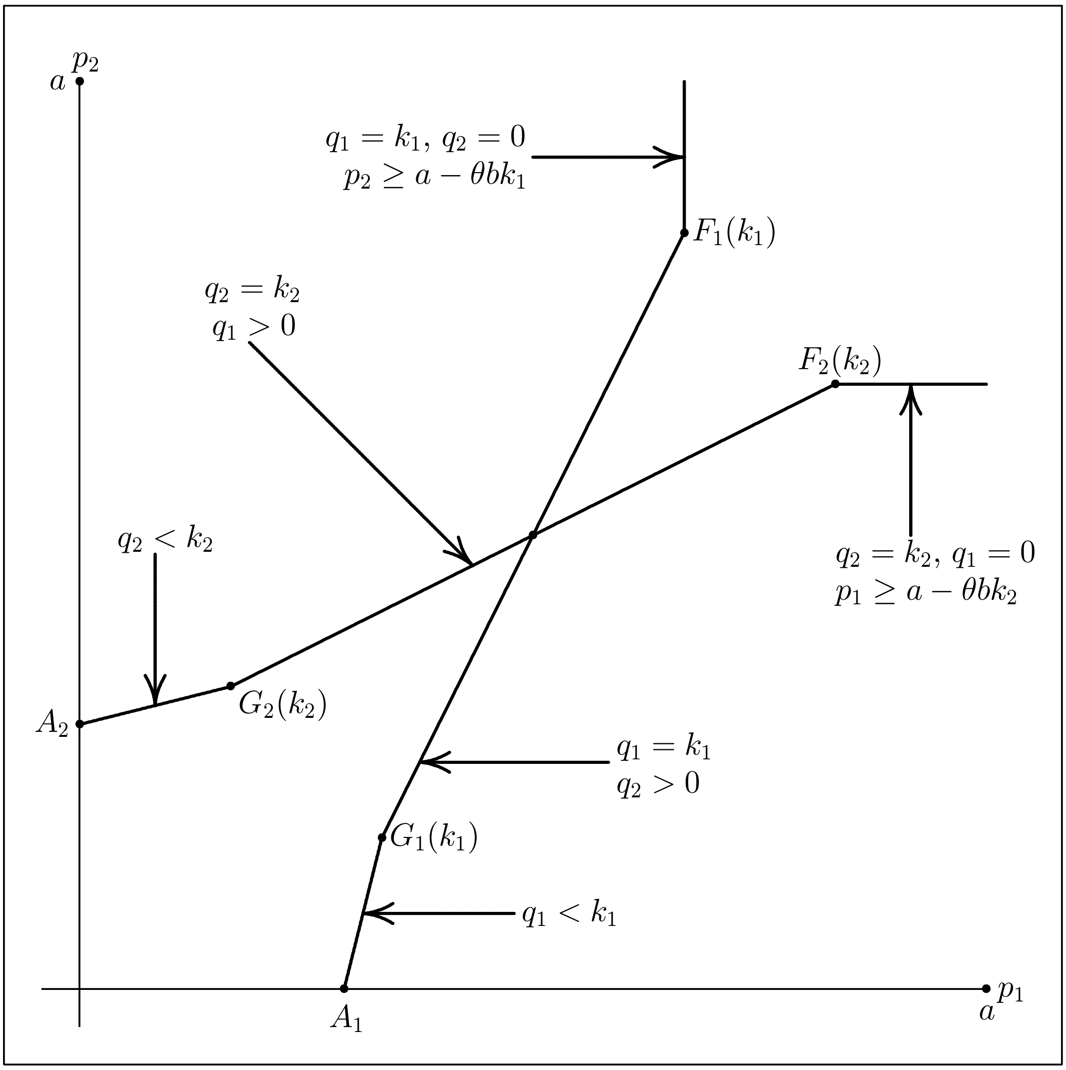

Figure 7.4: Second-stage (b2,b2) equilibrium for equilibrium capacities; both firms (b1,b2,b5) price reaction functions

8. Proof of *Lemma 1*

8.1. (b1,b4,b5)

Rewrite firm 1's branch one profit function, (5.2), in terms of deviations from marginal cost:

$$\pi_1 = (p_1 - c)\frac{(1-\theta)(a-c) - (p_1 - c) + \theta(p_2 - c)}{b(1-\theta^2)} - \rho k_1 \tag{8.1}$$

The first-order condition to maximize (8.1) with respect to p_1 is:

$$\frac{\partial \pi_1}{\partial p_1} = \frac{(1-\theta)(a-c) - 2(p_1 - c) + \theta(p_2 - c)}{b(1-\theta^2)} = 0. \tag{8.2}$$

Note that (8.2) implies that when the first-order condition holds, firm 1's output is

$$q_1 = \frac{(1-\theta)(a-c) - (p_1 - c) + \theta(p_2 - c)}{b(1-\theta^2)} = \frac{p_1 - c}{b(1-\theta^2)} \tag{8.3}$$

so that along this segment of its reaction function, firm 1's payoff is

$$\pi_1 = \frac{(p_1 - c)^2}{b(1-\theta^2)} - \rho k_1. \tag{8.4}$$

Solving (8.2) for p_1 gives the equation of the branch one ($q_1 < k_1$) segment of firm 1's reaction function:

$$p_1 = c + \frac{1}{2}[(1-\theta)(a-c) + \theta(p_2 - c)]. \tag{8.5}$$

For $p_2 = 0$,

$$p_1^{br}(0) = c + \frac{(1-\theta)(a-c) - \theta c}{2} = \frac{(1-\theta)a + c}{2}.$$

Firm 1 would shut down for $p_1 < c$. $p_1^{br}(0)$ is greater than or equal to c if varieties are sufficiently differentiated,

$$(1-\theta)(a-c) - \theta c \geq 0$$

or (7.3)

$$\theta \leq 1 - \frac{c}{a} = \frac{a-c}{a}.$$

Assume this condition is met.[2] Then in the (b1,b4,b5) case, the initial point of firm 1's price reaction function is

$$A_1\colon\ (p_{1A}, p_{2A}) = \left(c + \frac{(1-\theta)(a-c) - \theta c}{2}, 0\right) \tag{8.6}$$

(Figure 6.4).

Even though firm 1's branch one best response price rises as p_2 rises from $p_2 = 0$, p_1 rises relatively less than p_2, with the result that q_1 rises, and q_2 falls, moving up along firm 1's branch one price reaction function. This continues until p_2 reaches such a high level that q_2 falls to zero. As p_2 rises from this point, D_1 in Figure 6.4, firm 1's price rises, and q_1 falls, moving along the $q_2 = 0$ line (firm 1's branch four).

It follows that for firm 1's price reaction function to have the (b1,b4,b5) configuration, its capacity k_1 must permit it to produce the quantity demanded of it at point D_1, the intersection of firm 1's branch one and firm 1's branch four.

The system of equations formed by the equation of branch one, (8.5), and the equation of branch four, (5.8), both rewritten in terms of deviations from c, is

$$\begin{pmatrix} 2 & -\theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = (1-\theta)(a-c) \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \tag{8.7}$$

with solution

$$\begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = \frac{1-\theta}{2-\theta^2} \begin{pmatrix} 1+\theta \\ 2+\theta \end{pmatrix} (a-c). \tag{8.8}$$

Firm 1's branch one and branch four intersect at point

$$D_1\colon\ (p_{1D}, p_{2D}) = \left(c + \frac{1-\theta^2}{2-\theta^2}(a-c), c + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)\right). \tag{8.9}$$

From (8.3), the quantity demanded of firm 1 at point D_1 is

$$q_{1D} = \frac{p_1 - c}{b(1-\theta^2)} = \frac{1}{b(1-\theta^2)} \frac{1-\theta^2}{2-\theta^2}(a-c) = \frac{1}{2-\theta^2} \frac{a-c}{b}. \tag{8.10}$$

[2] If it is not, firm 1 shuts down for $p_2 = [c - (1-\theta)a]/\theta$ and the initial point of firm 1's price reaction function is $\left(c, \frac{c-(1-\theta)a}{\theta}\right)$.

Note that $q_{1D} > q_{m(c)}$, the unconstrained monopoly output with marginal cost c per unit

$$\frac{1}{2-\theta^2}\frac{a-c}{b} - \frac{a-c}{2b} = \frac{\theta^2}{2(2-\theta^2)}\frac{a-c}{b} \geq 0. \tag{8.11}$$

The condition for firm 1's price reaction function to have the (b1,b4,b5) configuration is then

$$k_1 \geq k_D. \tag{8.12}$$

Firm 1's best response price runs along the $q_2 = 0$ line until p_1 reaches the unconstrained monopoly price, which occurs for

$$c + \frac{1}{2}(a-c) = c + \frac{1}{\theta}\left((p_2 - c) - (1-\theta)(a-c)\right)$$

$$p_2 = c + \frac{1}{2}(2-\theta)(a-c) \equiv p_{2H}.$$

The second segment of firm 1's Bertrand best response function is the straight line connecting point D_1: (p_{1D}, p_{2D}) and point

$$H_1\colon\ (p_{1H}, p_{2H}) = \left(c + \frac{1}{2}(a-c), c + \frac{1}{2}(2-\theta)(a-c)\right). \tag{8.13}$$

For higher values of p_2, firm 1 charges the unconstrained monopoly price and sells the unconstrained monopoly quantity.

If (8.12) is met, firm 1's reaction function has three segments, branch one from point A_1 to point D_1, branch four from point D_1 to point H_1, and vertical at $p_1 = p_{1H}$ thereafter.

8.2. Branch two, branch five

At point A_1, $p_2 = 0$; the quantity demanded of firm 1 at point A_1 along firm 1's branch one ($q_1 < k_1$) is

$$q_{1A} = \frac{p_{1A} - c}{b(1-\theta^2)} = \frac{(1-\theta)(a-c) - \theta c}{2(1-\theta^2)b}. \tag{8.14}$$

q_{1A} is less than the unconstrained monopoly output of a single variety:

$$\frac{a-c}{2b} - \frac{(1-\theta)(a-c) - \theta c}{2(1-\theta^2)b} = \theta\frac{(1-\theta)a + \theta c}{2(1-\theta^2)b} \geq 0$$

Let

$$k_A = \frac{(1-\theta)(a-c)-\theta c}{2(1-\theta^2)b} \tag{8.15}$$

be the capacity that is just sufficient to allow the firm to produce q_{1A}.

If

$$k_1 \leq k_A, \tag{8.16}$$

firm 1 is on the capacity constrained (branch two) segment of its price reaction function until p_2 rises so much that q_2 falls to zero.

If $p_2 = 0$ along branch two, then from (5.6) firm 1's price is

$$p_1 = \theta p_2 + (1-\theta)a - b(1-\theta^2)k_1.$$

$$p_1 = (1-\theta)a - b(1-\theta^2)k_1 \equiv p_{1E}. \tag{8.17}$$

If $k_1 \leq k_A$, firm 1's reaction function begins at point

$$E_1(k_1) : (p_{1E}, p_{2E}) = \big((1-\theta)a - b(1-\theta^2)k_1, 0\big) \tag{8.18}$$

(Figure 6.1).

At what point do firm 1's branch two and branch four intersect? Solve the system of equations formed by (5.8) ($q_2 = 0$) and (5.6) ($q_1 = k_1$)

$$\begin{pmatrix} 1 & -\theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (1-\theta)a \begin{pmatrix} 1 \\ 1 \end{pmatrix} - b(1-\theta^2)k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.19}$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} - bk_1 \begin{pmatrix} 1 \\ \theta \end{pmatrix}. \tag{8.20}$$

The $q_2 = 0$ line and the $q_1 = k_1$ line intersect at point

$$F_1(k_1) : (p_{1F}, p_{2F}) = (a - bk_1, a - \theta bk_1) \tag{8.21}$$

(Figure 6.1).

Firm 1's branch two and branch four segments intersect at point $F_1(k_1)$, with coordinates given by (8.21).

If $q_2 = 0$, we obtain the same results if firm 1's constrained optimization problem is formulated with price or with quantity as firm 1's decision variable. We proceed in terms of quantity.

For $p_2 \geq p_{2F}$ and $k_1 \leq k_A$, firm 1 maximizes

$$(a - c - bq_1)q_1$$

subject to the capacity constraints

$$k_1 \geq q_1$$

and subject to the Kuhn-Tucker inequality for $q_2 = 0$ for the representative consumer constrained optimization problem,

$$p_2 \geq a - \theta b q_1.$$

A Lagrangian for firm 1's constrained optimization problem is

$$\mathcal{L}_1 = (a - c - bq_1)q_1 + \lambda_1(k_1 - q_1) + \lambda_2(p_2 - a + \theta b q_1).$$

The Kuhn-Tucker conditions are

$$a - c - 2bq_1 - \lambda_1 + \theta b \lambda_2 \leq 0 \quad q_1[a - c - 2bq_1 - \lambda_1 + \theta b \lambda_2] = 0 \quad q_1 \geq 0$$

$$k_1 - q_1 \geq 0 \quad \lambda_1(k_1 - q_1) = 0 \quad \lambda_1 \geq 0$$

$$p_2 - a + \theta b q_1 \geq 0 \quad \lambda_2(p_2 - a + \theta b q_1) = 0 \quad \lambda_2 \geq 0.$$

Suppose $q_1 = k_1 > 0$. Then

$$a - c - 2bq_1 - \lambda_1 + \theta b \lambda_2 = 0.$$

For this to be a solution, we must also have

$$p_2 \geq a - \theta b k_1 = p_{2F},$$

which condition is met. For $p_2 > a - \theta b k_1$, $\lambda_2 = 0$; then

$$\lambda_1 = 2b\left(\frac{a - c}{2b} - k_1\right) > 0.$$

When $k_1 \leq k_A$ and $p_2 \geq p_{2F}$, firm 1's best response is to set price $p_{1F} = a - bk_1$ and sell at capacity.

When capacity satisfies (8.16), the price reaction function has a branch two segment connecting point $E_1(k_1)$ to point $F_1(k_1)$ and is vertical thereafter.

Figure 6.1 shows firm 1's $(b2, b5)$ price reaction functions for three alternative levels of k_1. As k_1 falls, firm 1's branch two segment shifts right, and the point at which firm 1 shifts from its branch two to its branch four moves up the $q_2 = 0$ line.

8.3. Branch one, branch two, branch five and branch one, branch two, branch four,branch five

If $k_A \leq k_1 \leq k_D$, firm 1's reaction function begins on branch one, but its capacity is not sufficient to allow it to produce along branch one until it reaches branch four. If

$$k_A \leq k_1 \leq k_D, \tag{8.22}$$

then when p_2 rises sufficiently from $p_2 = c$, firm one moves from branch one to branch two.

What is the point of intersection of branch one and branch two? The equations of branch one and branch two are

$$2(p_1 - c) - \theta(p_2 - c) = (1 - \theta)(a - c)$$

and

$$p_1 - \theta p_2 = (1 - \theta)a - b(1 - \theta^2)k_1$$

respectively.

Rewritten in terms of deviations from c, the system of equations formed by the equations of firm 1's branch one and branch two is

$$\begin{pmatrix} 2 & -\theta \\ 1 & -\theta \end{pmatrix} \begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = (1 - \theta)(a - c) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - b(1 - \theta^2) \begin{pmatrix} 0 \\ k_1 \end{pmatrix}, \tag{8.23}$$

with solution

$$\begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = b(1 - \theta^2)k_1 \begin{pmatrix} 1 \\ 2/\theta \end{pmatrix} - (1 - \theta)(a - c) \begin{pmatrix} 0 \\ 1/\theta \end{pmatrix} \tag{8.24}$$

The point of intersection of branch one and branch two is

$$G_1(k_1)\colon\ (p_{1G}, p_{2G}) = \left(c + b(1 - \theta^2)k_1, c + \frac{2b(1 - \theta^2)k_1 - (1 - \theta)(a - c)}{\theta}\right) \tag{8.25}$$

(Figures 6.2 and 6.3).

By (8.11), $k_D > q_{m(c)}$. On the other hand, k_A is less than $q_{m(c)}$:

$$q_{m(c)} - k_A = \frac{a - c}{2b} - \frac{(1 - \theta)(a - c) - \theta c}{2(1 - \theta^2)b} =$$

$0 \leq k_1 \leq k_A$	(b2,b5)	$E_1(k_1) - F_1(k_1)-$ vertical
$k_A \leq k_1 \leq q_{m(c)}$	(b1,b2,b5)	$A_1 - G_1(k_1) - F_1(k_1)-$ vertical
$q_{m(c)} \leq k_1 \leq k_D$	(b1,b2,b4,b5)	$A_1 - G_1(k_1) - F_1(k_1)_1 - H_1-$ vertical
$k_D \leq k_1$	(b1,b4,b5)	$A_1 - D_1 - H_1-$ vertical

Table 8.1: Capacity and configuration of price reaction function

$$\frac{1}{2}\left(\frac{\theta}{1+\theta}\right)\frac{a-c}{b} + \frac{\theta c}{2(1-\theta^2)b} > 0.$$

For $k_A \leq k_1 \leq q_{m(c)}$, firm 1's reaction function has three segments, branch one from point A_1 to point $G_1(k_1)$, branch two from point $G_1(k_1)$ to point $F_1(k_1)$, and vertical (branch five) thereafter; see Figure 6.2.

