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Abstract

In this article we model production technology in a state-contingent framework. Our model analyzes production under uncertainty without being explicit about the nature of producer risk preferences. In our model producers' risk preferences are captured by the risk-neutral probabilities they assign to the different states of nature. Using a state-general state-contingent specification of technology we show that rational producers who encounter the same stochastic technology can make significantly different production choices. Further, we develop an econometric methodology to estimate the risk-neutral probabilities and the parameters of stochastic technology when there are two states of nature and only one of which is observed. Finally, we simulate data based on our state-general state-contingent specification of technology. Biased estimates of the technology parameters are obtained when we apply conventional ordinary least squares (OLS) estimator on the simulated data.

Keywords: CES, Cobb-Douglas, OLS, output-cubical, risk-neutral, state-allocable, state-contingent

JEL Classification: C15, C63, D21, D22, D81, Q10

Production under conditions of uncertainty is a central feature of economic reality, but one that is often ignored by economists. The central difficulty has been that the most appealing theoretical representation of an uncertain technology has been regarded as empirically intractable.

The first formal treatment of production under uncertainty was the general equilibrium analysis put forward independently by Arrow (1953) and Debreu (1952). To deal with uncertainty they introduced the concept of state-contingent commodities, whose realization is contingent on the occurrence of a particular state of nature. Once this ingenious but simple idea was established, all the tools developed for a deterministic world could be applied readily to decision-making under uncertainty. Chambers and Quiggin (2000) show that the duality methods of modern production theory are fully applicable to state-contingent production and conclude that “the state-contingent approach provides the best way to think about all problems involving uncertainty, including problems of consumer choice, the theory of the firm, and principal-agent relationships”.

However, empirical application of state-contingent theory in a production context has so far proven to be difficult. This is because of the fact that the *ex ante* production choices of firms are not fully observed. As a result most of the data needed for applying standard econometric methods are lost in unrealized states of nature. This problem is obscured in the most popular approach to modelling uncertain production, based on a stochastic production function (described by O’Donnell, Chambers, and Quiggin (2010) as an output-cubical technology). In the stochastic production framework, the choice of a scalar input level, along with a stochastic act of nature, determines output in every state of nature. Thus, observation of output in a single identifiable state of nature is sufficient to identify both the input choice and the output that would have been realized in any other state of nature. However, if this restrictive assumption does not hold, standard estimation techniques will yield systematically biased estimates of economic quantities of interest such as technical efficiency scores (O’Donnell, Chambers, and Quiggin 2010).

Recent efforts to estimate state-contingent technologies have involved predicting unobserved states of nature and/or *ex ante* production choices. By combining these predictions with observed input and output data it becomes possible to estimate the technology using conventional econometric techniques. For example, O'Donnell and Griffiths (2006) use a Bayesian finite mixtures approach to estimate unobserved states of nature and the parameters of an output-cubical state-contingent technology. Chavas (2008) estimates the parameters of a more flexible state-contingent technology by estimating a cost function defined over predicted state-contingent outputs. This paper develops an alternative approach for estimating flexible state-contingent technologies that obviates the need to predict unobserved state-contingent outputs. The technology we consider is a generalization of the flexible state-contingent production technology used in the simulation experiment of O'Donnell, Chambers, and Quiggin (2010).

In their simulation experiment O'Donnell, Chambers, and Quiggin (2010) use a single input and single output state-specific state-allocable specification of the technology. However, a state-specific state-allocable technology is too simplistic and such a representation of technology is seldom observed in a real world production process. A limitation of the state-specific state-allocable technology is that it assumes that the inputs are 'state-specific', that is, input allocated to a given state of nature contributes to output only in that particular state of nature. In this article we generalize O'Donnell, Chambers, and Quiggin (2010) model by proposing an alternative functional form that provides a better representation of real-world production technologies. Specifically, the proposed functional form represents a state-general state-contingent technology that allows substitution of output between the various states of nature. For a detailed discussion on various types of state-contingent techologies, see Rasmussen (2003).

We assume that all firms use the same stochastic technology but they may have different risk attitudes[1] and information sets, and *ex post* they may operate in different production environments. Firms maximize *ex ante* their preference function subject to stochastic technology constraint; in other words they are assumed to act rationally, thereby leaving no room for either technical or allocative inefficiency. Consequently we develop a parsimonious parametric model to describe rational producers' behaviour towards uncertainty.

Further, we show how to econometrically estimate this flexible state-contingent technology when inputs, realized output and the state of nature faced by firms are observed *ex post*. Using noiseless simulated data we demonstrate that our estimation methodology can be used to recover unknown parameters and other economic quantities of interest without error. Finally, we apply conventional OLS estimator to the simulated data and discover that it gives biased estimates of the parameters of the production technology.

Technology

We assume that all firms have access to a common stochastic production technology to produce a stochastic output designated by $\tilde{z} = (z_1, z_2)$, using deterministic input $x \in \mathbb{R}_+$. Nature resolves the uncertainty by choosing a state from a state space Ω. In our simulation experiments we assume for the sake of simplicity that there are two possible states of nature, so $\Omega = \{1,2\}$, but the analysis presented in this article can be extended to state space consisting of any arbitrary number of states of nature. We model production as a two period game with nature, with periods denoted as 0 and 1 respectively. In period 0, the producer allocates input x to the production process and in period 1 nature reveals the actual state of nature contained in the state space $\Omega = \{1,2\}$, and in the process determines the realized output.

O'Donnell, Chambers, and Quiggin (2010) model production using a state-specific state-allocable representation of technology where the input allocated to a specific state of nature $\{s\}$ is given by

$$x_s = a_s {z_s}^b, \quad s \in \Omega = \{1,2\} \tag{1}$$

Assuming that the firms are rational and efficient, the total input used in the production process in period 0 is the sum of the inputs allocated to each state of nature, that is

$$x = x_1 + x_2 = a_1 {z_1}^b + a_2 {z_2}^b \tag{2}$$

In their (O'Donnell, Chambers, and Quiggin (2010)) specification the state-allocable technology is state-specific, that is input $x_1 = a_1 {z_1}^b$ is allocated exclusively to state of nature $\{1\}$ and input $x_2 = a_2 {z_2}^b$ is allocated specifically to state of nature $\{2\}$. For example, if this technology is used to model agricultural production, it would imply that crop yield in a 'dry' season will be zero if no input is allocated to irrigation infrastructure. Our experience shows that this is not the case, that is crop yield in a 'dry' season will be low, but not zero, if no pre-season labour is allocated to irrigation infrastucture. State-allocable technology is too simplistic and such a representation of technology is seldom observed in a real world production process. Hence, we model production using a state-general state-contingent specification of technology.

