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Abstract

We study a round-robin tournament with n symmetric players where in each of the $n-1$ stages each of the players competes against a different player in the Tullock contest. Each player has a limited budget of effort that decreases within the stages proportionally to the effort he exerted in the previous stages. We show that when the prize for winning (value of winning) is equal between the stages, a player's effort is weakly decreasing over the stages. We also show how the contest designer can influence the players' allocation of effort by changing the distribution of prizes between the stages. In particular, we analyze the distribution of prizes over the stages that balance the effort allocation such that a player exerts the same effort over the different stages. In addition, we analyze the distribution of prizes over the stages that maximizes the players' expected total effort.

1 Introduction

The elimination tournament and the round-robin tournament are two common examples of multi-stage tournaments. In the elimination tournament, teams or individual players play pair-wise matches. The winner advances to the next round while the loser is eliminated from the competition. In the round-robin tournament, on the other hand, every individual player or team competes against all the others where in every stage a player plays a pair-wise match against a different opponent. Sportive events are commonly organized either as elimination tournaments (the ATP tennis tournaments), or as round-robin tournaments

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(professional football and basketball leagues) or as a combination of a round-robin tournament in the first part of the season and then as an elimination tournament in the second part (US-Basketball, NCAA College Basketball, the FIFA (soccer) World Cup Playoffs, the UEFA Champions' League).

The elimination tournament structure has been widely analyzed in the literature on contests. Rosen (1986) studied an elimination tournament with homogeneous players where the probability of winning a match is a stochastic function of the players' efforts. In particular, he analyzed the effect of the allocation of prizes on the players' allocation of effort. Gradstein and Konrad (1999) studied a rent-seeking contest à la Tullock (with homogenous players) and found that simultaneous contests are strictly superior to elimination tournaments if the contest's rules are discriminatory enough (as in an all-pay auction).[1] Groh et al. (2009) studied an elimination tournament with four asymmetric players where players are matched in the all-pay auction in each of the stages. Their analysis indicates that for the Gradstein-Konrad result to hold it is necessary that the multistage contest induces a positive probability that the two strongest players do *not* reach the final with probability one. If the two strongest players do reach the final with a positive probability, the elimination tournament has several advantages over other multi-stage tournaments particularly the simultaneous contest.

In contrast to elimination tournaments, the literature on round-robin tournaments seems to be quite sparse.[2] This paper attempts to fill this gap by studying a round-robin tournament with n symmetric players (teams) where in each of the $n-1$ stages, each of the players competes against a different player in the Tullock contest. Each player has a limited budget of effort that decreases within the stages proportionally to the effort he exerted in the previous stages, where for each effort unit that a player exerts, he loses $\alpha, 0 \leq \alpha \leq 1$ units of effort from his budget.

One of the main questions in multi-stage tournaments particularly in round-robin tournaments, is how the players' effort will be allocated over the different stages of the tournament and how the contest designer can influence this allocation. The contest theory literature offers different opinions about whether or not

[1] Harbaugh and Klumpp (2005) studied an elimination tournament with two types of players (strong and weak) where players are matched in the Tullock contest in every stage, but there is a resource constraint for the total effort.

[2] Ryvkin and Ortman (2008) studied a model of a noisy tournament by which they compared the predictive power of the simultaneous, elimination and round-robin tournaments.

players strategically allocate their effort in multi-stage tournaments. Ferrall and Smith (1999) used data from professional sport leagues in the US to show that teams do not strategically allocate their effort but actually exert as much effort as possible in each of the stages. On the other hand, Amegashie et al. (2007) as well as Matros (2006) showed that if players have fixed equal resources they spend more resources in the initial rounds than in the following ones. Likewise, Harbaugh and Klump (2005) showed in a two-stage tournament that weak players exert more effort in the first stage (semi-final) whereas strong players save more effort for the second stage (final). In our round-robin tournament we also find that a player strategically allocates his effort. In particular, when the prize for winning (value of winning) is equal between the stages, a player's effort is weakly decreasing over the stages, while if the value of α is sufficiently high (high fatigue) a player exerts the same level of effort in the first stages and from some stage on his effort decreases over the stages. Moreover, for $\alpha \geq 0.5$, independent of the number of players (stages), the players exert the same effort over the first $n-2$ stages but exert a smaller effort in the last stage. We also show that the smaller the value of α (fatigue), the smaller is the number of first stages in which the players exert the same effort. The intuition for this is clear since when fatigue is smaller a player has less incentive to save effort for the next stages and then he exerts the highest possible effort according to his budget of effort in every stage of the tournament.[3]

We further show that the players' allocation of effort can be influenced by the contest designer if he changes the distribution of prizes among the stages. If the contest designer wishes to balance the effort allocation such that a player will exert the same effort over all the stages, he should award the same prize for all the stages except the final stage in which the designer should award a higher prize. The intuition for this result is that by awarding a large prize in the last stage, the players have incentive to save effort in the previous stages for the competition in the last stage since winning in the last stage is very profitable. An interesting point is that in our model there is only one allocation of prizes that yields a balanced effort allocation. This allocation of prizes in fact is different than Rosen's well-known result (1986) according to which the rewards in later stages must be higher than the rewards in earlier stages in order to sustain a

