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Stochastic Approach to Index Numbers for Multilateral Price Comparisons and their Standard Errors

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Abstract

The main objective of the paper is to demonstrate that a number of widely used multilateral index numbers for international comparisons of purchasing power parities (PPPs) and real incomes can be derived using the stochastic approach. The paper shows that price index numbers from commonly used methods like the Ikle, the Rao-weighted and an additive multilateral system are all weighted least squares estimators of the parameters of the country-product-dummy (CPD) model. The advantage of the stochastic approach is that we can derive standard errors for the estimates of the purchasing power parities (PPPs). The PPPs and the parameters of the stochastic model are estimated using a weighted maximum likelihood procedure under different stochastic specification. Estimates of PPPs and their standard errors for OECD countries using the proposed methods are presented.

The paper also outlines a method of moments approach to the estimation of PPPs under the stochastic approach. The paper shows how the Geary-Khamis system of multilateral index numbers is a method of moments estimator of the parameters of the CPD model. The paper, therefore, provides a coherent stochastic framework for the Geary-Khamis system and derives standard errors of the Geary-Khamis PPPs.

1. **Introduction**

International comparisons of real income, consumption, investment and other national income aggregates rely on purchasing power parities (PPP) compiled under the auspices of the International Comparison Program (ICP) conducted by international organizations including the World Bank, OECD, EUROSTAT and the United Nations. It is well recognized that exchanges rates are not appropriate for the conversion of economic aggregates expressed in national currency units into a common currency unit.[1] PPPs which are designed to measure spatial price level differences across countries are being used. Results from the 2005 round of the ICP have been recently released by the World Bank and regional organizations like the Asian Development Bank and the African Development Bank.[2]

Purchasing power parities are computed using price data collected from the participating countries. PPP compilation within the ICP is undertaken at two levels, viz., at the basic heading level and at a more aggregated level[3]. At the basic heading level price data are aggregated without any weights to yield PPPs for various basic headings. The basic heading PPPs are then aggregated to yield PPPs for higher level aggregates like consumption, investment and gross domestic product. The main focus of the paper is on the step involving the aggregation above the basic heading level where weights for each basic heading are available for all the countries.

A range of methods have been proposed in the literature to compute purchasing power parities for aggregation above the basic heading level. Some of the more popular ones are Geary-Khamis (Geary, 1958, Khamis 1970), Ikle (1972), Country-Product-Dummy (CPD) (Rao 1990, 2004, 2005; Diewert, 2005), Elteto-Koves-Szulc (EKS) (see e.g. Rao 2004). Balk (1996) compared the analytical properties of more than 10 different aggregation methods using the test approach. Diewert (2005) has demonstrated that a number of

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[1] For a more detailed discussion of the issues involved, see the recent ICP Report of the Asian Development Bank (http://adb.org/Documents/Reports/Icp-Purchasing-Power-Expenditures/default.asp).

[2] Readers will find global results from the 2005 ICP from the World Bank website, www.worldbank.org/data/icp. The ICP website also has links to important material including the ICP Handbook and other research materials.

[3] See the ICP handbook for more details.

commonly used formulae can be derived using the CPD method and Rao (2005) established that the Rao (1990) method for computing PPPs is equivalent to the weighted CPD method. Thus a formal link between the stochastic approach to index numbers in the form of the CPD method and some of the more commonly used multilateral index number formulae has been established through the work of Diewert (2005) and Rao (2005). In the past there have been attempts to derive the Geary-Khamis method using stochastic approach (Rao and Selvanathan, 1992 and Diewert, 2005) but none of the attempts have been successful in providing a proper framework under the stochastic approach to derive the Geary-Khamis index and its standard errors. This problem is revisited and a solution is offered for the problem.

The PPPs compiled under the auspices of the ICP are widely used by researchers, analysts and policy makers in conducting studies on catch-up and convergence, measurement of regional and global inequality and poverty and on comparative national price levels. The published PPPs are used without explicit recognition of the fact that the PPPs are based on extensive price surveys and are the result of aggregation methods using expenditure weights from national accounts. The main reason for such use is the fact that there are no published measures of reliability, in the form of standard errors, of the published PPPs are available. To date there has been no major effort to develop methods for the compilation of measures of reliability associated with PPPs derived using various aggregation methods.

The main objective of the paper is to address this problem by offering a link between the CPD model from the stochastic approach and PPPs compiled using aggregation methods like the Ikle, Geary-Khamis and the Rao and other variants of the GK method. The paper shows that PPPs from these aggregation methods are the weighted likelihood estimators under different stochastic specification of the disturbance of the CPD model or as method of moments (MOM) estimators of parameters under different choice of the moment conditions. A result of particular interest is the one that shows that PPPs from the Geary-

Khamis method are the MOM estimators of the parameters of the CPD, thus offering for the first time a satisfactory derivation of the method using the stochastic approach.[4]

The paper is organized as follows. Section 2 establishes the basic notation and provides an overview of the main aggregation methods considered in this paper. Section 3 briefly describes the CPD model used in international comparisons and shows how different systems are equivalent to the weighted maximum likelihood estimators of the parameters of the CPD model under different stochastic assumptions. Section 4 is devoted to a discussion on the method of deriving standard errors for the estimated PPPs. Section 5 focuses on the method of moments estimation of parameters of the CPD model. In Section 6 we present estimated PPPs and their standard errors using OECD international comparisons data for the 1996 benchmark year. The paper is concluded with some remarks in Section 7.

2. Notation and Selected Multilateral Index Number Systems

Let p_{ij} and q_{ij} represent the price and the quantity of the jth commodity in the ith country respectively where $j = 1,...,M$ indexes the countries and $i = 1,...,N$ indexes the commodities. We assume that all the prices are strictly positive and all the quantities are non-negative with the minimum condition that for each i q_{ij} is strictly positive for at least one j; and for each j q_{ij} is strictly positive for at least one i. Also define PPP_j as purchasing power parity or the general price level in j-th country relative to a numeraire country and P_i as the world average price for the ith commodity. We also need the following systems of weights w_{ij} and w_{ij}^{*} in defining different systems of index numbers. These weights are defined as

$$w_{ij} = \frac{p_{ij}q_{ij}}{\sum_{i=1}^{N} p_{ij}q_{ij}} \quad \text{and} \quad w_{ij}^{*} = \frac{w_{ij}}{\sum_{j=1}^{M} w_{ij}} \tag{1}$$

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[4] Khamis (1984) and Rao and Selvanathan (1992) offer stochastic approach interpretation based on a partial approach. For example, they usea stochastic specification for the PPPs under the assumption of full knowledge of the international prices.

It is evident that $\sum_{i=1}^{N} w_{ij} = 1$ and $\sum_{j=1}^{M} w_{ij}^{*} = 1$.

We start with a description of the Geary-Khamis method which is the first multilateral system to make use of the twin concepts of purchasing power parities (PPP_j) and international average prices (P_i).