For $q_{m(c)} \leq k_1 \leq k_D$, firm 1's reaction function has four segments, branch one from point A_1 to point $G_1(k_1)$, branch two from point $G_1(k_1)$ to point $F_1(k_1)$, branch four from point $F_1(k_1)$ to point H_1, and vertical (branch five) thereafter; see Figure 6.3.

8.4. Summary

The results obtained above are summarized in Table 8.1.

9. Proof of *Lemma 2*

9.1. Cell (4,4)

Figure 9.1 shows an equilibrium with both firms on branch one of their price reaction functions when both firms have reaction functions with configuration (b1,b4,b5). This combination of price reaction functions occurs in the (4,4) cell of Figure 7.1.

When both firms have (b1,b4,b5) reaction functions, equilibrium occurs at the intersection of the branch one segments if point D_1 lies above firm 2's branch one and point D_2 lies to the right of firm 1's branch one.

From (8.9), the coordinates of point D_1 are

$$(p_{1D}, p_{2D}) = \left(c + \frac{1-\theta^2}{2-\theta^2}(a-c), c + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)\right).$$

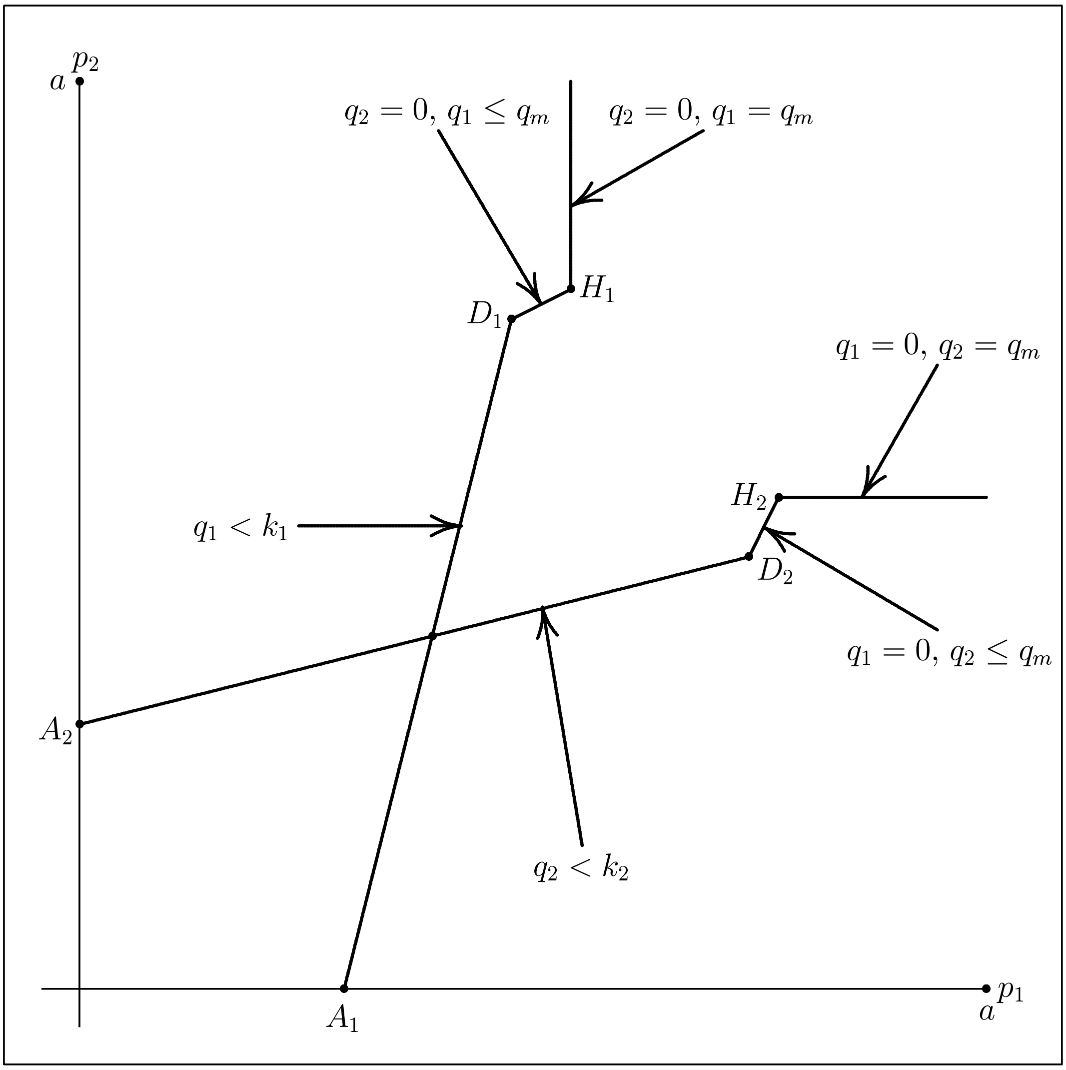

Figure 9.1: $(b1, b1)$ second-stage equilibrium, both firms $(b1, b4, b5)$ price reaction functions

The equation of the branch one segment of firm 2's price reaction function is

$$-\theta(p_1 - c) + 2(p_2 - c) = (1 - \theta)(a - c) \tag{9.1}$$

The condition for point D_1 to lie above firm 2's branch one is

$$-\theta(p_{1D} - c) + 2(p_{2D} - c) \geq (1 - \theta)(a - c) \tag{9.2}$$

Substituting the coordinates of point D_1, the condition is met if

$$-\theta\left[\frac{1-\theta^2}{2-\theta^2}(a-c)\right] + 2\left[\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)\right] \geq (1-\theta)(a-c)$$

$$-\theta\frac{1+\theta}{2-\theta^2} + 2\frac{2+\theta}{2-\theta^2} \geq 1$$

$$\frac{4+\theta-\theta^2}{2-\theta^2} \geq 1$$

$$4+\theta-\theta^2 \geq 2-\theta^2$$

$$2+\theta \geq 0,$$

which is always the case. In the same way, point D_2 is always to the right of firm one's branch two in cell (4,4). When both firms have (b1,b4,b5) price reaction functions, equilibrium always occurs at the intersection of the branch one segments.

9.2. Cell (3,4)

This configuration is shown in Figure 9.2. One condition for (b1,b1) equilibrium in cell (3,4) is that point D_2 be to the right of firm 1's branch 1; by the argument of Section 9.1, this condition is always met. The second condition is that point G_1 be above firm two's branch one.

The coordinates of point $G_1(k_1)$ are

$$(p_{1G}, p_{2G}) = \left(c + b(1-\theta^2)k_1, c + \frac{2b(1-\theta^2)k_1 - (1-\theta)(a-c)}{\theta}\right)$$

From (9.1), the condition for point G_1 to lie above firm 2's branch one is

$$-\theta(p_{1G} - c) + 2(p_{2G} - c) \geq (1 - \theta)(a - c); \tag{9.3}$$

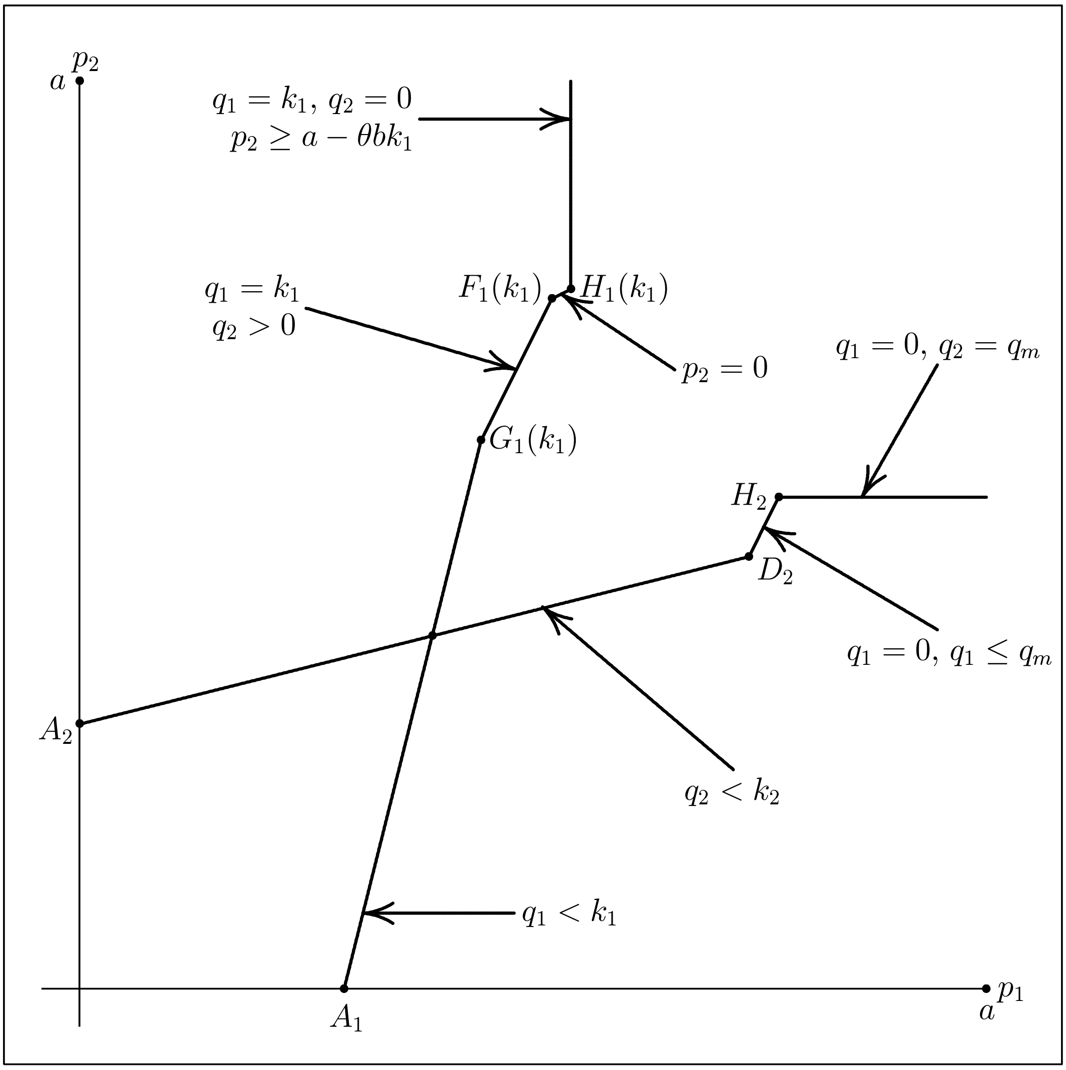

Figure 9.2: (b1,b1) equilibrium, cell (3,4): $k_m \leq k_1 \leq k_D$, $k_D \leq k_1$

or

$$-\theta(1-\theta^2)bk_1 + 2\frac{2(1-\theta^2)bk_1 - (1-\theta)(a-c)}{\theta} \geq (1-\theta)(a-c)$$

or

$$k_1 \geq \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} \equiv k^*_{B(c)}, \tag{9.4}$$

where, from (4.4), the capacity level $k^*_{B(c)}$ is the capacity level that is just sufficient to allow the firm to produce Bertrand equilibrium output when both firms have marginal cost c.

On the other hand, if

$$k_1 \leq k^*_{B(c)}$$

in cell (3,4), equilibrium is of type (b1,b2) (see Figure 9.3).

By similar arguments, in cell (4,3), equilibrium is of type (b1,b1) for $k_2 \geq k^*_{B(c)}$ and equilibrium is of type (b2,b1) for $k_2 \leq k^*_{B(c)}$.

9.3. Cell (2,4)

Second-stage equilibrium is of type (b1,b1) for $k_1 \geq k^*_{B(c)}$, as shown in Figure 9.4.

Second-stage equilibrium is of type (b2,b1) for $k_1 \leq k^*_{B(c)}$, as shown in Figure 9.5.

In the same way, second-stage equilibrium in cell (4,2) is of type (b1,b1) for $k_2 \geq k^*_{B(c)}$, and second-stage equilibrium is of type (b1,b2) in cell (4,2) for $k_2 \leq k^*_{B(c)}$.

9.4. Cell (1,4)

The conditions for (b2,b1) second-stage equilibrium in cell (1,4), as shown in Figure 9.6, are that point F_1 be above firm 2's branch one and that point D_2 be to the right of firm 1's branch 2.

The coordinates of point F_1 are

$$(p_{1F}, p_{2F}) = (a - bk_1, a - \theta bk_1),$$

and the condition for point F_1 to be above firm 2's branch one is

$$-\theta(p_{1F} - c) + 2(p_{2F} - c) \geq (1-\theta)(a-c)$$

$$-\theta(a - c - bk_1) + 2(a - c - \theta bk_1) \geq (1-\theta)(a-c)$$

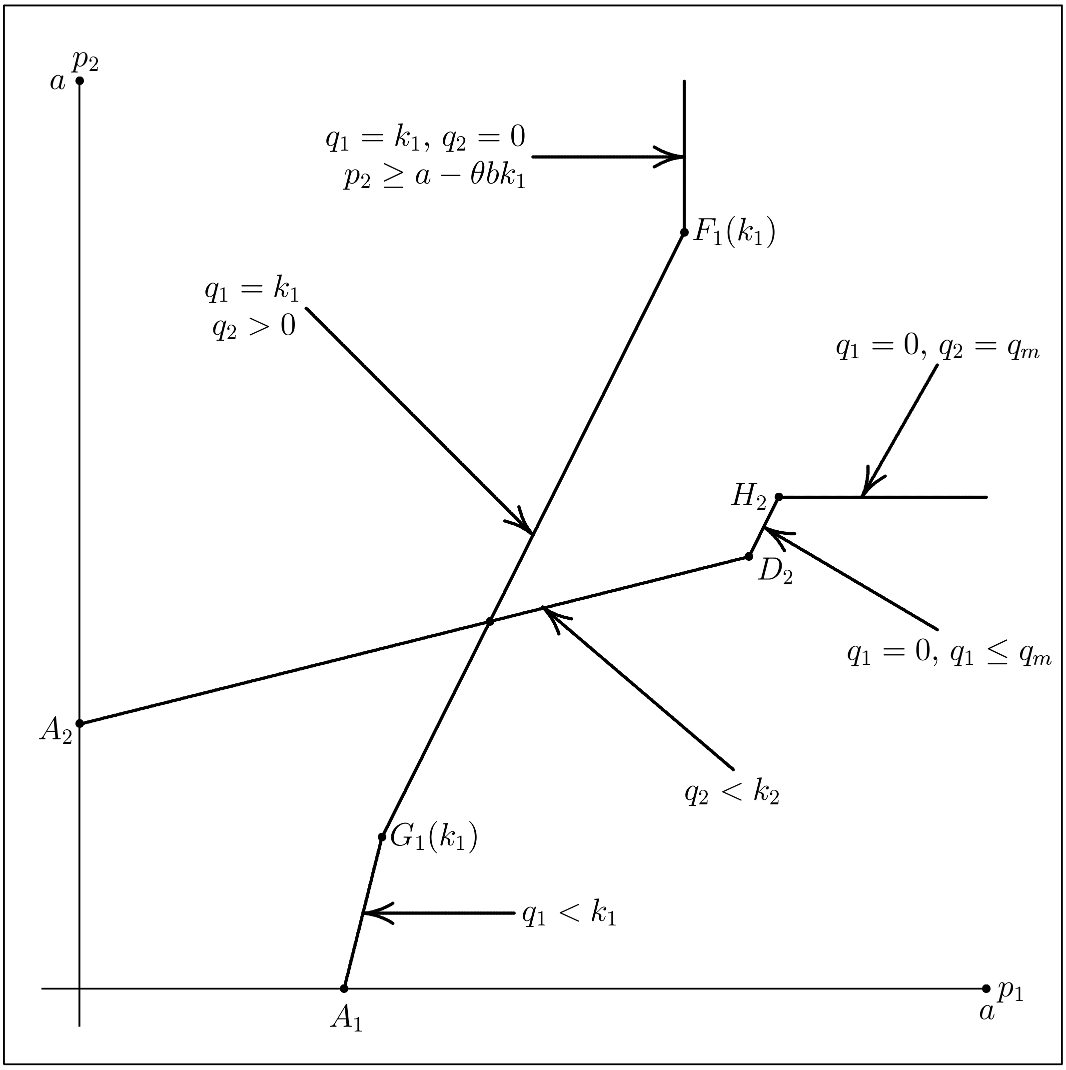

Figure 9.3: (b1,b2) equilibrium, cell (3,4): $k_m \leq k_1 \leq k_D$, $k_D \leq k_1$