[3] Ryvkin (2009) also studied the phenomenon of fatigue but in a different multi-stage tournament known as the best-of k contest where he models fatigue as a reduction in a player's probability of winning resulting from previous efforts. He found that agents are more likely to choose higher efforts in the later stages of competition which is exactly opposite to our findings.

non-decreasing effort along the elimination tournament.[4]

The designer's goal, however, may not necessarily be to balance the players' effort over the different stages of multi-stage tournaments. In many cases it can be to maximize the expected total effort. The optimal allocation of prizes that maximizes the total effort exerted by the players has been studied in various multi-stage tournaments. Fu and Lu (2009), for example, studied a multi-stage sequential elimination Tullock contest, and showed that in the optimal contest, a designer who wishes to maximize the players' total effort should eliminate one contestant at each stage until the final, and the winner of the final takes the entire prize sum. Moldovanu and Sela (2006) studied an elimination two-stage all-pay auction under incomplete information and showed that it is optimal for a designer who wishes to maximize the expected total effort to allocate the entire prize sum to the winner in the second (final) stage of the tournament. In our round-robin tournament, however, we show that if the designer wishes to maximize the expected total effort over all the stages he should award a series of prizes that decreases over the $n-2$ stages, with the prize in the last stage being either smaller or larger than the previous prizes. The reason for this result is that the highest total effort is obtained when each player exerts in each stage an effort that is equal to his budget of effort in that stage such that he does not save effort for the next stages. Since the rate of saving effort is decreasing in stages, the optimal incentive to prevent saving of effort should be decreasing as well and therefore the values of the prizes are decreasing over all the stages except the last one.

The model most related to our round-robin tournament is the Colonel Blotto game (see, for example, Roberson 2006 and Kvasov 2007) in which two players simultaneously distribute forces across n battlefields. The payoff of the Colonel Blotto game is the proportion of wins on the individual battlefields, while within each battlefield, the player who allocates the higher level of force wins.[5] In fact, our model without a variable budget of effort, i.e., when the budget of effort is not reduced over the stages, is precisely equivalent to the Colonel Blotto game. However, the variability of the players' budgets of effort plays a key role in the round-robin tournament we consider so that it is more complex than the Colonel Blotto game.

[4]Other works on allocation of resources in sequential contests include Konrad (2004), Warneryd (1998) and Klumpp and Polborn (2006).

[5]Robson (2005) studied the Colonel Blotto Game where in each battlefield there is a Tullock contest, and Snyder (1989) examined a related game where in each battlefield there is a contest of the type employed by Rosen (1986).

The rest of the paper is organized as follows: Section 2 introduces our round-robin tournament. Section 3 analyzes the players' effort distribution over its stages. Sections 4 studies the allocation of prizes that yields a balanced distribution of effort over the stages and Section 5 studies the optimal allocation of prizes that yields the highest total effort. Section 6 concludes.

2 The model

Consider a round-robin tournament with n players (teams). The players compete against each other such that in every stage $j, 1 \leq j \leq n-1$, a player competes against a different opponent in a Tullock contest for a prize equal to p_j. Thus, if players i and k compete against each other in stage j, player i exerts an effort x_j^i while player k exerts an effort x_j^k, then player i's expected utility in stage j is $u_j^i = p_j \frac{x_j^i}{x_j^i + x_j{}^k}$ and player k's expected utility is $u_j^k = p_j \frac{x_j^k}{x_j^i + x_j{}^k}$. Each player has a budget of v units of effort to allocate across the $n-1$ stages, with the budget of effort being reduced in the stages where for each effort unit that the player exerts he loses α units of effort from his budget, that is, $v_{j+1}^i = v_j^i - \alpha x_j^i$, $0 \leq \alpha \leq 1$ where v_j^i is player i's budget of effort in stage j. The player's effort in each stage is smaller or equal to his budget of effort in that stage. The only cost of effort for a player is the opportunity cost of having less effort for the other stages. The contest designer may influence the players' allocation of effort by awarding different prizes for different stages, while each player wishes to maximize the sum of the prizes over the stages.

3 The allocation of effort

We assume first that the prize for winning (value for winning) a match is the same over all the stages and that this value of the prize is normalized to be 1. It is important to note that in our round-robin tournament, the relations between the prizes in the different stages affect the players' allocation of effort. In other words, if we multiply the values of all the prizes by the same constant, the players' allocation of effort will not be changed. Thus, if a prize is equal to 1 in all the stages, each player will want to maximize the number of wins. Then, the maximization problem of player $i, i = 1, ..., n$ is:

$$
\begin{aligned}
& Max_{x_1^i,\ldots,x_{n-1}^i} \sum_{j=1}^{n-1} \frac{x_j^i}{x_j^i + \widetilde{x}_j^i} \qquad (1)\\
& s.t. \\
x_j^i \quad \leq \quad & v - \alpha \sum_{m=1}^{j-1} x_m^i \quad , \quad j = 1, ..., n-1
\end{aligned}
$$

where $\widetilde{x}_j^i$ is the effort of the opponent of player i in stage j. Note that since there is no alternative value for the player's effort in the last stage, each player in this stage (stage $n-1$) exerts an effort that is equal to his budget of effort. The solution of this maximization problem yields the players' allocation of effort.