We model production using a CES specification of technology, where the relationship between the total input[2] used across various states of nature and the *ex post* realization[3] of stochastic output is given by

$$x = (a_1 {z_1}^b + a_2 {z_2}^b)^{\gamma/b} \tag{3}$$

where the interpretation of parameter γ and b are discussed below, z_s is the amount of stochastic output produced in period 1 by employing x amount of non-stochastic input in period 0. $a_s \geq 0$ can be either interpreted as a technology parameter related to production of output in state of nature $\{s\}$ or it can be conceived as a realization of an unobserved random variable determined by nature *ex post*. O'Donnell, Chambers, and Quiggin (2010) specification of technology is a special case of our CES specification as (3) collapses to (2) when $\gamma = b$. Here, it is important to bear in mind that the term arises purely due to the stochastic process of production determined by nature and not as a consequence of any measurement error or researchers' ignorance about the particular functional form.

Whether the technology is state-allocable or state-general, when the total input used in the production process is fixed, the substitution between state-contingent outputs is brought about by re-allocating input among the various states of nature. In the case of state-specific state-allocable technology, the substitution between state-contingent outputs is exclusively accomplished by substituting inputs between various states of nature. But this may not be true in the case of state-general technology because if the input is state-general, then it is possible to produce output in a given state of nature even if no input is allocated to the corresponding state of nature.

Properties of State-General State-Contingent Technology

In order to produce z_1 when nature chooses state $\{1\}$ and z_2 when nature selects state $\{2\}$, the producer must commit in period 0 a minimum input $x = (a_1 {z_1}^b + a_2 {z_2}^b)^{\gamma/b}$. The convex transformation defining technically feasible production space employing a total input x is $t(z_1, z_2, x) = g(z_1, z_2) - x$, where $g(z_1, z_2) = (a_1 {z_1}^b + a_2 {z_2}^b)^{\gamma/b}$. While inefficient but technically feasible production choices are given by $(z_1, z_2) : t(z_1, z_2, x) < 0$, efficient production choices are represented by $(z_1, z_2) : t(z_1, z_2, x) = 0$. Hence, the input distance function[4] for this stochastic technology is $D_I(x, z_1, z_2) = \frac{x}{g(z_1, z_2)}$ and the output distance function is of CET (constant elasticity of transformation) form[5] (See Powell and Gruen 1967): $D_O(x, z_1, z_2) = x^{-\frac{1}{\gamma}} g(z_1, z_2)^{\frac{1}{\gamma}}$.

For a given normalized input price $w > 0$, the minimum cost of producing stochastic output $\tilde{z} = (z_1, z_2)$ is $c(w, z_1, z_2) = wg(z_1, z_2)$. In addition, the marginal cost of producing unit output in every state of nature is $w\gamma a_s (a_1 + a_2)^{\frac{\gamma - b}{b}}$, $s \in \Omega$. For given (fixed) amount of input x, the marginal rate of transformation between *ex post* outputs is $MRT = -\left(\frac{a_1}{a_2}\right)\left(\frac{z_1}{z_2}\right)^{b-1}$ and the elasticity of transformation between any pair of *ex post* outputs is $\sigma = \left|\frac{d \ln z_1/z_2}{d\, MRT}\right| = \frac{1}{1-b}$.

The parameter b is a transformation of elasticity of substitution and is referred to as substitution parameter (see Arrow et al. 1961). We impose the parametric restriction $b \geq 1$ to ensure that the input isoquants in state-contingent output space have the right curvature (they are concave when viewed from the origin). Different transformation curves (state-contingent production possibility frontier) are generated by fixing total input x at different levels. An increase in input x shifts the transformation curve out from the origin. The transformation curve is negatively sloped as the specification of technology allows for substitutability between stochastic *ex post* outputs. The parameter γ represents economy of scale. The technology exhibits increasing, constant or decreasing returns to scale.

The lowest admissible value of b is one; this implies an infinite elasticity of substitution and therefore straight-line isoquants, meaning *ex post* output is perfectly substitutable between states of nature. Re-arranging equation (3) we have

(4) $$x^{1/\gamma} = g(z_1, z_2)^{1/\gamma} = (a_1 {z_1}^b + a_2 {z_2}^b)^{1/b}$$

As $b \to \infty$ the elasticity of transformation converges to zero, implying that no substitution is possible between outputs in different states of nature. Taking limits on both sides of equation (4) as $b \to \infty$, we have

(5) $$\lim_{b \to \infty} x^{1/\gamma} = \lim_{b \to \infty} g(z_1, z_2)^{1/\gamma} = \lim_{b \to \infty} (a_1 {z_1}^b + a_2 {z_2}^b)^{1/b}$$

Applying limiting argument originally due to Hardy, Littlewood, and Polya (1934) we can re-write equation (5) as

(6) $$x \to \text{Max}\{{z_1}^\gamma, {z_2}^\gamma\}$$

Thus the *ex post* output in state $\{s\}$ can be expressed in terms of *ex ante* input requirement as

(7) $$z_s = x^{\frac{1}{\gamma}}, \quad s \in \Omega$$

where x is not allocable across state. Equation (7) represents an output-cubical technology and output-cubical stochastic production functions have been the foundation of stochastic frontier analysis.

Efficient Firm Behaviour

We assume that the firms seek to maximize their utility function $W(y)$ where $\mathbf{y} = (y_1, ..., y_S)$ and $y_s = z_s - wx,\ s \in \Omega$ is the *ex post* net return in the state of nature $\{s\}$. The utility function W is continuously differentiable, non-decreasing and quasi-concave in its arguments. This form of utility function is quite general and it contains the family of expected utility function in net returns as a special case. We further assume that the firms are technically efficient, i.e., they lie on the production possibility frontier. This is further ensured by the fact that the preferences are non-decreasing in net returns and that the technology proposed above is smooth.

Given that the state $\{s\}$ has been realized, the variables relevant to firms' welfare in the production problem are the committed (*ex ante*) input x in period 0 and realized (*ex post*) stochastic output $z_s \in \mathbb{R}_+$ in period 1. We further assume that the state-contingent utility function displays a degree of separability between input x committed prior to the realization of the state of nature and the net returns (profits) accumulated when the state of nature $\{s\}$ is realized.

The first order conditions for efficient firm behaviour can be written as

(8) $$\underset{z_1,\dots,z_S}{Max}\ \{W(\mathbf{y}) : D_I(x,\mathbf{z};\beta) \geq 1\}$$

where $D_I(x, \mathbf{z}, \boldsymbol{\beta})$ is the input distance function and $W(.)$ is the welfare function having the property $W_s \equiv \frac{\partial W(\mathbf{y})}{\partial y_s} \geq 0,\ s \in \Omega$. The first order conditions for efficient behaviour of firm are given by

(9) $$\pi_s - wm(z_s, \boldsymbol{\beta}_s, x) \leq 0 \quad \forall\, s \in \Omega$$

where $m(.)$ is non-negative function state-contingent output z_s, state-contingent technology parameters $\boldsymbol{\beta}_s$ and total input x applied to the production process and

(10) $$\pi_s \equiv \frac{W_s(\mathbf{y})}{\sum\limits_{s \in \Omega} W_s(\mathbf{y})} \in (0,1)$$

The monotonicity of the welfare (utility) function in net returns ensures that $\sum\limits_{s \in \Omega} \pi_s(\mathbf{y}) = 1$. π_s is referred to as risk-neutral probability in state of nature $\{s\}$, as it represents the subjective probability that a risk-neutral firm would require in order to make the same production choices (produce the same *ex post* output using the same amount of input *ex ante*) as a rational firm with preferences W.

