The University of Adelaide
School of Economics

Research Paper No. 2010-29
November 2010

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November 16, 2010

Abstract

We study the problem of allocating objects when monetary transfers are possible. We are interested in mechanisms that allocate the objects in an efficient way and induce the agents to report their true preferences. Within the class of such mechanisms, first we characterize *egalitarian-equivalent* mechanisms. Then, we add a bounded-deficit condition and characterize the corresponding class. Finally, we investigate the relations between *egalitarian-equivalence* and other fairness notions such as no-envy.

JEL Classifications: C79, D61, D63.

Key words: *fairness, allocation of indivisible goods and money, task assignments, strategy-proofness, the Vickrey-Clarke-Groves mechanisms, egalitarian-equivalence, no-envy, order preservation.*

1 INTRODUCTION

We study problems where a principal, which we call the "center", has to allocate heterogeneous tasks among agents based on the costs the agents incur for performing the tasks. The center may be a social planner (government) which pursues goals like efficiency and fairness. In order to induce the agents to report their costs truthfully, the center must offer them incentives: money transfers are made between the center and the agents. Agents have preferences over the sets of tasks and transfers. We assume that preferences are represented by quasi-linear utility functions, all tasks must be allocated, each task is assigned to only one agent; and there is no restriction on the number of tasks or the size of the transfer an agent can be assigned.

Examples to this task assignment problem include job and wage assignments and imposition of tasks on agents. Specific cases of the last example are government requisitions and eminent domain proceedings (see Yengin, 2010a). However, our results can be easily extended to a more general setting where heterogeneous desirable objects are allocated among a finite set of agents whose valuations for the objects are their private information and monetary transfers are allowed. Among the many real life examples are auctions, the allocation of donated goods and money among the needy, the allocation of inheritance among heirs, the allocation of landing rights to airlines, and the allocation of water entitlements, fishing and pollution permits.

*The first draft of this paper was written while I was a Ph.D. student at the University of Rochester. I am grateful to William Thomson for his guidance and advice. I also thank Paulo Barelli and Gábor Virág, and two anonymous referees for their helpful comments. An earlier version of this paper was titled "Fairness in Auctions and Task Assignments".

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A mechanism determines who is assigned which tasks and what the transfers are. Our aim is to design mechanisms that attain three essential goals: efficiency, immunity to manipulations, and equity.

In terms of efficiency, we are interested in mechanisms that assign the tasks such that the total cost incurred by the agents is minimal[1] (*assignment-efficiency).*

Agents may manipulate the allocation in their favor by misrepresenting their costs. Hence, an *assignment-efficient* mechanism can only minimize the actual total cost if the mechanism is immune to such manipulations. *Strategy-proofness* requires that truthful revelation of costs be a dominant strategy for all agents.

The equity concept we consider here, *egalitarian-equivalence* (Pazner and Schmeidler; 1978), is arguably, one of the main fairness notions that has been extensively studied in the fair allocation literature. Fairness is especially important when tasks are imposed on agents (against their will) as in eminent domain proceedings. Also, when agents have equal rights over the allocated objects, government is concerned about the fairness of the allocation. Examples to this case include allocation of donated goods among the needy, auctions held to allocate water licences among the farmers, allocation of pollution permits to factories and so on.

We have in mind situations where agents have equal rights over the resources or equal responsibilities over the completion of tasks. In cases like these, assigning each agent the same bundle might be seen as fair. However, since tasks are heterogeneous, such an allocation composed of identical bundles is not possible in general. Then, an alternative way to ensure fairness is to choose a feasible allocation that is Pareto-indifferent to an identical-bundle allocation. A mechanism is *egalitarian-equivalent* if for each economy, it chooses a feasible allocation that leaves each agent indifferent between her assigned bundle and a common *reference bundle* composed of a *reference set of tasks* and a *reference transfer.*[2]

Another perspective that would lead to *egalitarian-equivalence* as an equity concept is the liberal-egalitarian theory of justice:

Suppose every agent were assigned the same bundle. Then, the utility differences of agents would be solely due to the differences in their cost functions (i.e., preferences). If agents are held responsible for their costs (preferences), but not for the heterogeneity in the resources, then this allocation would be fair. Since an allocation composed of identical bundles may not exist, an alternative to support this liberal-egalitarian notion is to use an *egalitarian-equivalent* mechanism.[3]

It is well known that in the class of problems we study, the so called Vickrey-Clarke-Groves mechanisms (simply referred to as the Groves mechanisms) are the only mechanisms satisfying the first two goals we want to achieve: assignment-efficiency and *strategy-proofness.* Adding equity as an additional requirement leads to our first result: characterization of the class of *egalitarian-equivalent* Groves mechanisms. *Egalitarian-equivalence* can be strengthen by requiring the same set of tasks, say $\overline{A}$, to be the reference set of tasks for all economies ($\overline{A}-$*egalitarian-equivalence*). Our main Theorem demonstrates a surprising fact: although in general $\overline{A}-$*egalitarian-equivalence* is stronger than *egalitarian-equivalence,* when Groves mechanisms are considered, these two notions coincide. This result simplifies the design of *egalitarian-equivalent* Groves mechanisms.

By Green and Laffont (1977), no Groves mechanism balances the budget. Still, it is possible

[1]In case of desirable objects, the total value experienced by all agents should be maximized.

[2]The reference bundle is common for all agents in a given economy. For different economies, different reference bundles can be chosen.

[3]*Egalitarian-equivalence* can be related to the idea of "*equality of resources*" (Dworkin, 2000). For a more detailed philosophical motivation for *egalitarian-equivalence* based on the liberal-egalitarian distributive theory of justice, see Yengin (2010a).

to design *egalitarian-equivalent* Groves mechanisms that generate bounded budget deficits. Our second result characterizes the class of such mechanisms. Among the *egalitarian-equivalent* Groves mechanisms that respect the same upper bound on deficit, we specify the Pareto-dominant ones. These mechanisms are necessarily $\emptyset-$*egalitarian-equivalent*.

Finally, we analyze the relations between the *egalitarian-equivalent* Groves mechanisms and Groves mechanisms satisfying other fairness notions, such as no-envy or order-preservation. We find out that under assignment-efficiency and *strategy-proofness,* interesting logical relations hold between several fairness notions that do not exist otherwise.

The analysis of the Groves mechanisms from the fairness perspective has been the object of only few recent papers. In Yengin (2010a), we introduce the class of *Super-Fair* Groves mechanisms and characterize this class with several sets of fairness axioms (one of which is egalitarian-equivalence and no-envy). In the same model as ours where heterogeneous objects are allocated, Pápai (2003) characterizes *envy-free* (no agent prefers another agent's bundle to her own) Groves mechanisms. Porter, Shoham, and Tennenholtz (2004) introduce a class of Groves mechanisms that respect a welfare lower bound based on Rawl's maximin equity criterion ($k-$*fairness*) and Atlamaz and Yengin (2008) characterize this class. Yengin (2010b) considers several welfare bounds (including the *identical-preferences lower-bound*) and characterizes Groves mechanisms that respect them.

In Section 2, we present the model and define the Groves mechanisms. In Section 3, we characterize *egalitarian-equivalent* Groves mechanisms. In Section 4, we investigate the implications of imposing an upper-bound on the deficit. Section 5 analyses the relations between different classes of Groves mechanisms. Section 6 concludes. All proofs are in the Appendix.

2 Model

A finite set of indivisible tasks is to be allocated among a finite set of agents. All tasks must be allocated. An agent can be assigned either no task, a single task, or more than one task. Each task is assigned to only one agent. Let $\mathbb{A}$ be the finite set of tasks, with $|\mathbb{A}| \geq 1$.

There are $n \geq 2$ agents, let $N = \{1, 2, ..., n\}$ be the set of n agents. The number of agents may be smaller than, equal to, or greater than the number of tasks.

Let $2^{\mathbb{A}}$ be the set of subsets of $\mathbb{A}$. Each agent i has a cost function $c_i : 2^{\mathbb{A}} \to \mathbb{R}_+$ with $c_i(\emptyset) = 0$.[4] Let $\mathcal{C}$ be the set of such cost functions and $\mathcal{C}^N$ be the $n-$fold Cartesian product of $\mathcal{C}$.

A *cost profile* is a list $c \equiv (c_1, ..., c_n) \in \mathcal{C}^N$ where for each $i \in N$, $c_i \in \mathcal{C}$. A cost profile defines an *economy.* For each $i \in N$, let c_{-i} be the list of the cost functions of the agents in $N\backslash\{i\}$. For each $N' \subset N$, let $c_{-N'}$ be the list of the cost functions of the agents in $N\backslash N'$.

There is a perfectly divisible good we call "money". Let t_i denote agent i's consumption of the good. We call t_i agent i'*s transfer*: if $t_i > 0$, it is a transfer from the center to i; if $t_i < 0$, $|t_i|$ is a transfer from i to the center.

Agent i's utility when she is assigned the set of tasks $A_i \in 2^{\mathbb{A}}$ (note that A_i may be empty) and consumes $t_i \in \mathbb{R}$ is

$$u(A_i, t_i; c_i) = -c_i(A_i) + t_i.$$

Let $\mathcal{A} = \{(A'_i)_{i\in N} :$ for each $i \in N$, $A'_i \in 2^{\mathbb{A}}$, for each pair $\{i, j\} \subseteq N$, $A'_i \cap A'_j = \emptyset$, and $\bigcup\limits_{i\in N} A'_i = \mathbb{A}\}$.

An *assignment* is a list $(A_i)_{i\in N} \in \mathcal{A}$. A *transfer profile* is a list $(t_i)_{i\in N} \in \mathbb{R}^N$. An *allocation* is a list $(A_i, t_i)_{i\in N}$ where $(A_i)_{i\in N}$ is an assignment and $(t_i)_{i\in N}$ is a transfer profile.

[4]As usual, $\mathbb{R}_+$ denotes the set of non-negative real numbers.

A *mechanism* is a function $\varphi \equiv (A, t)$ defined over $\mathcal{C}^N$ that associates with each economy an allocation: for each $c \in \mathcal{C}^N$ and each $i \in N$, $\varphi_i(c) \equiv (A_i(c), t_i(c)) \in 2^{\mathbb{A}} \times \mathbb{R}$.