Proposition 1 *In the round-robin tournament, a player's equilibrium efforts are as follows:*

1) For $\alpha < \frac{1}{n-1}$,

$$x_j = v(1-\alpha)^{j-1} \quad , \quad j \leq 1, ..., n-1.$$

such that in every stage j*, a player exerts an effort that is equal to his budget of effort in that stage.*

2) For $\frac{1}{n-k+1} \leq \alpha < \frac{1}{n-k}, k = 2, 3, ..., n-1$,

$$
\begin{aligned}
x_j &= \frac{v}{(n-1)\alpha} \quad , \quad j = 1, ..., k-1 \\
x_j &= \frac{v(n-k)(1-\alpha)^{j-k}}{n-1} \quad , j = k, ..., n-1
\end{aligned}
$$

Proof. See Appendix. ■

The following example illustrates the equilibrium strategies given by Proposition 1 for the case of four players.

Example 1 *In the round-robin tournament with four players the players' equilibrium efforts are as follows:*

	x_1	x_2	x_3
$\alpha < \frac{1}{3}$	v	$v(1-\alpha)$	$v(1-\alpha)^2$
$\frac{1}{3} < \alpha \leq \frac{1}{2}$	$\frac{v}{3\alpha}$	$\frac{2v}{3}$	$\frac{2v(1-\alpha)}{3}$
$\alpha > \frac{1}{2}$	$\frac{v}{3\alpha}$	$\frac{v}{3\alpha}$	$\frac{v}{3}$

In the above figure we can see that for every value of $\alpha, x_1(\alpha) \geq x_2(\alpha) \geq x_3(\alpha)$*, namely, a player's effort decreases during the stages. We can also see that a player's effort in every stage* $x_j(\alpha), j = 1, 2, 3$ *is non-increasing in* α.

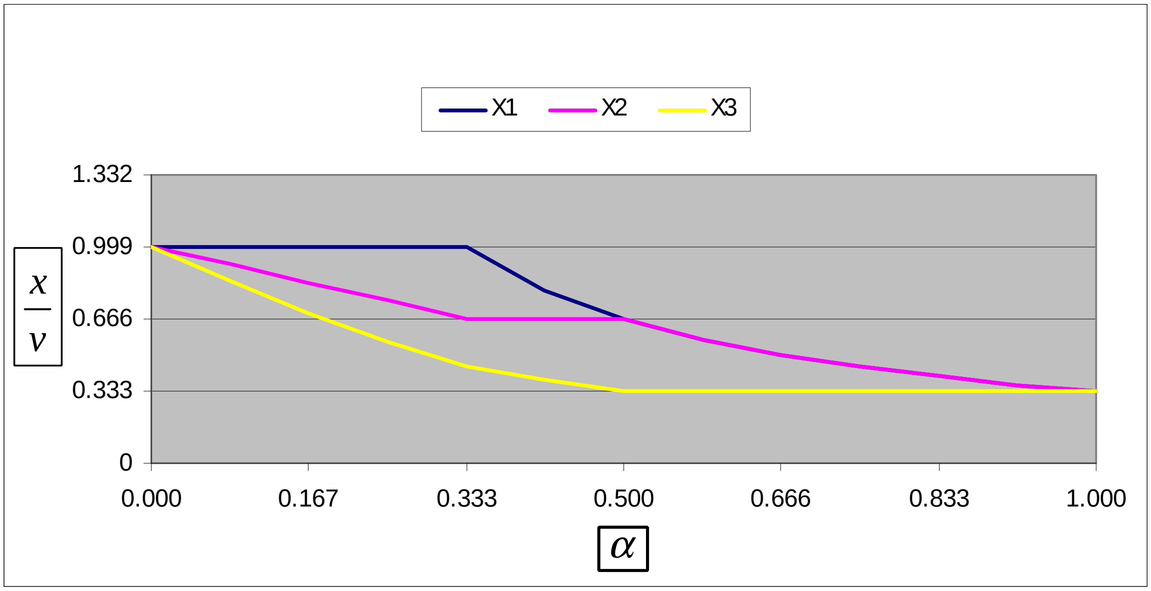

Figure 1: The allocation of effort in the tournament with four players

By Proposition 1, as well as in the above example, the equilibrium strategy forms a non-increasing sequence of effort $x_j \geq x_{j+1}$, $j = 1, ..., n-1$ over all the stages. In addition, for every level of α there is a critical stage $j^*(\alpha)$ such that in all the stages $j \geq j^*(\alpha)$ a player exerts an effort equal to his budget of effort in that stage, while in all the previous stages $j < j^*$, a player exerts the same effort which is smaller than his budget of effort in that stage.[6]

A player's total effort in the round-robin tournament with n players is given for every $\frac{1}{n-k+1} \leq \alpha < \frac{1}{n-k}, k = 2, 3, ..., n-1$ by

$$E_n = v\frac{k-1}{(n-1)\alpha} + \sum_{j=k}^{n-1} v\frac{(n-k)(1-\alpha)^{j-k}}{n-1} = v\frac{(n-1)-(n-k)(1-\alpha)^{n-k}}{(n-1)\alpha} \quad (2)$$

In our round-robin tournament with $n+1$ players for the same α (note that k is now larger by 1 than in the round-robin tournament with n players) the total effort of a player is

$$E_{n+1} = v\frac{n-(n-k)(1-\alpha)^{n-k}}{n\alpha}$$

[6] Similar results are obtained by Che and Gale (1997) in one-stage Tullock contests where each player has a different budget constraint.