Hence, the study of firms with a particular set of preferences actually boils down to analyzing the behaviour of risk-neutral firms with varying subjective probabilities. This further implies that while analyzing the behaviour of firms that are efficient, there is no need to explicitly take into account their risk attitudes (whether they are risk averse or risk lover).

Optimizing Behaviour in Two State Case

For our specification[6] of technology given by (3), the firms optimization problem can be written as:

(11) $$\max_{z_1,\ldots,z_S} \{W(\mathbf{y}) : x \geq (\sum_s a_s {z_s}^b)^{\gamma/b}\}$$

The first-order conditions for efficient firm behaviour are:

(12) $$\frac{\partial W(\mathbf{y})}{\partial y_s} - w \sum_s \frac{\partial W(\mathbf{y})}{\partial y_s} \frac{\partial x}{\partial z_s} \leq 0 \quad s \in \Omega = \{1,2\} \text{ or}$$

(13) $$\frac{\partial W(\mathbf{y})}{\partial y_s} - w \sum_s \frac{\partial W(\mathbf{y})}{\partial y_s} \frac{\partial (\sum\limits_s a_s {z_s}^b)^{\gamma/b}}{\partial z_s} \leq 0 \quad s \in \Omega = \{1,2\}$$

Dividing both sides of equation (13) by $\sum_{s\in\Omega} \partial W(\mathbf{y}) \big/ \partial y_s$ and using equation (10) we have

$$\pi_s - \gamma w a_s {z_s}^{b-1} x^{\frac{\gamma-b}{\gamma}} \leq 0 \quad s \in \Omega = \{1,2\} \tag{14}$$

where the risk-neutral probability π_s of a firm in state of nature $\{s\}$ is given by equation (10).

Adding the risk-neutral subjective probabilities given by (10), across all states of nature gives us the *efficient set*[7]

$$\Xi(w,x) = \{ \quad (z_1,z_2) : 1 - \gamma w x^{\frac{\gamma-b}{\gamma}} \sum_{s\in\Omega} a_s z_s^{b-1} \leq 0 \quad \} \tag{15}$$

and the set for which strict equality holds in the above equation is referred to as the *efficient frontier*. The *efficient frontier* represents the boundary of $\Xi(w)$and therefore it satisfies the first order conditions (with equality) given by (9). Hence, we can write the *efficient set* as:

$$\Xi_{eff}(w,x) = \{ \quad (z_1,z_2) : 1 - \gamma w x^{\frac{\gamma-b}{\gamma}} \sum_{s\in\Omega} a_s z_s^{b-1} = 0 \quad \} \tag{16}$$

If the firms base their risk-neutral probabilities on the technology used in the various states of nature, that is, if $\pi_s(\mathbf{y}) \propto a_s$, then they will choose to produce the same output no matter what state of nature is realized *ex post*. Let π_j and π_i be the risk-neutral probabilities in state of nature $\{j\}$ and $\{i\}$ respectively. Then based on (14) the ratio of these subjective probabilities for an efficient firm is

$$\frac{\pi_j}{\pi_i} = \frac{a_j z_j^{b-1}}{a_i z_i^{b-1}} \tag{17}$$

If $\frac{\pi_j}{\pi_i} = \frac{a_j}{a_i}$, then from (17) it must be the case that $z_j = z_i \quad \forall i,j \in \Omega$. Therefore, the riskless plan converges to $(z_1,z_2) = (x^{\frac{1}{\gamma}}, x^{\frac{1}{\gamma}})$ as $b \to \infty$.

The cost function is linear when the technology exhibits constant return to scale and allows for perfect substitutability between *ex post* output. In this special case when $\gamma = 1$ and $b = 1$ the *efficient set* is equal to non-negative orthant, provided $1 - w\sum_{s\in\Omega} a_s \leq 0$. The marginal return for non stochastically increasing the output in the direction of equal output ray is one, while the corresponding *ex ante* marginal cost is $w\sum_{s\in\Omega} a_s$. Hence the *efficient set* spans the non-negative orthant, which is $\Xi(w) = \mathbb{R}_+{}^2$, as long as the marginal return from increasing in the direction of equal output is strictly negative or zero. In the special case where $\gamma = b$ our model is identical to the model used in O'Donnell, Chambers, and Quiggin (2010).

Based on their expectations (given by their risk-neutral probabilities $\pi_s(\mathbf{y})$) about the future states of nature or their attitudes towards risk (whether they are risk lover or risk averse) or mixture of these two factors, different firms will end up on different points on the *efficient frontier*. For any rational firm having a general welfare (utility) function $W(\mathbf{y})$ the relationship between state-contingent output and the subjective risk-neutral probability can be derived by re-writing the first order condition given by (14) as

(18) $$z_s = (\frac{\pi_s}{\gamma w a_s})^{\frac{1}{b-1}} x^{\frac{b-\gamma}{\gamma(b-1)}}, \quad s \in \Omega = \{1,2\}$$

From (18) it follows that on the *efficient frontier* the output of a rational firm in any state of nature increases with an increase in the risk-neutral probability in the corresponding state of nature, provided $b > 1$. Again, (18) implies if $b \geq \gamma$ and $b > 1$ then the state-contingent output in period 1 increases with an increase in the total input allocated to the production process in period 0.

Firms choose risk free output combination if their risk-neutral probabilities are proportional to the technology in the corresponding state of nature, that is , $\pi_s \propto a_s$ and hence the output is given by

(19) $$z_s = (\frac{1}{\gamma w \sum_{s\in\Omega} a_s})^{\frac{1}{b-1}} x^{\frac{b-\gamma}{\gamma(b-1)}}, \quad \forall s \in \Omega = \{1,2\}$$

If $\frac{\pi_1}{\pi_2} > \frac{a_1}{a_2}$, then any rational firm will produce output $z_1 > z_2$ in period 1 by committing input $0 \leq x \leq a_1{}^{\frac{1}{1-\gamma}}(w\gamma)^{\frac{\gamma}{1-\gamma}}$ in period 0 and if $\frac{\pi_1}{\pi_2} < \frac{a_1}{a_2}$, then it will produce $z_1 < z_2$ in period 1 by using input $0 \leq x \leq a_2{}^{\frac{1}{1-\gamma}}(w\gamma)^{\frac{\gamma}{1-\gamma}}$ in period 0.

