Risk-hedging using options for an upgrading investment in a data network

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***Abstract*—In this paper, we illustrate how a mobile data network operator can plan an upgrading investment to anticipate explosions of the demand, taking into account the expected generated profit and the customers satisfaction. The former parameter grows with the demand, whereas the latter sinks if the demand is too high as throughput may collapse. As the equipment price decreases with time, it may be interesting to wait rather than to invest at once. We then propose a real option strategy to hedge against the risk that the investment has to take place earlier than expected. At last, we price this option with a backward dynamic programming approach, using recent improvements based on least-squares estimations.**

***Index Terms*—data network, investment, real option, dynamic programming, least-squares.**

I. Introduction

Today, it is expected that the data traffic will be significantly growing in mobile networks. The total amount of transferred data is supposed to grow exponentially, as it has been the case since the middle 90' for the Internet. To face these soaring volumes of data to be transferred, mobile operators must periodically upgrade their equipments to offer higher throughputs and avoid blocking problems. However as the demand does not increase steadily and must be considered as partly random, the expected profit is difficult to be forecast.

In this article, we consider upgrade investments in a HSDPA cellular network. Note that, when demand increases, the data communication duration becomes longer and longer for each user until the network is saturated. The individual throughput experienced in the network may become very small. On the other hand, as the demand rises, the operator increases its profit. When the network starts experiencing saturation problems, throughput and profit may fall. The operator must then upgrade its network by adding new frequency carriers, facing the following trade-off:

- The later the investment, the lower individual throughputs and customer satisfaction. Permanent non-satisfaction will result into churn and additional loss of profit.
- The sooner the investment, the more expensive the costs of upgrade elements.

This article aims at modeling analytically the trade-off. We first derive analytical values for capacity, individual throughput and satisfaction as a function of the demand, and use them to calculate operator's profit, taking into account randomness of the rising demand, and decrease of network element costs according to time. We second introduce a real options method to hedge against the risk that demand evolves in an unexpected way leading to a premature investment decision or a too late one. To perform that, we introduce an American call that allows its owner (the mobile operator) to buy an equipment at a fixed price, possibly less than the real one, until a maturity date. Given the profit analytical model and the option's parameters, we propose a dynamic programming method to price the option. At the same time, we obtain the expected best investment date.
Note that in a previous work [2], we already used the profit model to find the best investment date by an actualization algorithm. However, we did not use risk-hedging nor dynamic programming. Note that d'Halluin [1] presented a work on a method to determine the best investment date in a wireless network. His approach was based on dynamic programming, but he did not introduce any risk hedging method nor an option pricing. Conversely, Longstaff & Schwartz [11] introduced a pricing method for american options, but his work was not adapted to telecommunication networks investment.

The remainder of this paper is organized as follows: in a first section, we build an analytical model of the operator's profit based on an HSDPA network. Then we introduce in section II an american option to hedge against the risk aforementioned. We define the underlying asset and the option's payoff. To price the option we use a risk-neutral approach, whose mathematical justification lies in the appendix. In section III, we show how dynamic programming can help solving the pricing problem, and the best investment date problem as well. In section IV, we present the numerical results before concluding the paper.

II. The basic model: operator's profit and investment cost

The operator profit depends on the amount of data flowed by the network. It thus depends on the mean traffic demand per cell (in Mbits/sec/cell). We denote it by X_t, where the time $t = 0, 1...$ varies *discretely*, for example day by day. Note that profit does not necessarily increase when demand grows. It in fact also depends on the customer satisfaction, that we shall calculate hereafter.

A. Traffic Demand

We assume that the network is formed by circular cells of radius R, with a uniformly distributed demand. We have:

$$X_t = \lambda_t \times E[\xi],$$

where λ_t is the arrival rate per cell at the date t and $E[\xi]$ is the mean size of a typical data flow. In the following we assume that $E[\xi]$ remains stable during $[0, T]$, so that X_t is proportional to λ_t. This model is strictly equivalent to the case where the number of active users is constant, but they initiate connections more often.

To model the evolution of $(X_t)_{t\in\mathbb{N}}$, let us consider it as the daily *sampling* of a continuous stochastic process $(\tilde{X}(t, W_t))_{t\in\mathbb{R}^+}$. Usually, one monitors the 24 demands over one hour each, keeps the second or third highest, and multiply it by a given factor. As many random phenomena related to a social behavior (e.g. [3]), we assume that $\tilde{X}(t, W_t)$ is a geometric brownian motion (see [6] page 88):

$$\tilde{X}(t, W_t) = x_0 \mathrm{e}^{(\alpha - \sigma^2/2)t + \sigma W_t}, \quad t \in \mathbb{R}^+ ,$$

where W_t is a standard brownian motion, α is the trend of the demand and σ is its volatility.