Geary-Khamis method

The Geary-Khamis multilateral system due to Geary (1958) and Khamis (1970)[5] is a popular method of aggregation for international comparisons as it provides additively consistent international comparisons. The Geary-Khamis system is defined by the following system of interdependent system of equations:

$$PPP_j = \frac{\sum_{i=1}^{n} p_{ij} q_{ij}}{\sum_{i=1}^{n} P_i q_{ij}} \text{ for } j = 1,2,\dots, M; \text{ and } P_i = \frac{\sum_{j=1}^{m} \left(p_{ij} q_{ij} / PPP_j \right)}{\sum_{j=1}^{m} q_{ij}} \text{ for } i= 1,2,\dots N \quad (2)$$

For a given set of international prices, P_i, purchasing power parity of currency of country j is defined as the ratio of value of the commodity bundle of country j evaluated, respectively, at the national prices, p_{ij}, and at the international prices, P_i. Similarly, for a given set of PPPs, international average prices are defined as the unit price derived from the total expenditure on commodity i across all countries and the total quantity of the commodity.

The simultaneous equation system in (2) has a solution that is unique up to a factor of proportionality. Given observed prices and quantity data from all the countries, the system is generally solved using an iterative procedure. Kravis, Heston and Summers (1982) discuss various properties of the Geary-Khamis method and it remained as the principal aggregation method for international comparisons until the more recent phases of the ICP.[6]

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[5] Khamis has authored a number of papers that have delved deeply into various properties of the Geary-Khamis system.

[6] The EKS method is now preferred as the principal aggregation method and the recently completed 2005 round of the ICP is based on the EKS method.

A major criticism of the method surrounds the definition of the international price, in (2), which is essentially a quantity weighted average of the observed prices in different countries. As a result the GK international prices tend to resemble those observed in richer countries and the real incomes of poorer countries tended to be overstated.[7]

We consider two aggregation methods which use the same framework as the Geary-Khamis method but designed to address some of the main problems associated with the GK method.

Rao System for multilateral comparisons

Rao (1990) proposed a multilateral system derived through some modifications to the GK system. The Rao system replaces the quantity-share weights used in the definition of GK international prices by a system of weights that are based on expenditure shares. In addition, the system is defined using weighed geometric averages in the place of arithmetic averages used in the GK system. The system is defined as:

$$PPP_j = \prod_{i=1}^{N}\left(\frac{p_{ij}}{P_i}\right)^{w_{ij}} \text{ for } j=1,2,\ldots M; \text{ and } P_i = \prod_{j=1}^{M}\left(\frac{p_{ij}}{PPP_j}\right)^{w_{ij}^*} \text{ for } i=1,2,\ldots N \tag{3}$$

The system defined here is shown to have a non-trivial solution that is unique up to a factor of proportionality. In the case of binary comparisons, with *M*=2, the Rao index is similar to the Tornqvist index.[8] The use of expenditure share weights reduced the likelihood of Gerchenkron effect present in the GK system. However, the Rao system is not additively consistent.

Ikle System for multilateral comparisons

Ikle (1972) proposed an additively consistent system that is similar to the GK system and makes use of the twin concepts of PPPs and international prices. Following Balk (1996) the

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[7] This is usually referred to as the "Gerchenkron" effect.
[8] The binary index is essentially weighted geometric mean of price relatives where the weights are defined as harmonic means of expenditure share weights in the two countries.

Ikle (1972) system can be written in a form similar to equations (2) and (3). The system is given by:

$$\frac{1}{PPP_j} = \sum_{i=1}^{N} \left(\frac{p_i}{P_{ij}} w_{ij} \right) \text{ for } j=1,2,\dots\text{M and } \frac{1}{P_i} = \sum_{j=1}^{M} \left(\frac{PPP_j}{p_{ij}} w_{ij}^* \right) \text{ for } i=1,2,\dots\text{N} \tag{4}$$

The Ikle system also has a non-trivial solution which is unique up to a factor of proportionality. It is useful to note here that the international prices, P_i, are defined as weighted harmonic means of prices observed in different countries after conversion to a common currency unit. Thus there is an element of commonality between the GK, Ikle and the Rao systems in that they use, respectively, weighted arithmetic, harmonic and geometric averages of national prices. The Ikle system has not been used in international comparisons until the 2005 ICP round.[9]

A new multilateral system with expenditure share weighted arithmetic averages

Given the strong conceptual similarity between the GK, Ikle and the Rao systems, we consider a version of these indices based on arithmetic averages as is the case with the GK system but use it with expenditure share weights used in the Rao and Ikle systems. The system is simply defined as:

$$PPP_j = \sum_{i=1}^{N} \left(\frac{p_{ij}}{P_i} w_{ij} \right) \text{ for } j=1,2,\dots\text{M; and } P_i = \sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^* \text{ for } i=1,2,\dots\text{N} \tag{5}$$

The existence and uniqueness of solutions to system (5) is established in Hajarghast and Rao (2008).

3. The Country-Product-Dummy Model and Multilateral Index Number Systems

So far we have described four systems of multilateral systems that are strongly linked in their conceptual framework with the Geary-Khamis system. In the next section we show that these systems can be derived as estimators of parameters of the CPD model under different distributional assumptions.

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[9] The Ikle method was used in the African region during the 2005 ICP round.

The CPD model was first proposed by Summers (1973) as a method of filling missing values in price data for international comparisons. It was also the preferred method of aggregation of price data below the basic heading level in international comparisons (Kravis, Heston and Summers, 1982). In the 2005 ICP round it has been the recommended method of aggregation below the basic heading level. The CPD model is gaining popularity as an aggregation method for aggregation above the basic heading level (see Rao, 2004 and 2005; and Diewert, 2005). The CPD model is know considered as the principal method of aggregation under *stochastic approach*.

The CPD model postulates that the observed price of *i*-th commodity in *j*-th country, p_{ij}, is the product of three components: the purchasing power parity (i.e. PPP_j); the price level of the j-th commodity relative to other commodities (i.e. P_i) and a random disturbance term u_{ij} as follows

$$p_{ij} = P_i PPP_j u_{ij} \tag{6}$$

where u_{ij}'s are random disturbance terms which are independently and identically distributed. The parameters of the model (PPPs and Ps) can be estimated from (6). The original model proposed by Summers (1973) simply transforms the model into a log-linear form and apply ordinary least squares to estimate the parameters. The estimated parameters are then used in filling any missing price observations. Rao (2005) showed that the Rao system defined in (3) is identical to the weighted least squares estimator of the parameters of the CPD model. This result has provided a useful link between the CPD model and aggregation methods above the basic heading level.

In this section we prove that the Rao, Ikle and the new system can be derived as weighted maximum likelihood estimators of the parameters of the CPD model under different distributional assumptions for the disturbances, u_{ij}.