It is important to note that the output combination chosen by a risk-neutral firm having a certain belief (risk-neutral subjective probabilities) about future states of nature, could have been chosen by a risk averse (or risk loving) firm with a different set of subjective probabilities. For example, consider a producer who maximizes her expected welfare (utility) and ascribes probability p_1 to state of nature $\{1\}$. Assuming that producer has an exponential[8] utility function, her welfare function can be written as

$$W(y) = -p_1 \exp(-\lambda y_1) - (1-p_1)\exp(-\lambda y_2) \tag{20}$$

where $\lambda = -\frac{W''}{W'}$ represents the coefficient of absolute risk aversion (Arrow 1965; Pratt 1964). From the first order condition for (20) the risk-neutral probability in state $\{1\}$ is

$$\pi_1 = \frac{p_1 \exp(-\lambda z_1)}{p_1 \exp(-\lambda z_1) + (1-p_1)\exp(-\lambda z_2)} \tag{21}$$

Rearranging (21) in terms of p_1 we get

$$p_1 = \frac{1}{1 + \frac{1-\pi_1}{\pi_1}\exp(-\lambda(z_1 - z_2))} \tag{22}$$

Equation (22) implies that any rational risk averse firm that has unit coefficient of risk aversion, assigns a probability p_1 to state of nature $\{1\}$ and maximizes expected exponential utility over net return will produce the same output as a risk-neutral firm that has a risk-neutral probabilty π_1 in the corresponding state of nature.

Estimation Methodology

In many real world production processes we observe both realized state of nature and total input allocated to different states of nature. O'Donnell, Chambers, and Quiggin (2010) describe such a production system where in the presence of uncertainty sugar-cane producers face the choice of planting different varieties of sugar-cane depending on their expectations about future states of nature. Specifically, they have to decide between planting a high yielding variety which is susceptible to disease and a low yielding variety that is resistent to damage from disease. In their application input allocations correspond to land used in planting a different variety of sugar-cane, and realized state of nature is captured by the degree of disease infestation. Therefore, both input allocations as well as realized state of nature are observed *ex post*. In such empirical applications conventional techniques such as stochastic frontier analysis (SFA) and data envelopment analysis (DEA) can be readily used to estimate the parameters of the underlying production technology.

In some other applications the input allocations are observed but realized state of nature is not observed. For example, medical doctors are often aware of various kinds of influenza vaccines supplied by medical professionals to the patients (input allocations) but there is no way they can observe how many of the patients are actually exposed to different traits of influenza virus (realized state). In such cases, if the production technology is output-cubical then the technology parameters can be estimated (for example O'Donnell and Griffiths 2006) in finite mixtures framework.

This article develops methodology for estimating the parameters of the production technology in a third empirical context, namely when there are two observable states of nature but output[9] in only one of two states is observed.

Underpinning our estimation methodology is the assumption that firms are rational and technically efficient in production. For notational convenience we write equation (3) in a more general form as

$$x = f(\mathbf{z}, \boldsymbol{\beta}) \tag{23}$$

The rationality assumption means that an interior solution to the firms optimization problem is given by

$$\pi_s = wm(z_s, \boldsymbol{\beta}_s, x) \quad \forall\, s \in \Omega \tag{24}$$

where $m(z_s, \boldsymbol{\beta}_s, x)$ is function of state-contingent output z_s, total input applied in the production process x and parameter vector $\boldsymbol{\beta}_s$ representing the state-contingent technology.

Equation (24) is especially important for two reasons. First, if the inverse of $m(z_s, \boldsymbol{\beta}_s, x)$ exists then we can express state-contingent outputs as a function of normalized input prices, total input applied in the production process and risk-neutral probabilities:

$$z_s = m^{-1}\left(w^{-1}\pi_s, \boldsymbol{\beta}_s, x\right) \quad \forall\, s \in \Omega \tag{25}$$

Second, in the two-state case, equation (24) allows us to express risk-neutral probabilities as functions of normalized input prices, realized states of nature, and observed outputs:

$$\pi_1 = e_1\left[wm(q, \boldsymbol{\beta}_1, x)\right] + e_2\left[1 - wm(q, \boldsymbol{\beta}_2, x)\right] \tag{26}$$

and

$$\pi_2 = e_2\left[wm(q, \boldsymbol{\beta}_2, x)\right] + e_1\left[1 - wm(q, \boldsymbol{\beta}_1, x)\right] \tag{27}$$

where $e_s = 1$ if state of nature s is realized *ex post*, $s \in \Omega = 1, 2$ and 0 otherwise.

Equations (26) and (27) can be substituted into equation (25), and the result can then be substituted into equation (23). This yields a possibly nonlinear relationship between total inputs, normalized input prices, realized states of nature, observed outputs, as well as the unknown parameters of the production technology. Estimation involves embedding this relationship in a stochastic framework and applying an appropriate econometric estimator, such as nonlinear least squares (NLS). Importantly, equation (24) cannot be used on its own to recover the parameters of the technology. To see this, simply note that for any (z_s, β_s) pair there exists a π_s that will satisfy (24) exactly. This means that the parameters and risk-neutral probabilities cannot be separately identified unless additional information is introduced into the estimation process. In this article, this additional information comes in the form of equation (23).

Estimating Risk Neutral Probabilities and the Parameters of Technology

In terms of the quantities introduced so far, under the assumption that the firms are rational we have

(28) $$\pi_s - \gamma w a_s {z_s}^{b-1} x^{\frac{\gamma-b}{\gamma}} = 0 \quad s \in \Omega = \{1,2\}$$

(29) $$z_s = (\frac{\pi_s}{\gamma w a_s})^{\frac{1}{b-1}} x^{\frac{b-\gamma}{\gamma(b-1)}} \quad s \in \Omega = \{1,2\}$$

(30) $$\pi_1 = e_1[\gamma w a_1 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] + e_2[1 - \gamma w a_2 q^{b-1} x^{\frac{\gamma-b}{\gamma}}]$$

(31) $$\pi_2 = e_1[1 - \gamma w a_1 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] + e_2[\gamma w a_2 q^{b-1} x^{\frac{\gamma-b}{\gamma}}]$$

Substituting for z_1 and z_2 using (29) in (3) we have

(32) $$x = \{\textstyle\sum_{s\in\Omega} a_s (\frac{\pi_s}{\gamma w a_s})^{\frac{b}{b-1}}\}^{\frac{\gamma(b-1)}{b(\gamma-1)}}$$

Taking logarithm on both sides of (32) and substituting for risk-neutral probabilities in (32) using (28) and $\pi_s = 1 - \sum_{j\in\Omega\setminus\{s\}} \pi_j$; $s \in \Omega = \{1,2\}$, we have

(33) $$\ln q - \frac{1}{\gamma}\ln x + e_1\{\frac{1}{b}\ln a_1 + r_1(q,w,x,\boldsymbol{\beta})\} + e_2\{\frac{1}{b}\ln a_2 + r_2(q,w,x,\boldsymbol{\beta})\} = 0$$

where $r_i(q,w,x,\boldsymbol{\beta}) = \frac{1}{b}\ln(1 + \frac{a_j[\frac{1-wa_i\gamma q^{b-1}x^{\frac{\gamma-b}{\gamma}}}{wa_j\gamma}]^{\frac{b}{b-1}} x^{\frac{b(b-\gamma)}{\gamma(b-1)}}}{a_i q^b})$; $i \neq j \in \Omega = \{1,2\}$, $q = e_1 z_1 + e_2 z_2$ and $\boldsymbol{\beta} = (\gamma, b, a_1, a_2)$.