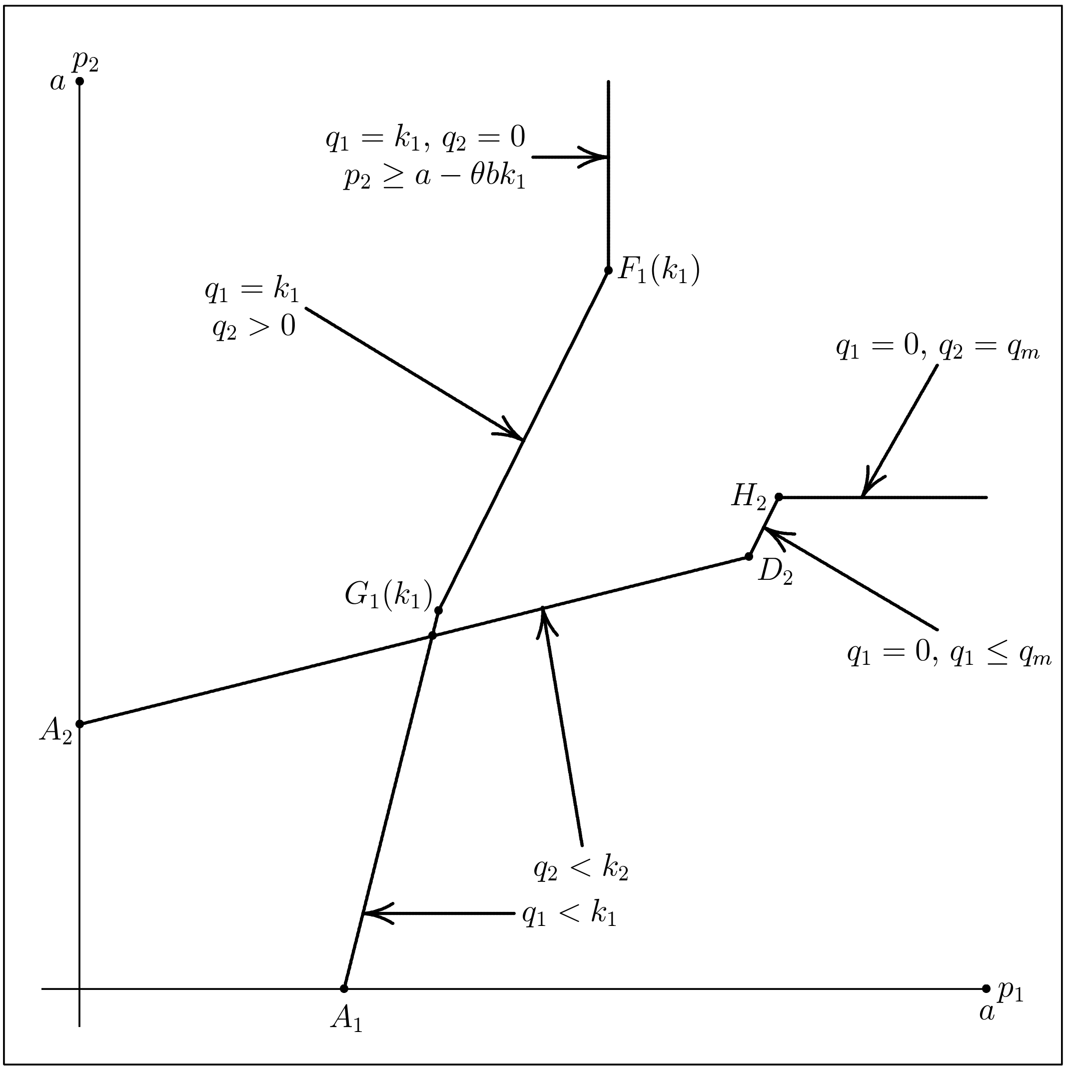

Figure 9.4: (b1,b1) equilibrium, firm 1 (b1,b2,b5) price reaction function, firm 2 (b1,b4,b5) price reaction function

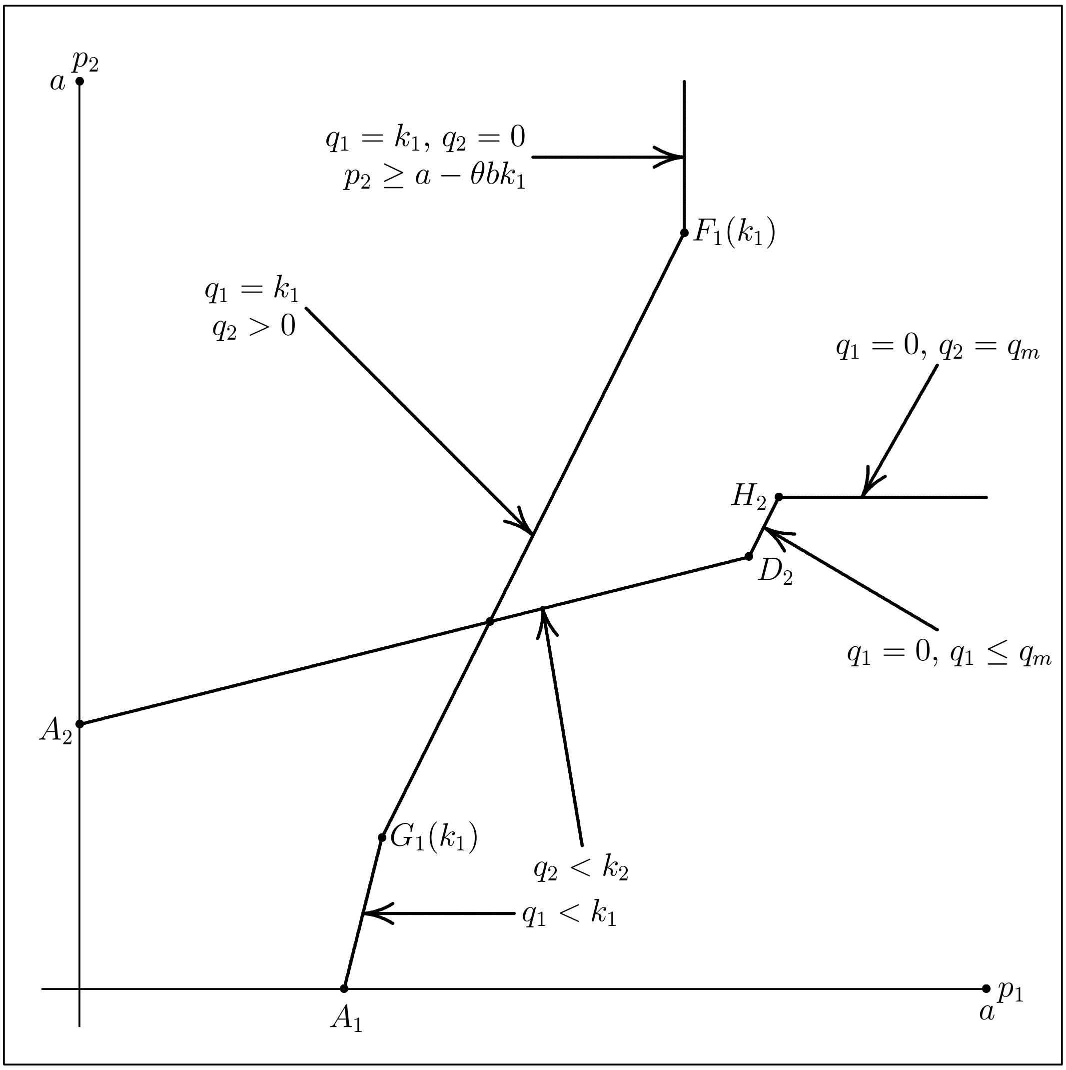

Figure 9.5: (b2,b1) equilibrium, firm 1 (b1,b2,b5) price reaction function, firm 2 (b1,b4,b5) price reaction function

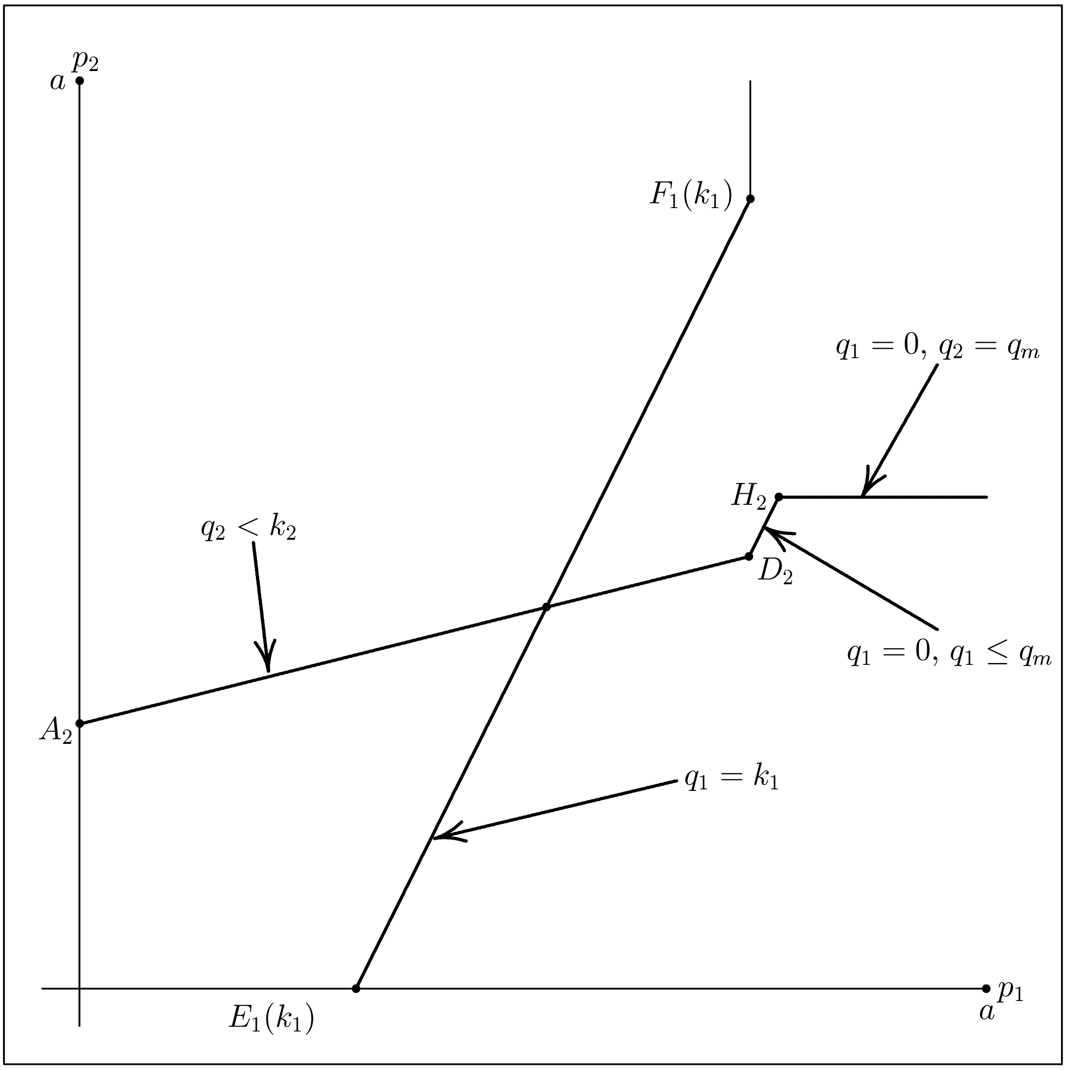

Figure 9.6: Firm 1's price reaction function, $k_1 \leq k_A$

$$
\begin{aligned}
-\theta(a-c)+\theta b k_1+2(a-c)-2\theta b k_1 &\geq (1-\theta)(a-c)\\
-(1-\theta)(a-c)-\theta(a-c)+2(a-c) &\geq \theta b k_1\\
k_1 &\leq \frac{a-c}{\theta b}. \qquad (9.5)
\end{aligned}
$$

$(a-c)/b$ is the long-run equilibrium output of a perfectly competitive industry; (9.5) will be met for all k_1 of interest.

The coordinates of point D_2 are

$$
(p_{1D}, p_{2D}) = \left(c + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c), c + \frac{1-\theta^2}{2-\theta^2}(a-c)\right).
$$

The condition for point D_2 to be to the right of firm 1's branch two is

$$
-\theta(p_{2D}-c)+p_{1D}-c \geq +(1-\theta)(a-c) - b(1-\theta^2)k_1.
$$

$$
-\theta\frac{1-\theta^2}{2-\theta^2}(a-c) + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c) \geq (1-\theta)(a-c) - b(1-\theta^2)k_1.
$$

$$
\begin{aligned}
b(1-\theta^2)k_1 &\geq (1-\theta)\left[1+\theta\frac{1+\theta}{2-\theta^2} - \frac{2+\theta}{2-\theta^2}\right](a-c).\\
b(1+\theta)k_1 &\geq 0
\end{aligned}
$$

and this condition is always met.

In cell (1,4), second-stage equilibrium is of type (b2,b1). In cell (4,1), second-stage equilibrium is of type (b1,b2).

9.5. Cell (3,3)

Cell (3,3) is defined by the inequalities $q_{m(c)} \leq k_1 \leq k_D$, $q_{m(c)} \leq k_2 \leq k_D$.

Figure 9.7 shows a second-stage equilibrium with both firms on branch one of their (b1,b2,b4,b5) reaction functions, producing less than capacity. This combination of price reaction functions occurs in the (3,3) cell of Figure 7.1.

The conditions for second-stage equilibrium to have this configuration are that point $G_1(k_1)$ be above firm 2's branch one and point $G_2(k_2)$ be to the right of firm 1's branch one. From Section 9.2, the conditions for this are $k_1 \geq k^*_{B(c)}$, $k_2 \geq k^*_{B(c)}$.

Since

$$
q_{m(c)} - k^*_{B(c)} = \frac{a-c}{2b} - \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} =
$$

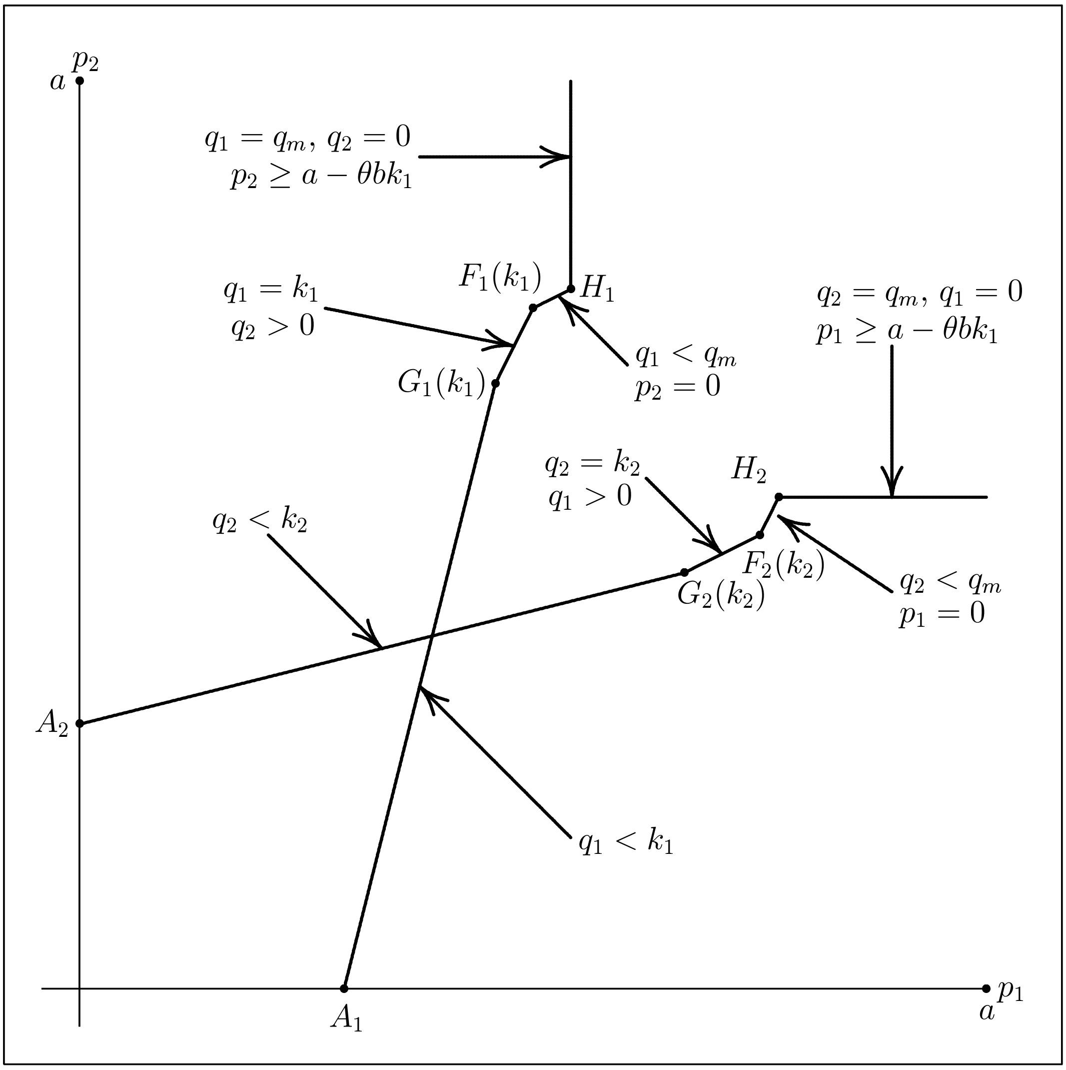

Figure 9.7: (b1,b1) equilibrium, both firms (b1,b2,b4,b5) price reaction functions

$$\frac{\theta(1-\theta)}{2(1+\theta)(2-\theta)}\frac{a-c}{b} \geq 0,$$

this condition is always satisfied, and in cell (3,3) the second-stage equilibrium is always of type is (b2,b2).