The difference of a player's total effort in both tournaments (with n and $n-1$ players) is

$$E_{n+1} - E_n = v\frac{(n-k)(1-\alpha)^{n-k}}{n(n-1)\alpha} > 0$$

Hence, a player's total effort increases in the number of players.

4 Balance of effort

In the previous section, we showed that a player exerts the same effort in the first stages of the round-robin tournament but he decreases his effort in the last stages. However, the designer of the tournament may wish to balance the players' effort in the tournament such that the players' effort will be the same over all the stages. Rosen (1986) showed in a different form of an elimination tournament that the designer should award a series of increasing rewards in order to sustain a non-decreasing effort along the tournament. In contrast, we show that in our round-robin tournament, in order to balance the players' effort over all the stages of the tournament, the prize should be the same for all the stages except for the last one, which should be larger than all the others.

If a player's effort is the same over all the stages, we have

$$x = x_{n-1} = v - \alpha\sum_{j=1}^{n-2} x_j = v - \alpha(n-2)x$$

Thus, the level of the effort in each stage is

$$x = \frac{v}{1+\alpha(n-2)} \tag{3}$$

In order to balance the players' effort for all the stages, the contest designer should award a prize in the last stage that is sufficiently larger than the prizes in the other stages. Then, winning in the last stage becomes very profitable for all the players and therefore they save effort for the last stage such that they do not completely use their budgets of effort in the first $n-2$ stages. Consequently, if the contest designer awards an identical prize normalized to 1 for each of the first $n-2$ stages and a prize p_{n-1} for winning in the last stage, the maximization problem of player $i, 1 \leq i \leq n$, is:

$$\begin{aligned} & Max_{x_1^i,...,x_{n-1}^i} \sum_{j=1}^{n-2} \frac{x_j^i}{x_j^i + \widetilde{x}_j^i} + p_{n-1} \frac{x_{n-1}^i}{x_{n-1}^i + \widetilde{x}_{n-1}^i} \\ & s.t. \\ x_j^i \quad \leq \quad & v - \alpha \sum_{m=1}^{j-1} x_m^i \quad , \quad j = 1, ..., n-1 \end{aligned} \tag{4}$$

where $\widetilde{x}_j^i$ is the effort of the opponent of player i in stage j. The solution of the above maximization problem can be given by (3) for all the stages if the prize allocation is as follows.

Proposition 2 *In the round-robin tournament, if the prize for winning each of the first $n-2$ stages is equal to 1 and the prize for winning the last stage is equal to $\frac{1}{\alpha}$, each player exerts the same effort over all the stages.*

Proof. See Appendix. ■

It is of interest to examine the relation between balanced effort and total effort. The following result demonstrates that the total effort cannot be maximized under the constraint of balanced effort.

Proposition 3 *In the round-robin tournament if the players' effort is the same over all the stages then the players' total effort is smaller than in the round-robin tournament when the prize for winning is the same over all the stages.*

Proof. See Appendix. ■

Proposition 3 invites the question of which allocation of prizes yields the highest total effort, which will be the focus of the next section.

5 Maximization of total effort

In this section we assume that the contest designer wishes to maximize the players' expected total effort. Then, the designer's maximization problem is:

$$\begin{aligned} & Max_{x_1,...,x_{n-1}} \sum_{j=1}^{n-1} x_j \\ & s.t. \\ x_j \quad \leq \quad & v - \alpha \sum_{m=1}^{j-1} x_m \quad , \quad j = 1, ..., n-1 \end{aligned}$$

where x_j is a player's effort in stage j. The designer can influence the players' allocations of effort by awarding different prizes for winning in the different stages. For stage $j,, j = 1, ..., n-1$, let p_j be the prize awarded by the designer for winning. Then, player i's maximization problem is:

$$\begin{aligned} & Max_{x_1^i,...,x_{n-1}^i} \sum_{j=1}^{n-1} \frac{p_j x_j^i}{x_j^i + \widetilde{x}_j^i} \\ & s.t. \\ x_j^i \quad \leq \quad & v - \alpha \sum_{m=1}^{j-1} x_m^i \quad , \quad j = 1, ..., n-1 \end{aligned}$$

The following result provides the minimal values of the prizes for winning that encourage the players to exert the highest possible effort in every stage of the tournament.