Since the risk-neutral probabilities must lie on a unit interval, we have the following restriction on parameters in equation (33):

(34) $$0 \leq e_1[\gamma w a_1 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] + e_2[1 - \gamma w a_2 q^{b-1} x^{\frac{\gamma-b}{\gamma}}] \leq 1$$

An associated econometric estimating equation is:

$$\ln q_{nt} = \frac{1}{\gamma}\ln x_{nt} - e_{1nt}\{\frac{1}{b}\ln a_1 + r_{1nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta})\} - e_{2nt}\{\frac{1}{b}\ln a_2 + r_{2nt}(q_{nt}, w_{nt}, x_{nt}, \boldsymbol{\beta})\} + v_{nt} \tag{35}$$

And the corresponding restriction on each observation in the sample is given by:

$$0 \leq e_{1nt}[\gamma w a_1 {q_{nt}}^{b-1} {x_{nt}}^{\frac{\gamma-b}{\gamma}}] + e_{2nt}[1 - \gamma w a_2 {q_{nt}}^{b-1} {x_{nt}}^{\frac{\gamma-b}{\gamma}}] \leq 1 \tag{36}$$

where the subscripts n and t represent firms and time periods respectively $(n = 1, ..., N; t = 1..., T)$, and v_{nt} is a random variable representing statistical noise.

Simulated Data

This section uses simulation methods to compare the performance of conventional estimators with the NLS estimator developed above. The input demand x is derived by substituting equation (29) into equation (3) and the state-contingent outputs z_1 and z_2 are generated using equation (29). The following equation expresses input demand in terms of the risk-neutral probabilities and the technology parameters:

$$x = [a_1(\frac{\pi_1}{a_1 w \gamma})^{\frac{b}{b-1}} + a_2(\frac{\pi_2}{a_2 w \gamma})^{\frac{b}{b-1}}]^{\frac{\gamma(b-1)}{b(\gamma-1)}} \tag{37}$$

Therefore in table 1 the input demand x is simulated using (37) and state-contingent outputs (z_1, z_2) are simulated using equation (29). In our simulation we assign equal probabilites to each state of nature. The realized state of nature and the output corresponding to this state of nature are listed in columns 6 and 7 respectively in table 1. Finally, the values of the unknown parameters used to generate this table were $b = 2$, $a_1 = 1.5$, $a_2 = 0.5$ and $\gamma = 1.25$.

Numerical Example Using Simulated Data

The case that interests us from the perspective of estimation is when state-contingent output (z_s), realized state of nature $(\{s\})$, total input (x) allocated to the production process and input price (w) are observed. This case is of empirical importance. For example, in agricultural production, in addition to observing inputs used and realized crop yield, we often also observe whether the season is 'wet' or 'dry'. In this case we use non-linear least squares (NLS) to estimate the parameters of technology using (35).

The econometric equation for the conventional ordinary least squares (OLS) estimator with Cobb-Douglas functional form can be written as

$$\ln(q_{nt}) = e_1[-\alpha \ln(a_1)] + e_2[-\alpha \ln(a_2)] + \alpha \ln(x_{nt}) + v_{nt} \tag{38}$$

where $e_j = 1$ if $j \in \{1,2\}$ is the realized state of nature (otherwise $e_j = 0$) and $q = e_1 z_1 + e_2 z_2$. The subscripts n and t in (38) represent firms and time periods respectively ($n = 1,...,25; t = 1$) and v_{nt} is a random variable representing statistical noise.

We apply conventional OLS estimator[10] to the simulated data shown in table 1. Estimates of the technology parameters using OLS estimator given by (38) is compared with NLS estimator given by (35). Table 2 shows that conventional OLS estimator provides biased estimates of the production technology parameters. But the NLS estimator that assumes a CES specification of technology exactly[11] recovers the parameters of technology with standard errors of zero. The associated risk-neutral probabilities and unobserved state-contingent outputs were also recovered without error.

Elasticity of scale represents an economically important characteristic of any production technology. For the conventional Cobb-Douglas and CES specifications the elasticity of scale is given by parameters α and γ^{-1} respectively. In table 2 we observe that the conventional OLS estimator performs badly in measuring elasticity of scale.

It is important to note that the problem is not with the conventional OLS estimator, but the bias arises due to mis-specification of the stochastic technology. This can be seen by considering a case where both state-contingent outputs z_1 and z_2 along with total input (x) allocated to the production process are observed. This case is implausible because in the real world only one state of nature is realized and data on outputs in the unrealized states of nature are lost. We consider this case purely for the sake of econometic estimation. In order to estimate the parameters of technology we re-write (3) as follows:

$$z_1{}^b = \frac{1}{a_1} x^{b/\gamma} (1 - \frac{a_2 z_2{}^b}{x^{b/\gamma}}) \tag{39}$$

Taking logarithm and on both sides of (39) and then dividing both sides of (39) by b we have

$$\ln(z_1) = -\frac{1}{b}\ln(a_1) + \frac{1}{\gamma}\ln(x) + \frac{1}{b}\ln(1 - \frac{a_2 z_2{}^b}{x^{b/\gamma}}) \tag{40}$$

And the corresponding econometric estimation equation can be written as

$$\ln(z_{1nt}) = -\frac{1}{b}\ln(a_1) + \frac{1}{\gamma}\ln(x_{nt}) + \frac{1}{b}\ln(1 - \frac{a_2 z_{2nt}{}^b}{x_{nt}{}^{b/\gamma}}) + v_{nt} \tag{41}$$

where the subscripts n and t represent firms and time periods respectively $(n = 1,...,25; t = 1)$ and v_{nt} is a random variable representing statistical noise.

Even when both state-contingent outputs are observed (unrealistic case) conventional OLS provides bias[12] estimates of the technology parameters. Again, when we estimate (41) assuming a state general specification technology given by CES functional form we get exact estimates of the production technology parameters with zero standard errors.

Simulation Experiment

To further explore the nature of bias for each of the technology parameters using conventional OLS estimator, we perform a simulation experiment. In the simulation experiment we fix the risk-neutral probabilities shown in the second column in table 1 in each of the $N = 10,000$ replications, but we allow each of the 25 firms to experience any of the two possible states of nature *ex post* with probability 0.5.

We observe from tables 3 and 4 that the bias in the estimates of technology parameters a_1 and a_2 is severe when technology exhibits high substitutability between state-contingent outputs, that is when $b = 1.1$. For example, in table 3 we observe that when technology exhibits decreasing returns to scale and high substitutability between state-contingent outputs, that is when $\gamma^{-1} = 0.8$ and $b = 1.1$, parameter a_1 has an estimated mean of $2.1671E + 122$ and estimated standard error of $5.3210E + 123$. Similarly, in table 4 we find that when technology exhibits increasing returns to scale and high substitutability between state-contingent outputs, that is when $\gamma^{-1} = 1.25$ and $b = 1.1$, parameter a_2 has an estimated mean of $2.6150E + 73$ and estimated standard error of $2.6128E + 75$.