9.6. Cell (2,3)

Figure 9.8 shows a (b2,b1) second-stage equilibrium when firm 1 has a (b1,b2,b5) price reaction function and firm 2 has a (b1,b2,b4,b5) price reaction function. This combination of price reaction functions occurs in the (2,3) cell of Figure 7.1.

The conditions for such an equilibrium are first that point $F_1(k_1)$ be above firm 2's branch one while point $G_1(k_1)$ is below firm 2's branch one, and second that point $G_2(k_2)$ be to the right of firm 1's branch two.

From Section 9.4, point $F_1(k_1)$ is above firm 2's branch one for all k_1 of interest. From Section 9.2, the condition for point $G_1(k_1)$ to be below firm 2's branch one is that $k_1 \leq k^*_{B(c)}$.

The coordinates of point $G_2(k_2)$ are

$$\left(c + \frac{2(1-\theta^2)bk_2 - (1-\theta)(a-c)}{\theta}, c + (1-\theta^2)bk_2\right).$$

The condition for point $G_2(k_2)$ to be to the right of firm 1's branch two is

$$p_1 - c - \theta(p_2 - c) \geq (1-\theta)(a-c) - (1-\theta^2)bk_1.$$

$$\frac{2(1-\theta^2)bk_2 - (1-\theta)(a-c)}{\theta} - \theta(1-\theta^2)bk_2 \geq (1-\theta)(a-c) - (1-\theta^2)bk_1.$$

$$2(1-\theta^2)bk_2 - (1-\theta)(a-c) - \theta^2(1-\theta^2)bk_2 \geq \theta(1-\theta)(a-c) - \theta(1-\theta^2)bk_1.$$

$$\theta(1-\theta^2)bk_1 + (1-\theta^2)(2-\theta^2)bk_2 \geq \theta(1-\theta)(a-c) + (1-\theta)(a-c)$$

$$\theta(1-\theta^2)bk_1 + (1-\theta^2)(2-\theta^2)bk_2 \geq (1-\theta^2)(a-c)$$

$$\theta k_1 + (2-\theta^2)k_2 \geq \frac{a-c}{b}$$

$$k_2 \geq \frac{1}{2-\theta^2}\left(\frac{a-c}{b} - \theta k_1\right) \equiv k^{br}_{B(c)}(k_1), \tag{9.6}$$

where $k^{br}_{B(c)}(k_1)$ is the capacity level that is just sufficient to allow firm 2 to produce its Bertrand best response output if firm 2's marginal cost is c and firm 1's output is k_1.

If (9.6) is met, second-stage equilibrium in cell (2,3) is of type (b2,b1). If $k_2 \leq k^{br}_{B(c)}(k_1)$, second-stage equilibrium in cell (2,3) is of type (b1,b1).

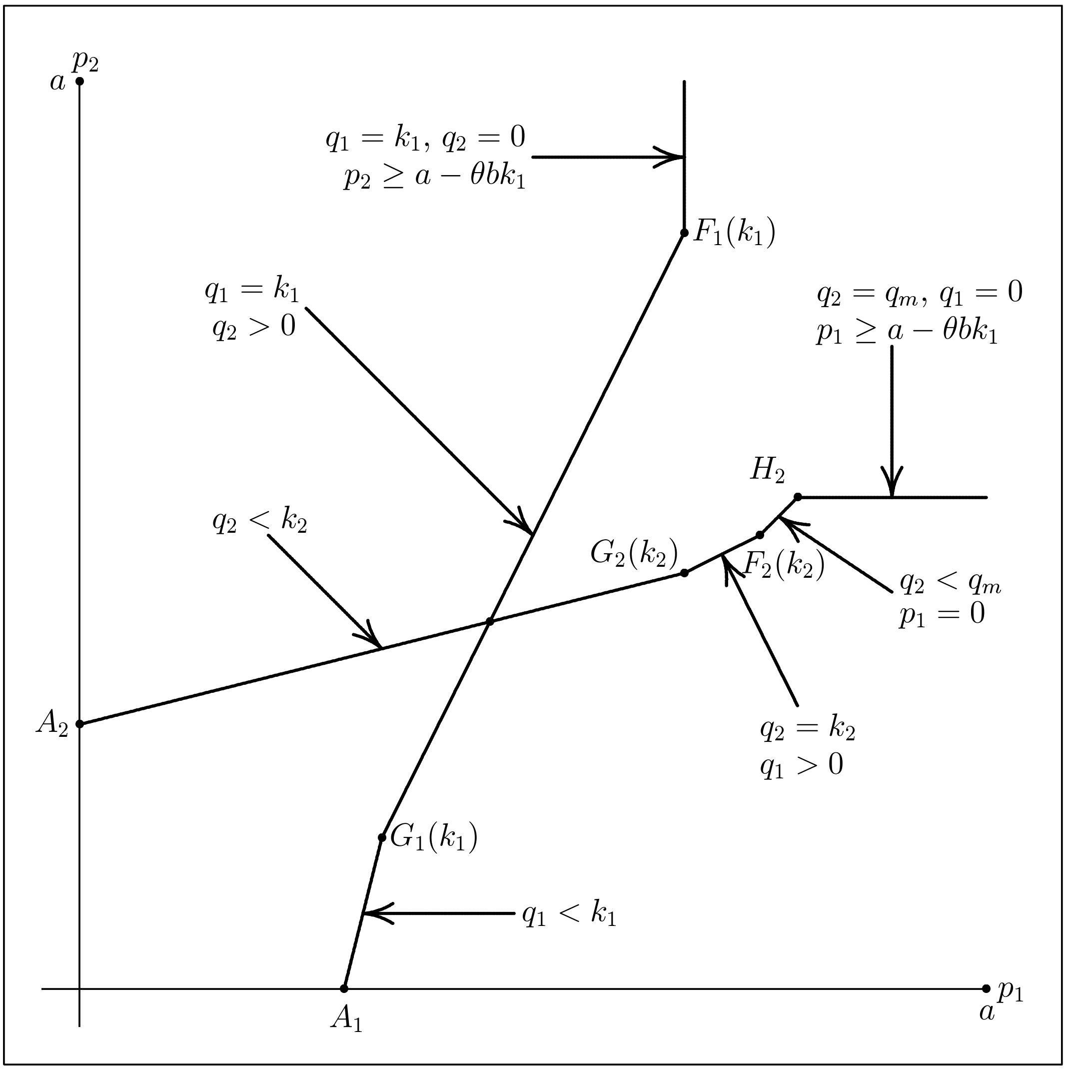

Figure 9.8: (b2,b1) second-stage equilibrium, firm 1 (b1,b2,b5) price reaction function, firm 2 (b1,b2,b4,b5) price reaction function

9.7. Cell (1,3)

Cell (1,3) is defined for $q_{m(c)} \leq k_2 \leq k_D$, $0 \leq k_1 \leq k_A$. The conditions for equilibrium to occur at the intersection of the branch two segments of the price reaction functions in cell (1,3) are that point $F_1(k_1)$ be above firm 2's branch two, that point $F_2(k_2)$ be to the right of firm 1's branch two, and that point $G_2(k_2)$ be to the left of firm 1's branch two.

The coordinates of point $F_1(k_1)$ are

$$(p_{1F}, p_{2F}) = (a - bk_1, a - \theta bk_1).$$

The equation of firm 2's branch two is

$$-\theta(p_1 - c) + p_2 - c = (1 - \theta)(a - c) - (1 - \theta^2)bk_2.$$

The condition for point $F_1(k_1)$ to be above firm 2's branch two is

$$-\theta(a - c - bk_1) + a - c - \theta bk_1 \geq (1 - \theta)(a - c) - (1 - \theta^2)bk_2$$

$$-\theta(a - c) + \theta bk_1 + a - c - \theta bk_1 \geq (1 - \theta)(a - c) - (1 - \theta^2)bk_2$$

$$k_2 \geq 0,$$

and this is satisfied for all k_2.

In the same way, point $F_2(k_2)$ is always to the right of firm 1's branch two.

From Section 9.6, the condition for point $G_2(k_2)$ to be to the left of firm 1's branch two is

$$k_2 \leq k^{br}_{B(c)}(k_1),$$

and if this condition is met, the second-stage equilibrium in cell (1,3) is of type (b2,b2). If instead $k_2 \geq k^{br}_{B(c)}(k_1)$, second-stage equilibrium in cell (1,3) is of type (b2,b1).

In the same way, in cell (3,2), second-stage equilibrium is of type (b1,b2) if $k_1 \leq k^{br}_{B(c)}(k_2)$ and of type (b2,b2) if $k_1 \geq k^{br}_{B(c)}(k_2)$.

9.8. Cell (2,2)

Cell (2,2) is defined by the inequalities $k_A \leq k_1 \leq q_{m(c)}$, $k_A \leq k_2 \leq q_{m(c)}$.

Figure 7.4 shows a second-stage (b2,b2) equilibrium in cell (2,2). The condition for this to occur is that point $F_1(k_1)$ be above firm 2's branch two, $G_1(k_1)$

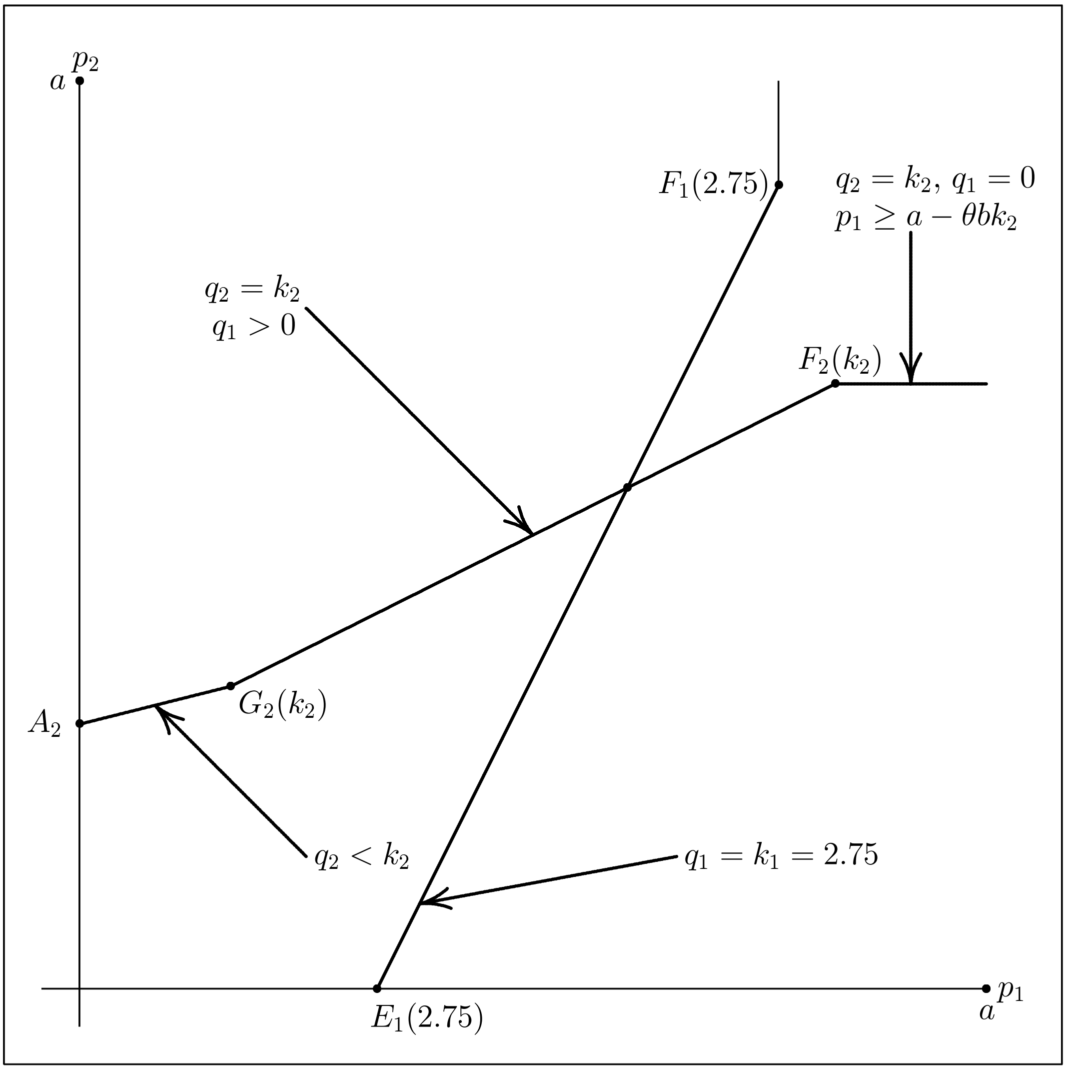

Figure 9.9: (b2,b2) second-stage equilibrium, cell(1,3)

below firm 2's branch two, $F_2(k_2)$ to the right of firm 1's branch two, and $G_2(k_2)$ to the left of firm 1's branch two.

From Section 9.7, $F_1(k_1)$ and $F_2(k_2)$ always have the required positions, while $G_1(k_1)$ and $G_2(k_2)$ have the required positions if

$$k_1 \leq k^*_{B(c)}(k_2),\ k_2 \leq k^*_{B(c)}(k_1). \tag{9.7}$$

If (9.7) is not met,

$$k_1 \geq k^*_{B(c)}(k_2),\ k_2 \geq k^*_{B(c)}(k_1), \tag{9.8}$$

cell (2,2) second-stage equilibrium is of type (b1,b1).

If point $F_1(k_1)$ is above firm 2's branch two, point $G_1(k_1)$ below firm 2's branch two, and point $G_2(k_2)$ to the right of firm 1's branch two, then second-stage equilibrium is of type (b2,b1). From Section 9.6, the condition for point $G_2(k_2)$ to be to the right of firm 1's branch two is

$$k_2 \geq k^{br}_{B(c)}(k_1).$$

Thus the conditions for (b2,b1) second-stage equilibrium in cell (2,2) are

$$k_1 \leq k^*_{B(c)}(k_2),\ k_2 \geq k^{br}_{B(c)}(k_1) \tag{9.9}$$

In the same way, the conditions for (b1,b2) second-stage equilibrium in cell (2,2) are

$$k_1 \geq k^{br}_{B(c)}(k_2),\ k_2 \leq k^*_{B(c)}(k_1). \tag{9.10}$$

9.9. Cell (1,2)

Cell (1,2) is defined by the inequalities $0 \leq k_1 \leq k_A$, $k_A \leq k_2 \leq q_{m(c)}$.

Figure 9.10 shows a (b2,b2) second-stage equilibrium in cell (1,2). The conditions for the type of equilibrium to occur are that point $F_1(k_1)$ be above firm 2's branch two, point $F_2(k_2)$ be to the right of firm 1's branch (which conditions are always satisfied, and that point $E_1(k_1)$ be below firm 2's branch two, while point $G_2(k_2)$ be to the right of firm 1's branch two.