3.1 CPD model with lognormal disturbances and the Rao system

We consider the case where u_{ij}'s are lognormally distributed. This means that $\ln u_{ij}$ is normally distributed, in this case with mean equal to zero and variance equal to σ^2. In this case we consider the model in its log-linear form

$$\ln p_{ij} = \ln P_i + \ln PPP_j + v_{ij} \quad \text{where} \quad v_{ij} = \ln u_{ij} \sim N(0, \sigma^2)$$

This log-linear equation can be equivalently expressed in the form of a linear regression model:

$$\ln p_{ij} = \sum_{i=1}^{N} \eta_i D_i + \sum_{j=1}^{M} \pi_j D_j * + v_{ij} \quad \text{where} \quad \eta_i = \ln P_i \quad \text{and} \quad \pi_j = \ln PPP_j \qquad (7)$$

where D_i is i-th commodity dummy variable which takes value equal to 1 for commodity i and 0 otherwise; and D_j* is j-th country dummy variable which takes value equal to 1 for a price observation belonging to country j and equal to 0 otherwise. Thus the explanatory variables in (7) are essentially country and product dummy variables and hence the model is known as the country-product-dummy model.

Under the lognormality of the disturbances, u_{ij}, in the original model, the maximum likelihood estimators of the parameters in the log-linear model are the same as the ordinary least squares estimators of the parameters since the disturbances, v_{ij}, are normally distributed. Now we consider the weighted regression model:

$$\sqrt{w_{ij}} \ln p_{ij} = \sum_{i=1}^{N} \eta_i \sqrt{w_{ij}} D_i + \sum_{j=1}^{M} \pi_j \sqrt{w_{ij}} D_j * + \sqrt{w_{ij}} v_{ij} \qquad (8)$$

Rao (2005) has shown that the least squares estimators of the parameters in the weighed CPD model (8) are identical to the solutions of the log-linear equations obtained from the Rao system in (3). Further it can be easily shown that, under lognormality of u_{ij} and normality of v_{ij} the weighted maximum likelihood estimator of the parameters in (7) are the same as the weighted least squares estimators obtained through (8).

The discussion here establishes the result that under the lognormality of the disturbances, the weighted maximum likelihood estimators of the parameters are identical to the PPP_j's and P_i's from the Rao (1990) system defined in (3).

3.2 Gamma distribution and the new Index

Here we start with the CPD model and assume that u_{ij}s follows a gamma distribution[10] as follows

$$u_{ij} \sim Gamma(r,r) \tag{9}$$

where r is a parameter to be estimated. We combine the CPD model in (6) and the distributional assumption (9) to write[11]

$$\frac{p_{ij}}{P_i PPP_j} \sim Gamma(r,r) \tag{10}$$

The choice of the same parameter r for the two parameters of the Gamma distribution ensures that the expected value of the disturbance term is equal to 1.[12] Now outline the weighted maximum likelihood method and establish the required equivalence.

Our purpose here is to estimate parameters (i.e. P_i, PPP_j and r) using a maximum likelihood procedure. From the definition of the gamma density function we can easily show that

$$p_{ij} \sim \frac{r^r}{\Gamma(r)} \frac{p_{ij}^{r-1}}{P_i PPP_j^{r}} e^{-r\frac{p_{ij}r}{P_i PPP_j}} \tag{11}$$

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[10] The choice of the Gamma distribution is guided by the fact that observed prices, after conversion to a common currency, have a skewed distribution. The assumption of lognormal distribution also implies a skewed distribution for log-prices.

[11] One may notice the close association of the proposed model to what is known as a generalized linear model with gamma distribution. A generalized linear gamma regression may be defined as (see McCullagh and Nelder 1989) $y_i/\mathbf{x}_i\boldsymbol{\beta} \sim Gamma(r,r)$. Our model is a nonlinear version of such a model.

[12] For further details on the lognormal, gamma and inverse-gamma distributions used here, the reader is referred to Johnson, Kotz and Balakrishnan (1994).

Therefore the log of density function can be written as

$$LnL_{ij} \propto r\ln r - \ln\Gamma(r) + (r-1)\ln p_{ij} - \ln P_i - r\ln PPP_j - r\frac{p_{ij}r}{P_i PPP_j} \tag{12}$$

We can proceed with this (log-) density function and obtain estimates of the parameters of interest using the standard maximum likelihood procedure but we would like to incorporate the weights into the model as well. Use of weights is consistent with standard index number approach of weighting price relatives by their expenditure shares. This is also the approach used by Rao (2005) where weighted least squares method is employed.

One way of doing this is to use a weighted likelihood estimation procedure. Let's define the weighted likelihood function as

$$WL = \prod_{i=1}^{N}\prod_{j=1}^{M} L_{ij}^{w_{ij}/M} \tag{13}$$

and therefore the weighted log-likelihood function becomes

$$LnWL = \sum_{i=1}^{N}\sum_{j=1}^{M}\frac{w_{ij}}{M} LnL_{ij} \tag{14}$$

Then our weighted log- likelihood function becomes

$$\ln WL \propto (r-1)\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln p_{ij} - r\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln P_i - r\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln PPP_j -$$
$$r\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{p_{ij}w_{ij}}{P_i PPP_j} + r\ln r(\sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij}) - \ln\Gamma(r)\sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij} \tag{15}$$

Note that the above function may not represent a density function therefore we don't interpret the estimation procedure as a maximum likelihood procedure. We rather interpret it as an M-estimation procedure (for more on M-Estimators and their properties see chapter 12 of Wooldridge 2002 or chapter 5 of Cameron and Trivedi 2005).

Maximization of this objective function is not particularly difficult. The only potential problem is the presence of a gamma function in the likelihood function however most of the existing software such as LIMDEP and GAUSS can handle maximization of the functions containing gamma functions fairly easily.

We can also derive the first order conditions from maximization of the above likelihood function as follows

$$-\frac{r\sum_{j=1}^{M} w_{ij}}{P_i} + \frac{r}{P_i^2}\sum_{j=1}^{M}\frac{p_{ij}w_{ij}}{PPP_j} = 0$$

$$-\frac{r\sum_{i=1}^{N} w_{ij}}{PPP_j} + \frac{r}{PPP_j^2}\sum_{i=1}^{N}\frac{p_{ij}w_{ij}}{P_i} = 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln p_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln P_i - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln PPP_j - \sum_{i=1}^{N}\sum_{j=1}^{M}\frac{p_{ij}w_{ij}}{P_i PPP_j} + M + M\ln r - M\frac{\partial}{\partial r}\ln\Gamma(r) = 0$$

After some algebraic manipulations, we can rewrite the above sets of equations as

$$P_i - \sum_{j=1}^{M}\frac{p_{ij}w_{ij}^{*}}{PPP_j} = 0$$

$$PPP_j - \sum_{i=1}^{N}\frac{p_{ij}w_{ij}}{P_i} = 0 \tag{16}$$

$$\frac{\partial}{\partial r}\ln\Gamma(r) - \ln r = \frac{1}{M}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln p_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln P_i - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln PPP_j - \sum_{i=1}^{N}\sum_{j=1}^{M}\frac{p_{ij}w_{ij}}{P_i PPP_j} + M\right)$$

We observe that the first two equations in (16) are the same as the system of equations we introduced as the new system defined in (5) and these equations do not depend upon the value of r.