In tables 3 and 4 we find the bias in technology parameters a_1 and a_2 is least when the technology exhibits low degree of substitutability between state-contingent outputs, that is $b = 11$. For example, in table 3 we observe that when technology exhibits increasing returns to scale and low degree of substitutability between state-contingent outputs, that is when $\gamma^{-1} = 0.8$ and $b = 11$, parameter a_1 has an estimated mean of 0.9229 and estimated standard error of 0.6195. Again, in table 4 we find that when technology exhibits decreasing returns to scale and low degree of substitutability between state-contingent outputs, that is when $\gamma^{-1} = 1.25$ and $b = 11$, technology parameter a_2 has an estimated mean of 0.6949 and estimated standard error of 0.4383.

Again, we observe in table 5 that the bias in the estimates of elasticity of scale parameter γ^{-1} is least when technology exhibits low degree of substitutability between state-contingent outputs, that is $b = 11$. For example, in table 5 when technology exhibits low degree of substitutability between state-contingent outputs, that is $b = 11$, the estimated means of elasticity of scale parameter for decreasing and increasing returns to scale are 0.6600 and 1.4207 respectively and the corresponding standard errors are 0.1842 and 0.2291 respectively.

Finally, we plot the estimated probability density functions of a_1, a_2 and γ^{-1} in figures 1, 2 and 3 respectively. These pdf plots clearly indicate the nature of bias in the estimation of the technology parameter.

Conclusion

Representation of the production technology and the description of firm behaviour are the two critical elements of the model. The production technology defines deterministic (observed) input and stochastic output combinations that are technically feasible. Given this particular nature of technology that firms have access to, their optimal production choices are determined by their risk attitudes and beliefs involving the relative probabilities of different states of nature. In this article we model producer behaviour towards uncertainty and derive the risk-neutral probibilities assigned to different states of nature.

An estimation methodology is developed in order to estimate parameters of technology and producers' risk-neutral probabilities. We then simulate data based on our CES specification of state-general state-contingent technology. Biased estimates of the technology parameters are obtained using conventional ordinary least squares (OLS) estimator on the simulated data. Since the simulated data does not have any measurement error, the source of the measurement bias arises due to mis-specification of the underlying stochastic technology. Hence, the bias in the estimates of technology parameters cannot be attributed to the conventional OLS estimators.

Finally, a simulation experiment is performed to determine the nature of bias in parameter estimates. We find that the estimation bias for the productivity parameter in each of the two states of nature and the elasticity of scale parameter is minimum when technology exhibits low degree of substitutability between state-contingent outputs.

Notes

[1]Producers risk attitudes can be captured by the shape of their *ex post* utility function.

[2]The input in CES functional form is state general.

[3]Only one of the two possible state-contingent outputs is observed.

[4]The input distance function is defined as $D_I(x,z,\boldsymbol{\beta}) = \max\{\rho : x/\rho \text{ can produce } z\}$. Let ρ^* be the maximum factor by which a firm can contract its input and still produce the same output. That is $g(z,\boldsymbol{\beta}) - x/\rho^* = 0$. It follows that $D_I(x,z,\boldsymbol{\beta}) = x/g(z,\boldsymbol{\beta})$.

[5]The family of CET production possibility schedule is algebraically identical to CES isoquants, apart from the difference in the sign determining their concavity.

[6]This state-contingent production function closely resembles the conventional multi-input and single output CES production function. In conventional representation of CES production function, the output produced is expressed as a function of multiple input used in the production process. In the CES type state-contingent production function given by (3) the total input applied to the production process is expressed as a function of state-contingent outputs.

[7]This is the definition given by Chambers and Quiggin (2000)

[8]Exponential utility function allows net returns to be both negative as well as positive.

[9]Also the inputs allocated to each of the two states of nature are unobserved, irrespective of whether the technology is state-allocable or state-general.

[10]First $\ln(q)$ is regressed on state dependant constants and $\ln(x)$ and state dependant constants c_1 and c_2 are estimated along with coefficient (c_3) of $\ln(x)$. Then, a_1 and a_2 are

derived using the transformation $a_1 = \exp(-c_1/c_3)$ and $a_2 = \exp(-c_1/c_3)$ respectively. Finally the standard errors for a_1 and a_2 are computed using delta method.

[11]This result should not come as a surprise because the data was generated using this CES specification and there was no noise added to the data.

[12]In this case the input is a function of the two state-contingent outputs and the functional form is Cobb-Douglas.

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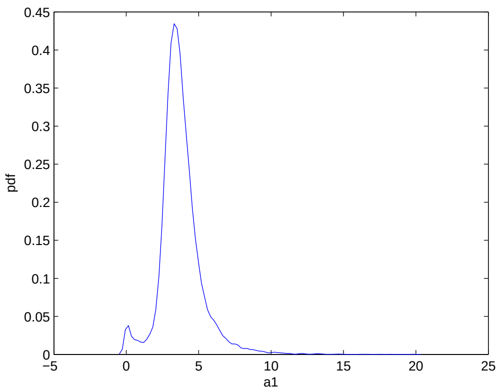
Figures



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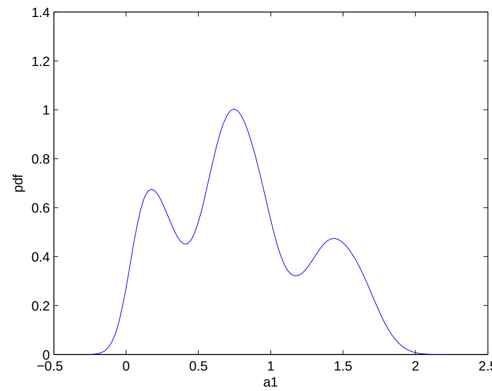

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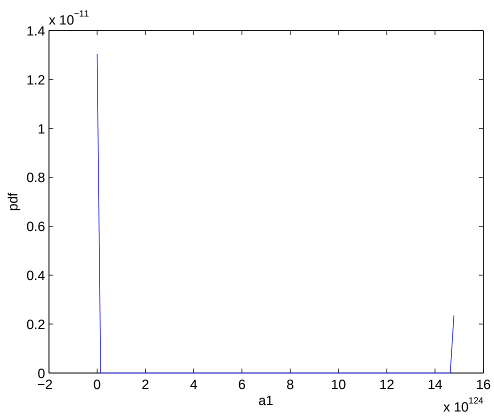

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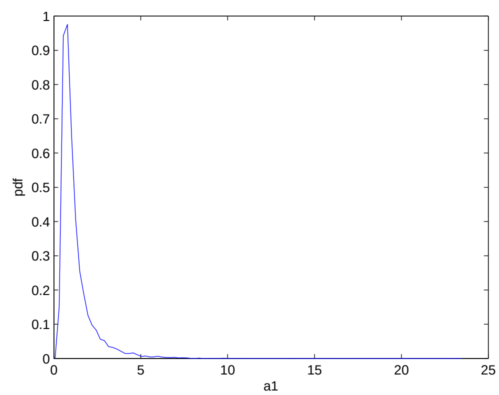