For each $c \in \mathcal{C}^N$, let $W(c)$ be the minimal total cost among all assignments. That is,

$$W(c) = \min \left\{ \sum_{i \in N} c_i(A'_i) : (A'_i)_{i \in N} \in \mathcal{A} \right\}.$$

2.1 The Groves Mechanisms

Since the utility of each agent is increasing in her transfer, and her transfer and the total transfer can be of any size, every allocation is Pareto-dominated by some other allocation with higher transfers. That is, no allocation is Pareto-efficient. Still, we can define a notion of efficiency restricted to the assignment of the tasks. Since utilities are quasi-linear, given an economy, an allocation that minimizes the total cost incurred by all agents is Pareto-efficient for that economy among all allocations with the same, or smaller, total transfer. A mechanism is *assignment-efficient* if it chooses only such allocations.

Assignment-Efficiency: For each $c \in \mathcal{C}^N$,

$$\sum_{i \in N} c_i(A_i(c)) = W(c).$$

For each $c \in \mathcal{C}^N$, let $\mathcal{A}^*(c)$ be the set of efficient assignments for c.

An *assignment-efficient* mechanism assigns the tasks so that the actual total cost is minimal if the agents report their true costs. *Strategy-proofness* requires that no agent ever benefits by misrepresenting her costs (Gibbard, 1973; Satterthwaite, 1975).

Strategy-proofness[5]: For each $c \in \mathcal{C}^N$, each $i \in N$, and each $c'_i \in \mathcal{C}$,

$$u(\varphi_i(c); c_i) \geq u(\varphi_i(c'_i, c_{-i}); c_i).$$

The so called Groves mechanisms were introduced by Vickrey (1961), Clarke (1971), and Groves (1973). A Groves mechanism chooses, for each economy, an efficient assignment of the tasks. In the literature, Groves mechanisms are sometimes defined as correspondences that select all the efficient assignments in an economy. We work with single-valued Groves mechanisms and assume that each Groves mechanism is associated with a tie-breaking rule that determines which of the efficient assignments (if there are more than one) is chosen. Let $\mathcal{T}$ be the set of all possible tie-breaking rules and τ be a typical element of this set.

The transfer of each agent determined by a Groves mechanism has two parts. First, each agent pays the total cost incurred by all other agents at the assignment chosen by the mechanism. Second, each agent receives an amount of money that does not depend on her own cost. For each $i \in N$, let h_i be a real-valued function defined over $\mathcal{C}^N$ such that for each $c \in \mathcal{C}^N$, h_i depends only on c_{-i}. Let $h = (h_i)_{i \in N}$ and $\mathcal{H}$ be the set of all such h.

The Groves mechanism associated with $h \in \mathcal{H}$ and $\tau \in \mathcal{T}$, $\mathbf{G}^{h,\tau}$: Let $G^{h,\tau} \equiv (A^\tau, t^{h,\tau})$ be such that for each $c \in \mathcal{C}^N$ and each $i \in N$,

$$A^\tau(c) \in \mathcal{A}^*(c)$$

[5]See Sprumont (1995), Barberà (2001), Thomson (2005), for surveys on strategy-proofness.

and

$$t_i^{h,\tau}(c) \quad = \quad -\sum_{j\in N\setminus\{i\}} c_j(A_j^\tau(c)) + h_i(c_{-i}).$$

The following result will be of much use.

Lemma 1. *For each $c \in \mathcal{C}^N$ and each $i \in N$,*

$$u(G_i^{h,\tau}(c); c_i) = -W(c) + h_i(c_{-i}).$$

By Lemma 1, for each $h \in \mathcal{H}$, the mechanisms in $\{G^{h,\tau}\}_{\tau\in\mathcal{T}}$ are *Pareto-indifferent*.[6] Hence, the particular tie-breaking rule used is irrelevant in the determination of the utilities.

The following theorem justifies our interest in Groves mechanisms.

Theorem A *A mechanism is assignment-efficient and strategy-proof if and only if it is a Groves mechanism.*

Proof: Since $\mathcal{C}^N$ is convex, the proof follows from Holmström (1979). □

3 Egalitarian-Equivalence

If agents are equally responsible for the completion of the tasks, then assigning them identical bundles would be fair. But such an identical-bundle allocation is not possible in general due to the heterogeneity in tasks. Fortunately, in our model, we can always find an allocation such that each agent is indifferent between her assigned bundle and a common *reference bundle* (consisting of a *reference set of tasks* and a *reference transfer*). *Egalitarian-equivalence* (Pazner and Schmeidler, 1978) requires that only such allocations be chosen.

Egalitarian-equivalence: For each $c \in \mathcal{C}^N$, there is a set of tasks (which may be empty) $R \in 2^{\mathbb{A}}$ and a transfer $r \in \mathbb{R}$ such that for each $i \in N$,

$$u(\varphi_i(c); c_i) = u(R, r; c_i).$$

To characterize the *egalitarian-equivalent* Groves mechanisms, we need the following notation.

Notation 1

(a) For each $i \in N$, let $c_i^0 \in \mathcal{C}$ be such that for each $A \in 2^{\mathbb{A}}$, $c_i^0(A) = 0$. Let $c^0 = (c_i^0)_{i\in N}$.

(b) Let $\Pi \equiv \{\pi | \pi : N \to \{1, 2, ..., n\}$ is a bijection$\}$.

For each $i \in N$, let $\pi^i \in \Pi$ be a permutation where agent i comes last. For each $i \in N$, let $\Pi^i = \{\pi^i \in \Pi : \pi^i(i) = n\}$ be the set of all such permutations.

For instance, if $N = \{1, 2\}$, then there are two possible permutations of agents, π^1 and π^2 : in π^1, agent 2 comes first and in π^2, she comes last. Hence, $\Pi = \{\pi^1, \pi^2\}$ where $\pi^1(2) = 1$, $\pi^1(1) = 2$, $\pi^2(1) = 1$, and $\pi^2(2) = 2$.

(c) For each $c \in \mathcal{C}^N$, each $\pi \in \Pi$, and each $j \in N$, let $c^{\pi,j} \in \mathcal{C}^N$ be such that for each $i \in N$,

$$c_i^{\pi,j} = \begin{cases} c_i & \text{if } \pi(i) \le \pi(j), \\ c_i^0 & \text{if } \pi(i) > \pi(j). \end{cases}$$

[6]Let $c \in \mathcal{C}^N$. Allocations $(A_i, t_i)_{i\in N}$ and $(A'_i, t'_i)_{i\in N}$ are *Pareto-indifferent* for c if and only if for each $i \in N$, $u(A_i, t_i; c_i) = u(A'_i, t'_i; c_i)$. The mechanisms φ and φ' are *Pareto-indifferent* if for each $c \in \mathcal{C}^N$ and each $i \in N$, $u(\varphi_i(c); c_i) = u(\varphi'_i(c); c_i)$.

That is, $c^{\pi,j}$ is an economy where we permute the agents according to π, and any agent i who comes after agent j in this permutation has a cost function c_i^0, and all other agents' cost functions are same as in cost profile c. For instance, let $\pi^* \in \Pi$ be such that for each $i \in N$, $\pi^*(i) = i$. Then, for each $j \in N$, $c^{\pi^*,j} = (c_1, c_2, .., c_{j-1}, c_j, c_{j+1}^0, c_{j+2}^0, ..., c_n^0)$. $\diamond$

If a Groves mechanism is *egalitarian-equivalent*, then for each $c \in \mathcal{C}^N$, it selects an allocation that is Pareto-indifferent to a reference bundle $(R(c), r(c))$. That is, there are functions $R: \mathcal{C}^N \to 2^{\mathbb{A}}$ and $r: \mathcal{C}^N \to \mathbb{R}$ which associate each economy with a reference set of tasks and a reference transfer, respectively. The following result is a direct implication of *egalitarian-equivalence* and Lemma 1:

Lemma 2. *A Groves mechanism $G^{h,\tau}$ is egalitarian-equivalent if and only if there is a reference-set-of-tasks-function $R: \mathcal{C}^N \to 2^{\mathbb{A}}$ and a reference-transfer-function $r: \mathcal{C}^N \to \mathbb{R}$ such that for each $c \in \mathcal{C}^N$ and each $i \in N$,*

$$h_i(c_{-i}) = W(c) + r(c) - c_i(R(c)). \tag{1}$$

Suppose we pick any two arbitrary functions $R: \mathcal{C}^N \to 2^{\mathbb{A}}$ and $r: \mathcal{C}^N \to \mathbb{R}$ and use them to compute the right-hand-side (RHS) of (1). Then, we can not guarantee that for each $c \in \mathcal{C}^N$ and each $i \in N$, $h_i(c_{-i})$ is independent of c_i. Hence, Lemma 2 does not hold for any two arbitrary functions R and r. To guarantee the independence of h_i from c_i, functions R and r should satisfy condition (2) below:

For each $c \in \mathcal{C}^N$, each $i \in N$, and each $\widehat{c} \in \mathcal{C}^N$ such that $c_{-i} = \widehat{c}_{-i}$, since $h_i(c_{-i}) = h_i(\widehat{c}_{-i})$, by (1),

$$W(c) + r(c) - c_i(R(c)) = W(\widehat{c}_i, c_{-i}) + r(\widehat{c}_i, c_{-i}) - \widehat{c}_i(R(\widehat{c}_i, c_{-i})). \tag{2}$$

If we characterize the set of admissible R and r functions for which (2) holds, then we can construct any *egalitarian-equivalent* Groves mechanism by using (1). To design such admissible functions R and r associated with an *egalitarian-equivalent* Groves mechanism, we can check whether (2) holds for each $c \in \mathcal{C}^N$, each $i \in N$, and each $\widehat{c} \in \mathcal{C}^N$ such that $c_{-i} = \widehat{c}_{-i}$. A less time consuming alternative method is to fix a benchmark economy such as c^0 and utilize the $n-$step iteration method we describe below.

The n-step Process:

Let $G^{h,\tau}$ be an *egalitarian-equivalent* Groves mechanism. Then, by Lemma 2, there are $R: \mathcal{C}^N \to 2^{\mathbb{A}}$ and $r: \mathcal{C}^N \to \mathbb{R}$ such that (1) holds. Pick any economy $c \in \mathcal{C}^N$. Starting from c^0, we can reach c in n steps, at each step $k \leq n$, changing the cost function of agent k from c_k^0 to c_k. Let c^k denote the economy obtained at step $k \leq n$. That is, by Notation 1c, $c^k \equiv c^{\pi^*,k}$.

Note that at each step $k \leq n$, $c_{-k}^{k-1} = c_{-k}^k$. Hence, by (2), we will connect the reference transfer of c^{k-1} to that of c^k.