Thus we have shown that the new multilateral system based on weighted arithmetic averages is identical to the weighted maximum likelihood estimator of the CPD model with disturbances following a gamma distribution.

3.3 Inverse Gamma Distribution and the Ikle Index

We follow the same approach as in Section 3.2 above in the derivation of the Ikle index from the CPD model. In particular we show that the weighted least squares estimator of the parameters of the CPD model when the disturbances follow inverse-Gamma distribution. In order to use the inverse-Gamma distribution, we rewrite the CPD model in (6) slightly differently. We use the reciprocal of the price and obtain:

$$\frac{1}{p_{ij}} = \frac{1}{P_i PPP_j} u_{ij} \tag{17}$$

where u_{ij}s are random disturbance terms which are independently and identically and as before they are assumed to follow a gamma distribution[13]

$$u_{ij} \sim Gamma(r,r) \tag{18}$$

where r is a parameter to be estimated. Model in equation (17) differs from the model in equation (10) mainly in the specification of the disturbance term and how it enters the equation. One of the possible advantages of this model is that we do not have the inverse relationship between variance of p_{ij} and w_{ij}. We combine (17) and (18) to write

$$\frac{1}{p_{ij}} \propto \frac{r^r}{\Gamma(r)} \frac{(P_i PPP_j)^r}{p_{ij}^{r-1}} e^{-r\frac{P_i PPP_j}{p_{ij}}} \tag{19}$$

Following the same procedure as we used in Section 3.2, we may obtain the likelihood function as

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[13] Since the disturbance term in (17) is the reciprocal of the disturbance term in the original CPD model (6), the assumption in (18) is same as the assumption that disturbance term in (6) follows inverse-Gamma distribution.

$$lnL \propto -(r-1)\sum_{i=1}^{n}\sum_{j=1}^{n}w_{ij}\ln p_{ij} + r\sum_{i=1}^{n}\sum_{j=1}^{n}w_{ij}\ln P_i + r\sum_{i=1}^{n}\sum_{j=1}^{n}w_{ij}\ln PPP_j -$$

$$r\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{P_iPPP_jw_{ij}}{p_{ij}} + r\ln r(\sum_{i=1}^{N}\sum_{j=1}^{M}w_{ij}) - \ln\Gamma(r)\sum_{i=1}^{N}\sum_{j=1}^{M}w_{ij} \quad (20)$$

Taking derivative with respect to **PPP** and **P** yields the Ikle system of equations

$$\frac{1}{PPP_j} = \sum_{i=1}^{N}\left(\frac{P_i}{P_{ij}}w_{ij}\right)$$

$$\frac{1}{P_i} = \sum_{j=1}^{M}\frac{PPP_j}{P_{ij}}w_{ij}^* \quad (21)$$

Thus we have shown that the Ikle system is the same as the weighted least squares estimators of the parameters of the CPD model under the assumption of inverse-Gamma for the disturbances.

Results shown in Sections 3.1 to 3.3 establish that the Rao, Ikle and the new system are all weighted least squares estimators of the parameters of the CPD model that are distinguished by the differences in the distributions of the disturbance of the CPD model. Therefore, we have been able to show that all these index numbers belong to a class of index numbers based on the stochastic approach. Unfortunately we have not been able to identify a distribution for the disturbance term under which the Geary-Khamis method could be derived. However, we show in Section 5 that the GK system can also be derived from the CPD model by showing that the GK system is equivalent to the method of moments estimator of the parameters of the CPD models. We will return to this shortly.

4. Computation of Standard Errors

We have emphasized that the advantage of the stochastic approach to index numbers and the use of CPD is to obtain standard errors for estimated indices. One might think that standard errors from conventional weighted least square or weighted maximum likelihood provided by standard software can be used for this purpose. But such standard errors are not valid if these are not derived using proper expressions. Since we have shown that

various systems of multilateral index numbers can be derived using CPD model, it remains for us to derive the expressions to be used in deriving the standard errors. In order to derive standard errors for PPPs and international prices, P_i's, we make use of results available for M-estimators discussed in econometric literature.

We start with a general discussion of M- estimators and their variances. An M-Estimator $\hat{\boldsymbol{\theta}}$ is defined as an estimator that maximizes an objective function of the following form (See e.g. Cameron and Trivedi 2005)

$$Q_N(\boldsymbol{\theta}) = \frac{1}{N} \sum_{i=1}^{N} h_i(y_i, \mathbf{x}_i; \boldsymbol{\theta}) \tag{22}$$

where y_i and $\mathbf{x}_i$ represent dependent and independent variables respectively. $\boldsymbol{\theta}$ is the vector of parameters to be estimated. The function Q is the same as the weighted likelihood function in logarithmic form given in equations (15) and (20) above.

Following Cameron and Trivedi (2005), it has been shown that $\hat{\boldsymbol{\theta}}$ has the following asymptotic distribution

$$\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} \aleph[\mathbf{0}, \mathbf{A}_0^{-1} \mathbf{B}_0 \mathbf{A}_0^{-1}]$$

where

$$\mathbf{A}_0 = \text{plim} \frac{1}{N} \sum_{i=1}^{N} \left. \frac{\partial^2 h_i}{\partial \boldsymbol{\theta}' \partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta}_0} \tag{23}$$

$$\mathbf{B}_0 = \text{plim} \frac{1}{N} \sum_{i=1}^{N} \left. \frac{\partial h_i}{\partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta}_0} \left. \frac{\partial h_i}{\partial \boldsymbol{\theta}'} \right|_{\boldsymbol{\theta}_0}$$

In practice, a consistent estimator can be obtained as

$$\mathbf{VAR}(\hat{\boldsymbol{\theta}}) = \frac{1}{N} \hat{\mathbf{A}}^{-1} \hat{\mathbf{B}} \hat{\mathbf{A}}^{-1} \tag{24}$$

where

$$\hat{\mathbf{A}} = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial^2 h}{\partial \boldsymbol{\theta}' \partial \boldsymbol{\theta}} \bigg|_{\hat{\boldsymbol{\theta}}} \tag{25}$$

$$\hat{\mathbf{B}} = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial h_i}{\partial \boldsymbol{\theta}} \bigg|_{\hat{\boldsymbol{\theta}}} \frac{\partial h_i}{\partial \boldsymbol{\theta}'} \bigg|_{\hat{\boldsymbol{\theta}}} \tag{26}$$

In some special cases like the maximum likelihood or nonlinear least square with homoscedastic errors it can be shown that $\mathbf{A}_0^{-1} = -\mathbf{B}_0$. In such cases the variance formula can be simplified to

$$\mathbf{VAR}(\hat{\boldsymbol{\theta}}) = -\frac{1}{N} \hat{\mathbf{A}}^{-1} \tag{27}$$

Many software programs use this formula as their default standard error formula. But in case of the problem studied in this paper this formula lead to incorrect standard errors for the estimated parameters and we must use the more general formula given by (23).