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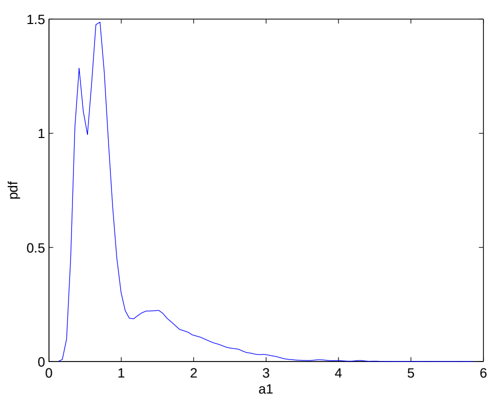
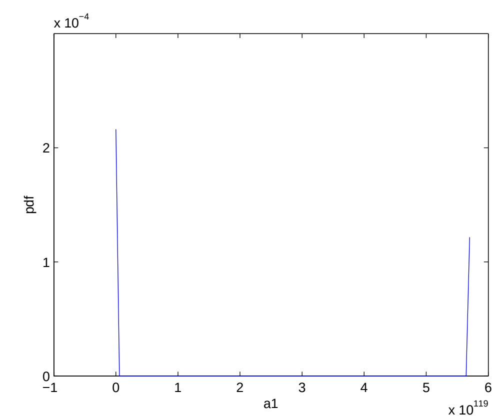
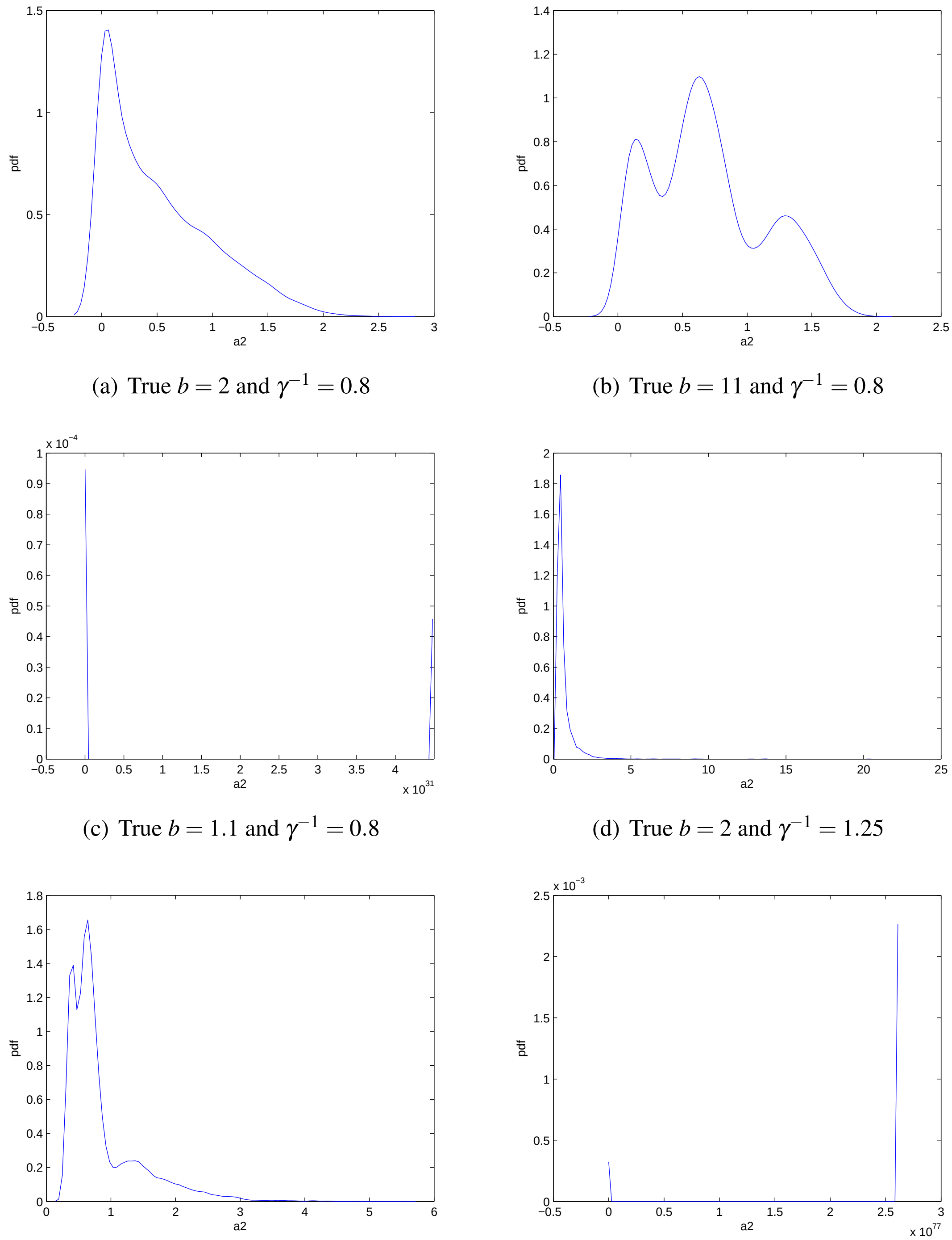


(e) True $b = 11$ and $\gamma^{-1} = 1.25$



(f) True $b = 1.1$ and $\gamma^{-1} = 1.25$

Figure 1: Estimated probability density of technology parameter a_1



(a) True $b = 2$ and $\gamma^{-1} = 0.8$

(b) True $b = 11$ and $\gamma^{-1} = 0.8$

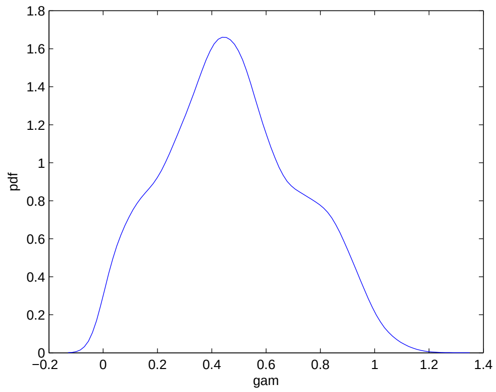
(c) True $b = 1.1$ and $\gamma^{-1} = 0.8$

(d) True $b = 2$ and $\gamma^{-1} = 1.25$

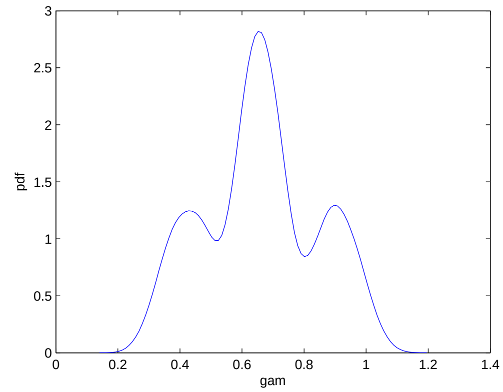
(e) True $b = 11$ and $\gamma^{-1} = 1.25$