B. Recapitulation of the HSDPA model

To evaluate the customer satisfaction, let us first recall the flow throughput $\gamma_t(r)$ a user can expect at distance r from the center of the cell as carried out in [5]. The resource of a single downlink data channel is time-shared between active users. Denote by ϕ_u the fraction of time the BS transmits to user u, with $\sum_u \phi_u$ = 1. The data rate of user u is then $C_u(r) = C(r) \times \phi_u$, where $C(r)$ is the peak data rate, obtained in the absence of any other user in the cell, i.e., for ϕ_u = 1. When there are x users, the "fair power" sharing is defined by $\phi_u = \frac{1}{x}$. We have:

$$C(r) = min\left(C_0, \frac{Z}{b} \times \frac{\Gamma(r)}{\eta + I(r)}\right),$$

where:

- C_0 is the maximum peak rate (which depends on channel bandwidth and coding efficiency)
- Z is the cell chip rate
- b is a lower bound for energy-per-bit to noise density ratio (E_b/N_0)
- $\Gamma(r)$ is the path loss between the BS and user u
- η is the thermal noise to received power ratio
- $I(r)$ is the interference to received power ratio.

Fig. 1 shows the peak data rate in a hexagonal cell. We only take interference from immediate adjacent cells in account, and we assume that the path loss $\Gamma(r)$ decreases according to $1/r^4$.

We define the cell load as (see [5]):

$$\rho_t = \frac{X_t}{\pi R^2} \int_0^R \frac{2\pi r dr}{C(r)}.$$

- if $\rho_t > 1$, the cell is overloaded and one can show ([5]) that the number of active users grows indefinitely; any individual data rate tends to zero, and the cell is saturated

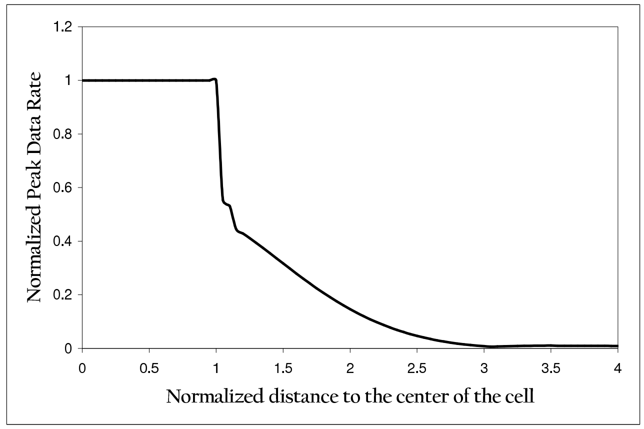

Fig. 1. Peak data rate against distance to the center of the cell. The distance and the rate are normalized w.r.t. the cell's radius and to C_0. Z/b = 3 Mbits/sec, η = 10%.

- if $\rho_t < 1$, the cell is underloaded and the number of active users tends to a finite stationary regime.

We are naturally led to introduce:

$$X_{max} = \pi R^2 \left(\int_0^R \frac{2\pi r dr}{C(r)}\right)^{-1}, \qquad (1)$$

such that $\rho_t = X_t/X_{max}$. Hence $\rho_t < 1 \iff X_t < X_{max}$. Let us denote the flow throughput of users at distance r by $\gamma_t(r)$ (it is the ratio of the mean flow size to the mean flow duration). Then, it can be shown (see [5]) that, if $\rho_t < 1$:

$$\gamma_t(r) = C(r)(1 - \rho_t). \qquad (2)$$

By assumption, the mean flow size (the numerator of $\gamma_t(r)$) does not vary significantly, whereas the mean flow duration (the denominator) does as the load increases. Thus, if we want to compute the mean flow throughput, the significant number to calculate is the *harmonic* mean of $\gamma_t(r)$ over the cell (balanced by the proportion of active users between r and $r + dr$). In other words, we have to calculate the arithmetic mean of $1/\gamma_t(r)$ over the cell. From (2) we deduce:

$$\overline{\gamma_t} = R^2 \left(\int_0^R \frac{2r dr}{\gamma_t(r)}\right)^{-1} = X_{max} - X_t,$$

and of course, in overload, $\overline{\gamma_t}$ = 0. Finally, we can summarize the whole calculation by:

$$\overline{\gamma_t} = (X_{max} - X_t)^+, \qquad (3)$$

where $x^+ = max(x, 0)$. Note that $\overline{\gamma_t} = 0$ if and only if the cell is saturated.

C. Customers satisfaction

Now we can compute the customer satisfaction, which can reasonably be supposed to depend on $\overline{\gamma_t}$. Since subjective satisfactions have been shown to be more sensitive to small variations at low throughputs than at high throughputs, Enderlé and Lagrange propose in [8] to model the customer satisfaction as a negative exponential function of the throughput:

$$S_t = \mathrm{e}^{-\beta/(X_{max}-X_t)^+}.$$

For example, β can be chosen as: $\beta = log(2) \cdot \gamma_{1/2}$, where $\gamma_{1/2}$ is the throughput value ensuring a satisfaction of 50%. Once again, note that $S_t = 0$ if and only if $X_t \geq X_{max}$, i.e., if and only if the cell is saturated.