Step 1: Consider $c^0 = (c_1^0, c_2^0, ..., c_n^0)$ and $c^1 = (c_1, c_2^0, ..., c_n^0)$.
Since $c_{-1}^0 = c_{-1}^1$, $h_1(c_{-1}^0) = h_1(c_{-1}^1)$. By (2), $W(c^0)+r(c^0)-c_1^0(R(c^0)) = W(c^1)+r(c^1)-c_1(R(c^1))$. Note that in c^0 and c^1, the minimal total cost is achieved by assigning all the tasks to one of the agents, say i, whose cost function is c_i^0. Hence, $W(c^0) = W(c^1) = 0$. Also, $c_1^0(R(c^0)) = 0$. Hence, $r(c^1) = r(c^0) + c_1(R(c^1))$.

Let $k \leq n - 1$.

Step k: Consider $c^{k-1} = (c_1, c_2, ..., c_{k-1}, c_k^0, c_{k+1}^0, ..., c_n^0)$ and $c^k = (c_1, c_2, ..., c_k, c_{k+1}^0, c_{k+2}^0, ..., c_n^0)$.

Since $c_{-k}^{k-1} = c_{-k}^{k}$, by (2), $r(c^k) = r(c^{k-1}) + W(c^{k-1}) - W(c^k) + c_k(R(c^k)) - c_k^0(R(c^{k-1}))$. Note that $W(c^{k-1}) = W(c^k) = c_k^0(R(c^{k-1})) = 0$. Hence,

$$r(c^k) = r(c^{k-1}) + c_k(R(c^k)).$$

Since this equation holds for each $1 \leq k \leq n-1$, by recursive substitution, we get

$$r(c^{n-1}) = r(c^0) + \sum_{j=1}^{n-1} c_j(R(c^j)). \tag{3}$$

Step n: Consider $c^{n-1} = (c_1, c_2, ..., c_{n-1}, c_n^0)$ and $c^n = c$.
Since $c_{-n}^{n-1} = c_{-n}$, by (2), $r(c) = r(c^{n-1}) + W(c^{n-1}) - W(c) + c_n(R(c)) - c_n^0(R(c^{n-1}))$. Note that $W(c^{n-1}) = c_n^0(R(c^{n-1})) = 0$. Hence, by (3),

$$r(c) = r(c^0) + \sum_{j \in N} c_j(R(c^{\pi^*,j})) - W(c). \tag{4}$$

◇

Instead of π^*, we could have used any permutation of agents and applied the same $n-$step process (for the formal proof, see the proof of Proposition 1). For instance, if $n \geq 9$ and we use π where $\pi(3) = 1$, $\pi(9) = 2$ etc., then, at step 1, we change the cost function of agent 3 from c_3^0 to c_3, and in the second step, we change the cost function of agent 9, and so on. Hence (4) holds for any $\pi \in \Pi$. That is, for each $\pi \in \Pi$,

$$r(c) = r(c^0) + \sum_{j \in N} c_j(R(c^{\pi,j})) - W(c). \tag{5}$$

By (5), for each pair $\{\pi, \pi'\} \subseteq \Pi$,

$$\sum_{j \in N} c_j(R(c^{\pi,j})) = r(c) + W(c) - r(c^0) = \sum_{j \in N} c_j(R(c^{\pi',j})).$$

Hence, the functions $R : \mathcal{C}^N \to 2^{\mathbb{A}}$ for which Lemma 2 holds must be as in the following definition.

Definition 1. Let $R : \mathcal{C}^N \to 2^{\mathbb{A}}$ be such that for each $c \in \mathcal{C}^N$ and each pair $\{\pi, \pi'\} \subseteq \Pi$,

$$\sum_{j \in N} c_j(R(c^{\pi,j})) = \sum_{j \in N} c_j(R(c^{\pi',j})). \tag{6}$$

Let $\mathcal{R}$ be the set of all functions R satisfying (6). ◇

Lemma 2 states that if a Groves mechanism is *egalitarian-equivalent*, then there are functions R and r such that (1) holds. We showed that (1) implies (2) which in turn implies, through the $n-$step process, (5) and (6). Hence, by substituting (5) into (1), we get the following result. Let $\rho \equiv r(c^0)$.

Lemma 3. *If a Groves mechanism $G^{h,\tau}$ is egalitarian-equivalent, then there is $R \in \mathcal{R}$ and a real number $\rho \in \mathbb{R}$ such that for each $c \in \mathcal{C}^N$, each $i \in N$, and each $\pi \in \Pi$,*

$$h_i(c_{-i}) = \rho + \sum_{j \in N} c_j(R(c^{\pi,j})) - c_i(R(c)). \tag{7}$$

At first glance, since R depends on c, RHS of (7) seems to depend on c_i (the same can be said for equation (1)). This would contradict the fact that $h_i(c_{-i})$ is independent of c_i. However, this is not the case due to the n-step process and (6). Next remark explains this fact (see Appendix for the proof).

Remark 1. Let $h \in \mathcal{H}$ be as in (7). Then, for each $i \in N$ and each pair $\{c, \widehat{c}\} \subset \mathcal{C}^N$ with $c_{-i} = \widehat{c}_{-i}$, $h_i(c_{-i}) = h_i(\widehat{c}_{-i})$.

Note that for each $c \in \mathcal{C}^N$, each $i \in N$, and each $\pi^i \in \Pi^i$, $c^{\pi^i, i} = c$. If we use π^i in (7), we get equation (8) in the next definition.

Definition 2. For each $R \in \mathcal{R}$, each real number $\rho \in \mathbb{R}$, and each $\tau \in \mathcal{T}$, let $G^{R,\rho,\tau} \equiv G^{h,\tau}$ where h is such that for each $c \in \mathcal{C}^N$ and each $i \in N$,

$$h_i(c_{-i}) = \rho + \sum_{j \in N \setminus \{i\}} c_j(R(c^{\pi^i, j})). \tag{8}$$

Let $\mathcal{G} \equiv \{G^{R,\rho,\tau} |\ R \in \mathcal{R},\ \rho \in \mathbb{R},\ \tau \in \mathcal{T}\}$ be the class of all such mechanisms.

The following example illustrates (8).

Example 1. Let $N = \{1, 2\}$. Let $c = (c_1, c_2) \in \mathcal{C}^N$. Then, $c^{\pi^1,\,1} = c$, $c^{\pi^1,\,2} = (c_1^0, c_2)$, $c^{\pi^2,\,1} = (c_1, c_2^0)$, and $c^{\pi^2,\,2} = c$.

Let $G^{R,\rho,\tau} \in \mathcal{G}$. By (8), $h_1(c_2) = \rho + c_2(R(c_1^0, c_2))$ and $h_2(c_1) = \rho + c_1(R(c_1, c_2^0))$.

Each mechanism $G^{R,\rho,\tau}$ in $\mathcal{G}$ selects, for each economy $c \in C^N$, an allocation that is Pareto-indifferent to the reference bundle $(R(c), r(c))$ where the reference set of tasks is chosen by $R \in \mathcal{R}$ and the reference transfer satisfies (5) with $\rho = r(c^0)$. That is, if we pick a real number ρ and fix $\rho = r(c^0)$, and pick a function R satisfying (6), then a Groves mechanism associated with these ρ and R, through (8), must be *egalitarian-equivalent.* Also, any *egalitarian-equivalent* Groves mechanism can be constructed this way.

Proposition 1. *Class $\mathcal{G}$ is the class of all egalitarian-equivalent Groves mechanisms.*

In Lemma 2, we did not have any information about the form of the admissible functions R and r for which (1) works. We saw that arbitrary functions R and r can not be used. So (1) was not very informative. However, by Proposition 1, we just need to search for a function R that satisfies (6) to construct an *egalitarian-equivalent* Groves mechanism using (8).

So far, we have not said anything about the exact form of the functions in $\mathcal{R}$ except for that they satisfy (6). Note that $\mathcal{R}$ is non-empty since it includes all functions R which select the same set of tasks as the reference set for all economies. If such a function is used, we have the following strengthening of *egalitarian-equivalence:*

Let $\overline{A} \in 2^{\mathbb{A}}$.

$\overline{A}-$**Egalitarian-equivalence:** For each $c \in \mathcal{C}^N$, there is $r \in \mathbb{R}$ such that for each $i \in N$,

$$u(\varphi_i(c); c_i) = u((\overline{A}, r); c_i).$$

A Groves mechanism $G^{h,\tau}$ is $\overline{A}-$*egalitarian-equivalent* if and only if there is $\rho \in \mathbb{R}$ such that $G^{h,\tau} \equiv G^{R,\rho,\tau}$ where for each $c \in \mathcal{C}^N$, $R(c) = \overline{A}$. In particular, for each $c \in \mathcal{C}^N$, each $\pi \in \Pi$, and each $j \in N$, $R(c^{\pi,j}) = \overline{A}$. Hence, by (8), we have the following Corollary to Proposition 1.

Corollary 1. *A Groves mechanism $G^{h,\tau}$ is $\overline{A}-$egalitarian-equivalent if and only if there is a real number $\rho \in \mathbb{R}$ such that for each $c \in \mathcal{C}^N$ and each $i \in N$,*

$$h_i(c_{-i}) = \rho + \sum_{j \in N \setminus \{i\}} c_j(\overline{A}). \tag{9}$$

For each $\overline{A} \in 2^{\mathbb{A}}$, let $\mathcal{G}^{\overline{A}}$ be the class of all $\overline{A}-$*egalitarian-equivalent* mechanisms and $\mathcal{G}^* = \{\mathcal{G}^{\overline{A}} : \overline{A} \in 2^{\mathbb{A}}\}$.

Since $\overline{A}-$*egalitarian-equivalence* implies *egalitarian-equivalence*, if we pick $\overline{A} \in 2^{\mathbb{A}}$ and a real number $\rho \in \mathbb{R}$, we can design an *egalitarian-equivalent* Groves mechanism using (9). Our main Theorem next states that we can design all *egalitarian-equivalent* Groves mechanisms in this way. Hence, we obtain a surprising result: even though in general, $\overline{A}-$*egalitarian-equivalence* is stronger than *egalitarian-equivalence*, a Groves mechanism is *egalitarian-equivalent* if and only if it is $\overline{A}-$*egalitarian-equivalent* for some $\overline{A} \in 2^{\mathbb{A}}$. That is, $\mathcal{G} \equiv \mathcal{G}^*$.