For example if we apply formula (27) to the estimates from a weighted least squares regression we obtain following formula

$$\mathbf{VAR}(\hat{\boldsymbol{\theta}}) = \hat{\sigma}^2 (\mathbf{X'\Omega X})^{-1} \tag{28}$$

where $\boldsymbol{\Omega}$ is a diagonal matrix with weights on its diagonal which coincide the standard formula for weighted least square when there is heteroscedasticity in error term. However the correct formula for the variance estimator to be used in the case where we used weighted least squares when the disturbances are homoskedastic, is given by:

$$\mathbf{VAR}(\boldsymbol{\theta}) = \hat{\hat{\sigma}}^2 (\mathbf{X'\Omega X})^{-1} (\mathbf{X'\Omega'\Omega X}) (\mathbf{X'\Omega X})^{-1} \tag{29}$$

where $\hat{\hat{\sigma}}^2$ is obtained from the un-weighted regression. This formula is similar to that suggested in Rao (2004) for the computation of standard errors for the weighted CPD method.

In Section 6 we present estimated PPPs from different methods along with their standard errors derived under different stochastic assumptions discussed in Section 3. Before that we turn to the derivation of the GK system from the CPD model.

5. Derivation of Geary-Khamis System Using the CPD Model

We recall that the Geary-Khamis system in equation (2) is given by:

$$PPP_j = \frac{\sum_{i=1}^{n} p_{ij}q_{ij}}{\sum_{i=1}^{n} P_i q_{ij}} \text{ for } j = 1,2,\ldots, \text{M}; \text{ and } P_i = \frac{\sum_{j=1}^{m} \left(p_{ij}q_{ij} \Big/ PPP_j \right)}{\sum_{j=1}^{m} q_{ij}} \text{ for } i = 1,2,\ldots \text{N}$$

In the past there have been several attempts to cast the G-K method in a stochastic framework so that standard errors can be derived. One of the early attempts was due to Rao and Selvanathan (1992) but their approach is limited since the standard errors for PPPs were derived conditional on the knowledge of the international prices, P_i's. Recently, Diewert (2005) attempted to derive the Geary-Khamis bilateral index using the stochastic approach based on the CPD method but the derivation is based on several ad hoc steps. In this paper, we show that the Geary-Khamis *PPP*'s are the method of moments estimators of the parameters of the CPD specification discussed in earlier sections of the paper. In particular, the approach used here recognizes the non-additive nature of the CPD model and proposes the method of moments approach. These aspects are presented in the following subsections. In section 5.1 we discuss how a non-additive nonlinear system of equations can be estimated using a generalized method of moments. Section 5.2 applies this approach to the CPD model which is a non-additive model and shows how the arithmetic and the Geary-Khamis indices can be derived using this approach. A numerical illustration which presents the G-K *PPP*'s and their standard errors is included in Section 6.

5.1 Estimation of non-additive nonlinear models

In establishing a relationship between the GK method and the CPD model, we consider the CPD model as a non-additive model and then look at the problem of estimation of the parameters of the non-additive model using the method of moments estimation technique.

Consider the following nonlinear regression model

$$r(y_i, \mathbf{x}_i, \boldsymbol{\beta}) = u_i \qquad (30)$$

where y_i represent the dependent variable, u_i represents the random errors, $r(y_i, \mathbf{x}_i, \boldsymbol{\beta})$ is a nonlinear function and $\mathbf{x_i}$ is a $1 \times L$ vector, $\boldsymbol{\beta}$ is a $K \times 1$ column vector, $i = 1,...., N$ indexes the number of observations and we also assume that $E(u_i) = 0$. We make a further assumption that the model is non-additive[14] which means it can not be written as

$$y_i - g(\mathbf{x}_i, \boldsymbol{\beta}) = u_i \qquad (31)$$

Parameters of an additive model can be estimated using a nonlinear least squares approach but it can be shown that the use of least square criterion does not provide consistent estimators for non-additive models (see e.g. Cameron and Trivedi 2005).

How a non-additive model can be estimated? We consider the method of moments estimation of the parameters of the model. An obvious starting point is to base the estimation of parameters in (31) on the moment conditions $E(\mathbf{X'u}) = \mathbf{0}$ where $\mathbf{X}$ is the $N \times L$ matrix containing $\mathbf{x}_i$s and $\mathbf{u}$ is an $N \times 1$ vector containing u_is. However other moment conditions can be used. More generally we can base the estimation on the following K moment conditions:

$$E(\mathbf{R(x,\beta)}^{'} \mathbf{u}) = \mathbf{0} \qquad (32)$$

---

[14] It is easy to check that the CPD model is non-additive model using the definition below.

where R is a $N \times K$ vector of functions of $\mathbf{X}$ and $\boldsymbol{\beta}$. By construction there are as many moment conditions as parameters therefore a method of moment estimator can be obtained by solving following sample moment conditions

$$\frac{1}{N} \mathbf{R(X,\hat{\boldsymbol{\beta}})'} \mathbf{r(y,X,\hat{\boldsymbol{\beta}})} = \mathbf{0} \tag{33}$$

This estimator is asymptotically normal with variance matrix

$$Var(\hat{\boldsymbol{\beta}}_{MM}) = \hat{\sigma}^2 \left[\hat{\mathbf{D}}'\hat{\mathbf{R}}\right]^{-1} \hat{\mathbf{R}}'\hat{\mathbf{R}} \left[\hat{\mathbf{R}}'\hat{\mathbf{D}}\right]^{-1} \tag{34}$$

where $\hat{\mathbf{D}} = \dfrac{\partial \mathbf{r(y,X,\boldsymbol{\beta})}}{\partial \boldsymbol{\beta}'}\bigg|_{\hat{\boldsymbol{\beta}}}$, $\hat{\mathbf{R}} = \mathbf{R(X,\hat{\boldsymbol{\beta}})}$ and $\hat{\sigma}^2 = \dfrac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{N}$

The main issue in the above estimation problem is the specification of the moment conditions defined by $\mathbf{R(X,\boldsymbol{\beta})}$. It has been shown (see e.g. Davidson and Mackinnon 2004) that the most efficient choice is

$$\mathbf{R(X,\boldsymbol{\beta})}^* = E\left[\frac{\partial \mathbf{r(y,X,\boldsymbol{\beta})'}}{\partial \boldsymbol{\beta}} \mid \mathbf{X}\right] \tag{35}$$

In general the expectation term in the right hand side can not be derived unless we make very strong distributional assumptions but fortunately for the type of models we consider in this paper it is tractable.