D. Daily Profit

Let us recall that X_t is sampled day by day, for example during the second or the third highest peak hour. Within 24 hours, the operator transmits $\mu \min(X_t, X_{max})$ to active users, where μ is a multiplicative factor between the peak hour and the whole day. Typically, we can consider that the peak hour represents 25 % of the total daily transfer. This leads to $\mu \approx 4 \cdot 3600 \approx$ 14000 sec. Since taxation is applied to the volume of transfers and not to the duration, the gross daily profit per cell is given by:

$$\pi^{gross} = \delta\, \mu \min(X_t, X_{max}), \quad (4)$$

where δ is the transfer price (say in \$/Mbit). However the gross profit should be weighed by the customer satisfaction to account for the quality of the communications. The *net profit* is thus calculated as the product of π^{gross} by S_t:

$$\pi^{net} = \delta\, \mu \min(X_t, X_{max}) \mathrm{e}^{-\beta/(X_{max}-X_t)^+}.$$

If $S_t = 0$, i.e. if the cell is saturated, the net profit is null. If the satisfaction is maximal, i.e., $S_t = 1$, the net profit is equal to the gross profit (4). To sum up we have:

$$\boxed{\begin{array}{ll} \pi_t = \delta\, \mu\, X_t \mathrm{e}^{-\beta/(X_{max}-X_t)} & \textit{if } X_t < X_{max} \\ \pi_t = 0 & \textit{otherwise}. \end{array}}$$

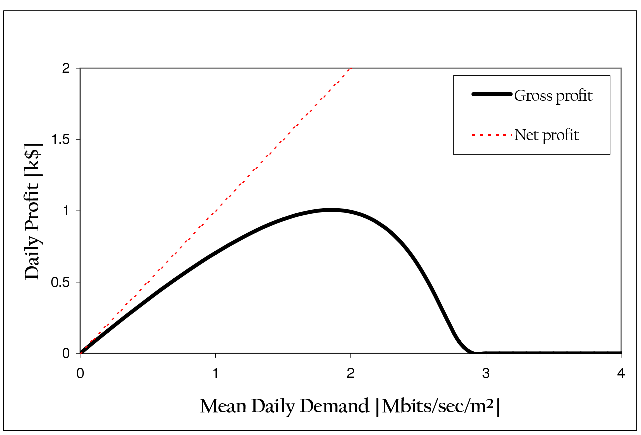

Fig. 2. Daily Profit generated by demand. X_{max} = 3 Mbits/sec/cell, δ = 0.1 \$/Mbit, μ = 14000 sec, β = 0.7 Mbit/sec (which corresponds to $\gamma_{1/2} \approx 1$ Mbit/sec).

Intuitively, as the demand rises, X_t will increase as will the profit (Fig. 2). Then, the profit will decrease because the unsatisfaction effect becomes dominant.

E. Upgrading Investment

As can be seen from Figure 2, if no upgrading action is taken, the profit will progressively tend to zero. Once the operator decides to upgrade, he can install additional transmitters operating on different frequency bands. In such a case we obtain a higher value of X_{max}, so that:

$$\pi'_t = \delta\, \mu \min(X_t, X'_{max}) \mathrm{e}^{-\beta/(X'_{max}-X_t)^+}.$$

The upgrading cost is a decreasing function of time. The decrease of the cost is due to many factors, for example the R&D progress, and also the serialization in the manufacturing chain. Moore's law states that electronic devices' capacity doubles every 18 months. So in this paper we assume it decreases exponentially:

$$K(t) = K_0 \mathrm{e}^{-\epsilon t},$$

where ϵ is the depreciation rate.

F. Total Profit

Let us introduce date T, at which the investment becomes obsolete (in other words, the proposed investment cannot be undertaken after T).If we denote the investment date by t_0 ($0 < t_0 < T$), the total profit $\Pi_T(t_0)$ actualized at the date $t = T$ is:

$$\Pi_T(t_0) = \sum_{t=0}^{t_0-1} \mathrm{e}^{\zeta(T-t)} \pi_t + \sum_{t=t_0}^{T} \mathrm{e}^{\zeta(T-t)} \pi'_t, \quad (5)$$

where ζ is the actualization rate. For simplicity, we assume that ζ is constant during the period $[0, T]$.

III. Risk-hedging using an American option

A. Externalizing the financial risk

As shown above, there is a tradeoff between the growth of the demand (encouraging to invest) and the depreciation of the equipment cost (encouraging to wait). Then the risk is to be led to invest while the equipment is still expensive. In this section we show how to hedge against this risk using an American option. This option, acquired from a third party like a bank, gives us the right to buy the equipment at price $K^\star$ instead of $K(t)$, until date $t^\star = t(K^\star)$ (see Fig. 3). Let us recall that the operator has the right but not the obligation to exercise this option, but has to pay in return a *premium* to the bank, denoted by P. If he has to invest before date $t^\star$, he will exercise the option, give $K^\star$ to the equipment provider and the bank will pay the difference. Otherwise, he will not exercise the option and he will lose the premium, but he still can invest.

In this section we will try to answer the two following questions:

- when is this option going to be exercised ?
- how much does it cost (i.e. calculate P) ?