Theorem 1. *A Groves mechanism is egalitarian-equivalent if and only if it belongs to $\mathcal{G}^*$.*

Let us give some intuition for the proof of Theorem 1. The proof extensively relies on equation (6), in particular, the fact that (6) implies the following: for any given c_{-i}, it is always possible to find a cost function c_i such that $R(c^{\pi,i}) = R(c^{\pi',i})$. For instance, when $N = \{1, 2\}$, equation (6) implies that (i) $c_1(R(c_1, c_2)) + c_2(R(c_1^0, c_2)) = c_1(R(c_1, c_2^0)) + c_2(R(c_1, c_2))$. For a given c_2, there exists c_1 such that equation (i) is violated unless $R(c_1, c_2) = R(c_1, c_2^0)$, which in turn implies that $c_2(R(c_1^0, c_2)) = c_2(R(c_1, c_2))$.

Note that Theorem 1 does not imply that the R function associated with a Groves mechanism in $\mathcal{G}^*$ must be constant. For instance, consider $R : \mathcal{C}^N \to 2^{\mathbb{A}}$ such that $R(c^0) = \mathbb{A}$ and for each $c \in \mathcal{C}^N \setminus \{c^0\}$, $R(c) = \emptyset$. Then, for each $c \in \mathcal{C}^N$ and each $i \in N$, $h_i(c_{-i}) = \rho$ and $G^{h,\tau}$ is $\emptyset-$*egalitarian-equivalent*

In the class of two-agent economies, the transfer function of an $\mathbb{A}-$*egalitarian-equivalent* Groves mechanism resembles that of a Pivotal mechanism. Remember that a Groves mechanism $G^{h,\tau}$ is Pivotal[7] if for each $c \in \mathcal{C}^N$ and each $i \in N$, $h_i(c_{-i}) = W(c_{-i})$. Since for each $N = \{i, j\}$ and each $c \in \mathcal{C}^N$, $W(c_{-i}) = c_j(\mathbb{A})$, we have the following result:

Remark 2. On the class of two-agent economies, a Groves mechanism $G^{h,\tau}$ is $\mathbb{A}-$*egalitarian-equivalent* if and only if there is $\rho \in \mathbb{R}$ such that for each $c \in \mathcal{C}^N$ and each $i \in N$,

$$h_i(c_{-i}) = \rho + W(c_{-i}).$$

4 Egalitarian-Equivalent Mechanisms with Bounded Deficits

If the center wants to use a mechanism that allocates the objects efficiently and induces the agents to report their true preferences, then it has to select a Groves mechanism. However, one often is also interested in the amount of budget deficit or surplus that is generated by the mechanism. It is well-known that no Groves mechanism balances the budget (Green and Laffont, 1977). However, we can design Groves mechanisms that respect an upper bound on the deficit generated.

[7] In the literature, these mechanisms are also known by the following names: Vickrey mechanisms, Clarke mechanisms, and Second-price sealed-bid auctions.

Suppose that the center is willing to incur a deficit up to a certain amount $T \in \mathbb{R}$ (T may be negative). The following condition requires that the deficit in any economy, no matter what the costs of the agents in N, should never exceed T.

$T-$**Bounded-Deficit** **($T-$BD):** For each $c \in \mathcal{C}^N$,

$$\sum_{i \in N} t_i(c) \leq T.$$

The revenue (budget surplus) of the center is equal to the negative of the total transfer (budget deficit). Hence, if $G^{h,\tau}$ satisfies $T-$*bounded-deficit*, then for each $c \in \mathcal{C}^N$, $-\sum_{i \in N} t_i^{h,\tau}(c) \geq -T$. That is, the center is guaranteed to generate a revenue at least as much as $-T$. Note that such a guarantee is absent when a Pivotal mechanism is used.

In a related model to ours, Ohseto (2004) characterizes the class of $\emptyset-$*egalitarian-equivalent* Groves mechanisms that respect $T-$*bounded-deficit.*

Ohseto's model differs from the one we study in three important respects: in his model, the indivisible goods are homogenous (i.e., the cost of each task is same for an agent), the number of goods is strictly less than the number of agents, and each agent can be assigned at most one indivisible good.

Ohseto's Theorem 3 (stated for the undesirable objects) is the following:

A Groves mechanism $G^{h,\tau}$ satisfies $\emptyset-$egalitarian-equivalence and $T-$bounded deficit if and only if for each $c \in \mathcal{C}^N$ and each $i \in N$, $h_i(c_{-i}) \leq \frac{T}{|N|}$.

We can show that not only his result still holds in our more general setting, but moreover, that even if we weaken the $\emptyset-$*egalitarian-equivalence* to *egalitarian-equivalence*, we still characterize the same class of Groves mechanisms as stated in our next result.

Note that a Groves mechanism $G^{h,\tau}$ is $\emptyset-$*egalitarian-equivalent* if and only if $G^{h,\tau} \in \mathcal{G}^{\emptyset}$. Then, the transfer function is of the following form: there is $\rho \in \mathbb{R}$ such that for each $c \in \mathcal{C}^N$ and each $i \in N$, $h_i(c_{-i}) = \rho$ and $t_i^{h,\tau}(c) = \rho - \sum\limits_{j \in N \setminus \{i\}} c_j(A_j^\tau(c))$.

For each $\rho \in \mathbb{R}$, let $\mathcal{S}^\rho \equiv \{G^{h,\tau} |$for each $c \in \mathcal{C}^N$ and each $i \in N$, $h_i(c_{-i}) = \rho$; and $\tau \in \mathcal{T}\}$.

Proposition 2. *A Groves mechanism $G^{h,\tau}$ satisfies egalitarian-equivalence and $T-$bounded-deficit if and only if there is $\rho \in \mathbb{R}$ such that $G^{h,\tau} \in \mathcal{S}^\rho$ and*

$$\rho \leq \frac{T}{|N|}. \tag{10}$$

By Corollary 1, an interesting implication of Proposition 2 is the following: *assignment-efficiency, strategy-proofness, egalitarian-equivalence,* and $T-$*bounded-deficit* together imply $\emptyset-$*egalitarian-equivalence.*

Remark 3. If an *egalitarian-equivalent* Groves mechanism satisfies $T-$*bounded-deficit*, then it is $\emptyset-$*egalitarian-equivalent.*

Note that a mechanism generates no-deficit if it satisfies $T-$*bounded deficit* where $T = 0$. The following result follows from Proposition 2.

Corollary 2. *A Groves mechanism $G^{h,\tau}$ satisfies egalitarian-equivalence and generates no-deficit if and only if there is $\rho \in \mathbb{R}$ such that $G^{h,\tau} \in \mathcal{S}^\rho$ and*

$$\rho \leq 0.$$

By Proposition 2, one can rank *egalitarian-equivalent* Groves mechanisms according to the maximal budget deficit (or the minimal budget surplus) they may generate. Hence, the center can select the mechanism that fits in its targets regarding the bounds on deficit or surplus.

Moreover, among all *egalitarian-equivalent* Groves mechanisms that respect a given upper bound T on deficit (that is among all mechanisms in $\mathcal{S}^{\rho}$ such that $\rho \leq \frac{T}{|N|}$), the ones for which (10) holds as an equality Pareto-dominates the others.[8] These Pareto-dominant mechanisms are also the ones which have the minimal surplus.

Corollary 3. ***(i)*** *Mechanisms in $\mathcal{S}^{\frac{T}{|N|}}$ Pareto-dominate all egalitarian-equivalent Groves mechanisms that satisfy $T-$bounded-deficit.*

Mechanisms in $\mathcal{S}^{\frac{T}{|N|}}$ have minimal surplus among all egalitarian-equivalent Groves mechanisms that satisfy $T-$bounded-deficit.

(ii) *Mechanisms in $\mathcal{S}^0$ Pareto-dominate all egalitarian-equivalent Groves mechanisms that generate no-deficit.*

Mechanisms in $\mathcal{S}^0$ have minimal surplus among all egalitarian-equivalent Groves mechanisms that generate no-deficit.

Besides the bounds on deficit, the center may also be interested in bounds on the welfare of the agents. The following welfare lower bound incorporates the notion that it is unfair for an agent, if the agent is assigned all the tasks but has to pay the center. Hence, the utility an agent would experience if she was assigned all of the tasks and zero transfer should be a lower-bound on her welfare.

The Stand-Alone Lower-Bound: For each $c \in \mathcal{C}^N$ and each $i \in N$, $u(\varphi_i(c); c_i) \geq -c_i(\mathbb{A})$.

The following result follows from Proposition 2b in Yengin (2010b).

Remark 4. A Groves mechanism satisfies the *stand-alone lower-bound* and *no-deficit* if and only if it belongs to $\mathcal{S}^0$.

5 Other Fairness Notions and Logical Relations

When *assignment-efficiency* and *strategy-proofness* are imposed, several logical relations hold between fairness definitions that do not exist otherwise. Before we state these logical relations, let us present these definitions.

Perhaps, the most basic fairness notion is to require that whenever two agents have the same characteristics (e.g., the same cost functions), they should be treated equally.

Equal Treatment of Equals: For each pair $\{i, j\} \subseteq N$ and each $c \in \mathcal{C}^N$ such that $c_i = c_j$,

$$u(\varphi_i(c); c_i) = u(\varphi_j(c); c_j).$$

The following property requires that the allocation is invariant with respect to the relabelling of agents.

Anonymity: For each bijection $\pi : N \to N$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$\varphi_i(c) = \varphi_{\pi(i)}((c_{\pi(j)})_{j \in N}).$$

[8]Let $c \in \mathcal{C}^N$. Allocation $(A_i, t_i)_{i \in N}$ *Pareto-dominates* $(A'_i, t'_i)_{i \in N}$ for c if and only if for each $i \in N$, $u(A_i, t_i; c_i) \geq u(A'_i, t'_i; c_i)$ with strict inequality for some $i \in N$. The mechanism φ *Pareto-dominates* φ' if for each $c \in \mathcal{C}^N$ and each $i \in N$, $u(\varphi_i(c); c_i) \geq u(\varphi'_i(c); c_i)$ and there are $c \in \mathcal{C}^N$ and $i \in N$ such that $u(\varphi_i(c); c_i) > u(\varphi'_i(c); c_i)$.

Note that if a Groves mechanism $G^{h,\tau}$ is *anonymous*, then, for each pair $\{i,j\} \subseteq N$, $h_i = h_j$. Obviously, *anonymity* implies *equal treatment of equals.* However, for Groves mechanisms, we show in Proposition 3i that these two properties actually characterize the same class.