(f) True $b = 1.1$ and $\gamma^{-1} = 1.25$

Figure 2: Estimated probability density of technology parameter a_2



(a) True $b = 2$ and $\gamma^{-1} = 0.8$



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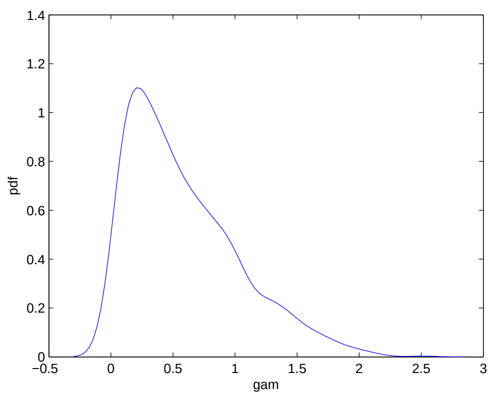
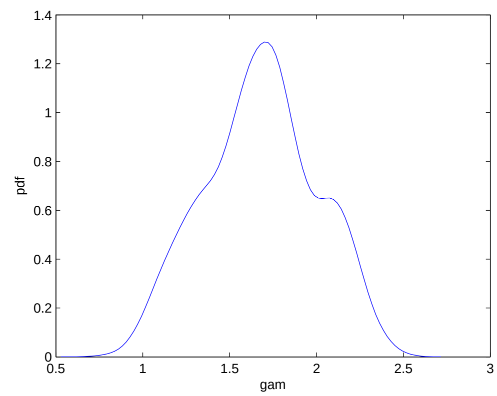
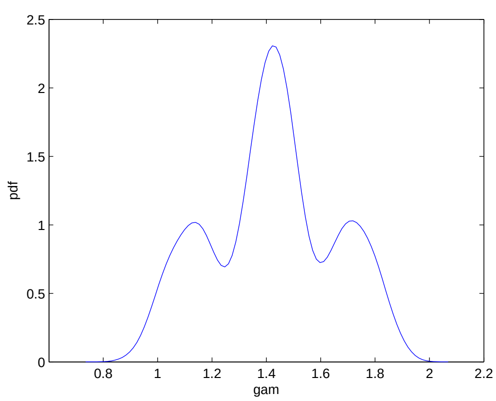


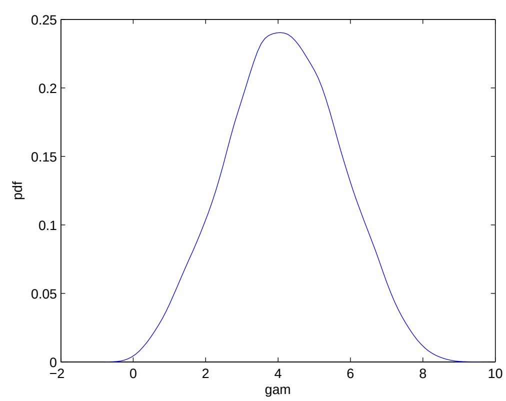
(c) True $b = 1.1$ and $\gamma^{-1} = 0.8$



(d) True $b = 2$ and $\gamma^{-1} = 1.25$



(e) True $b = 11$ and $\gamma^{-1} = 1.25$



(f) True $b = 1.1$ and $\gamma^{-1} = 1.25$

Figure 3: Estimated probability density of elasticity of scale parameter γ^{-1}

Tables

Table 1: Simulated Data:$(a_1, a_2) = (1.5, 0.5), b = 2, \gamma = 1.25, w = 0.5$

Firm	π_1	x	z_1	z_2	s	z_s
1	0.030	50.978	0.339	32.836	2	32.836
2	0.042	47.940	0.457	31.256	1	0.457
3	0.147	27.455	1.144	19.919	1	1.144
4	0.244	15.953	1.371	12.746	1	1.371
5	0.246	15.772	1.373	12.626	2	12.626
6	0.306	11.172	1.389	9.449	2	9.449
7	0.320	10.305	1.384	8.820	1	1.384
8	0.369	7.772	1.347	6.910	2	6.910
9	0.380	7.298	1.336	6.538	1	1.336
10	0.418	5.889	1.292	5.396	1	1.292
11	0.479	4.235	1.215	3.964	2	3.964
12	0.500	3.805	1.189	3.567	2	3.567
13	0.504	3.730	1.184	3.497	1	1.184
14	0.546	3.060	1.139	2.842	2	2.842
15	0.548	3.033	1.137	2.814	2	2.814
16	0.549	3.019	1.136	2.801	2	2.801
17	0.566	2.807	1.121	2.580	1	1.122
18	0.595	2.506	1.101	2.249	1	1.101
19	0.657	2.075	1.086	1.701	2	1.701
20	0.704	1.906	1.106	1.395	2	1.395
21	0.750	1.854	1.159	1.159	2	1.159
22	0.791	1.895	1.238	0.982	1	1.238
23	0.864	2.192	1.476	0.697	1	1.476
24	0.944	2.928	1.919	0.341	1	1.919
25	0.979	3.434	2.189	0.141	2	0.141

Table 2: Parameter Estimates Using OLS and NLS estimators

True Value	OLS Estimates	NLS Estimates
$a_1 = 1.5$	3.9406	1.5000
	(2.6947)	(0.0000)
$a_1 = 0.5$	0.2805	0.5000
	(0.4109)	(0.0000)
$b = 2$		2.0000
		(0.0000)
$\gamma^{-1} = 0.8$	$\alpha = 0.4056$	0.8000
	(0.1887)	(0.0000)

Note: Standard errors are shown in parenthesis

Table 3: Sample statistics for estimated technology parameter a_1

True $(a_1 = 1.5, a_2 = 0.5, \gamma^{-1}, b)$	Mean	Std. Dev	Min	Max
(0.8,2)	3.7537	1.4759	0.0000	19.8475
(0.8,11)	0.7956	0.4679	0.0089	1.9545
(0.8,1.1)	2.1671E+122	5.3210E+123	3.7221	1.4782E+125
(1.25,2)	1.2328	1.0454	0.3126	23.2493
(1.25,11)	0.9229	0.6195	0.2997	5.6889
(1.25,1.1)	5.7097E+115	5.7048E+117	0.2241	5.7000E+119

Table 4: Sample statistics for estimated technology parameter a_2

True $(a_1 = 1.5, a_2 = 0.5, \gamma^{-1}, b)$	Mean	Std. Dev	Min	Max
(0.8,2)	0.5048	0.4815	0.0000	2.5796
(0.8,11)	0.6949	0.4383	0.0085	1.8888
(0.8,1.1)	3.8061E+28	1.3058E+30	0.0000	4.4798E+31
(1.25,2)	0.6444	0.6020	0.1588	20.3998
(1.25,11)	0.8575	0.5792	0.2913	5.5585
(1.25,1.1)	2.6150E+73	2.6128E+75	0.0174	2.6105E+77

Table 5: Sample statistics for estimated elasticity of scale parameter γ^{-1}

True ($a_1 = 1.5, a_2 = 0.5, \gamma^{-1}, b$)	Mean	Std. Dev	Min	Max
(0.8,2)	0.4755	0.2441	0.0000	1.2209
(0.8,11)	0.6600	0.1842	0.2342	1.1015
(0.8,1.1)	0.5857	0.4423	0.0301	2.5072
(1.25,2)	1.6798	0.3250	0.6959	2.5497
(1.25,11)	1.4207	0.2291	0.8500	1.9542
(1.25,1.1)	4.1551	1.5576	0.0418	8.8157