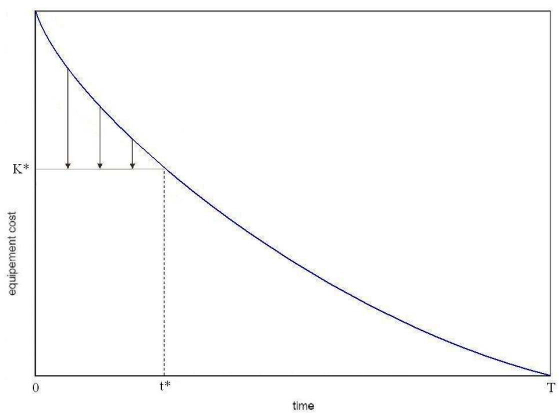

Fig. 3. exponential decrease of the cost

B. Introducing the American option

When is the option going to be exercised ? It depends on the additional profit expected from investing to upgrade the network: at least, this additional profit has to be greater than $K^\star$. At date t, it can be expressed as follows:

$$S_t = \mathbb{E}\left[\int_t^T \mathrm{e}^{-\zeta(s-t)}(\pi'(s,W_s)-\pi(s,W_s))\,ds \;\middle|\; \mathcal{F}_t\right]. \quad (6)$$

Facing the decision to invest or not, the operator's strategy is to compare the profit realized if investing with the value of waiting, typically to check that the traffic is not going to decrease unexpectedly which would make the upgrading expenditure a sunk cost. This appears to be the classical problem of finding the exercise strategy for an American option, with the following features:

- $t^\star$ as the option's maturity
- $K^\star$ as the exercise price or strike
- S_t as the underlying asset
- $(S_t - K^\star)^+$ as the option's payoff, denoted by $Z(t)$:

$$Z(t) = \max\{S_t - K^\star, 0\} \quad (7)$$

C. Pricing of the American option

The resolution of this problem appeals to classical stochastics theory and the risk-neutralization approach: see [10][6].

1) Preliminaries: to detail this approach, let us introduce two progressively measurable processes μ_t and κ_t, respectively the expected total return on the asset and its volatility, so that:

$$dS_t/S_t = \mu_t\,dt + \kappa_t\,dW_t \quad (8)$$

along with the market price of risk:

$$\theta_t = \kappa_t^{-1}(\mu_t - \zeta).$$

We obtain expressions of μ_t, κ_t and θ_t in the appendices A to C, where we show that:

$$\theta_t = -\frac{\pi'(t,W_t)-\pi(t,W_t)}{\frac{\partial v}{\partial x}(t,W_t)},$$

with:

$$v(t,x) = \int_t^T \mathbb{E}\left[\mathrm{e}^{-\zeta(s-t)}(\pi'(s,W_s)-\pi(s,W_s))|W_t = x\right]\,ds. \quad (9)$$

Note that applying the risk-neutralization approach will also require Novikov's condition (see [7], page 65), which states that:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2\,dt\right)\right] < +\infty. \quad (10)$$

In the appendix D, we show that Novikov's condition is verified in our specific case.

2) The risk-neutralization approach: under this condition, let $\mathcal{S}([t,t^\star])$ be the set of stopping times with values in $[t,t^\star]$ and define the following process known as the Snell envelope:

$$Y_t = \sup_{\tau\in\mathcal{S}([t,t^\star])} \mathbb{E}_{\mathbb{Q}^\star}\left[\mathrm{e}^{-\zeta\tau}Z(\tau)\middle|\mathcal{F}_t\right]. \quad (11)$$

Here, $\mathbb{Q}^\star$ is the risk-neutral probability, whose density w.r.t. $\mathbb{P}$, the historical probability, is:

$$\frac{d\mathbb{Q}^\star}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^{t^\star}\theta_s^2\,ds - \int_0^{t^\star}\theta_s\,dW_s\right).$$

In fact, since we will have to simulate trajectories of the asset beyond date $t^\star$ (until date T), we will rather choose the probability $\mathbb{Q}$, whose density w.r.t. $\mathbb{P}$ is:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L_T = \exp\left(-\frac{1}{2}\int_0^T\theta_s^2\,ds - \int_0^T\theta_s\,dW_s\right). \quad (12)$$

Note that $\mathbb{Q}$ is indeed a probability measure, since $\mathbb{E}_\mathbb{P}[L_T] = 1$, as θ_t verifies Novikov's condition (see previous paragraph). Note also that $\mathbb{Q}^\star$ is the restriction of $\mathbb{Q}$ to $\mathcal{F}_{t^\star}$ (see [6], Theorem 9.1.2.), so that (11) still holds with $\mathbb{Q}$ if $t \le t^\star$. Then the premium of the option at any time $t \in [0,t^\star]$ is given by [10]:

$$\Pi_t = \mathbb{E}_\mathbb{Q}\left[\mathrm{e}^{-\zeta(\tau(t)-t)}Z(\tau(t))\middle|\mathcal{F}_t\right], \quad (13)$$

where $\tau(t)$ is the solution of the maximization in (11). $\tau(t)$ is interpreted as the optimal exercise strategy of the option calculated at date t [(1)].