Suppose that for each subset of tasks, agent i incurs a cost that is at least as high as what agent j incurs. If j were assigned a lower utility than i, it would be as if j were penalized for having lower costs. The following property is meant to prevent this situation.

If $c_i(A) \geq c_j(A)$ for each $A \in 2^{\mathbb{A}}$, then we write $c_i \geq c_j$.

Order Preservation[9]**:** For each pair $\{i,j\} \subseteq N$ and each $c \in \mathcal{C}^N$ such that $c_i \geq c_j$,

$$u(\varphi_i(c); c_i) \leq u(\varphi_j(c); c_j).$$

Another central fairness notion is that, each agent should find her bundle at least as desirable as the bundle of any other agent (Foley, 1967).

No-envy: For each pair $\{i,j\} \subseteq N$ and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq u(\varphi_j(c); c_i).$$

Pápai (2003)[10] proves that on the *unrestricted* domain, no Groves mechanism is *envy-free.* On the *subadditive* domain of cost profiles[11], she characterizes the class of *envy-free* Groves mechanisms. It is easy to see that in general, *no-envy* implies *equal treatment of equals.* Observation 3 in Pápai (2003) states that all *envy-free* Groves mechanism are *anonymous.* We show further that under *assignment-efficiency* and *strategy-proofness*, *no-envy* implies *order preservation*, which in turn implies *anonymity.*

The following Proposition states the inclusion relations between several classes of Groves mechanisms.

Proposition 3. ***(i)*** *A Groves mechanism satisfies* equal treatment of equals *if and only if it is* anonymous.

(ii) *If a Groves mechanism* preserves order*, then it is* anonymous.

(iii) *If a Groves mechanism is* egalitarian-equivalent*, then it* preserves order.

(iv) *If a Groves mechanism is* envy-free*, then it* preserves order.

(v) *If a Groves mechanism is* $\emptyset-$egalitarian-equivalent*, then on the* additive *and the* subadditive *domains, it is* envy-free.

An alternative statement of Proposition 3ii is as follows: *assignment-efficiency, strategy-proofness,* and *order preservation* together imply *anonymity.* Other parts of Proposition 3 can also be stated in a similar way.

[9]The same property appears in Atlamaz and Yengin (2008). Also, a similar property appears in Porter, Shoham, and Tennenholtz (2004) under the name of "no- competence penalty".

[10]Pápai (2003) studies *envy-free* Groves mechanisms in the same model as ours. The only immaterial difference between her model and ours is that in her model, objects are desirable. Her results still hold if we adapt them to our costly objects setting.

[11]If for each pair $\{A, A'\} \subseteq 2^{\mathbb{A}}$, $c_i(A \cup A') \leq (=)\ c_i(A) + c_i(A')$, then c_i is *subadditive* (*additive*).

6 Concluding Remarks

In Section 3, we saw that even though in general, *egalitarian-equivalence* is a weaker requirement than $\overline{A}-$*egalitarian-equivalence*, for Groves mechanisms, these two requirements are identical. This result simplifies the design of *egalitarian-equivalent* Groves mechanisms. The transfers of an $\overline{A}-$*egalitarian-equivalent* Groves mechanism $G^{h,\tau}$ have the following form: for each $c \in \mathcal{C}^N$ and each $i \in N$,

$$\begin{aligned} t_i^{h,\tau}(c) &= \rho + \sum_{j \in N \setminus \{i\}} [c_j(\overline{A}) - c_j(A_j^\tau(c))], \\ &= \rho + c_i(A_i^\tau(c)) - c_i(\overline{A}) + \sum_{j \in N} c_j(\overline{A}) - W(c). \end{aligned}$$

This transfer has two parts: first, i receives a fixed payment ρ from the center. Then, she receives the sum of the differences in the cost of each agent $j \in N \setminus \{i\}$ to perform $\overline{A}$ and j's assigned task $A_j^\tau(c)$. One can interpret this transfer as follows:

Each agent i is held responsible for her cost of performing the reference set of tasks $\overline{A}$, hence each agent i pays the cost $c_i(\overline{A})$ and is reimbursed for her actual cost $c_i(A_i^\tau(c))$. Hence, agents are treated as if they are assigned the same set of tasks $\overline{A}$. As a result, the center pays to agents a total reimbursement of $\sum_{j \in N} c_j(A_j^\tau(c)) = W(c)$ and collects from agents a total of $\sum_{j \in N} c_j(\overline{A})$. The difference between these two amounts is paid back to each agent as a rebate. Also, the center collects from each agent, a lump-sum tax of $-\rho$ (note that ρ can be negative).

The results in Section 4 indicate that if the center wants to use a Pareto-dominant *egalitarian-equivalent* Groves mechanism that restricts the budget deficit to be no more than an amount $T \in \mathbb{R}$, then the reference set of tasks $\overline{A}$ has to be $\emptyset$ and $\rho = T/n$. Such a mechanism would be *envy-free* on the *additive* and the *subadditive* domains, due to Proposition $3v$ in Section 5. Also, any *egalitarian-equivalent* Groves mechanism is *anonymous* and *order-preserving*. Hence, *egalitarian-equivalent* Groves mechanisms satisfy other appealing fairness notions as well and can be designed to limit budget deficits.

Let us compare the mechanisms characterized in our paper with other classes of Groves mechanisms characterized in the literature. It is easy to see that the class of $k-$*fair* mechanisms introduced by Porter et al (2004) and characterized by Atlamaz and Yengin (2008) is disjoint with the class of *egalitarian-equivalent* Groves mechanisms (just compare the respective transfer functions). There is no inclusion relationship between the class of *egalitarian-equivalent* Groves mechanisms and the class of *envy-free* Groves mechanisms characterized by Pápai (2003) on the *subadditive* domain. However, Yengin (2010a) shows that on the *subadditive* domain, if a Groves mechanism is *egalitarian-equivalent* and *envy-free,* then (i) on the domain of economies with at least three agents, it is $\emptyset-$*egalitarian-equivalent,* (ii) on the domain of two-agent economies, it is either $\emptyset-$*egalitarian-equivalent* or $\mathbb{A}-$*egalitarian-equivalent.*

7 Appendix

Proof of Proposition 1:
We first show that each $G^{R,\rho,\tau} \in \mathcal{G}$ is *egalitarian-equivalent*. Let $G^{R,\rho,\tau} = G^{h,\tau}$. Then, h satisfies (8). For each $c \in \mathcal{C}^N$, each $i \in N$, and each $\pi^i \in \Pi^i$, let $r(c) = -W(c) + \rho + \sum_{j \in N} c_j(R(c^{\pi^i,j}))$.

Then, for each $c \in \mathcal{C}^N$, each $i \in N$, and each $\pi^i \in \Pi^i$, since $c^{\pi^i,i} = c$, by (8), we have

$$\begin{aligned} -W(c) + h_i(c_{-i}) &= -W(c) + \rho + \sum_{j \in N \setminus \{i\}} c_j(R(c^{\pi^i,j})), \\ &= -W(c) + \rho + \sum_{j \in N \setminus \{i\}} c_j(R(c^{\pi^i,j})) + c_i(R(c^{\pi^i,i})) - c_i(R(c)). \\ &= r(c) - c_i(R(c)). \end{aligned} \tag{11}$$

Hence, by Lemma 1 and (11), for each $c \in \mathcal{C}^N$ and each $i \in N$, there is a set of tasks $R(c)$ and a transfer $r(c)$ such that $u(G_i^{h,\tau}(c); c_i) = u(R(c), r(c); c_i)$. Hence, $G^{h,\tau}$ is *egalitarian-equivalent*.

Now, we show that each *egalitarian-equivalent* Groves mechanism must belong to $\mathcal{G}$. Let $G^{h,\tau}$ be an *egalitarian-equivalent* Groves mechanism. By Lemma 2, there are $R : \mathcal{C}^N \to 2^{\mathbb{A}}$ and $r : \mathcal{C}^N \to \mathbb{R}$ such that (1) holds. Let $c \in \mathcal{C}^N$.
For each $\pi \in \Pi$ and each $j \in N$, let $b^{\pi,j} \in \mathcal{C}^N$ be such that for each $i \in N$,

$$b_i^{\pi,j} = \begin{cases} c_i & \text{if } \pi(i) \leq \pi(j) - 1, \\ c_i^0 & \text{if } \pi(i) > \pi(j) - 1. \end{cases}$$

Note that for each $\pi \in \Pi$ and each $j \in N$, $c_j^{\pi,j} = c_j$, $b_j^{\pi,j} = c_j^0$, and $c_{-j}^{\pi,j} = b_{-j}^{\pi,j}$.
By (1), for each $\pi \in \Pi$ and each $j \in N$,

$$h_j(c_{-j}^{\pi,j}) = W(c^{\pi,j}) + r(c^{\pi,j}) - c_j(R(c^{\pi,j})), \text{ and} \tag{12}$$

$$h_j(b_{-j}^{\pi,j}) = W(b^{\pi,j}) + r(b^{\pi,j}) - c_j^0(R(b^{\pi,j})). \tag{13}$$

Since for each $\pi \in \Pi$ and each $j \in N$, $c_{-j}^{\pi,j} = b_{-j}^{\pi,j}$, by (12) and (13),

$$r(c^{\pi,j}) = -W(c^{\pi,j}) + W(b^{\pi,j}) + r(b^{\pi,j}) + c_j(R(c^{\pi,j})) - c_j^0(R(b^{\pi,j})). \tag{14}$$

Starting from c^0, we reach c using the $n-$step process described in Section 3. At each step $k \leq n$, find $r(c^{\pi,j})$ with $\pi(j) = k$.
Note that at each step $k \leq n$, for each pair $\{j, l\} \subseteq N$ such that $\pi(j) = k = \pi(l) + 1$, we have $b^{\pi,j} = c^{\pi,l}$.
Hence, at each step $k \leq n$, in equation (14), instead of $r(b^{\pi,j})$, insert the value of $r(c^{\pi,l})$ found at step $k-1$.
Note that for each $j \in N$ with $\pi(j) = 1$, $b^{\pi,j} = c^0$; and for each $j \in N$ with $\pi(j) = n$, $c^{\pi,j} = c$.
Hence, by recursive substitution of (14), we obtain for each $\pi \in \Pi$,

$$\begin{aligned} r(c) &= -W(c) + W(c^0) + r(c^0) + \sum_{j \in N} c_j(R(c^{\pi,j})) - \sum_{j \in N} c_j^0(R(b^{\pi,j})), \\ &= -W(c) + r(c^0) + \sum_{j \in N} c_j(R(c^{\pi,j})). \end{aligned} \tag{15}$$

Since (15) holds for any $\pi \in \Pi$, we have for each pair $\{\pi, \pi'\} \subseteq \Pi$, $\sum_{j \in N} c_j(R(c^{\pi,j})) = r(c) + W(c) - r(c^0) = \sum_{j \in N} c_j(R(c^{\pi',j}))$. Hence, $R \in \mathcal{R}$. Let $\rho = r(c^0)$. By Lemma 2 and (15), for each $\pi \in \Pi$,

$$h_i(c_{-i}) = W(c) + r(c) - c_i(R(c)) = \rho + \sum_{j \in N} c_j(R(c^{\pi,j})) - c_i(R(c)). \tag{16}$$

If we use $\pi^i \in \Pi^i$ in (16), since $c^{\pi^i,i} = c$, we obtain (8). Hence, $G^{h,\tau} \in \mathcal{G}$. This completes the proof. ■

Proof of Theorem 1:

Obviously, if a Groves mechanism is $\overline{A}-$*egalitarian-equivalent* for some $\overline{A} \in 2^{\mathbb{A}}$, then it is *egalitarian-equivalent.*

Conversely, let $G^{h,\tau}$ be *egalitarian-equivalent.* Then, by Lemma 3, (7) holds. Let $R \in \mathcal{R}$ and $\rho \in \mathbb{R}$ be such that (7) holds.