Point $E_1(k_1)$, which is on the horizontal axis, is always below firm 2's branch two. From Section 9.6, the condition for point $G_2(k_2)$ to be to the right of firm 1's branch two is

$$k_2 \geq k^{br}_{B(c)}(k_1). \tag{9.11}$$

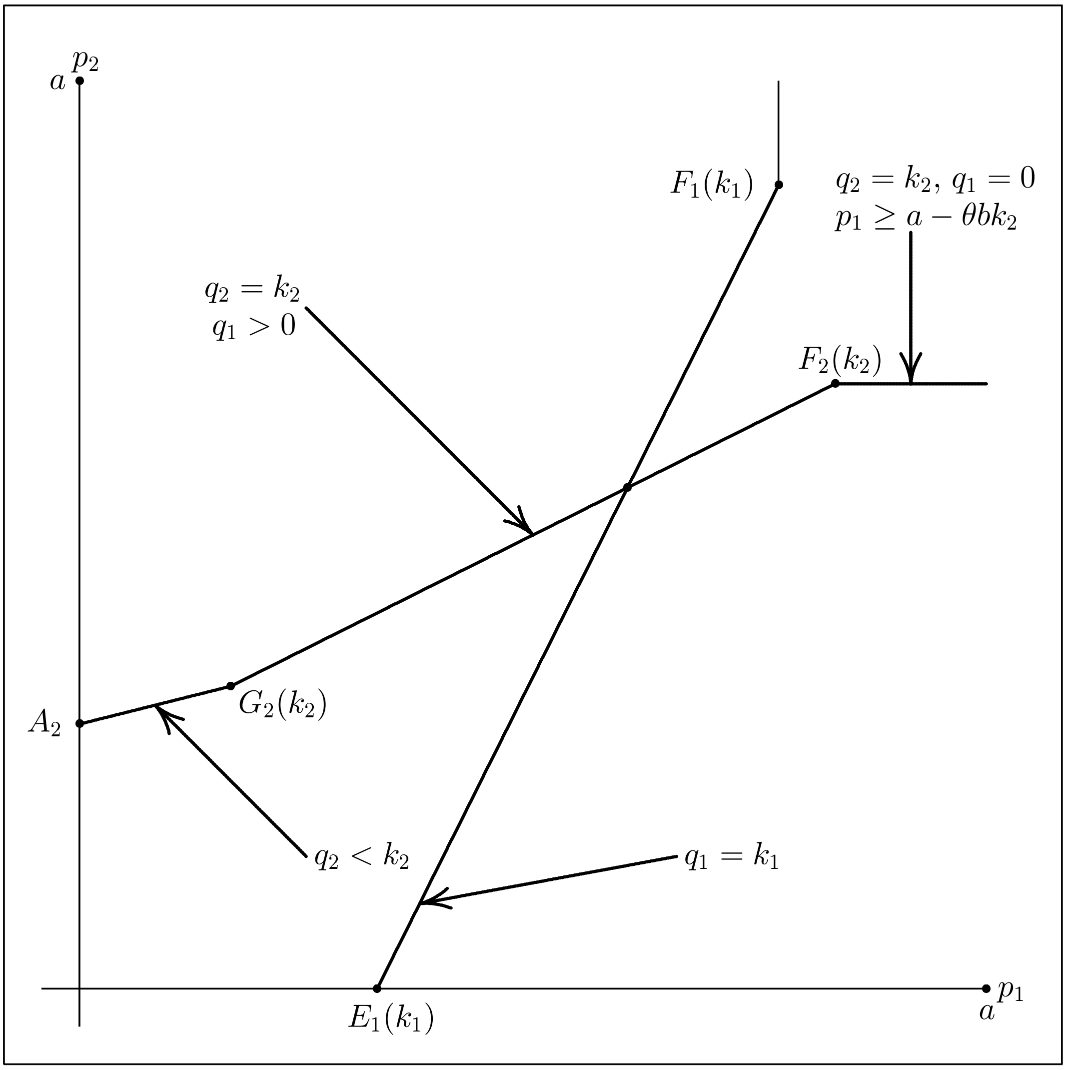

Figure 9.10: Firm 1's price reaction function, $k_1 \leq k_A$

If on the other hand

$$k_2 \leq k_{B(c)}^{br}(k_1), \tag{9.12}$$

second stage equilibrium in cell (1,2) is of type (b2,b2).

In the same way, if

$$k_1 \geq k_{B(c)}^{br}(k_2). \tag{9.13}$$

then second-stage equilibrium in cell (2,1) is of type (b2,b2), while if

$$k_1 \leq k_{B(c)}^{br}(k_2), \tag{9.14}$$

then second-stage equilibrium in cell (2,1) is of type (b1,b2).

9.10. Cell (1,1)

Cell (1,1) is defined by the inequalities $0 \leq k_1 \leq k_A$, $0 \leq k_2 \leq k_A$.

Figure 9.11 shows a second-stage (b2,b2) equilibrium in cell (1,1). The conditions for cell (1,2) second-stage equilibrium to have this form are that point $F_1(k_1)$ be above firm 2's branch two while point $F_2(k_2)$ is to the right of firm 1's branch two. These conditions are always satisfied.

10. Proof of *Lemma 3*

10.1. $(b1, b1)$

In $(b1, b1)$ equilibrium, equilibrium prices are the Bertrand equilibrium prices with marginal cost equal to c:

$$p_{b1b1} = c + \frac{1-\theta}{2-\theta}(a-c); \tag{10.1}$$

(see (4.3)); equilibrium quantities are

$$q_{b1b1} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b}; \tag{10.2}$$

equilibrium payoffs are

$$\pi_{ib1b1} = \frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_i. \tag{10.3}$$

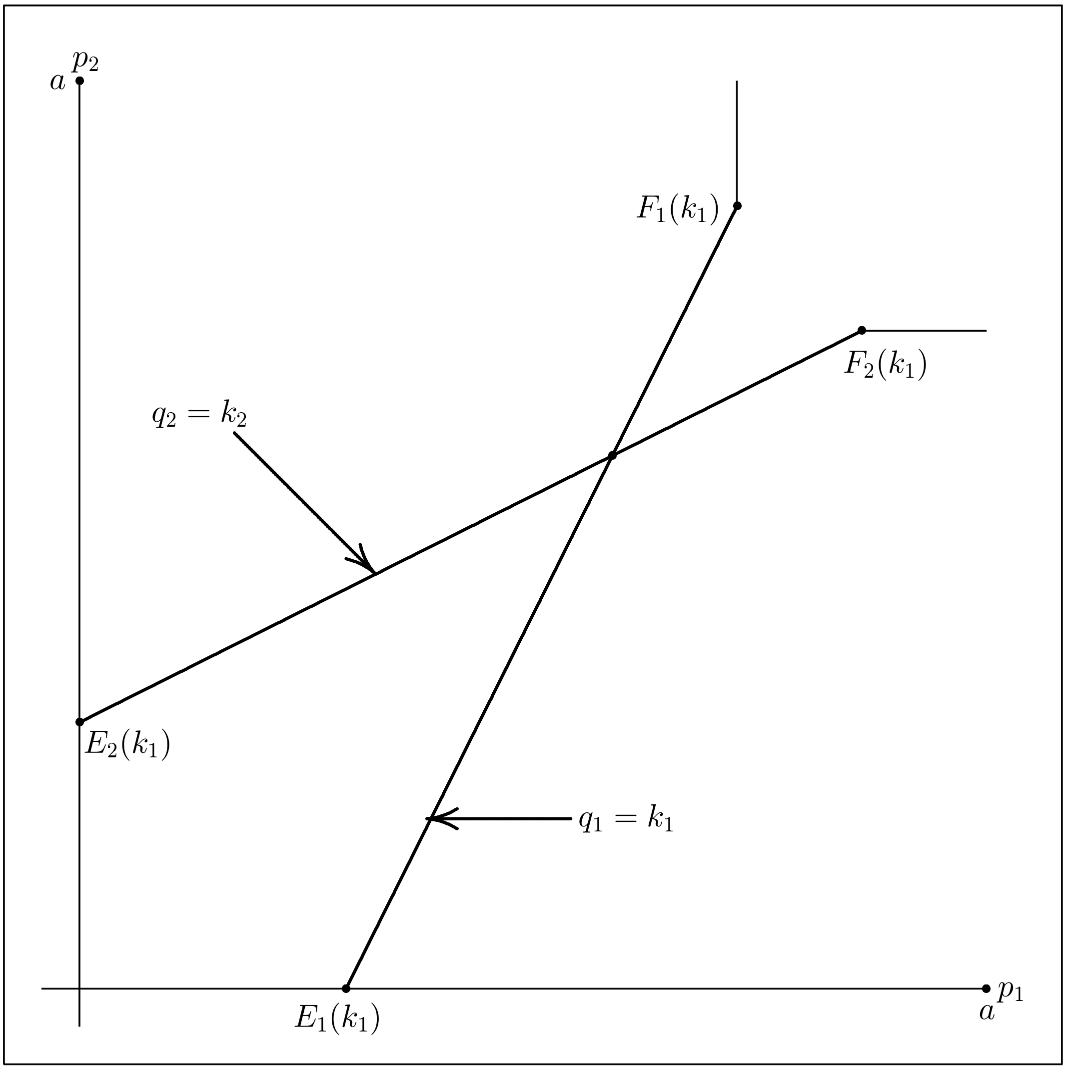

Figure 9.11: $(b2, b2)$ equilibrium, both firms $(b2, b5)$ reaction functions

10.2. $(b2, b2)$

If both firms are on their quantity constraint lines, equilibrium prices are

$$\begin{pmatrix} 1 & -\theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (1-\theta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} a - b(1-\theta^2) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \tag{10.4}$$

with solution

$$p_{1b2b2} = a - b(k_1 + \theta k_2) \tag{10.5}$$

$$p_{2b2b2} = a - b(\theta k_1 + k_2), \tag{10.6}$$

which simply recovers the equations of the inverse demand curves (2.11), (2.12).

Equilibrium outputs are k_1 and k_2. Second-stage payoff functions are

$$\pi_{1b2b2} = (a - c - \rho - b(k_1 + \theta k_2))k_1 \tag{10.7}$$

$$\pi_{2b2b2} = (a - c - \rho - b(\theta k_1 + k_2))k_2. \tag{10.8}$$

10.3. $(b2, b1)$

If point $C_1(k_1)$ is below the $q_2 < k_2$ line,

$$k_1 \leq k^*_{B(c)}, \tag{10.9}$$

and point $C_2(k_2)$ is to the right of the $q_1 = k_1$ line,

$$k_2 \geq k^{br}_{B(c)}(k_1), \tag{10.10}$$

then equilibrium occurs where firm 1's branch two intersects firm 2's branch one. This case is symmetric with the $(b1, b2)$ equilibrium.

Firm 1 is capacity constrained; the equation of firm 1's price reaction function is (5.6)

$$p_1 = \theta p_2 + (1-\theta)a - b(1-\theta^2)k_1,$$

or, rewritten in terms of deviations from c,

$$p_1 - c - \theta(p_2 - c) = (1-\theta)(a-c) - b(1-\theta^2)k_1.$$

Firm is not capacity constrained; the equation of firm 2's price reaction function is (9.1)

$$-\theta(p_1 - c) + 2(p_2 - c) = (1-\theta)(a-c).$$

Equilibrium prices solve

$$\begin{pmatrix} 1 & -\theta \\ -\theta & 2 \end{pmatrix}\begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = (1-\theta)(a-c)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - b(1-\theta^2)k_1\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

leading to

$$p_{1b2b1} = c + \frac{(1-\theta)(2+\theta)(a-c) - 2(1-\theta^2)bk_1}{2-\theta^2} \tag{10.11}$$

$$p_{2b2b1} = c + \frac{1-\theta^2}{2-\theta^2}(a - c - \theta bk_1) \tag{10.12}$$

Equilibrium quantities demanded are

$$q_{1b2b1} = k_1$$

$$q_{2b2b1} = \frac{1}{2-\theta^2}\frac{a-c-\theta bk_1}{b}. \tag{10.13}$$

Second-stage payoffs are

$$\pi_{1b2b1} = \left(\frac{(1-\theta)(2+\theta)(a-c) - 2(1-\theta^2)bk_1}{2-\theta^2} - \rho\right)k_1 \tag{10.14}$$

$$\pi_{2b2b1} = \frac{1-\theta^2}{(2-\theta^2)^2}\frac{(a-c-\theta bk_1)^2}{b} - \rho k_2 \tag{10.15}$$

10.4. $(b1, b2)$

The equation of the branch one segment of firm 1's price reaction function is

$$2(p_1 - c) - \theta(p_2 - c) = (1-\theta)(a-c) \tag{10.16}$$

The equation of the branch two segment of firm 2's reaction function is

$$p_2 = \theta p_1 + (1-\theta)a - b(1-\theta^2)k_2. \tag{10.17}$$

Rewrite (10.17) in terms of deviations from c:

$$-\theta(p_1 - c) + p_2 - c = (1-\theta)(a-c) - b(1-\theta^2)k_2$$

The system of equations is

$$\begin{pmatrix} 2 & -\theta \\ -\theta & 1 \end{pmatrix}\begin{pmatrix} p_1 - c \\ p_2 - c \end{pmatrix} = (1-\theta)(a-c)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - (1-\theta^2)bk_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10.18}$$

Equilibrium prices in $(b1, b2)$ equilibrium are

$$p_{1b1b2} = c + \frac{1-\theta^2}{2-\theta^2}(a - c - \theta b k_2) \tag{10.19}$$

$$p_{2b1b2} = c + \frac{(1-\theta)(2+\theta)(a-c) - 2(1-\theta^2)bk_2}{2-\theta^2} \tag{10.20}$$

Firm 1 is on branch one of its reaction function; from (8.3), the quantity demanded of firm 1 is

$$q_{1b1b2} = \frac{p_1 - c}{b(1-\theta^2)} = \frac{1}{b(1-\theta^2)}\frac{1-\theta^2}{2-\theta^2}(a - c - \theta b k_2)$$

$$= \frac{1}{2-\theta^2}\frac{a - c - \theta b k_2}{b}. \tag{10.21}$$

Firm 1's second-stage payoff is

$$\pi_{1b1b2} = \frac{1-\theta^2}{(2-\theta^2)^2}\frac{(a - c - \theta b k_2)^2}{b} - \rho k_1. \tag{10.22}$$

Firm 2's second-stage payoff is

$$\pi_{2b1b2} = (p_{2b1b2} - c - \rho)k_2 =$$

$$\left[\frac{(1-\theta)(2+\theta)(a-c) - 2(1-\theta^2)bk_2}{2-\theta^2} - \rho\right]k_2 \tag{10.23}$$

11. Proof of *Lemma 4*

By the way in which the first-stage payoff functions are derived, they must be continuous in capacities. This has been verified, but the details of these parts of the proof are omitted.

11.1. Lower region: $k_2 \leq k^*_{B(c)}$

Let $k_2 \leq k^*_{B(c)}$. Firm 1's payoff function is

$$\pi_1(k_1, k_2) =$$

$$\begin{cases} (a-c-\rho-b(k_1+\theta k_2))k_1 & 0 \leq k_1 \leq k^{br}_{B(c)}(k_2) \quad (b2,b2) \\ \frac{1-\theta^2}{(2-\theta^2)^2}\frac{(a-c-\theta b k_2)^2}{b} - \rho k_1. & k^{br}_{B(c)}(k_2) \leq k_1 \quad (b1,b2) \end{cases}. \tag{11.1}$$

If $k_1 \leq k^{br}_{B(c)}(k_2)$, second-stage equilibrium occurs where both firms are on branch two, producing at capacity.

By (10.5), firm 1's equilibrium price is given by the equation of its inverse demand curve, writing capacities in place of quantities demanded:

$$p_1 = a - b(k_1 + \theta k_2)$$

Its payoff is

$$\pi_1(k_1,k_2) = (p_1 - c - \rho)k_1 = (a - c - \rho - b(k_1 + \theta k_2))k_1. \tag{11.2}$$

If $k_1 \geq k^{br}_{B(c)}(k_2)$, then in second-stage equilibrium, firm 1 is on its branch one ($q_1 \leq k_1$), while firm 2 is on its branch two ($q_2 = k_2$). From (10.22), firm 1's equilibrium payoff is

$$\pi_1(k_1,k_2) = \frac{1-\theta^2}{(2-\theta^2)^2}\frac{(a-c-\theta b k_2)^2}{b} - \rho k_1. \tag{11.3}$$

For the numerical example, firm 1's payoff function for $k_2 = 4$ is shown in Figure 11.1.