IV. The dynamic programming solution

As stated above, the problem is to find the stopping time maximizing the option's payoff under risk neutrality (Eqn. (11)). However, it is impossible to compute $Z(t)$ analytically, so we make use of a dynamic programming approach, as in [11]. We recall that it consists in dividing the problem into two binary decisions at the final date $t^\star$: the "immediate" one and its generated value, and the "delaying" one and its

[1]Note that if the option were a european option, the price at date t would be:

$$\Pi_t = \mathbb{E}_\mathbb{Q}\left[\mathrm{e}^{-\zeta(t^\star-t)}Z(t^\star)\middle|\mathcal{F}_t\right]$$

(see [6] page 65). But here, our option is an American option, so we have to generalize this result and to use the Snell envelope.

continuation value. Then moving backward, and repeating the same binary decision, we obtain the expected optimal time which lies in an expected interval in which the investment should be undertaken [4]. We must then, at each moment, find two different values: the option's payoff in case of investment and the continuation value in case of waiting.

A. *Monte-Carlo simulations to generate the underlying asset*

Calculating S_t involves a complex integration (Eqn. (6)) that cannot be performed analytically. We then use Monte-Carlo simulations as follows:

- first we compute $v(t,x)$ with Eqn. (9) for $t \in [0, t^\star]$ and $x \in [w_{min}, w_{max}]$ [2].
- then we discretize time: $t = t_0 \dots t_N$ with $t_0 = 0$ and $t_N = t^\star = N\ \delta t$. After that we simulate J trajectories of S_t under $\mathbb{Q}$: the *j-th* trajectory is denoted by (S^j) and has the value S_n^j at time $t_n = n\ \delta t$. More precisely, we simulate (under $\mathbb{Q}$) J trajectories of the historical brownian (W^j) [3], and then we compute $S_n^j = v(t_n, W_n^j)$ by interpolating $v(t,x)$. This is far more efficient than computing directly the integral, especially if we want to simulate a large number of trajectories, since we do not have to compute v each time again. To know how we interpolate a surface, see appendix F.

B. *Continuation value and decision tree algorithm*

At time $t^\star$, the operator invests if $Z_N > 0$. More generally, at a time $t_n < t^\star$, the operator has two alternative choices: either invest now and get Z_n, or wait and get the expected continuation value, denoted by C_n. The generated cash-flow is then given by:

$$F_n = \max\{Z_n, C_n\}.$$

We already know Z_n by (7). As for C_n, we use the Least Squares Monte-Carlo (LSM) approach defined by Longstaff and Schwartz [11]. This approach consists in writing the expected continuation value C_n as a general function of S_n (in our case we took a 2-degree polynom), taking information from the J cash-flows at t_{n+1} and using the fact that:

$$C_n(S) = \mathrm{e}^{-\zeta \delta t} \mathbb{E}[F_{n+1} | S_n = S],$$

where F_{n+1} is the (random) cash-flow of the option at t_{n+1}. To obtain recursively C_n, we can write the following algorithm:

- at t_N, for each trajectory $j = 1 \dots J$, calculate the cash-flow $F_N^j = Z_N^j$.
- move one period back to t_{N-1}. For each (S^j), check if the option is "in the money", i.e. if $Z_{N-1}^j > 0$. If it is the case, calculate the continuation value C_{N-1}^j using the cash-flow if investment is delayed: $C_{N-1}^j = \mathrm{e}^{-\zeta \delta t} F_N^j$. Estimate then the general expression of $C_{N-1}(S)$ by the LSM algorithm. This consists in regressing the found values C_{N-1}^j on a constant, S and S^2, as in [11] (see appendix E). Let us denote the estimated expression by $\hat{C}_{N-1}(S)$. The estimated cash-flow at $N-1$ is then given by:
$$F_{N-1}^j = \max\{Z_{N-1}^j, \hat{C}_{N-1}(S_{N-1}^j)\}. \qquad (14)$$
If it is optimal to exerce at t_{N-1}, then by convention F_N^j becomes 0 (because the option can only be exercised once).
- for each time t_n, repeat the same process until $n = 0$.

Let us denote by O_n the set of the j such that $Z_n^j = 0$, and by I_n the set of the j such that $Z_n^j > 0$. Here is a summary of the whole algorithm:

1. simulate J trajectories (S^j) under $\mathbb{Q}$
2. for $j = 1 \dots J$, put $F_N^j = Z_N^j$
3. for $n = (N-1) \dots 1, 0$:
   3.1. for $j = 1 \dots J$, calculate Z_n^j:
   - if $Z_n^j = 0$, $j \in O_n$
   - if $Z_n^j > 0$, $j \in I_n$

   3.2. process O_n and I_n separately:

$\forall j \in O_n$:	$\forall j \in I_n$:
put $F_n^j = \mathrm{e}^{-\zeta \delta t} F_{n+1}^j$	- regress $C_n^j = \mathrm{e}^{-\zeta \delta t} F_{n+1}^j$ on 1, S and S^2 to obtain a 2-degree polynom $\hat{C}_n(S)$ - put $F_n^j = \max\{Z_n^j, \hat{C}_n(S)\}$ - if $Z_n^j > \hat{C}_n(S)$, then n is the new investment date, so put $F_m^j = 0 \quad \forall m > n$

C. *Option premium*

Averaging the F_0^j, and using (13) and the law of large numbers, we obtain the premium Π_0 of the option:

$$\Pi_0 \approx \frac{1}{N} \sum_{j=1}^{J} F_0^j.$$

D. *Expected investing time*

Investigating our decision tree, it can happen that for some j we do not decide to invest before $t^\star$. Then we will be lead to invest between $t^\star$ and T[4]. For such trajectories, we do not know when the investment takes place. Furthermore, even for the other trajectories, additional information between $t^\star$ and T can be useful to adjust the value of the investment date. For these two reasons, we decide to simulate S_t further until

[2] to bind efficiently the brownian motion, see Appendix D.

[3] to perform that, assuming that the probability of our random generator is $\mathbb{Q}$, we simulate a standard brownian motion $(W_t^{\mathbb{Q}})$, and then using Girsanov's theorem (see [6], Theorem 9.4.5.), we build by recursion a new brownian motion (W_t) under $\mathbb{P}$, such that $W_t = W_t^{\mathbb{Q}} - \int_{0 \le s < t} \theta(s, W_s)\ ds$.

[4] Note that in theory, it could happen that we never decide to invest, even after T. However, given the deterministic trend of the demand, this would mean that W_t remains extremely low. Considerations on the brownian motion (see appendix D) ensure that in practice it will not happen.

T [5]. Thus we perform one more time a backward dynamic algorithm, that time between 0 and T, using:

$$\begin{cases} Z(t) = (S_t - K(t))^+ & \text{if } t > t^\star \\ Z(t) = (S_t - K^\star)^+ & \text{otherwise} \end{cases}. \qquad (15)$$

Finally, we obtain for each of the J trajectories a best investment date T^j_{inv}. If $T^j_{\text{inv}} > t^\star$, it means that we have invested without exercising the original option, whereas if $T^j_{\text{inv}} \leq t^\star$, it means that we have exercised the option. Averaging the T^j_{inv}, we obtain the expected investing time under the risk-neutral probability $\mathbb{E}_{\mathbb{Q}}[T_{\text{inv}}]$. But for us, it is more relevant to calculate $\mathbb{E}_{\mathbb{P}}[T_{\text{inv}}]$. Using (12), we obtain:

$$\mathbb{E}_{\mathbb{P}}[T_{\text{inv}}] = \mathbb{E}_{\mathbb{Q}}\left[\frac{T_{\text{inv}}}{L_T}\right] \approx \frac{1}{N}\sum_{j=1}^{J}\frac{T^j_{\text{inv}}}{L^j_T}. \qquad (16)$$

V. Numerical results

In order to illustrate our algorithm, we applied it using the free simulator Scilab (see [9]). We considered a HSDPA pure data network with a random growing demand, as described in section II-A. We used the following parameters for our computation:

- the investment can take place until T = 150 days.
- the equipment can be purchased at the initial price K_0 = 300000 \$, and its price decreases with a rate ϵ of 50% per year.
- the actualization rate ζ is fixed to 5% per year.
- the traffic demand starts at x_0 = 1.2 Mbit/sec/cell, and increases with a drift fixed to α = 0.54% per day. Its volatility is fixed to 0.01 day$^{-1/2}$. Its maximal value is fixed to X_{max} = 3 Mbit/sec/cell before the investment, and to X'_{max} = 8 Mbit/sec/cell after the investment.
- the data transfer price is fixed to δ = 0.1 \$/Mbit.
- we take a satisfaction parameter of β = 0.7 Mbit/sec/cell.
- we simulate 10000 different trajectories of the asset.

A. Option's price

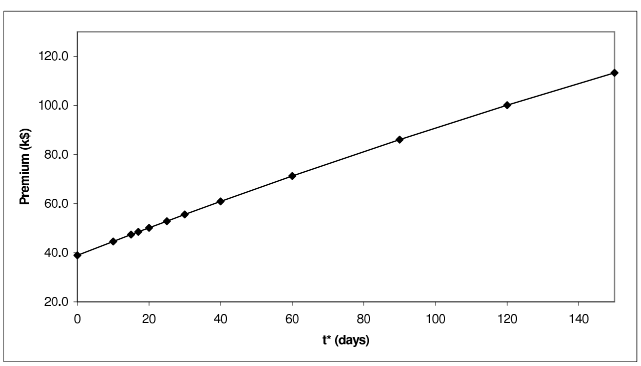

Fig. 4. Price of the option

On Figure (4), we represent the price of the option versus $t^\star$. Recall that the price is obtained with equation (13), where $t^\star$ implicitly appears in function Z (see equation (7)). It appears that the price increases with $t^\star$. This was expected, since the longer the option's maturity is, the higher the risk for the bank is, and then the more expensive the option is.