5.2 Estimation of PPPs under the optimal choice of moment conditions and standard errors using MOM

To obtain PPP s and their standard errors based on an the CPD model using MOM, we follow Rao (2005) and Diewert (2005) again to postulate that the observed price of j-th commodity in i-th country, p_{ij}, is the product of three components: the purchasing power parity (i.e. PPP_j); the price level of the j-th commodity relative to other commodities (i.e. P_i) and a random disturbance term as follows

$$p_{ij} = P_i PPP_j u_{ij}^*$$ (36)

where $u_{ij}^* s$ are random disturbance terms which are independently and identically distributed.[15] We also assume that $E(u_{ij}^*) = 1$. Model in equation (36) can be written in the following equivalent form

$$\frac{p_{ij}}{P_i PPP_j} - 1 = u_{ij}$$ (37)

with $E(u_{ij}) = 0$. This is now in the form of a non-additive nonlinear regression model as introduced in the previous section and therefore we can use the estimation method in the previous section. Using the theory discussed in the previous section, the equations to be solved can be written as

$$\frac{1}{nm} \mathbf{R'r = 0}$$ (38)

where $\mathbf{R'}$ is an $(n+m) \times (n \times m)$ matrix and it can be shown that most efficient choice of R according to (35) is defined as follows

---

[15] We use u_{ij}^* instead of u_{ij} in order to facilitate the specification of the non-additive model shown in (37).

and

$$r_{ij} = \frac{p_{ij}}{P_i PPP_j} - 1 \tag{39}$$

Considering the fact that

$$E\left[\frac{p_{ij}}{P_i PPP_j}\right] = 1 \tag{40}$$

We can write the equations in the following matrix form

$$
\begin{bmatrix}
-\frac{1}{P_1} & \diagdown & 0 & -\frac{1}{P_1} & \diagdown & 0 & \cdots & -\frac{1}{P_n} & \diagdown & 0 \\
0 & & \diagdown & 0 & & \ddots & & 0 & & \diagdown \\
& & -\frac{1}{P_n} & & -\frac{1}{P_n} & & \cdots & & -\frac{1}{P_n} & \\
-\frac{1}{PPP_1} & \cdots & -\frac{1}{PPP_1} & 0 & & & \ddots & & & 0 \\
0 & & & & & & & & & \\
0 & & & & & 0 & & & & 0 \\
0 & & & & & 0 & & -\frac{1}{PPP_m} & \cdots & -\frac{1}{PPP_m}
\end{bmatrix}
\begin{bmatrix}
\frac{p_{11}}{P_1 PPP_1} - 1 \\
\frac{p_{12}}{P_2 PPP_1} - 1 \\
\vdots \\
\vdots \\
\vdots \\
\frac{p_{nm}}{P_n PPP_m} - 1
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
\vdots \\
\vdots \\
\vdots \\
0
\end{bmatrix}
$$

We can write the normal equations as follows

$$\begin{cases}-\dfrac{1}{P_i}\displaystyle\sum_{j=1}^{M}\left(\dfrac{p_{ij}}{P_iPPP_j}-1\right)=0\\[4mm]-\dfrac{1}{PPP_j}\displaystyle\sum_{i=1}^{N}\left(\dfrac{p_{ij}}{P_iPPP_j}-1\right)=0\end{cases}\Rightarrow\begin{cases}P_i=\dfrac{1}{m}\displaystyle\sum_{j=1}^{m}\left(\dfrac{p_{ij}}{P_iPPP_j}\right)\\[4mm]PPP_j=\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left(\dfrac{p_{ij}}{P_iPPP_j}\right)\end{cases}\tag{41}$$

According to the theory in the previous section the variance for the estimated price indexes can be obtained by

$$Var(\hat{\boldsymbol{\beta}}_{MM})=\hat{\sigma}^2\left[\hat{\mathbf{D}}'\hat{\mathbf{R}}\right]^{-1}\hat{\mathbf{R}}'\hat{\mathbf{R}}\left[\hat{\mathbf{R}}'\hat{\mathbf{D}}\right]^{-1}\tag{42}$$

where

$$\mathbf{D'}=\begin{bmatrix}-\dfrac{p_{11}}{P_1^2PPP_1} & 0 & -\dfrac{p_{12}}{P_1^2PPP_2} & 0 & \cdots & -\dfrac{p_{1m}}{P_1^2PPP_m} & 0\\[3mm] 0 & & 0 & & \cdot & & 0\\[3mm] & -\dfrac{p_{n1}}{P_n^2PPP_1} & & -\dfrac{p_{n2}}{P_n^2PPP_2} & \ldots & & -\dfrac{p_{nm}}{P_n^2PPP_m}\\[4mm] -\dfrac{p_{11}}{P_1PPP_1^2} \;\cdots\; -\dfrac{p_{n1}}{P_nPPP_1^2} & & 0 & & & & 0\\[3mm] 0 & & & & & & \\[2mm] & & & & & & 0\\[3mm] 0 & & & & 0 & -\dfrac{p_{1m}}{P_1PPP_{1m}^2}\;\cdots\;-\dfrac{p_{nm}}{P_nPPP_m^2}\end{bmatrix}$$

So far we haven't introduced weights in our price index. One way doing this is to define the **R** matrix as follows

$$\mathbf{R}'=\begin{bmatrix} -\dfrac{w_{11}}{P_1} & \mathbf{0} & -\dfrac{w_{12}}{P_1} & \mathbf{0} & \cdots & -\dfrac{w_{1m}}{P_n} & \mathbf{0} \\[2ex] \mathbf{0} & -\dfrac{w_{n1}}{P_n} & \mathbf{0} & -\dfrac{w_{n2}}{P_n} & \cdots & \mathbf{0} & -\dfrac{w_{nm}}{P_n} \\[2ex] -\dfrac{w_{11}}{PPP_1} & \cdots & -\dfrac{w_{n1}}{PPP_1} & 0 & & & 0 \\[1ex] 0 & & & & & & \\[1ex] & & & & & & 0 \\[1ex] 0 & & & 0 & -\dfrac{w_{1m}}{PPP_m} & \cdots & -\dfrac{w_{nm}}{PPP_m} \end{bmatrix}$$

This definition for $\mathbf{R}$ matrix results in the following system of equations which coincides the weighted version of the arithmetic index

$$\begin{cases} PPP_j = \displaystyle\sum_{i=1}^{N}\left(\dfrac{p_{ij}}{P_i}w_{ij}\right) \\[3ex] P_i = \displaystyle\sum_{j=1}^{M}\dfrac{p_{ij}}{PPP_j}w_{ij}^{*} \end{cases} \qquad (43)$$

This set of equations is the same equations that defined the new system based on the expenditure share weighted arithmetic means to define PPPs and P_i's. This is exactly the arithmetic index introduced earlier in equation (5) in Section 2 of this paper.