Proposition 4 *In the round-robin tournament, players will exert the highest total effort if the prizes for winning satisfy*

$$\begin{aligned} p_{n-1} \quad &= \quad 1 \\ p_j \quad &\geq \quad \frac{\alpha}{(1-\alpha)^{n-j-1}} \ , \ j = 1, ..., n-2 \end{aligned}$$

Proof. See Appendix. ■

By Proposition 4, the sequence of the prizes for winning in the first $n-2$ stages is decreasing such that $p_j = \frac{p_{j+1}}{1-\alpha}$,$1 \leq j \leq n-2$. However, the value of the prize in the last stage might be larger or smaller than the other prizes and this relation depends on the parameter α. For sufficiently small values of α (low fatigue), all the prizes for winning in the $n-2$ first stages will be smaller than the prize in the last stage, and for sufficiently large values of α (high fatigue) all the prizes in the first $n-2$ stages will be larger than the prize in the last stage.

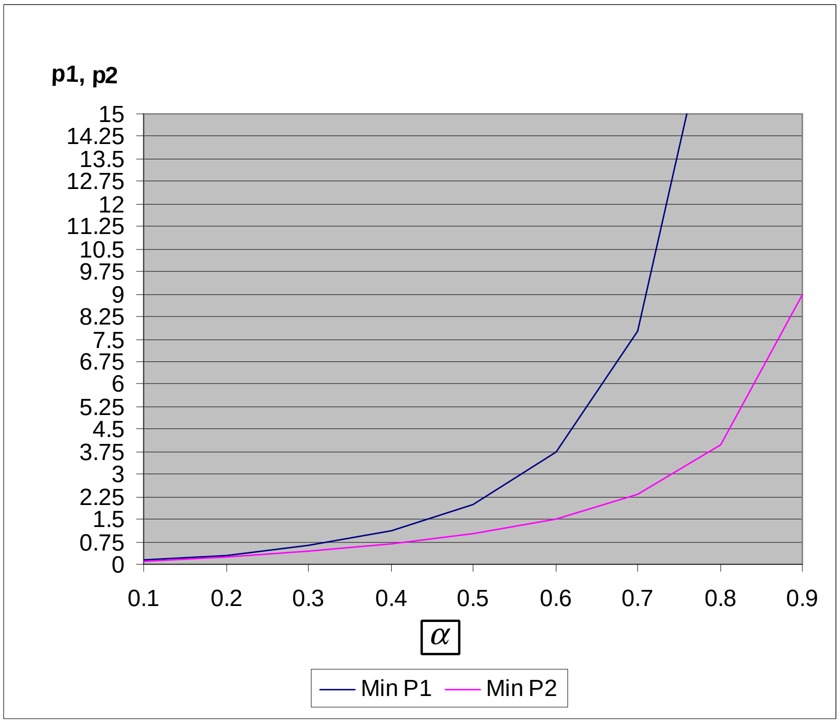

Figure 2: The optimal values of the prizes in the tournament with four players

Example 2 *In the round-robin tournament with four players in order to obtain the highest total effort the prizes for winning should satisfy:*

$$\begin{aligned} p_3 &= 1 \\ p_2 &\geq \frac{\alpha}{1-\alpha} \\ p_1 &\geq \frac{\alpha(1+p_2)}{1-\alpha} \end{aligned}$$

It can be easily verified that $p_1 \geq p_2$*. Moreover, by Figure 2 we can see that the difference between these prizes* $p_1 - p_2$ *increases in* α*, and both prizes* p_1 *and* p_2 *are smaller than* p_3 *for small values of* α *but are larger than* p_3 *for high values.*

We can see here the conflict between the two different goals of the contest designer: on the one hand, he wishes to balance the players' effort, but on the other, he wishes to maximize their effort. By Proposition 4, in order to maximize the total effort, the minimal value of all the prizes in the first $n-2$ stages should be $\frac{\alpha}{(1-\alpha)^{n-2}}$, where the value of the prize in the last stage is equal to 1. Then we obtain that the the ratio between the last stage's prize and the other prizes is $\frac{1}{\frac{\alpha}{(1-\alpha)^{n-2}}} = \frac{(1-\alpha)^{n-2}}{\alpha} < \frac{1}{\alpha}$, which is not enough to balance the player's effort over all the stages of the tournament.

6 Concluding remarks

We studied a round-robin tournament with n symmetric players where in each of the $n-1$ stages, a player competes against a different opponent in the Tullock contest. We showed that by awarding an identical prize for winning in all the stages except the last one in which a larger prize is awarded, the contest designer can balance the players' effort allocation, such that each player's effort will be the same over all the stages. We also showed that by allocating a series of decreasing prizes for winning over the first $n-2$ stages, the contest designer can maximize the players' expected total effort. These results were obtained for symmetric players who have the same values of winning in the round-robin tournament. The generalization of these results to an asymmetric round-robin tournament where players have different values of winning could be interesting, but the analysis would be extremely difficult to carry out.