Pick $c_1^* \in \mathcal{C}$ such that for each pair $\{A, A'\} \subseteq 2^{\mathbb{A}}$ with $A \neq A'$,

$$c_1^*(A) \neq c_1^*(A'). \tag{17}$$

Let $\overline{A} \equiv R(c_1^*, c_{-1}^0)$. We will show that for each $c \in \mathcal{C}^N$ and each $i \in N$,

$$h_i(c_{-i}) = \rho + \sum_{j \in N \backslash \{i\}} c_j(\overline{A}).$$

The proof is in two parts. Let $c \in \mathcal{C}^N$.

Part 1:

Let $\pi^1 \in \Pi^1$. Note that for each $j \in N\backslash\{1\}$, $c^{\pi^1,j} = (c_1^0, c_{-1}^{\pi^1,j})$. That is, $c^{\pi^1,\, j}$ is independent of c_1. Let

$$\begin{aligned} \underline{\varphi} &\equiv \sum_{j \in N\backslash\{1\}} \min_{A_j \in 2^{\mathbb{A}}} \{c_j(R(c_1^0, c_{-1}^{\pi^1,\, j})) - c_j(A_j)\}, \\ \overline{\varphi} &\equiv \sum_{j \in N\backslash\{1\}} \max_{A_j \in 2^{\mathbb{A}}} \{c_j(R(c_1^0, c_{-1}^{\pi^1,\, j})) - c_j(A_j)\}. \end{aligned}$$

Let $\varphi = \max\{|\underline{\varphi}|, |\overline{\varphi}|\}$ and $\gamma = \varphi + 1 > 0$. Hence, $\gamma > |\underline{\varphi}|$ and $\gamma > |\overline{\varphi}|$. Then, for each $\{A_j : j \in N\backslash\{1\}\} \subseteq 2^{\mathbb{A}}$,

$$-\gamma < \underline{\varphi} \leq \sum_{j \in N\backslash\{1\}} \{c_j(R(c_1^0, c_{-1}^{\pi^1,\, j})) - c_j(A_j)\} \leq \overline{\varphi} < \gamma. \tag{18}$$

Note that γ depends on c_{-1}, but not on c_1.

Let $f : 2^{\mathbb{A}}\backslash\{\emptyset\} \to \{1, 2, ..., 2^{|\mathbb{A}|} - 1\}$ be a bijection.

Let $\widetilde{c}_1 \in \mathcal{C}$ be such that for each $A \in 2^{\mathbb{A}}\backslash\{\emptyset\}$, $\widetilde{c}_1(A) = f(A)\gamma$.

Note that for each pair $\{A, A'\} \subseteq 2^{\mathbb{A}}$ with $A \neq A'$, since $f(A) \neq f(A')$,

$$|\widetilde{c}_1(A) - \widetilde{c}_1(A')| \geq \gamma \text{ and } \widetilde{c}_1(A) \neq \widetilde{c}_1(A'). \tag{19}$$

Let $\widetilde{c} = (\widetilde{c}_1, c_{-1})$. Note that for each $j \in N\backslash\{1\}$, $(c_1^0, c_{-1}^{\pi^1,j}) = \widetilde{c}^{\,\pi^1,j}$. Hence, by (18) and (19), for each $l \in N\backslash\{1\}$, each $\pi^l \in \Pi^l$, and each pair $\{A, A'\} \subseteq 2^{\mathbb{A}}$ with $A \neq A'$,

$$|\widetilde{c}_1(A) - \widetilde{c}_1(A')| \geq \gamma > |\sum_{j \in N\backslash\{1\}} [c_j(R(\widetilde{c}^{\,\pi^1,\, j})) - c_j(R(\widetilde{c}^{\,\pi^l,\, j})]|.$$

That is, if $A \neq A'$, then for each $l \in N\backslash\{1\}$ and each $\pi^l \in \Pi^l$, $\widetilde{c}_1(A) - \widetilde{c}_1(A') \neq \sum_{j \in N\backslash\{1\}} [c_j(R(\widetilde{c}^{\,\pi^1,\, j})) - c_j(R(\widetilde{c}^{\,\pi^l,\, j}))]$.

Hence, $\widetilde{c}_1(A) - \widetilde{c}_1(A') = \sum_{j \in N\backslash\{1\}} [c_j(R(\widetilde{c}^{\,\pi^1,\,j})) - c_j(R(\widetilde{c}^{\,\pi^l,\,j}))]$ implies that

$$A = A' \text{ and } \sum_{j \in N\backslash\{1\}} c_j(R(\widetilde{c}^{\,\pi^1,\,j})) = \sum_{j \in N\backslash\{1\}} c_j(R(\widetilde{c}^{\,\pi^l,\,j})). \tag{20}$$

Since $R \in \mathcal{R}$, by (6), for each $l \in N\backslash\{1\}$ and each $\pi^l \in \Pi^l$, $\sum_{j \in N} \widetilde{c}_j(R(\widetilde{c}^{\,\pi^l,\,j})) = \sum_{j \in N} \widetilde{c}_j(R(\widetilde{c}^{\,\pi^1,\,j}))$. By rearranging this equality,

$$\widetilde{c}_1(R(\widetilde{c}^{\,\pi^l,\,1})) - \widetilde{c}_1(R(\widetilde{c})) = \sum_{j \in N\backslash\{1\}} [c_j(R(\widetilde{c}^{\,\pi^1,\,j})) - c_j(R(\widetilde{c}^{\,\pi^l,\,j}))].$$

This equality and (20) together imply that for each $l \in N\backslash\{1\}$ and each $\pi^l \in \Pi^l$,

$$R(\widetilde{c}^{\,\pi^l,\,1}) = R(\widetilde{c}) \text{ and } \sum_{j \in N\backslash\{1\}} c_j(R(\widetilde{c}^{\,\pi^1,\,j})) = \sum_{j \in N\backslash\{1\}} c_j(R(\widetilde{c}^{\,\pi^l,\,j})). \tag{21}$$

By abuse of notation, let us represent a permutation $\pi \in \Pi$ as a list that shows the ordering of the agents: $\pi = (\pi^{-1}(1), \pi^{-1}(2), ..., \pi^{-1}(n))$. For instance, if $\pi = (i,j)$, then $\pi(i) = 1$ and $\pi(j) = 2$.

Consider $\widetilde{\pi}^n = (1,2,3,4,...,n)$. Note that $\widetilde{c}^{\,\widetilde{\pi}^n,\,n} = \widetilde{c}$ and for each $j \in N\backslash\{1\}$, $\widetilde{c}_j = c_j$. By (21), $R(\widetilde{c}^{\,\widetilde{\pi}^n,\,1}) = R(\widetilde{c})$. Hence, by (7),

$$\begin{aligned} h_1(\widetilde{c}_{-1}) &= \rho + \widetilde{c}_1(R(\widetilde{c}^{\,\widetilde{\pi}^n,\,1})) + \sum_{j \in N\backslash\{1,n\}} c_j(R(\widetilde{c}^{\,\widetilde{\pi}^n,\,j})) + c_n(R(\widetilde{c}^{\,\widetilde{\pi}^n,\,n})) - \widetilde{c}_1(R(\widetilde{c})), \\ &= \rho + \sum_{j \in N\backslash\{1,n\}} c_j(R(\widetilde{c}^{\,\widetilde{\pi}^n,\,j})) + c_n(R(\widetilde{c})). \end{aligned} \tag{22}$$

Claim: For each $j \in N\backslash\{1,n\}$, there exists $\pi^n \in \Pi^n$ such that

$$\widetilde{c}^{\,\widetilde{\pi}^n,\,j} = \widetilde{c}^{\,\pi^n,\,1}. \tag{23}$$

Proof of Claim: Let $j \in N\backslash\{1,n\}$. Then, $\widetilde{c}^{\,\widetilde{\pi}^n,\,j} = (\widetilde{c}_1, c_2, c_3, ..., c_j, c^0_{j+1}, ..., c^0_n)$. Let $\pi^n = (2,3,...,j,1,j+1,...,n)$. Then, $\widetilde{c}^{\,\pi^n,\,1} = \widetilde{c}^{\,\widetilde{\pi}^n,\,j}$.