Comparing (11.2) and (4.6), for $k_1 \leq k^{br}_{B(c)}(k_2)$, firm 1's payoff function has the same functional form as firm 1's payoff function in the standard Cournot duopoly model with product differentiation and marginal cost $c+\rho$, (4.6), and the capacity that maximizes firm 's payoff in the left-hand region $k_1 \leq k^{br}_{B(c)}(k_2)$, which with a certain abuse of notation we denote as

$$k^{br}_{1Cour}(k_2) = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right), \tag{11.4}$$

has the same functional form as the Cournot best response output with marginal cost $c+\rho$, (4.7).

In order for (11.4) to give the value that maximizes firm 1's equilibrium profit in the $(b2,b2)$ region, the capacity level identified by (11.4) must be in the $(b2,b2)$ region. The condition for this is

$$k_1(k_2) = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right) \leq k^{br}_{B(c)}(k_2) \tag{11.5}$$

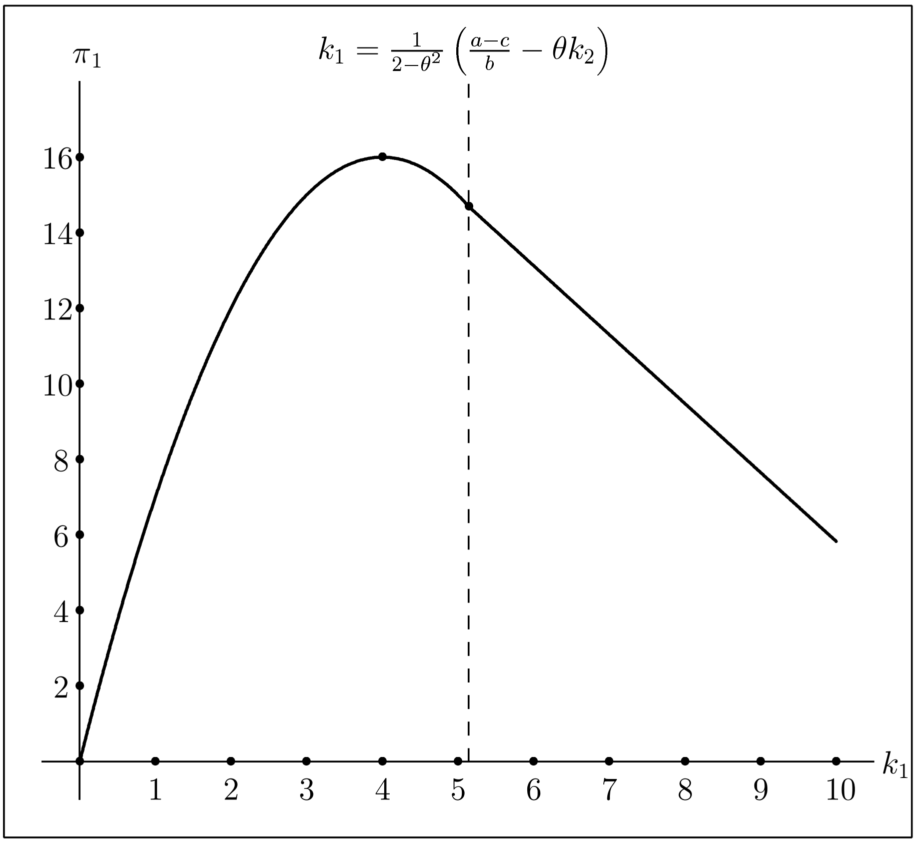

Figure 11.1: Firm 1's profit function, $k_2 = 4$

$$k_2 \leq \frac{\theta^2(a-c)+(2-\theta^2)\rho}{\theta^3 b} \equiv k_{LLint} \tag{11.6}$$

We know that in the region now under analysis

$$k_2 \leq k^*_{B(c)} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b}.$$

Comparing k_{LLint} and $k^*_{B(c)}$,

$$k_{LLint} - k^*_{B(c)} =$$

$$\frac{2-\theta^2}{\theta}\left[\frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} + \frac{1}{\theta^2}\frac{\rho}{b}\right] > 0.$$

Hence in the case we are now considering

$$k_2 \leq k^*_{B(c)} \leq k_{LLint},$$

and the global maximum of (11.2) occurs within the $(b2, b2)$ region.

To the right of the line $k_1 = k^{br}_{B(c)}(k_2)$ firm 1's payoff, (11.3),

$$\frac{1-\theta^2}{(2-\theta^2)^2}\frac{(a-c-\theta b k_2)^2}{b} - \rho k_1,$$

is its payoff for (b1,b2) equilibrium. This is maximized by making k_1 as small as possible within the relevant region, that is, by setting $k_1 = k^*_{B(c)}$. Since the payoff function is continuous, and rising moving to the left from $k_1 = k^{br}_{B(c)}(k_2)$, the global maximum of the payoff function for the left-hand segment occurs within the left-hand segment, and firm 1 maximizes its payoff for $k_2 \leq k^*_{B(c)}$ by setting the capacity (11.4),

$$k^{br}_{1Cour} = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right),$$

which is the equation of the segment of firm 1's capacity best response function $0 \leq k_2 \leq k^*_{B(c)}$.

11.2. Upper region: $\frac{1}{2-\theta^2}\frac{a-c}{b} \leq k_2$

$$\pi_1(k_1, k_2) =$$

$$\begin{cases} \left(\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)-\rho-2\frac{1-\theta^2}{2-\theta^2}bk_1\right)k_1 & 0 \leq k_1 \leq k^*_{B(c)} \quad (b2,b1) \\ \frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b}-\rho k_1 & k^*_{B(c)} \leq k_1 \quad (b1,b1) \end{cases} \tag{11.7}$$

Firm 1's payoff function in the left-hand region is its payoff in $(b2, b1)$ equilibrium,

$$\pi_{1b2b1} = \left[\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)-\rho-2\frac{1-\theta^2}{2-\theta^2}bk_1\right]k_1. \tag{11.8}$$

The global maximum of (11.8) is at

$$k^{br}_{1b2b1} = \frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}. \tag{11.9}$$

This is interior to the left-hand region, for which $k_1 \leq k^*_{B(c)} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b}$:

$$k^*_{B(c)} - k^{br}_{1b2b1} =$$

$$\frac{1}{4(1+\theta)}\left[\frac{\theta^2}{2-\theta}\frac{a-c}{b}+\frac{2-\theta^2}{1+\theta}\frac{\rho}{b}\right] > 0. \tag{11.10}$$

k^{br}_{1b2b1} may be negative, if ρ is sufficiently large. We will assume that $k^{br}_{1b2b1} > 0$. This assumption will be used several times below to determine the signs of various expressions.

Firm 1's payoff at (11.9) is

$$\pi_{1TL} = 2\frac{1-\theta^2}{2-\theta^2}b\left(k^{br}_{1b2b1}\right)^2 =$$

$$\frac{b}{8}\frac{2-\theta^2}{1-\theta^2}\left[\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{(2-\theta^2)b}\right]^2 \tag{11.11}$$

Firm 1's payoff in the right-hand region, (10.3), is its payoff in $(b1, b1)$ equilibrium,

$$\frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b}-\rho k_1,$$

and this is maximized by making k_1 as small as possible within the relevant range, that is, by setting $k_1 = k^*_{B(c)}$. By continuity of the payoff function and the fact that the payoff function rises moving left from the boundary $k_1 = k^*_{B(c)}$, the global maximum in the top region occurs at (11.9).

Figure 11.2 shows firm 1's payoff function for $k_2 \geq \frac{1}{2-\theta^2}\frac{a-c}{b}$.

Figure 11.3 shows the lower $(0 \leq k_2 \leq k^*_{B(c)})$ and upper $(\frac{1}{2-\theta^2}\frac{a-c}{b} \leq k_2 \leq \frac{a-c-\rho}{b})$ segments of the capacity reaction functions.

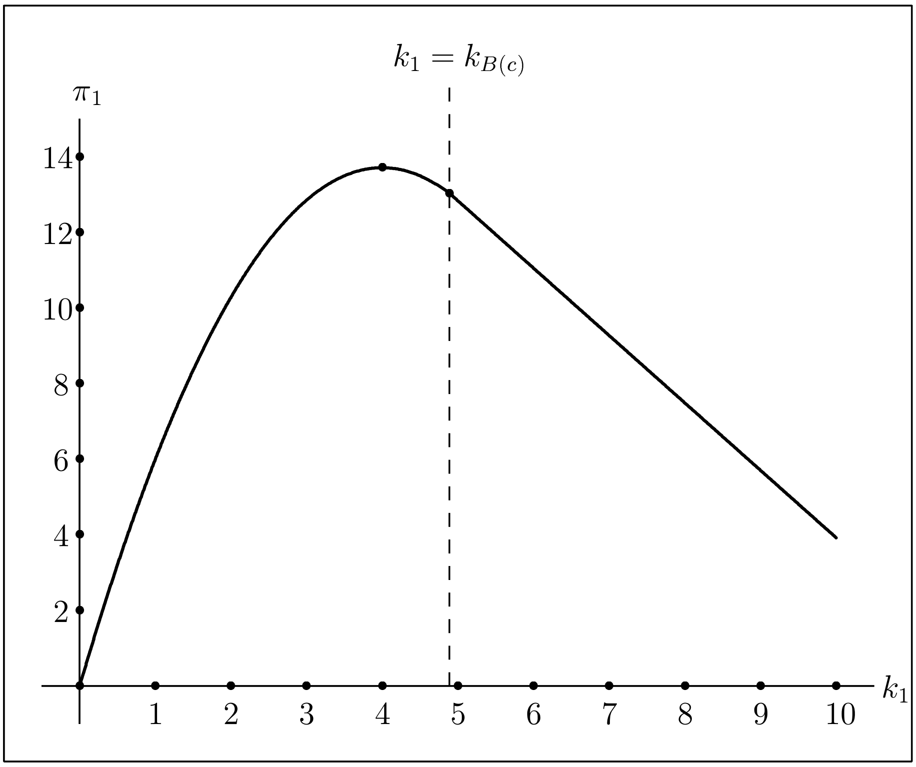

Figure 11.2: Firm 1's profit function, $k_2 \geq \frac{1}{2-\theta^2}\frac{a-c}{b}$

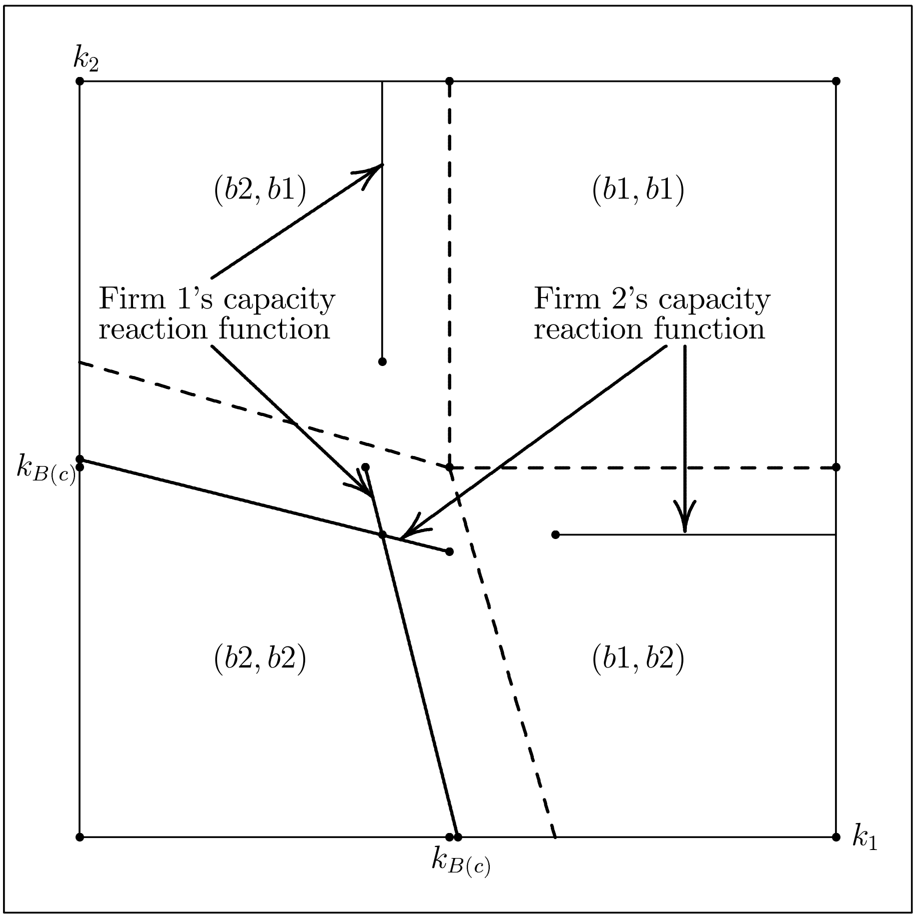

Figure 11.3: Capacity reaction functions, upper and lower segments, Kreps & Scheinkman model with product differentiation

11.3. Middle region: $k^*_{B(c)} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}$

The equation of the boundary between the $(b2, b2)$ region and the $(b2, b1)$ region is

$$\theta k_1 + (2-\theta^2)k_2 = \frac{a-c}{b} \tag{11.12}$$

or

$$k_2 = k^{br}_{B(c)}(k_1)$$

or

$$k_1 = \left[k^{br}_{B(c)}\right]^{-1}(k_2).$$

If $k_1 = 0$ along this line, then $k_2 = \frac{1}{2-\theta^2}\frac{a-c}{b}$. For $k^*_{B(c)} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}$, firm 1's payoff function has three segments:

$$\pi_1(k_1, k_2) = \tag{11.13}$$

$$\begin{cases} (a-c-\rho-b(k_1+\theta k_2))k_1 & 0 \leq k_1 \leq \left[k^{br}_{B(c)}\right]^{-1}(k_2) & (b2, b2) \\ \left(\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)-\rho-2\frac{1-\theta^2}{2-\theta^2}bk_1\right)k_1 & \left[k^{br}_{B(c)}\right]^{-1}(k_2) \leq k_1 \leq k^*_{B(c)} & (b2, b1) \\ \frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_1 & k^*_{B(c)} \leq k_1 & (b1, b1) \end{cases}$$

11.3.1. Left segment: $0 \leq k_1 \leq \left[k^{br}_{B(c)}\right]^{-1}(k_2)$

In this region, firm 1's payoff is that of $(b2, b2)$ equilibrium; this case has been analyzed in Section 11.1.

Firm 1's profit-maximizing capacity is (11.4),

$$k^{br}_{1Cour} = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right),$$

if k^{br}_{1Cour} lies with the left-hand range of the middle region $0 \leq k_1 \leq \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right]$. The condition for k^{br}_{1Cour} to be within the left-hand range of the middle region is

$$\frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right) \leq \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right],$$

or

$$k_2 \leq \frac{(2-\theta)(a-c)+\theta\rho}{(4-3\theta^2)b} \equiv k_{MLint}. \tag{11.14}$$

In the middle region

$$k^*_{B(c)} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}.$$

Compare k_{MLint} with the upper end of the range of k_2 that defines the middle region:

$$\frac{1}{2-\theta^2}\frac{a-c}{b} - k_{MLint} = \frac{\theta}{4-3\theta^2}\left[\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{2-\theta^2}\right] > 0,$$

where the sign depends on the assumption that (11.4), firm one's best-response capacity in the upper region, is positive.

Now compare k_{MLint} with the lower end of the range of k_2 that defines the middle region:

$$k_{MLint} - k^*_{B(c)} =$$

$$\frac{\theta}{4-3\theta^2}\left[\frac{\theta^2}{(1+\theta)(2-\theta)}\frac{a-c}{b} + \frac{\rho}{b}\right] > 0.$$

Hence

$$k^*_{B(c)} < k_{MLint} < \frac{1}{2-\theta^2}\frac{a-c}{b}.$$

For firm 2 capacity levels falling in the range

$$k^*_{B(c)} \leq k_2 \leq k_{MLint}, \tag{11.15}$$

k^{br}_{1Cour} is interior to the left-hand segment of the middle region, firm 1's payoff function has an interior maximum at k^{br}_{1Cour} on the left-hand segment, and firm 1's payoff at this maximum is

$$\frac{b}{4}\left(\frac{a-c-\rho}{b} - \theta k_2\right)^2$$

In contrast, for

$$k_{MLint} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b},$$

the local maximum of firm 1's payoff function on the range $0 \leq k_1 \leq \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right]$ is at the right-hand border of this range, $k_1 = \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right]$.