5.3 Derivation of the Geary-Khamis PPPs and standard errors

Consider again estimation of the following non-additive CPD model

$$\dfrac{p_{ij}}{P_i PPP_j} - 1 = u_{ij}$$

As we discussed in the previous sections we can base our estimation on the following moment conditions

$$E\left[\mathbf{R}'\mathbf{u}\right] = \mathbf{0}$$

and accordingly following sample moment conditions

$$\frac{1}{nm}\mathbf{R'r} = \mathbf{0}$$

Different definitions for $\mathbf{R}$ can lead to different estimators. As long as $\mathbf{R}$ is not correlated with $\mathbf{u}$ the estimator is consistent. We make a slight modification in the definition of $\mathbf{R}$ in the previous section as follows

$$\mathbf{R'} = \begin{bmatrix} -\dfrac{1}{P_1} & 0 & \dfrac{1}{P_1} & 0 & \cdots & -\dfrac{1}{P_1} & 0 \\ 0 & \dfrac{1}{P_n} & 0 & -\dfrac{1}{P_n} & \cdots & & \dfrac{1}{P_n} \\ -\dfrac{P_1}{PPP} \cdots -\dfrac{P_n}{PPP_1} & 0 & \cdots & & & 0 \\ 0 & & & & & 0 \\ 0 & & 0 & -\dfrac{P_1}{PPP_m} & \cdots & -\dfrac{P_n}{PPP_m} \end{bmatrix}$$

It is easy to see that $\mathbf{R}$ is not correlated with $\mathbf{u}$ because $\mathbf{P}$ and $\mathbf{PPP}$ are constant parameters of the model to be estimated. (Note also that P_is are close to one and therefore this matrix does not differ very much from the one in the last section). This definition for $\mathbf{R}$ results in the following equations

$$\begin{cases} \displaystyle\sum_{i=1}^{n} P_i\left(\frac{p_{ij}}{P_i PPP_j} - 1\right) = 0 \\ \displaystyle\sum_{j=1}^{N} 1\left(\frac{p_{ij}}{P_i PPP_j} - 1\right) = 0 \end{cases} \Rightarrow \begin{cases} PPP_j = \dfrac{\displaystyle\sum_{i=1}^{n} p_{ij}}{\displaystyle\sum_{i=1}^{n} P_i} \\ P_i = \dfrac{1}{m}\displaystyle\sum_{j=1}^{m}\left(\frac{p_{ij}}{PPP_j}\right) \end{cases}$$

But this is the un-weighted Geary-Khamis price index. We can derive the weighted price index by defining

$$
\mathbf{R'} = \begin{bmatrix}
-\dfrac{q_{11}}{P_1} & \quad \mathbf{0} & -\dfrac{q_{12}}{P_1} & \quad \mathbf{0} & \cdots & -\dfrac{q_{1m}}{P_n} & \quad \mathbf{0} \\[2ex]
\mathbf{0} & -\dfrac{q_{n1}}{P_n} & \mathbf{0} & -\dfrac{q_{n2}}{P_n} & \cdots & \mathbf{0} & -\dfrac{q_{nm}}{P_n} \\[2ex]
-\dfrac{q_{11}P_1}{PPP_1} \cdots -\dfrac{q_{n1}P_n}{PPP_1} & 0 & & & & 0 \\
0 & & & & & \\
& & & & & 0 \\
0 & & 0 & & -\dfrac{q_{1m}P_1}{PPP_m} & \cdots & -\dfrac{q_{nm}P_n}{PPP_m}
\end{bmatrix}
$$

This results in the following system of equations

$$
\begin{cases}
PPP_j = \dfrac{\displaystyle\sum_{i=1}^{n} p_{ij}q_{ij}}{\displaystyle\sum_{i=1}^{n} P_i q_{ij}} \\[4ex]
P_i = \dfrac{\displaystyle\sum_{j=1}^{m} \left(p_{ij}q_{ij} \Big/ PPP_j \right)}{\displaystyle\sum_{j=1}^{m} q_{ij}}
\end{cases}
$$

which is identical to the equations that define the Geary-Khamis system given in equation (2) in Section 2. Thus it is clear that the G-K PPPs and P_i's are the method of moments (weighted) estimators of the parameters of the CPD model.

As usual the standard errors for the estimated indexes can be obtained using following formula

$$
Var(\hat{\boldsymbol{\beta}}_{MM}) = \hat{\sigma}^2 \left[\hat{\mathbf{D}}'\hat{\mathbf{R}} \right]^{-1} \hat{\mathbf{R}}'\hat{\mathbf{R}} \left[\hat{\mathbf{R}}'\hat{\mathbf{D}} \right]^{-1}
$$

where $\mathbf{D}_{ij}$s are the same as in the previous section.

The result established in this section provides for the very first time a proper derivation of the GK system using stochastic approach. The MOM estimator derived here relates to the estimation of both PPPs and Ps simultaneously. This is more general than the partial approach used in Rao and Selvanathan (1992). This result also provides a method of estimating standard errors for PPPs from the GK method.

6. Empirical Application Using OECD Data

In this section we present estimated PPPs and their standard errors derived using the three methods of aggregation discussed in the paper and the 1996 OECD data. The price information that we have is in the form of PPPs at the basic heading level for 158 basic headings, with US dollar used as the numeraire currency. In addition we have expenditure, in national currency units, for each basic heading in all the OECD countries. These nominal expenditures provide the expenditure share data used in deriving the weighted maximum likelihood estimators under alternative stochastic specification of the disturbances.

For weighted CPD estimates we have used the weighted least squares methodology as explained in Rao (2005). For Ikle and the new index we used the weighted maximum likelihood approach described in Section 2.

Table: MLE estimates of PPPs and SE's

Country	MLE Estimates					
	New Index		CPD		Ikle	
	PPP	S.E	PPP	S.E	PPP	S.E.
GER	1.887	0.136	2.034	0.144	2.187	0.147
FRA	6.092	0.429	6.554	0.455	7.035	0.466
ITA	1425.96	109.727	1504.02	115.509	1584.381	119.196
NLD	1.921	0.150	2.056	0.155	2.205	0.156
BEL	35.491	2.577	37.890	2.698	40.450	2.728

LUX	33.578	2.488	35.816	2.618	38.191	2.700
UK	0.603	0.043	0.642	0.044	0.682	0.045
IRE	0.637	0.051	0.669	0.055	0.696	0.060
DNK	8.525	0.586	9.131	0.615	9.762	0.631
GRC	180.470	13.452	188.482	13.891	196.640	14.005
SPA	112.414	8.304	118.546	8.606	124.799	8.738
PRT	126.043	10.400	129.037	10.994	130.317	12.002
AUT	12.770	0.881	13.730	0.928	14.728	0.948
SUI	2.050	0.168	2.183	0.177	2.320	0.180
SWE	9.424	0.686	10.075	0.720	10.758	0.742
FIN	6.159	0.432	6.598	0.453	7.070	0.462
ICE	86.828	7.000	89.541	6.975	92.329	6.810
NOR	8.807	0.684	9.238	0.736	9.642	0.764
TUR	6304.23	579.128	6321.42	544.907	6357.003	506.991
AUS	1.264	0.099	1.333	0.103	1.407	0.104
NZL	1.464	0.111	1.530	0.113	1.596	0.115
JAP	182.031	13.622	187.429	14.282	192.392	14.780
CAN	1.168	0.090	1.229	0.094	1.295	0.096
USA	1.0		1.0		1.0	

Results shown in the table clearly demonstrate the feasibility and comparability of the new approaches to the estimation of PPPs. As it can be seen, PPPs and their standard errors based on CPD, Ikle and the new index are all numerically close to each other. An additional phenomenon to note is that the PPPs based on the weighted CPD (or from the log-normal specification for the disturbances) appear to be bounded by PPP estimates from the new index and the Ikle index. However this is only a coincidence and when a different country (e.g. Australia) is used as the reference country no special patterns emerged.