By (21), for each $\pi^n \in \Pi^n$, $R(\widetilde{c}^{\,\pi^n,\,1}) = R(\widetilde{c})$. This equality and (23) together imply that for each $j \in N\backslash\{1,n\}$, $R(\widetilde{c}^{\,\widetilde{\pi}^n,\,j}) = R(\widetilde{c})$. Hence, by (22), $h_1(\widetilde{c}_{-1}) = \rho + \sum_{j \in N\backslash\{1\}} c_j(R(\widetilde{c}))$. This equality and the fact that $c_{-1} = \widetilde{c}_{-1}$ together imply that

$$h_1(c_{-1}) = \rho + \sum_{j \in N\backslash\{1\}} c_j(R(\widetilde{c})). \tag{24}$$

Part 2:

Let $i \in N\backslash\{1\}$ and $\pi^i \in \Pi^i$. Note that for each $j \in N\backslash\{i\}$, $c^{\pi^i,j} = (c^0_i, c^{\pi^i,j}_{-i})$ is independent of c_i. Let

$$\begin{aligned} \underline{\alpha} &\equiv \sum_{j \in N\backslash\{i\}} \min_{A_j \in 2^{\mathbb{A}}} \{c_j(R(c^0_i, c^{\pi^i,\,j}_{-i})) - c_j(A_j)\}, \\ \overline{\alpha} &\equiv \sum_{j \in N\backslash\{i\}} \max_{A_j \in 2^{\mathbb{A}}} \{c_j(R(c^0_i, c^{\pi^i,\,j}_{-i})) - c_j(A_j)\}. \end{aligned}$$

Let $\alpha = \max\{|\underline{\alpha}|, |\overline{\alpha}|\}$. Let $\widetilde{c}_1 \in \mathcal{C}$ be constructed as in Part 1. Let

$$\lambda = \max\left\{\alpha, \max_{A \in 2^{\mathbb{A}}}\{|\widetilde{c}_1(R(\widetilde{c}_1, c^0_{-1})) - \widetilde{c}_1(A)|\}, \max_{A \in 2^{\mathbb{A}}}\{|c^*_1(R(c^*_1, c^0_{-1})) - c^*_1(A)|\}\right\} + 1 > 0. \quad (25)$$

Let $g : 2^{\mathbb{A}}\backslash\{\emptyset\} \to \{1, 2, ..., 2^{|\mathbb{A}|} - 1\}$ be a bijection.

For each $A \in 2^{\mathbb{A}}\backslash\{\emptyset\}$, let $\widehat{c}_i(A) = g(A)\lambda$.

Let $\widehat{c} = (\widehat{c}_i, c_{-i})$. Note that for each $j \in N\backslash\{i\}$, $(c^0_i, c^{\pi^i, j}_{-i}) = \widehat{c}^{\,\pi^i,\, j}$.

Similar to Part 1, by (25), for each $l \in N\backslash\{i\}$, each $\pi^l \in \Pi^l$, and each pair $\{A, A'\} \subseteq 2^{\mathbb{A}}$ with $A \neq A'$,

$|\widehat{c}_i(A') - \widehat{c}_i(A)| \geq \lambda > \max\{|\sum_{j \in N\backslash\{i\}}[c_j(R(\widehat{c}^{\,\pi^i,\, j})) - c_j(R(\widehat{c}^{\,\pi^l,\, j})]|, |\widetilde{c}_1(R(\widetilde{c}_1, c^0_{-1})) - \widetilde{c}_1(A)|, |c^*_1(R(c^*_1, c^0_{-1})) - c^*_1(A)|\}$.

Hence, $\widehat{c}_i(A) - \widehat{c}_i(A') = \sum_{j \in N\backslash\{i\}}[c_j(R(\widehat{c}^{\,\pi^i,\, j})) - c_j(R(\widehat{c}^{\,\pi^l,\, j})]$ implies that

$$A = A' \text{ and } \sum_{j \in N\backslash\{i\}} c_j(R(\widehat{c}^{\,\pi^i,\, j})) = \sum_{j \in N\backslash\{i\}} c_j(R(\widehat{c}^{\,\pi^l,\, j}). \quad (26)$$

and $\widehat{c}_i(A') - \widehat{c}_i(A) = \widetilde{c}_1(R(\widetilde{c}_1, c^0_{-1})) - \widetilde{c}_1(A)$ implies that

$$A = A' \text{ and } \widetilde{c}_1(R(\widetilde{c}_1, c^0_{-1})) = \widetilde{c}_1(A). \quad (27)$$

and $\widehat{c}_i(A') - \widehat{c}_i(A) = c^*_1(R(c^*_1, c^0_{-1})) - c^*_1(A)$ implies that

$$A = A' \text{ and } c^*_1(R(c^*_1, c^0_{-1})) = c^*_1(A). \quad (28)$$

Part 2a: Since $R \in \mathcal{R}$, by (6), for each $l \in N\backslash\{i\}$ and each $\pi^l \in \Pi^l$, $\sum_{j \in N} \widehat{c}_j(R(\widehat{c}^{\,\pi^l,\, j})) = \sum_{j \in N} \widehat{c}_j(R(\widehat{c}^{\,\pi^i,\, j}))$. That is,

$$\widehat{c}_i(R(\widehat{c}^{\,\pi^l,\, i})) - \widehat{c}_i(R(\widehat{c})) = \sum_{j \in N\backslash\{i\}}[c_j(R(\widehat{c}^{\,\pi^i,\, j})) - c_j(R(\widehat{c}^{\,\pi^l,\, j}))].$$

This equality and (26) together imply that for each $l \in N\backslash\{i\}$ and each $\pi^l \in \Pi^l$,

$$R(\widehat{c}^{\,\pi^l,\, i}) = R(\widehat{c}). \quad (29)$$

Consider Part 1 that follows after equation (21). By a parallel argument to that part (instead of $\widetilde{\pi}^n$, use $\widehat{\pi}^1$ with $\widehat{\pi}^1(i) = 1$)), using (29), we obtain

$$h_i(c_{-i}) = \rho + \sum_{j \in N\backslash\{i\}} c_j(R(\widehat{c})). \quad (30)$$

Part 2b: Consider $\overline{c} = (\widetilde{c}_1, \widehat{c}_i, c^0_{-\{1,i\}})$ and $\overline{\overline{c}} = (c^*_1, \widehat{c}_i, c^0_{-\{1,i\}})$.

Let $\overline{\pi}^1 \in \Pi^1$ be such that $\overline{\pi}^1(i) = n-1$ (e.g., if $i=2$, then $\overline{\pi}^1 = (3,4,...,n,i,1)$). Let $\overline{\pi}^i \in \Pi^i$ be such that $\overline{\pi}^i(1) = n-1$. By (6),

$$\sum_{j\in N}\overline{c}_j(R(\overline{c}^{\,\overline{\pi}^1,\, j})) = \sum_{j\in N}\overline{c}_j(R(\overline{c}^{\,\overline{\pi}^i,\, j})) \text{ and } \sum_{j\in N}\overline{\overline{c}}_j(R(\overline{\overline{c}}^{\,\overline{\pi}^1,\, j})) = \sum_{j\in N}\overline{\overline{c}}_j(R(\overline{\overline{c}}^{\,\overline{\pi}^i,\, j})). \tag{31}$$

Since for each $j \in N\backslash\{1,i\}$, $\overline{c}_j = \overline{\overline{c}}_j = c^0_j$, by rearranging (31),

$$\begin{aligned}\widehat{c}_i(R(\widehat{c}_i, c^0_{-i})) - \widehat{c}_i(R(\overline{c})) &= \widetilde{c}_1(R(\widetilde{c}_1, c^0_{-1})) - \widetilde{c}_1(R(\overline{c})), \text{ and} &(32)\\ \widehat{c}_i(R(\widehat{c}_i, c^0_{-i})) - \widehat{c}_i(R(\overline{\overline{c}})) &= c^*_1(R(c^*_1, c^0_{-1})) - c^*_1(R(\overline{\overline{c}})). &(33)\end{aligned}$$

Equalities (27) and (32) together imply $R(\widehat{c}_i, c^0_{-i}) = R(\overline{c})$ and $\widetilde{c}_1(R(\widetilde{c}_1, c^0_{-1})) = \widetilde{c}_1(R(\overline{c}))$. These equalities and (19) together imply

$$R(\widehat{c}_i, c^0_{-i}) = R(\widetilde{c}_1, c^0_{-1}) = R(\overline{c}). \tag{34}$$

Similarly, (28) and (33) together imply $R(\widehat{c}_i, c^0_{-i}) = R(\overline{\overline{c}})$ and $c^*_1(R(c^*_1, c^0_{-1})) = c^*_1(R(\overline{\overline{c}}))$. These equalities and (17) together imply

$$R(\widehat{c}_i, c^0_{-i}) = R(c^*_1, c^0_{-1}) = R(\overline{\overline{c}}). \tag{35}$$

Consider $\widehat{\pi}^1 \in \Pi^1$ such that $\widehat{\pi}^1(i) = 1$. Then,

$$\widehat{c}^{\,\widehat{\pi}^1,\, i} = (\widehat{c}_i, c^0_{-i}). \tag{36}$$

By (29), $R(\widehat{c}^{\,\widehat{\pi}^1,\, i}) = R(\widehat{c}_i, c_{-i})$. Hence, by (36),

$$R(\widehat{c}_i, c^0_{-i}) = R(\widehat{c}_i, c_{-i}). \tag{37}$$

Consider $\widetilde{\pi}^n \in \Pi^n$ such that $\widetilde{\pi}^n(1) = 1$. Then,

$$\widetilde{c}^{\,\widetilde{\pi}^n,\, 1} = (\widetilde{c}_1, c^0_{-1}). \tag{38}$$

By (21), $R(\widetilde{c}^{\,\widetilde{\pi}^n,\, 1}) = R(\widetilde{c}_1, c_{-1})$. Hence, by (38),

$$R(\widetilde{c}_1, c^0_{-1}) = R(\widetilde{c}_1, c_{-1}). \tag{39}$$

By (34), (35), (37), and (39),

$$R(\widehat{c}) = R(\widetilde{c}) = R(c^*_1, c^0_{-1}). \tag{40}$$

Note that $R(c^*_1, c^0_{-1}) = \overline{A}$. Then, by (24), (30), and (40), for each $i \in N$,

$$h_i(c_{-i}) = \rho + \sum_{j\in N\backslash\{i\}} c_j(\overline{A}). \tag{41}$$

Since $R(c^*_1, c^0_{-1})$ is independent of $c \in \mathcal{C}^N$, and we applied Parts 1 and 2 to an arbitrary $c \in \mathcal{C}^N$, for each $c \in \mathcal{C}^N$ and each $i \in N$, $h_i(c_{-i}) = \rho + \sum_{j\in N\backslash\{i\}} c_j(\overline{A})$. Hence, $G^{h,\tau}$ is $\overline{A}$*−egalitarian-equivalent.* ■

Proof of Proposition 2:
Let $T \in \mathbb{R}$, $\rho \in \mathbb{R}$, $G^{h,\tau} \in \mathcal{S}^{\rho}$, and $\rho \leq \frac{T}{|N|}$. By (9), $G^{h,\tau}$ is $\emptyset-$*egalitarian-equivalent*. Note that for each $c \in \mathcal{C}^N$,

$$\sum_{i \in N} t_i^{h,\tau}(c) = -(n-1)W(c) + n\rho.$$

Since $\rho \leq \frac{T}{n}$, $\sum_{i \in N} t_i^{h,\tau}(c) \leq -(n-1)W(c) + T \leq T$. Hence, $G^{h,\tau}$ satisfies $T-$*bounded-deficit.*