11.3.2. Middle segment: $\left[k^{br}_{B(c)}\right]^{-1}(k_2) \leq k_1 \leq k^*_{B(c)}$

Firm 1's payoff function in the middle segment of the middle region is its payoff in $(b2, b1)$ equilibrium,

$$\pi_{1b2b1} = \left[\frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c) - \rho - 2\frac{1-\theta^2}{2-\theta^2}bk_1\right]k_1.$$

The global maximum occurs for (11.9)

$$k^{br}_{1b2b1} = \frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}.$$

We know from our discussion of the upper region (see (11.10) and the associated text) that

$$k^{br}_{1b2b1} < k^*_{B(c)}.$$

The condition for

$$\frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right] \leq k^{br}_{1b2b1},$$

so that k^{br}_{1b2b1} lies within the middle range of the region, is

$$k_2 \geq \frac{4+2\theta-\theta^2}{4(1+\theta)(2-\theta^2)}\frac{a-c}{b} + \frac{\theta}{4(1-\theta^2)}\frac{\rho}{b} \equiv k_{MMint}. \tag{11.16}$$

k_{MMint} can also be written

$$k_{MMint} = \frac{1}{2-\theta^2}\left(\frac{a-c}{b} - \theta k^{br}_{1b2b1}\right) \tag{11.17}$$

We know that

$$k^*_{B(c)} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}.$$

Compare k_{MMint} and $k^*_{B(c)}$:

$$k_{MMint} - k^*_{B(c)} = \frac{\theta}{4(1+\theta)}\left[\frac{\theta^2}{(2-\theta)\left(2-\theta^2\right)}\frac{a-c}{b} + \frac{1}{1-\theta}\frac{\rho}{b}\right] > 0$$

Compare k_{MMint} and $\frac{1}{2-\theta^2}\frac{a-c}{b}$:

$$\frac{1}{2-\theta^2}\frac{a-c}{b} - k_{MMint} = \frac{\theta}{4(1+\theta)}\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{(1+\theta)(2-\theta^2)} > 0$$

where once again the sign depends on the assumption that (11.4), firm one's best-response capacity in the upper region, is positive.

Hence

$$k^*_{B(c)} \leq k_{MMint} \leq \frac{1}{2-\theta^2}\frac{a-c}{b}.$$

For

$$k^*_{B(c)} \leq k_2 \leq k_{MMint},$$

the maximum of firm 1's profit function on the range

$$\frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right] \leq k_1 \leq k^*_{B(c)}$$

occurs at the left-hand boundary,

$$k_1 = \frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right],$$

while for

$$k_{MMint} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b},$$

firm 1's profit function on the range

$$\frac{1}{\theta}\left[\frac{a-c}{b} - (2-\theta^2)k_2\right] \leq k_1 \leq k^*_{B(c)}$$

has an interior maximum at k^{br}_{1b2b1}:

$$k^{br}_{1b2b1} = \frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b},$$

and firm 1's payoff at this capacity level is

$$2\frac{1-\theta^2}{2-\theta^2}b\left(k^{br}_{1b2b1}\right)^2$$

11.3.3. Right segment: $k^*_{B(c)} \leq k_1$

Firm 1's payoff in the right-hand region,

$$\frac{1-\theta}{(1+\theta)(2-\theta)^2}\frac{(a-c)^2}{b} - \rho k_1,$$

is maximized by making k_1 as small as possible while remaining in the region, $k_1 = k^*_{B(c)}$.

11.3.4. Middle region: overall

On the left-hand segment of the middle region, for firm 2 capacity levels falling in the range

$$k^*_{B(c)} \leq k_2 \leq k_{MLint}, \tag{11.18}$$

firm 1's payoff function has an interior maximum on the range $0 \leq k_1 \leq \left[k^{br}_{B(c)}\right]^{-1}(k_2)$, at

$$k^{br}_{1Cour} = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right),$$

and firm 1's payoff at this maximum is

$$\frac{b}{4}\left(\frac{a-c-\rho}{b} - \theta k_2\right)^2$$

In contrast, for

$$k_{MLint} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}, \tag{11.19}$$

the local maximum of firm 1's payoff function on the range $0 \leq k_1 \leq \left[k^{br}_{B(c)}\right]^{-1}(k_2)$ is at the right-hand border of this range, $k_1 = \left[k^{br}_{B(c)}\right]^{-1}(k_2)$.

In the middle segment of the middle region, for

$$k^*_{B(c)}\frac{a-c}{b} \leq k_2 \leq k_{MMint}, \tag{11.20}$$

the maximum of firm 1's profit function on the range

$$\left[k^{br}_{B(c)}\right]^{-1}(k_2) \leq k_1 \leq k^*_{B(c)}$$

	left	middle	global
$k^*_{B(c)} \leq k_2 \leq k_{MMint}$	interior max	left boundary	left interior max
$k_{MMint} \leq k_2 \leq k_{MLint}$	interior max	interior max	?
$k_{MLint} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}$	right boundary	interior max	middle interior max

Table 11.1: Maxima, middle region

occurs at the left-hand boundary, $k_1 = \frac{1}{\theta}\left(\frac{a-c}{b} - (2-\theta^2)k_2\right)$while for

$$k_{MMint} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}, \tag{11.21}$$

firm 1's profit function on the range

$$\left[k^{br}_{B(c)}\right]^{-1}(k_2) \leq k_1 \leq k^*_{B(c)}$$

has an interior maximum at

$$k^{br}_{1b2b1} = \frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b},$$

and firm 1's payoff at this capacity level is

$$2\frac{1-\theta^2}{2-\theta^2}b\left(\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}\right)^2$$

Compare k_{MLint} and k_{MMint}:

$$k_{MLint} - k_{MMint} = \frac{\theta^3}{4(1+\theta)\left(4-3\theta^2\right)}\left[\frac{(1-\theta)(2+\theta)(a-c)-\left(2-\theta^2\right)\rho}{\left(2-\theta^2\right)}\right] > 0.$$

For

$$k^*_{B(c)} \leq k_2 \leq k_{MMint},$$

the global maximum of firm 1's payoff function is $k^{br}_{1Cour}(k_2)$, which therefore is the best-response capacity for $0 \leq k_2 \leq k_{MMint}$.

For

$$k_{MLint} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b},$$

the global maximum of firm 1's payoff function is k^{br}_{1b2b1},

$$k^{br}_{1b2b1} = \frac{(1-\theta)(2+\theta)(a-c) - (2-\theta^2)\rho}{4(1-\theta^2)b},$$

which therefore is firm 1's best response capacity for $k_{MLint} \leq k_2$.

Figure 11.4 extends the segments of the reaction functions shown in Figure 11.3 to include the upper and lower segments of the middle range of the reaction functions.

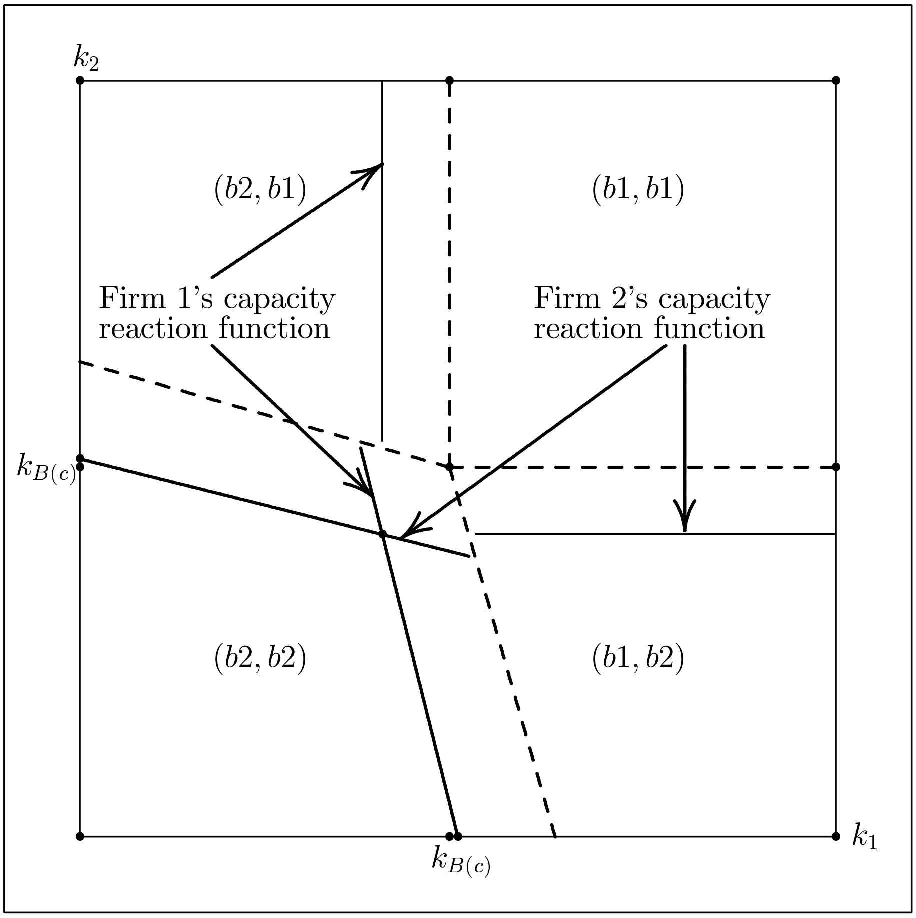

Figure 11.4: Capacity reaction functions, upper and lower segments, upper and lower segments of middle range, Kreps & Scheinkman model with product differentiation

For

$$k_{MMint} \leq k_2 \leq k_{MLint}$$

firm 1's payoff function has two local maxima, $k_{1Cour}^{br}(k_2)$ on the left segment and k_{1b2b1}^{br} on the middle segment.

Firm 1's payoff at $k_{1Cour}^{br}(k_2)$ is

$$\frac{b}{4}\left(\frac{a-c-\rho}{b} - \theta k_2\right)^2.$$

Firm 1's payoff at k_{1b2b1}^{br} is

$$2\frac{1-\theta^2}{2-\theta^2}b\left(k_{1b2b1}^{br}\right)^2 = 2\frac{1-\theta^2}{2-\theta^2}b\left(\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}\right)^2$$

We need to compare firm 1's payoffs at the two local maxima to determine the global maximum.

If $k_2 = k_{MMint}$,

$$k_{1Cour}^{br}(k_2) = \frac{1}{2}\left[\frac{a-c-\rho}{b} - \theta\left(\frac{4+2\theta-\theta^2}{4(1+\theta)(2-\theta^2)}\frac{a-c}{b} + \frac{\theta}{4(1-\theta^2)}\frac{\rho}{b}\right)\right] =$$

$$\frac{4-3\theta^2}{2(2-\theta^2)}k_{1b2b1}^{br}$$

and firm 1's payoff at the left-hand local maximum is

$$b\left[\frac{4-3\theta^2}{2(2-\theta^2)}k_{1b2b1}^{br}\right]^2.$$

When $k_2 = k_{MMint}$, the difference between firm 1's payoffs at the left and middle local maxima is

$$b\left[\frac{4-3\theta^2}{2(2-\theta^2)}k_{1b2b1}^{br}\right]^2 - 2\frac{1-\theta^2}{2-\theta^2}b\left(k_{1b2b1}^{br}\right)^2 = \frac{\theta^4}{4(2-\theta^2)^2}b\left(k_{1b2b1}^{br}\right)^2 > 0.$$

At the smallest level of k_2 for which firm 1's payoff function has two local maxima, the global maxima is in the center segment.

Now compare payoffs at the two local maxima for the largest value of k_2 for which firm 1's payoff function has two local maxima, k_{MLint}. For $k_2 = k_{MLint}$

$$k_{1Cour}^{br}(k_2) = \frac{1}{2}\left[\frac{a-c-\rho}{b} - \theta\left(\frac{(2-\theta)(a-c)+\theta\rho}{(4-3\theta^2)b}\right)\right] =$$

$$4\frac{1-\theta^2}{4-3\theta^2}k^{br}_{1b2b1}$$

and firm 1's payoff is

$$b\left[4\frac{1-\theta^2}{4-3\theta^2}bk^{br}_{1b2b1}\right]^2$$

and the difference between firm 1's payoffs at the middle and left local maxima is

$$b\left[\frac{4-3\theta^2}{2(2-\theta^2)}k^{br}_{1b2b1}\right]^2 - b\left[4\frac{1-\theta^2}{4-3\theta^2}k^{br}_{1b2b1}\right]^2 =$$

$$b\left\{\left[\frac{4-3\theta^2}{2(2-\theta^2)}\right]^2 - \left[4\frac{1-\theta^2}{4-3\theta^2}\right]^2\right\}\left(k^{br}_{1b2b1}\right)^2 =$$

$$b\left(\frac{4-3\theta^2}{2(2-\theta^2)}+4\frac{1-\theta^2}{4-3\theta^2}\right)\left(\frac{4-3\theta^2}{2(2-\theta^2)}-4\frac{1-\theta^2}{4-3\theta^2}\right)\left(k^{br}_{1b2b1}\right)^2 =$$

$$b\left(\frac{4-3\theta^2}{2(2-\theta^2)}+4\frac{1-\theta^2}{4-3\theta^2}\right)\left(\frac{1}{2}\frac{\theta^4}{\left(2-\theta^2\right)\left(4-3\theta^2\right)}\right)\left(k^{br}_{1b2b1}\right)^2 > 0$$

The payoff at the center local maxima falls as k_2 rises. Payoffs at the two local maxima are equal for

$$2\frac{1-\theta^2}{2-\theta^2}b\left(k^{br}_{1b2b1}\right)^2 = \frac{b}{4}\left(\frac{a-c-\rho}{b}-\theta k_2\right)^2$$

$$k_2 = k_S = \frac{1}{\theta}\left(\frac{a-c-\rho}{b} - 2\sqrt{2\frac{1-\theta^2}{2-\theta^2}k^{br}_{1b2b1}}\right) =$$

$$= \frac{1}{\theta}\left\{\frac{a-c-\rho}{b} - 2\sqrt{2\frac{1-\theta^2}{2-\theta^2}\left[\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}\right]}\right\}$$

Overall in the middle region, firm 1's capacity best response function is

$$k_1 = \begin{cases} k^{br}_{1Cour} = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right) & \text{for } k^*_{B(c)} \le k_2 \le k_S \\ k^{br}_{1b2b1} = \frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b} & \text{for } k_{S)} \le k_2 \le \frac{1}{2-\theta^2}\frac{a-c}{b} \end{cases}$$

Figures 11.5 and 11.6 illustrate the profit function in the middle segment of the middle region for the numerical example.

In this instance,

$$k_{MMint} = \frac{36}{7} = 5.1429$$
$$k_S = 5.1869$$
$$k_{MLint} = \frac{68}{13} = 5.2308$$

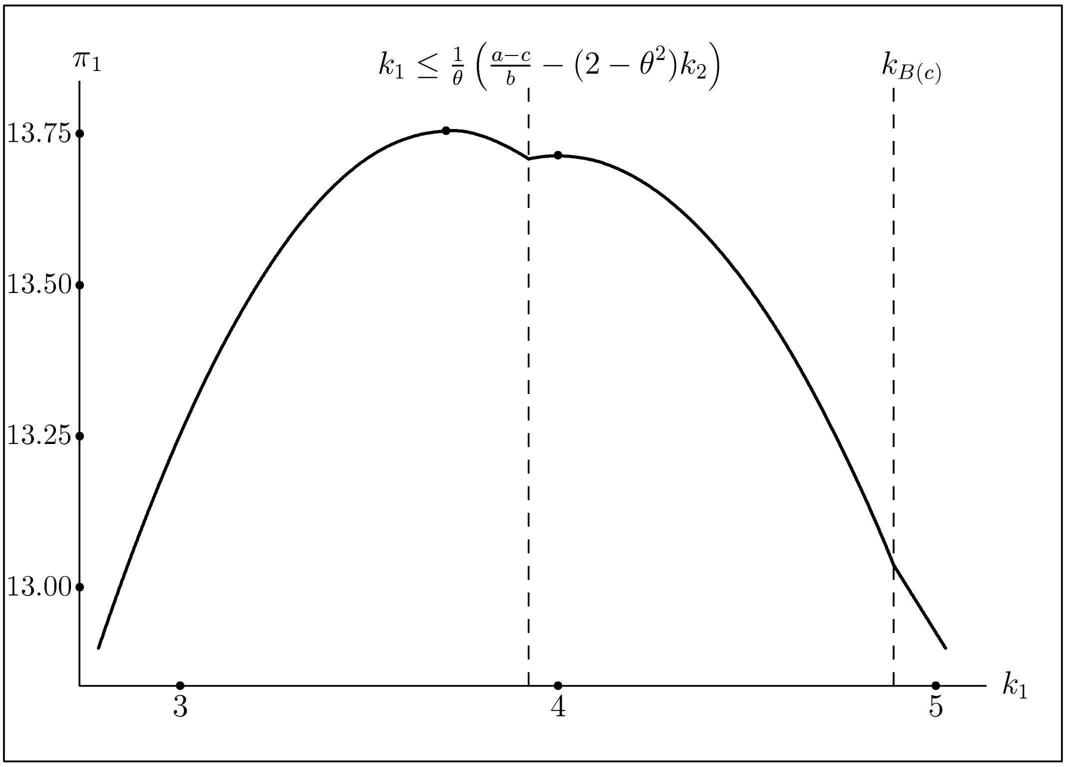

Figure 11.5: Firm 1's profit function, $k_2 = 5.1469$

Figure 11.5 shows firm 1's payoff function for $k_2 = 5.1469$, which is midway between k_{MMint} and k_S. The payoff function has two local maxima, and the global maximum is on the left.