[5]That is the reason why we chose $\mathbb{Q}$ instead of $\mathbb{Q}^\star$.

B. Investment date

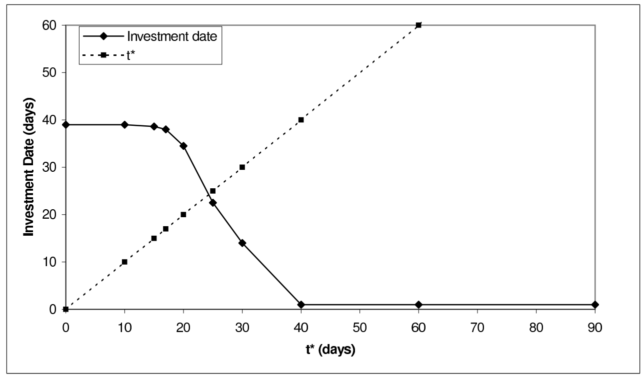

Fig. 5. Investment date

On Figure (5), we represent the investment date versus $t^\star$. Recall that the date is obtained with equation (16), where $t^\star$ appears in generalized function Z (see equation (15)), and may be prior to the investment's date. It appears that the investment date is very low for higher values of $t^\star$. This happens because $K^\star$ is very low, thus it is all the more interesting to invest early. Theoretically, as $K^\star$ decreases slowly toward 0, the investment date decreases accordingly until reaching 0 for $t^\star = \infty$. However, as investment can only occur on a daily basis, this cannot be obviously observed on Figure (5), unless we make computations for huge values of T, the final date. Unfortunately, this is not reasonable, since these computations will be extremely heavy.

However, the lower the option's maturity is, the later the investment takes place. The investment date may even be later than $t^\star$. In that case, the option is not exercised. This can be explained as follows: when $t^\star$ is low, $K^\star$, the equipment's exercise price, is quite high. Thus, the option is not really interesting. Rapidly the equipment's real price will sink under $K^\star$, and within that short period it is better to take the risk of waiting.

VI. Conclusion

In this work, we proposed a model for risk hedging when dealing with investment under uncertainty in telecommunication networks. In such a case, the risk comes from the random evolution of the demand, possibly resulting in unexpected explosions of the trafic leading to network saturation. To hedge again this risk, the operator would buy an option from some financial parts that gives him the right but not the obligation of buying equipments at a given price, until a maturity date. We calculate, using backward dynamic programming and a least square approach, the premium of the option and the expected investment date. Our results show that the option price increases with the exercise date, whereas the mean investment date sinks. As a future work, we aim at considering the case where multiple investments are possible: adding more than one band, or implementing a more efficient technology (e.g. 3G LTE).

APPENDIX

The purpose of this technical annexe is to prove that the mathematical conditions for applying a risk neutralization approach to price the American option are fulfilled. Precisely, there will be three main steps: 1- study the regularity of the function $v(t,x)$ from which the underlying asset is derived, and give a differential equation checked by its derivatives, 2- deduce from Itō's lemma applied to v the expression of the market price of risk θ_t, and 3- verify Novikov condition on θ_t thanks to numerical simulations.

A. An explicit expression for $v(t,x)$

Expression (6) of the underlying asset can be re-stated as follows:

$$S_t = v(t, W_t),$$

where we have introduced the function:

$$v(t,x) = \mathbb{E}\left[\int_t^T \phi(t,s,W_s)\,ds \middle| W_t = x\right]$$

with:

$$\phi(t,s,w) = \mathrm{e}^{-\zeta(s-t)}(\pi'(s,w) - \pi(s,w)).$$

In this section, we aim at giving a fully explicit expression for the function $v(t,x)$, in order to study its properties in the following of the annexe. For this purpose, let us first swap sum and expectation in the expression of v [6]. We obtain:

$$v(t,x) = \int_t^T \mathbb{E}\left[\phi(t,s,x+W_s-W_t)|W_t = x\right]\,ds,$$

and, since the increments of W_s are independent:

$$\begin{aligned} v(t,x) &= \int_t^T \mathbb{E}\left[\phi(t,s,x+W_s-W_t)\right]\,ds \\ &= \int_0^{T-t} \mathbb{E}\left[\phi(t,t+s,x+W_{t+s}-W_t)\right]\,ds. \end{aligned}$$

Let us introduce another two functions:

$$f(t,s,w) = \phi(t,t+s,w)$$

and:

$$u(t,s,x) = \int_{\mathbb{R}} f(t,s,w) g(x,s,w)\,dw, \qquad (17)$$

where $g(x,s,.)$ is the gaussian density with mean x and variance s. We finally get:

$$v(t,x) = \int_0^{T-t} u(t,s,x)\,ds. \qquad (18)$$

B. Regularity and differential equation for v

Let us first recall that:

$$\begin{cases} f(t,s,w) = \mathrm{e}^{-\zeta s}(\pi'(t+s,w) - \pi(t+s,w)) \\ g(x,s,w) = \frac{1}{\sqrt{2\pi s}}\,\mathrm{e}^{-\frac{(w-x)^2}{2s}} \end{cases}$$