Conversely, let $G^{h,\tau}$ be an *egalitarian-equivalent* Groves mechanism that satisfies $T-$*bounded-deficit.*
By $T-$*bounded-deficit*, for each $c \in \mathcal{C}^N$,

$$\sum_{i \in N} t_i^{h,\tau}(c) = -(n-1)W(c) + \sum_{i \in N} h_i(c_{-i}) \leq T. \tag{42}$$

By Theorem 1, there is $\overline{A} \in 2^{\mathbb{A}}$ such that $G^{h,\tau} \in \mathcal{G}^{\overline{A}}$. By (9) and (42), there is $\rho \in \mathbb{R}$ such that for each $c \in \mathcal{C}^N$, $-(n-1)W(c) + \sum_{i \in N} \sum_{j \in N \backslash \{i\}} c_j(\overline{A}) \leq T - n\rho$. Since $\sum_{i \in N} \sum_{j \in N \backslash \{i\}} c_j(\overline{A}) = (n-1)\sum_{j \in N} c_j(\overline{A})$, for each $c \in \mathcal{C}^N$,

$$\sum_{j \in N} c_j(\overline{A}) - W(c) \leq \frac{T - n\rho}{(n-1)}. \tag{43}$$

If $c = c^0$, then the left-hand-side of inequality (43) is 0. Hence, there is $\varepsilon \geq 0$ such that $\varepsilon = \frac{T-n\rho}{(n-1)}$. Then, by (43), for each $c \in \mathcal{C}^N$,

$$\sum_{j \in N} c_j(\overline{A}) \leq \varepsilon + W(c). \tag{44}$$

Let $\{i, l\} \subseteq N$ and $\widehat{c} \in \mathcal{C}^N$ be such that for each $A \in 2^{\mathbb{A}} \backslash \{\emptyset\}$, $\widehat{c}_i(A) > \varepsilon$ and $\widehat{c}_l(\mathbb{A}) = 0$. Then, for each $A \in 2^{\mathbb{A}} \backslash \{\emptyset\}$, $\sum_{j \in N} \widehat{c}_j(A) > \varepsilon$ and $W(\widehat{c}) = \widehat{c}_l(\mathbb{A}) = 0$. Thus, by (44), $\overline{A} = \emptyset$. That is, $G^{h,\tau}$ is $\emptyset-$*egalitarian-equivalent.* Then, by (9), $G^{h,\tau} \in \mathcal{S}^{\rho}$.
Since $G^{h,\tau}$ is $\emptyset-$*egalitarian-equivalent,* then, by (43), for each $c \in \mathcal{C}^N$,

$$n\rho - T \leq (n-1)W(c). \tag{45}$$

That is, $n\rho - T \leq \min_{c \in \mathcal{C}^N} \{(n-1)W(c)\}$. Since $\min_{c \in \mathcal{C}^N} W(c) = W(c^0) = 0$, by (45), we have $\rho \leq \frac{T}{n}$.
This completes the proof. ■

Proof of Corollary 3:

(i) Let $G^{h,\tau}$ be a Groves mechanism that satisfies *egalitarian-equivalence* and $T-$*bounded-deficit*. By Proposition 2, $G^{h,\tau}$ is such that for each $c \in \mathcal{C}^N$ and each $i \in N$, $h_i(c_{-i}) = \rho \leq \frac{T}{|N|}$. Then, by Lemma 1, $u(G_i^{h,\tau}(c); c_i) = -W(c) + \rho$. Since utility is increasing in ρ, $G^{h,\tau}$ Pareto-dominates all Groves mechanisms that satisfy *egalitarian-equivalence* and $T-$*bounded-deficit* if and only if $\rho = \frac{T}{|N|}$. That is, $G^{h,\tau} \in \mathcal{S}^{\frac{T}{|N|}}$.

For each $c \in \mathcal{C}^N$, the budget surplus is $-\sum_{i \in N} t_i^{h,\tau}(c) = (n-1)W(c) - \sum_{i \in N} h_i(c_{-i})$. Hence, the surplus is minimal if and only if $\sum_{i \in N} h_i(c_{-i})$ is maximal. Note that $\sum_{i \in N} h_i(c_{-i})$ is maximal if and only if $\rho = \frac{T}{|N|}$. Hence, $G^{h,\tau}$ generates minimal surplus among all Groves mechanisms that satisfy *egalitarian-equivalence* and $T-$*bounded-deficit* if and only if $G^{h,\tau} \in \mathcal{S}^{\frac{T}{|N|}}$.

(ii) The proof follows from part (i) and the fact that *no-deficit* is $T-$*bounded-deficit* where $T = 0$. ■

Proof of Proposition 3:

(i) It is easy to see that *anonymity* implies *equal treatment of equals*. Conversely, let $G^{h,\tau}$ be Groves mechanism that satisfies *equal treatment of equals*. Let $c \in \mathcal{C}^N$. We will show that for each $i \in N$ and each bijection $\pi : N \to N$, $h_i(c_{-i}) = h_{\pi(i)}(\pi(c)_{-\pi(i)})$ where $\pi(c) \equiv (c_{\pi(l)})_{l \in N}$. Let $\{i,j\} \subseteq N$ and $\widehat{c} \in \mathcal{C}^N$ be such that $\widehat{c}_i = c_j$ and $\widehat{c}_{-i} = c_{-i}$. Since $\widehat{c}_i = \widehat{c}_j$, by *equal treatment of equals* and Lemma 1,

$$h_i(\widehat{c}_{-i}) = h_j(\widehat{c}_{-j}). \tag{46}$$

Next, let $\pi' : N \to N$ be a bijection such that $\pi'(i) = j$, $\pi'(j) = i$, and for each $l \in N\backslash\{i,j\}$, $\pi'(l) = l$. Let $\pi'(c) \equiv (c_{\pi'(l)})_{l \in N}$. Note that $\widehat{c} = \pi'(\widehat{c})$. Hence, $\widehat{c}_{-j} = \pi'(\widehat{c})_{-\pi'(i)} = \pi'(c)_{-\pi'(i)}$. These equalities, (46), and the fact that $\widehat{c}_{-i} = c_{-i}$ together imply that $h_i(c_{-i}) = h_{\pi'(i)}(\pi'(c)_{-\pi'(i)})$. Note that for any bijection $\pi : N \to N$, starting from c, we can obtain $\pi(c)$ by carrying out of a finite sequence of pair-wise switching of labels of two agents one of whom is always in the i^{th} position. Hence, $G^{h,\tau}$ is *anonymous*.

(ii) Let $G^{h,\tau}$ be an *order preserving* Groves mechanism. Since *order preservation* implies *equal treatment of equals*, by Proposition 3 (i), $G^{h,\tau}$ is *anonymous*.

(iii) Let $G^{h,\tau}$ be an *egalitarian-equivalent* Groves mechanism. Let $\{i,j\} \subseteq N$ and $c \in \mathcal{C}^N$ be such that $c_i \geq c_j$. Then, by (7) (which is equivalent to (8)), $h_i(c_{-i}) \leq h_j(c_{-j})$. By Lemma 1, $G^{h,\tau}$ *preserves order*.

(iv) Let $G^{h,\tau}$ be an *envy-free* Groves mechanism. Assume, by contradiction, that $G^{h,\tau}$ does not *preserve order*. Then, there are $\{i,j\} \subseteq N$ and $c \in \mathcal{C}^N$ such that $c_i \geq c_j$ and $u(G_i^{h,\tau}(c); c_i) > u(G_j^{h,\tau}(c); c_j)$. By Lemma 1, $h_i(c_{-i}) > h_j(c_{-j})$. Then,

$$\begin{aligned} c_i(A_i^\tau(c)) - W(c) + h_i(c_{-i}) &> c_j(A_i^\tau(c)) - W(c) + h_j(c_{-j}), \\ -c_j(A_i^\tau(c)) - [W(c) - c_i(A_i^\tau(c))] + h_i(c_{-i}) &> -W(c) + h_j(c_{-j}), \\ -c_j(A_i^\tau(c)) + t_i^{h,\tau}(c) &> -W(c) + h_j(c_{-j}). \end{aligned}$$

This inequality and Lemma 1 together imply $u(G_i^{h,\tau}(c); c_j) > u(G_j^{h,\tau}(c); c_j)$, which contradicts *no-envy*.

(v) Let $G^{h,\tau}$ be an $\emptyset-$*egalitarian-equivalent* Groves mechanism. Let $\mathcal{C}_{ad}$ be the domain of *additive* cost functions and $\mathcal{C}_{sub}$ be the *subadditive* domain. Let $\mathcal{C} \in \{\mathcal{C}_{ad}, \mathcal{C}_{sub}\}$. Assume, by contradiction, that $G^{h,\tau}$ is not *envy-free* on $\mathcal{C}$. Then, there are $\{i,j\} \subseteq N$ and $c \in \mathcal{C}^N$ such that $u(G_i^{h,\tau}(c); c_i) < u(G_j^{h,\tau}(c); c_i)$. This inequality and Lemma 1 together imply

$$-W(c) + h_i(c_{-i}) < -c_i(A_j^\tau(c)) + t_j^{h,\tau}(c). \tag{47}$$

By $\emptyset-$*egalitarian-equivalence* and Corollary 1, there is $\rho \in \mathbb{R}$ such that

$$h_i(c_{-i}) = h_j(c_{-j}) = \rho. \tag{48}$$

By Lemma 1, $-c_j(A_j^\tau(c)) + t_j^{h,\tau}(c) = -W(c) + h_j(c_{-j})$. This equality, (47), and (48) together imply $c_i(A_j^\tau(c)) < c_j(A_j^\tau(c))$. Hence,

$$c_i(A_i^\tau(c)) + c_i(A_j^\tau(c)) < c_i(A_i^\tau(c)) + c_j(A_j^\tau(c)). \tag{49}$$

Note that on the *subadditive* domain, $c_i(A_i^\tau(c) \cup A_j^\tau(c)) \leq c_i(A_i^\tau(c)) + c_i(A_j^\tau(c))$, which holds as an equality on the *additive* domain. This inequality and (49) together imply that it is less costly to assign both $A_i^\tau(c)$ and $A_j^\tau(c)$ to agent i rather than assigning these sets to i and j, respectively. This contradicts that $A^\tau(c)$ is an efficient assignment. ■

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