7 Appendix

7.1 Proof of Proposition 1

The maximization problem of player $i, i = 1, ..., n$ is:

$$\begin{aligned} & Max_{x_1^i,\dots,x_{n-1}^i} \sum_{j=1}^{n-1} \frac{x_j^i}{x_j^i + \widetilde{x}_j^i} \qquad (5) \\ & s.t. \\ x_j^i \quad \leq \quad & v - \alpha \sum_{m=1}^{j-1} x_m^i \quad , \quad j = 1, ..., n-1 \end{aligned}$$

where $\widetilde{x}_j^i$ is the effort of the opponent of player i in stage j. Let $\frac{1}{n-k+1} \leq \alpha < \frac{1}{n-k}, k = 2, 3, ..., n-1$, and assume that the last $n-k$ constraints are binding while the first $k-1$ constraints are not (this assumption will be confirmed in the following). Then, the Lagrangian is given by

$$L = \sum_{j=1}^{n-1} \frac{x_j^i}{x_j^i + \widetilde{x}_j^i} - \sum_{j=k}^{n-1} \lambda_j (x_j^i - v + \alpha \sum_{m=1}^{j-1} x_m^i)$$

where $\lambda_j, j = k, ..., n-1$ are the Lagrange multipliers. The first-order conditions are:

$$\begin{aligned} \frac{dL}{dx_j^i} &= \frac{\widetilde{x}_j^i}{(x_j^i + \widetilde{x}_j^i)^2} - \alpha \sum_{m=k}^{n-1} \lambda_m = 0 \quad , \quad j = 1, ..., k-1 \\ \frac{dL}{dx_j^i} &= \frac{\widetilde{x}_j^i}{(x_j^i + \widetilde{x}_j^i)^2} - \lambda_j - \alpha \sum_{m=j+1}^{n-1} \lambda_m = 0 \ , \ \ j = k, ..., n-1 \\ \frac{dL}{d\lambda_j} &= x_j^i - v + \alpha \sum_{m=1}^{j-1} x_m^i = 0 \quad , \quad j = k, ..., n-1 \end{aligned} \tag{6}$$

If the first $k-1$ constraints are not binding, then player i's effort in each of the first $k-1$ stages is given by

$$x_j^i = \frac{v}{(n-1)\alpha} \quad , \ \ j = 1, ..., k-1 \tag{7}$$

Then by (7), the first constraint in the maximization problem (5) $\frac{v}{(n-1)\alpha} < v$ is satisfied iff $\alpha > \frac{1}{n-1}$ and the second constraint $\frac{v}{(n-1)\alpha} < v - \alpha\frac{v}{(n-1)\alpha} = \frac{v(n-2)\alpha}{(n-1)\alpha}$ is satisfied iff $\alpha > \frac{1}{n-2}$. Similarly the constraint $j, 3 \leq j \leq k-1$, $\frac{v}{(n-1)\alpha} < v - (j-1)\alpha\frac{v}{(n-1)\alpha}$ is satisfied iff $\alpha > \frac{1}{n-j}$. Thus, our assumption that $\frac{1}{n-k+1} \leq \alpha < \frac{1}{n-k}$ is a necessary and sufficient condition that exactly the first $k-1$ constraints are not binding.

If the last $n-k$ constraints are binding, then by (7) we obtain that player i's efforts in the last $n-k$ stages are given by

$$x_j^i = v - \alpha \sum_{i=1}^{j-1} x_i = \frac{(n-k)v(1-\alpha)^{j-k}}{n-1} \quad j = k, ..., n-1$$

Using the first order conditions (6) we can see that the last $n-k$ constraints are binding since for all $j = k, ..., n-1$

$$\begin{aligned} \lambda_j &= \frac{\widetilde{x}_j^i}{(x_j^i + \widetilde{x}_j^i)^2} - \alpha \sum_{m=j+1}^{n-1} \lambda_m \geq \frac{\widetilde{x}_j^i}{(x_j^i + \widetilde{x}_j^i)^2} - \alpha \sum_{m=k}^{n-1} \lambda_m \\ &= \frac{\widetilde{x}_j^i}{(x_j^i + \widetilde{x}_j^i)^2} - \frac{\widetilde{x}_1^i}{(x_1^i + \widetilde{x}_1^i)^2} = \frac{1}{4x_j^i} - \frac{1}{4x_1^i} > 0 \end{aligned}$$

In the above calculation of λ_j , $j = k, ..., n-1$, we used the symmetry of the players, namely, $x_j^i = \widetilde{x}_j^i$ and the relations between the player's efforts over all the stages, namely, $x_1^i = x_2^i =, ..., = x_{k-1}^i > x_j^i$ for all $j = k, ..., n-1$.

Similarly to the above analysis, if $\alpha \leq \frac{1}{n-1}$ we obtain that all the $n-1$ constraints in the maximization problem (5) are binding, and the solution of player i's maximization problem is:

$$x_j^i = v - \alpha \sum_{n=1}^{j-1} x_m^i = v(1-\alpha)^{j-1} \quad 1 \le j \le n-1.$$

Q.E.D.