Figure 11.6 shows firm 1's payoff function for $k_2 = 5.2089$, which is midway between k_S and k_{MLint}. The payoff function has two local maxima, and the global maximum is on the right.

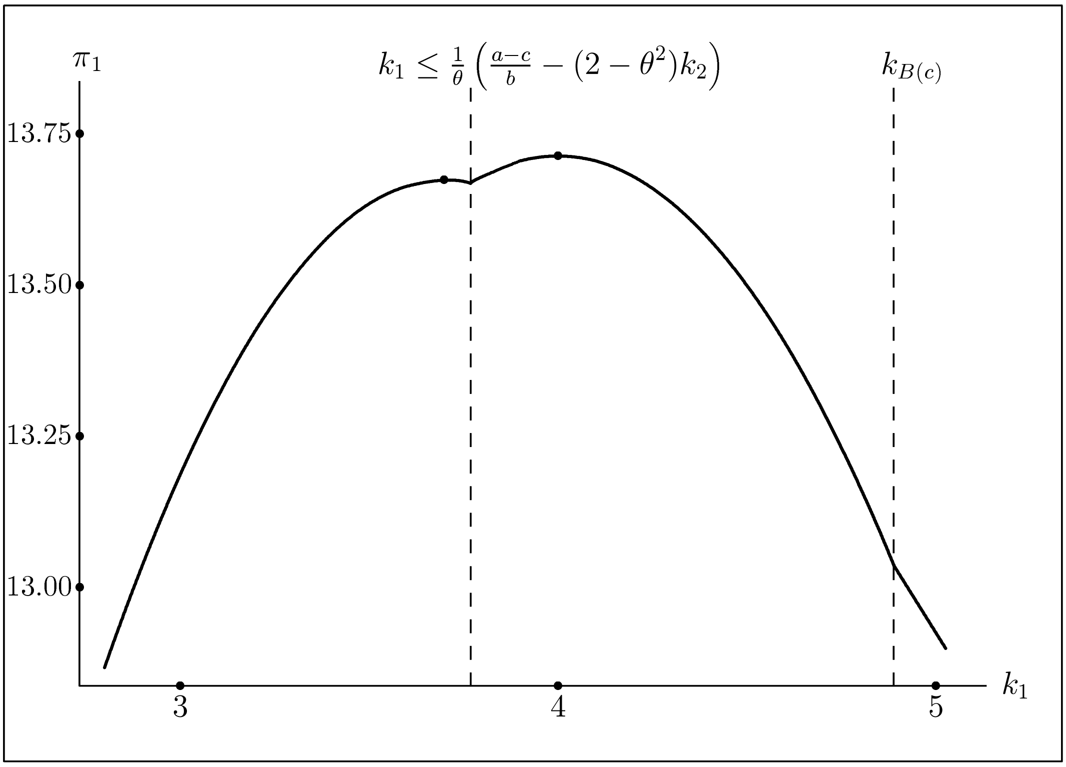

Figure 11.6: Firm 1's profit function, $k_2 = 5.2089$

For the numerical example, at the switch point between the two branches of the capacity reaction function, firm 1's best response capacity switches from

$$k_1 = \frac{1}{2}\left(\frac{12 - 1 - 1}{1} - \frac{1}{2}(5.1869)\right)$$

$$k_1 = 3.7033$$

to

$$k_2 = k_{1b2b1}^{br} = 4$$

The capacity reaction functions are shown in Figure 7.3. Figure 7.4 shows the price reaction functions for the continuation game when the noncooperative equilibrium capacity levels are chosen in the first stage.

12. Numerical Example

The figures are drawn for a particular set of parameter values,

- $a = 12$ (price-axis intercept of the inverse demand curve)
- $b = 1$ (absolute value of the slope of the inverse demand curve)
- $c = 1$ (marginal production cost)
- $\rho = 1$ (cost per unit of capacity)
- $\theta = 1/2$ (product differentiation parameter).

The units in which capacity is measured are normalized so that one unit of capacity allows production of one unit of output.

The upper limit on θ is

$$\frac{a - c}{a} = \frac{12 - 1}{12} = \frac{11}{12} = 0.9167.$$

The capacity levels k_A and k_D are evaluated for the numerical example in Table 12.1.

Values for the nodes of the segments of the price reaction function are given in Table 12.2.

The capacity limits for the various configurations of price reaction functions are

k_A	$\frac{(1-\theta)(a-c)-\theta c}{2(1-\theta^2)b}$	$\frac{(1-\frac{1}{2})(12-1)-\frac{1}{2}(1)}{2(1-\frac{1}{4})(1)} = \frac{10}{3} = 3.3333$
k_D	$\frac{1}{2-\theta^2}\frac{a-c}{b}$	$\frac{1}{2-\frac{1}{4}}\frac{12-1}{1} = \frac{44}{7} = 6.2857$

Table 12.1: Critical capacity values

A_1	$\left(c + \frac{(1-\theta)(a-c)-\theta c}{2}, 0\right) = \left(\frac{7}{2}, 0\right)$
D_1	$\left(c + \frac{1-\theta^2}{2-\theta^2}(a-c), c + \frac{(1-\theta)(2+\theta)}{2-\theta^2}(a-c)\right) = \left(\frac{40}{7}, \frac{62}{7}\right)$
E_1	$\left((1-\theta)a - b(1-\theta^2)k_1, 0\right) = \left(6 - \frac{3}{4}k_1, 0\right)$
F_1	$(a - bk_1, a - \theta bk_1) = \left(12 - k_1, 12 - \frac{1}{2}k_1\right)$
G_1	$\left(c + (1-\theta^2)bk_1, c + \frac{2(1-\theta^2)bk_1-(1-\theta)(a-c)}{\theta}\right) = \left(1 + \frac{3}{4}k_1, -10 + 3k_1\right)$
H_1	$\left(c + \frac{1}{2}(a-c), c + \frac{1}{2}(2-\theta)(a-c)\right) = \left(\frac{13}{2}, \frac{37}{4}\right)$

Table 12.2: Nodes of firm 1's price reaction function, numerical example one, numerical example one

$k_i \leq k_A = 3.3333$	(b2,b5) reaction function
$3.333 = k_A \leq k_i \leq q_{m(c)} = 5.5$	(b1,b2,b5) reaction function
$5.5 = q_{m(c)} \leq k_i \leq k_D = 6.2857$	(b1,b2,b4,b5) reaction function
$6.2857 = k_D \leq k_i \leq \frac{a-c-\rho}{b} = 10$	(b1,b4,b5) reaction function

The (b2,b5) reaction function arises for $k_1 \leq k_A = 3.3333$. The coordinates of the points in Figure 6.1 are given in Table 12.3.

For $3.333 \leq k_1 \leq q_{m(c)} = 5.5$, firm 1's price reaction function is of form (b1,b2,b5). The points used in Figure 6.1 are given in Table 12.4.

For $5.5 = q_{m(c)} \leq k_i \leq k_D = 6.2857$, firm 1's price reaction function is of form (b1,b2,b4,b5).

For $6.2857 = k_D \leq k_i \leq \frac{a-c-\rho}{b} = 10$, firm 1's price reaction function is of form (b1,b4,b5). The values for Figure 6.4 are

k_1	$E(k_1)$	$F(k_1)$
2.75	$(3.9375, 0)$	$(9.25, 10.625)$
3.125	$(3.65625, 0)$	$(8.875, 10.4375)$
3.3	$(3.525, 0)$	$(8.7, 10.35)$

Table 12.3: Points $E(k_1)$ and $F(k_1)$, alternative values of k_1

k_1	A_1	$G(k_1)$	$F(k_1)$
4.00	$(3.5, 0)$	$(4, 2)$	$(8, 10)$
5.5	$(3.5, 0)$	$(5.125, 6.5)$	$(6.5, 9.25)$

Table 12.4: Points $F(k_1)$ and $G(k_1)$, alternative values of k_1

k_1	A_1	$G(k_1)$	$F(k_1)$	H_1
6.00	$(3.5, 0)$	$(5.5, 8)$	$(6, 9)$	$(6.5, 9.25)$
6.25	$(3.5, 0)$	$(5.6875, 8.75)$	$(5.75, 8.875)$	$(6.5, 9.25)$

Table 12.5: Points $F(k_1)$, $G(k_1)$, and $H(k_1)$, alternative values of k_1

- $A_1 = (3.5, 0)$
- $D_1 = (5.7143, 8.8571)$
- $H_1 = (6.5, 9.25)$

For the numerical example

$$k^*_{B(c)} = \frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} = \frac{1}{(1+\frac{1}{2})(2-\frac{1}{2})}\frac{12-1}{1} = \frac{44}{9} = 4.8889.$$

(b1,b2) equilibrium
For the numerical example, prices are

$$p_{1b1b2} = c + \frac{1-\theta^2}{2-\theta^2}(a-c-\theta b k_2) = 1 + \frac{1-\frac{1}{4}}{2-\frac{1}{4}}\left((12-1) - \frac{1}{2}(4)\right) = 4.8571$$

$$p_{2b1b2} = c + \frac{(1-\theta)(2+\theta)(a-c) - 2(1-\theta^2)bk_1}{2-\theta^2}$$

$$= 1 + \frac{(1-\frac{1}{2})(2+\frac{1}{2})(12-1) - 2(1-\frac{1}{4})(4)}{2-\frac{1}{4}} = 5.4286$$

In the numerical example, when $k_2 = 4$, the k_1 coordinate of the border between the $(b2, b2)$ and $(b1, b2)$ regions is

$$k_1 = \frac{1}{2-\frac{1}{4}}\left(\frac{12-1}{1} - \frac{1}{2}(4)\right) = \frac{36}{7} = 5.1429.$$

and the value of profit at this point is 14.69388.

For the numerical example, the equation of the best response function over the range $0 \leq k_2 \leq \frac{44}{9} = 4.8889$ is

$$k_1 = \frac{1}{2}\left(\frac{12-1-1}{1} - \frac{1}{2}k_2\right)$$

$$k_1 = 5 - \frac{1}{4}k_2$$

Endpoints are $(0, 5)$ and $(3.7778, 4.8889)$.

For the numerical example, the top region is for

$$k_2 \geq \frac{1}{2-\frac{1}{4}}\frac{12-1}{1} = \frac{44}{7} = 6.2857$$

and the value of (11.4) is

$$\frac{(1-\frac{1}{2})(2+\frac{1}{2})(12-1) - (2-\frac{1}{4})(1)}{4(1-\frac{1}{4})(1)} = 4$$

For the numerical example, (11.14) is

$$k_2 \leq \frac{(2-\frac{1}{2})(12-1)+\frac{1}{2}(1)}{(4-3\left(\frac{1}{4}\right))(1)} = \frac{68}{13} = 5.2308,$$

while the range of k_2 is

$$\frac{1}{(1+\theta)(2-\theta)}\frac{a-c}{b} = k^*_{B(c)} \leq k_2 \leq \frac{1}{2-\theta^2}\frac{a-c}{b}$$

$$\frac{1}{(1+\frac{1}{2})(2-\frac{1}{2})}\frac{12-1}{1} \leq k_2 \leq \frac{1}{2-\frac{1}{4}}\frac{12-1}{1}$$

$$\frac{44}{9} \leq k_2 \leq \frac{44}{7}$$

$$4.8889 \leq k_2 \leq 6.2857$$

For

$$4.8889 \leq k_2 \leq 5.2308,$$

firm 1's payoff function has a local maximum at k_{1Cour}^{br},

$$k_{1Cour}^{br} = \frac{1}{2}\left(\frac{a-c-\rho}{b} - \theta k_2\right)$$

$$k_1(k_2) = 5 - \frac{1}{4}k_2$$

and its payoff at this capacity is

$$\frac{b}{4}\left(\frac{a-c-\rho}{b} - \theta k_2\right)^2$$

$$\frac{1}{16}\left(20 - k_2\right)^2$$

For

$$\frac{68}{13} = 5.2308 \leq k_2 \leq 6.2857 = \frac{44}{7},$$

the global maximum on the range $0 \leq k_1 \leq \left[k_{B(c)}^{br}\right]^{-1}(k_2)$ occurs at $k_1 = \left[k_{B(c)}^{br}\right]^{-1}(k_2)$, and firm 1's payoff at this point is

$$\frac{1}{\theta b}\left[-\frac{1-\theta}{\theta}(a-c) - \rho + \frac{2}{\theta}(1-\theta^2)bk_2\right]\left[a-c-(2-\theta^2)bk_2\right] =$$

$$\frac{2(1-\theta^2)(2-\theta^2)}{\theta^2}b\left(k_2 - \frac{(1-\theta)(a-c)+\theta\rho}{2(1-\theta^2)b}\right)\left(\frac{1}{2-\theta^2}\frac{a-c}{b} - k_2\right)$$

For the numerical example, this is

$$\frac{1}{\left(\frac{1}{2}\right)(1)}\left(-\frac{1-\frac{1}{2}}{\frac{1}{2}}(12-1) - 1 + \frac{2}{\left(\frac{1}{2}\right)}(1-\frac{1}{4})(1)k_2\right)\left(12-1-(2-\frac{1}{4})(1)k_2\right)$$

$$= \frac{21}{2}\left(k_2 - 4\right)\left(\frac{44}{7} - k_2\right)$$

For the numerical example,

$$k_{MMint} = \frac{4+2\theta-\theta^2}{4(1+\theta)(2-\theta^2)}\frac{a-c}{b} + \frac{\theta}{4(1-\theta^2)}\frac{\rho}{b}$$

$$\frac{4+2\left(\frac{1}{2}\right)-\frac{1}{4}}{4(1+\left(\frac{1}{2}\right))(2-\frac{1}{4})}\frac{11}{1} + \frac{\left(\frac{1}{2}\right)}{4(1-\frac{1}{4})}\frac{1}{1}$$

$$= \frac{36}{7} = 5.1429$$

$$k_{1Cour}^{br}(k_2) = 5 - \frac{1}{4}k_2$$

is the equation of firm 1's capacity best response function for $(0,5)$ to $(3.7143, 5.1429)$.

$$k_{MLint} = \frac{(2-\theta)(a-c)+\theta\rho}{(4-3\theta^2)b}$$

$$\frac{(2-\frac{1}{2})(12-1)+\frac{1}{2}(1)}{(4-\frac{3}{4})(1)}$$

$$= \frac{68}{13} = 5.2308$$

and

$$k_1 = \frac{(1-\frac{1}{2})(2+\frac{1}{2})(12-1)-(2-\frac{1}{4})(1)}{4(1-\frac{1}{4})(1)} = 4$$

is firm 1's best response capacity for $k_2 \geq 5.2308$.

$$k_S = \frac{1}{\theta}\left\{\frac{a-c-\rho}{b} - 2\sqrt{2\frac{1-\theta^2}{2-\theta^2}\left[\frac{(1-\theta)(2+\theta)(a-c)-(2-\theta^2)\rho}{4(1-\theta^2)b}\right]}\right\} \geq k_2$$

Evaluate this for the numerical example:

$$\frac{1}{\left(\frac{1}{2}\right)}\left(\frac{12-1-1}{1} - 2\sqrt{2\frac{1-\frac{1}{4}}{2-\frac{1}{4}}\left(\frac{(1-\frac{1}{2})(2+\frac{1}{2})(12-1)-(2-\frac{1}{4})(1)}{4(1-\frac{1}{4})(1)}\right)}\right) \geq k_2$$

$$20 - \frac{16}{7}\sqrt{42} \geq k_2$$

$$5.1869 \geq k_2$$

13. References

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