Table 2 shows the estimated PPPs and their standard errors based on: (i) arithmetic index using MOM; and (ii) Geary -Khamis using the method introduced in this paper. The

standard errors of the arithmetic index based on the MLE approach discussed in Sections 4 and 5 of this paper are also presented.

Table2: Estimates of PPPs and SE's

	Arithmetic Index	MOM SE Arithmetic	MLE SE Arithematic	G-K Index	MOM SE G-K
GER	1.887	0.109442	0.136	2.08316	0.15474
FRA	6.092	0.606755	0.429	6.679491	0.516194
ITA	1425.96	79.25337	109.727	1537.168	129.5046
NLD	1.921	0.11156	0.150	2.032161	0.156602
BEL	35.491	1.946125	2.577	38.70436	2.700867
LUX	33.578	2.454269	2.488	36.7877	3.446165
UK	0.603	0.036311	0.043	0.679564	0.053761
IRE	0.637	0.037709	0.051	0.657754	0.056569
DNK	8.525	0.591807	0.586	9.457703	0.872669
GRC	180.470	9.271153	13.452	187.3352	13.14857
SPA	112.414	7.726502	8.304	122.1712	10.59001
PRT	126.043	6.56711	10.400	124.7745	9.307088
AUT	12.770	0.731266	0.881	14.40264	1.098328
SUI	2.050	0.146331	0.168	2.220059	0.179608
SWE	9.424	0.726701	0.686	10.56069	1.024583
FIN	6.159	0.404593	0.432	6.895726	0.638499
ICE	86.828	6.142211	7.000	90.02853	9.473389
NOR	8.807	0.457666	0.684	9.119335	0.764748
TUR	6304.23	393.9744	579.128	5967.556	549.1221
AUS	1.264	0.08598	0.099	1.351173	0.106996
NZL	1.464	0.106893	0.111	1.545069	0.140098
JAP	182.031	12.52263	13.622	179.0048	15.83708

CAN	1.168	0.085695	0.090	1.271441	0.115112
USA	1.0			1	

The results from the table are consistent with the expectations. The standard errors for the arithmetic index using GMM is slightly more efficient than MLE. This could be because GMM is robust to the choice of distribution for the error term and the standard errors for the Geary-Khamis using the method proposed here are higher than the other two which is expected because it is not the most efficient estimator based on our stochastic specification.

Which disturbance specification?

It is clear from the empirical results presented here that it is possible to derive PPPs from different methods by simply varying the distribution of the disturbance term. Or alternatively use a method of moments estimator which does not rely on any distributional assumptions. We have not yet established a formal test procedure which can be used in selecting a distribution from lognormal, Gamma and inverse-Gamma distributions based on the observed price data. In Figure 1 below we provide a graphical representation of different distributional assumptions and compare them with the least squares residuals obtained from the log-linear version of the CPD model in Section 3.1.
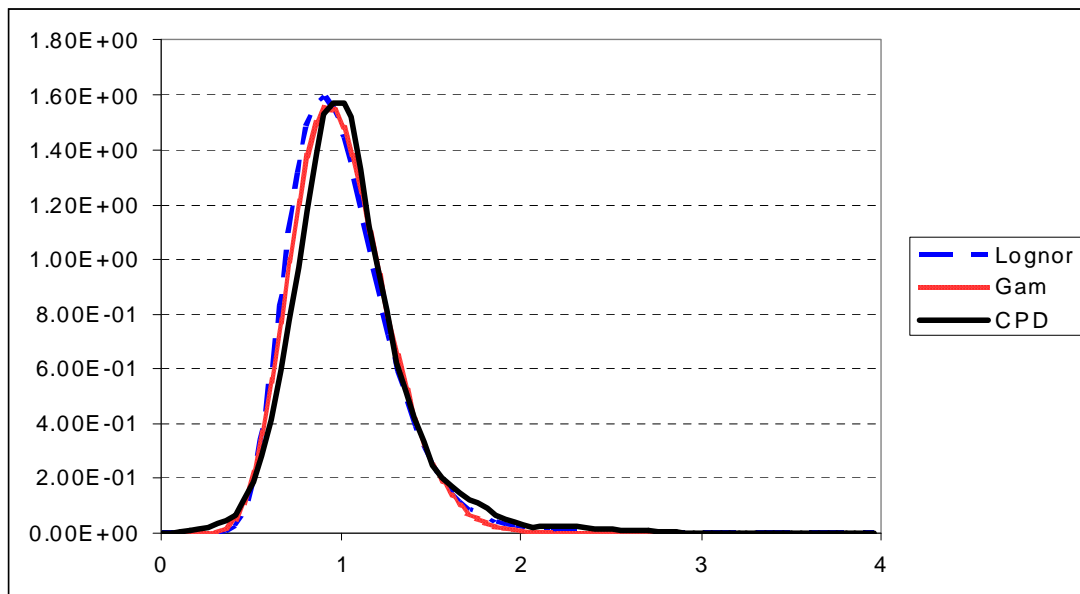

Figure: Distribution of the disturbances of the CPD model

The density function under CPD model simply represents the residuals derived using the OLS estimators of the parameters of the CPD model without any distributional assumptions. The distributions implied by lognormal and Gamma distributions are also presented. From the figure it appears that the Gamma distribution provides a better approximation to the disturbances from the OLS. An implication of this is that if we were to select the Gamma distribution to represent the distribution of the disturbances of the CPD model, then we should be using the arithmetic version of the GK system using expenditure share weights. However, this is an issue that requires further research.

7. Concluding Remarks

The paper has proposed a straightforward extension to two known multilateral methods due to Ikle (1972) and Rao (1990). The new index uses weighted arithmetic averages to define PPPs and international prices, P_i's, instead of harmonic and geometric averages used respectively in Ikle and Rao specifications. The paper has also established that all the three indexes can be shown to be the weighted maximum likelihood estimators of the CPD model when the disturbances follow lognormal, gamma or the inverse gamma distributions respectively. Derivation of the indices using the stochastic approach makes it possible to derive appropriate standard errors for the Ikle and the new index proposed here. Further, given that all these indexes are generated by the same CPD model but with alternative disturbance specifications it allows us to test for the distributional assumptions underlying these three methods and use such specification tests to choose between alternative methods. Further work is necessary to see if it is possible to explore other specifications for the distribution of the disturbance and the index number formulae resulting from such specifications. The paper also outlines the approach necessary to compute the true standard errors of PPPs when weighted maximum likelihood methods are used.

The paper has also shown that the commonly used Geary-Khamis PPPs can be derived from the CPD model and the stochastic approach described here. In particular, the G-K PPPs are shown to be weighted method of moments (MOM) estimators of the parameters of the CPD model.

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