7.2 Proof of Proposition 2

Player i's maximization problem is:

$$\begin{aligned} & Max_{x_1^i,...,x_{n-1}^i} \sum_{j=1}^{n-2} \frac{x_j^i}{x_j^i + \widetilde{x}_j^i} + p_{n-1} \frac{x_{n-1}^i}{x_{n-1}^i + \widetilde{x}_{n-1}^i} \\ & s.t. \\ x_j^i \quad \le \quad & v - \alpha \sum_{m=1}^{j-1} x_m^i \quad , \quad j = 1, ..., n-1 \end{aligned} \tag{8}$$

where $\widetilde{x}_j^i$ is the effort of the opponent of player i in stage j. We assume that all the first $n-2$ constraints are not binding but the last constraint is binding (this assumption will be confirmed in the following). Then, the Lagrangian is given by

$$L = \sum_{j=1}^{n-2} \frac{x_j^i}{x_j^i + \widetilde{x}_j^i} + p_{n-1} \frac{x_{n-1}^i}{x_{n-1}^i + \widetilde{x}_{n-1}^i} - \lambda_{n-1}(x_{n-1}^i - v + \alpha \sum_{m=1}^{n-2} x_m^i)$$

where λ_{n-1} is the Lagrange multiplier. The first-order conditions are:

$$\begin{aligned} \frac{dL}{dx_j^i} &= \frac{\widetilde{x}_j^i}{(x_j^i + \widetilde{x}_j^i)^2} - \lambda_{n-1}\alpha = 0 \ , \ \ j = 1, ..., n-2 \\ \frac{dL}{dx_{n-1}^i} &= p_{n-1} \frac{\widetilde{x}_{n-1}^i}{(x_{n-1}^i + \widetilde{x}_{n-1}^i)^2} - \lambda_{n-1} = 0 \end{aligned} \tag{9}$$

From the comparison of the first $n-2$ first-order conditions and the symmetry of the players $x_j^i = \widetilde{x}_j^i$, $j = 1, ..., n-2$ we have

$$x_1^i = x_2^i = ... = x_{n-2}^i$$

If we divide the first first-order condition by the last one we obtain

$$\frac{x_{n-1}^i}{p_{n-1} x_1^i} = \alpha$$

If $x_{n-1} = x_1$, the value of winning in the last period is equal to

$$p_{n-1} = \frac{1}{\alpha}$$

By the last-order condition given by (9) we obtain that

$$x = x^i_{n-1} = v - \alpha \sum_{j=1}^{n-2} x^i_j = v - \alpha(n-2)x$$

Thus, the level of the effort in each stage is given by

$$x = \frac{v}{1+\alpha(n-2)}$$

In order to show that all the first $n-2$ constraints in the maximization problem (8) are not binding, it is sufficient to show that

$$x^i_{n-2} = \frac{v}{1+\alpha(n-2)} \leq v - \alpha \sum_{m=1}^{n-3} x^i_m = \frac{v(\alpha+1)}{1+\alpha(n-2)}$$

Thus, all the first $n-2$ constraints are not binding for all $\alpha > 0$ and since

$$\lambda_{n-1} = \frac{p_{n-1}}{4x} = \frac{1+\alpha(n-2)}{4\alpha v} > 0$$

the last constraint is binding accordingly to our assumption. $Q.E.D.$

7.3 Proof of Proposition 3

By (2), each player's total effort in a tournament with equal prizes over the stages (each prize is normalized to be 1) is

$$E_p(k) = v\frac{(n-1)-(n-k)(1-\alpha)^{n-k}}{(n-1)\alpha}$$

where $\frac{1}{n-k+1} \leq \alpha < \frac{1}{n-k}$, $k = 2, ..., n-1$ (for $k = 1$, $0 \leq \alpha < \frac{1}{n-1}$).

By (3), each player's total effort in a tournament with equal efforts over the stages (the prizes are $p_j = 1, j = 1..., n-2$ and $p_{n-1} = \frac{1}{\alpha}$) is

$$E_e = v\frac{n-1}{1+\alpha(n-2)}$$

For $k = n-1$ we have

$$E_p(k) - E_e = v(\frac{n-1-(1-\alpha)}{(n-1)\alpha} - \frac{n-1}{1+\alpha(n-2)}) = v\frac{(n-2)(1-\alpha)^2}{(n-1)\alpha(1+\alpha(n-2))} > 0$$

Suppose that $E_p(k) - E_e > 0$ for all $n-1 \geq k \geq \widetilde{k}$. We will show by induction that this inequality holds for

$\widetilde{k}-1$. The difference between a player's efforts for $k=\widetilde{k}-1$ is

$$\begin{aligned} E_p(\widetilde{k}-1)-E_e &= v(\frac{(n-1)-(n-\widetilde{k}+1)(1-\alpha)^{n-\widetilde{k}+1}}{(n-1)\alpha}-\frac{n-1}{1+\alpha(n-2)}) \\ &= v(\frac{(n-1)-(n-\widetilde{k})(1-\alpha)^{n-\widetilde{k}}}{(n-1)\alpha}-\frac{n-1}{1+\alpha(n-2)})+\frac{(1-\alpha)(\alpha(n-\widetilde{k})+\alpha-1)}{(n-1)\alpha} \end{aligned}$$

By the induction assumption, we need to show that

$$\frac{(1-\alpha)(\alpha(n-\widetilde{k})+\alpha-1)}{(n-1)\alpha}\geq 0$$

Thus, it is sufficient to show that

$$\alpha(n-\widetilde{k}+1)-1\geq 0$$

Since by definition $\frac{1}{n-k+1}\leq\alpha<\frac{1}{n-k}$ we obtain that the last inequality holds. *Q.E.D.*

7.4 Proof of Proposition 4

We first show that the maximal effort of each player is obtained when the highest possible effort is exerted in every stage, namely, each player exerts an effort in every stage that is equal to his budget of effort in that stage. Then, the designer's maximization problem is:

$$\begin{aligned} & Max_{x_1,\ldots,x_{n-1}}\sum_{j=1}^{n-1}x_j \\ & s.t. \\ x_j &\leq v-\alpha\sum_{m=1}^{j-1}x_m \quad , \quad j=1,...,n-1 \end{aligned}$$

We assume that all the constraints are binding (this assumption will be confirmed in the following). The Lagrangian is given by

$$L_1=\sum_{j=1}^{n-1}x_j-\delta_1(x_1-v)-\sum_{j=2}^{n-1}\delta_j(x_j-v+\alpha\sum_{m=1}^{j-1}x_m)$$

where $\delta_j, j=1,...,n-1$ are the Lagrange multipliers. The first-order conditions are:

$$\begin{aligned} \frac{dL_1}{dx_j} &= 1-\alpha-\delta_j-\alpha\sum_{m=j+1}^{n-1}\delta_m=0 \ , \ j=1,...,n-1 \\ \frac{dL_1}{d\delta_j} &= x_j-v+\alpha\sum_{m=1}^{j-1}x_m=0 \quad , \quad j=1,....,n-1 \end{aligned}$$

If all the constraints are binding, the solution of this system of equations is

$$\begin{aligned}x_j &= v(1-\alpha)^{j-1} \ , \ \ j=1,...,n-1 \\ \delta_j &= (1-\alpha)^{n-j} \ \ , \ j=1,...,n-1\end{aligned}$$

Since $\delta_j > 0$ for all $j = 1, ..., n-1$ we obtain the desired result.

Next, we will find the value of prizes such that each player will exert in every stage an effort that is equal to his budget of effort in that stage. Player i's maximization problem is:

$$\begin{aligned} & Max_{x_1^i,...,x_{n-1}^i} \sum_{j=1}^{n-1} \frac{p_j x_j^i}{x_j^i + \widetilde{x}_j^i} \qquad (10) \\ & s.t. \\ x_j^i &\leq v - \alpha \sum_{m=1}^{j-1} x_m^i \ \ , \ \ j=1,...,n-1 \end{aligned}$$

The Lagrangian is given by

$$L_2 = \sum_{j=1}^{n-1} \frac{p_j x_j^i}{x_j^i + \widetilde{x}_j^i} - \lambda_1(x_1 - v) - \sum_{j=2}^{n-1} \lambda_j (x_j - v + \alpha \sum_{m=1}^{j-1} x_m)$$

The first-order conditions are:

$$\begin{aligned}\frac{dL_2}{dx_j^i} &= \frac{p_j \widetilde{x}_j^i}{(x_j^i + \widetilde{x}_j^i)^2} - \lambda_j - \alpha \sum_{m=j+1}^{n-1} \lambda_m = 0 \ \ , \ \ j=1,...,n-1 \qquad (11) \\ \frac{dL_2}{d\lambda_j} &= x_j^i - v + \alpha \sum_{m=1}^{j-1} x_m^i = 0 \ \ , \ \ \ j=1,....,n-1\end{aligned}$$

If all the constraints are binding, player i's efforts over all the stages are given by:

$$x_j^i = v - \alpha \sum_{m=1}^{j-1} x_m^i = v(1-\alpha)^{j-1} \ \ , \ \ j=1,...,n-1 \qquad (12)$$

By (11) and (12) we obtain that

$$\lambda_{n-1} = \frac{1}{4v(1-\alpha)^{n-2}} > 0$$

and by induction we obtain that if for all $n-2 \leq j < 1$

$$\begin{aligned}\lambda_j &= \frac{p_j}{4v(1-\alpha)^{j-1}} - \frac{\alpha}{4v(1-\alpha)^{n-2}} = \frac{p_j(1-\alpha)^{n-j-1} - \alpha}{4v(1-\alpha)^{n-2}} \\ p_j &= \frac{\alpha}{(1-\alpha)^{n-j-1}}\end{aligned}$$

then

$$\lambda_{j+1} = \frac{p_{j+1}(1-\alpha)^{n-(j+1)-1} - \alpha}{4v(1-\alpha)^{n-2}}$$

where

$$\lambda_{j+1} \geq 0 \text{ iff } p_{j+1} \geq \frac{\alpha}{(1-\alpha)^{n-(j+1)-1}}$$

Hence, if the prizes satisfy

$$\begin{aligned} p_{n-1} &= 1 \\ p_j &= \frac{\alpha}{(1-\alpha)^{n-j-1}} \text{ , } j = 1, ..., n-2 \end{aligned}$$

all the constraints in the maximization problem (10) are binding. *Q.E.D.*

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