Lemma 1: *$\pi(s,w)$ and $\pi'(s,w)$ are $C^\infty(]0,+\infty[\times\mathbb{R})$, bounded, with bounded derivatives*

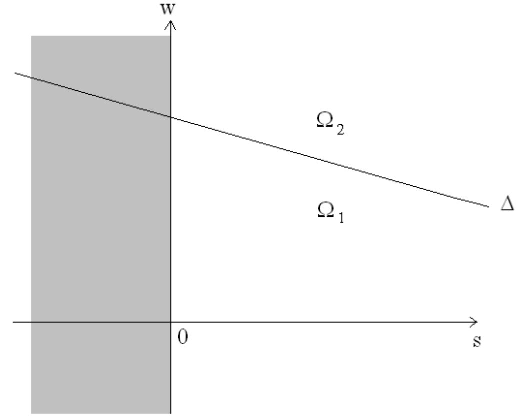

Fig. 6. Partition of the plane into two subsets Ω_1 and Ω_2.

Proof: let us prove the property with π^1 for example.

- First we prove that π is $C^\infty(]0,+\infty[\times\mathbb{R})$. For all $s_0 \in]0,+\infty[$, let us introduce $w_0 = h(s_0)$, the number such that $X(s_0,w_0) = X_{max}$:

  $$\sigma w_0 = \log(X_{max}/x_0) - (\alpha - \sigma^2/2)s_0$$

  The points (s_0,w_0) define a line Δ (see Fig. 6). Let us also introduce the two subsets of $]0,+\infty[\times\mathbb{R}$:

  $$\begin{cases} \Omega_1 = \{(s,w)/w < h(s)\} \\ \Omega_2 = \{(s,w)/w \geq h(s)\} \end{cases}$$

  These two subsets are situated respectively under and above Δ. On Ω_1, $X(s,w) < X_{max}$ and we have:

  $$\pi(s,w) = \delta x_0 \mathrm{e}^{(\alpha-\frac{\sigma^2}{2})s+\sigma w}\mathrm{e}^{-\beta/(X_{max}-x_0\mathrm{e}^{(\alpha-\frac{\sigma^2}{2})s+\sigma w})},$$

  and on Ω_2, $X(s,w) \geq X_{max}$ and $\pi = 0$.
  On Ω_1, since:

  $$\begin{cases} \frac{\partial}{\partial s}X = (\alpha - {}^{\sigma^2}\!/_2)X \\ \frac{\partial}{\partial w}X = \sigma X \end{cases},$$

  one can show by recursion over $n = p+q$ that the derivatives of π can be written:

  $$\frac{\partial^n}{\partial s^p \partial w^q}\pi = \frac{P_{p,q}(X)}{(X_{max}-X)^{2n}}\,\mathrm{e}^{-\beta/(X_{max}-X)} \qquad (19)$$

  where $P_{p,q}$ is a polynom. Hence the denominator is counterbalanced by the second exponential term in the expression of the derivative of π, so that the derivatives all tend to 0 in the neighborhood of Δ and the transition between Ω_1 and Ω_2 is C^∞.

- Secondly we prove that each derivative of π is bounded. Expression (19) is a continuous function of X on $[0,X_{max}[$, and is also continuous at X_{max}. Hence, it is bounded for $X \in [0,X_{max}]$. Since:

  $$X(\Omega_1) =]0,X_{max}] \subset [0,X_{max}],$$

  this achieves the proof.

[6]This is possible because ϕ is positive, since $0 \leq \pi \leq \pi'$

Lemma 2: *for any two compact sets $C \subset]0,+\infty[, C' \subset \mathbb{R}$, we have the following majorations:*

$$\begin{array}{ll} \forall s \in C, & \left|\frac{\partial^i g}{\partial s^i}\right| \leq \varphi^i_{x,C}(w) \\ \forall x \in C', & \left|\frac{\partial^i g}{\partial x^i}\right| \leq \psi^i_{s,C'}(w) \end{array}$$

where $\varphi^i_{x,C}$ and $\psi^i_{s,C'}$ are summable over $\mathbb{R}$.

Proof: we only prove the first majoration, the second one is exactly similar. One can show by recursion that $\frac{\partial^i g}{\partial s^i}$ can be written:

$$\frac{\partial^i g}{\partial s^i} = g(x,s,w) \sum_{k=0}^{d_i} a_{i,k}(s) w^k, \qquad (20)$$

where each $a_{i,k}$ varies continuously with s. We want to bound (20) when s varies within a compact set $C = [a,b] \subset]0,+\infty[$. Since:

$$\left|\frac{\partial^i g}{\partial s^i}\right| \leq g(x,s,w) \sum_{k=0}^{d_i} |a_{i,k}(s)||w|^k,$$

we deduce:

$$\left|\frac{\partial^i g}{\partial s^i}\right| \leq \frac{1}{\sqrt{2\pi a}} \mathrm{e}^{-\frac{(w-x)^2}{2b}} \sum_{k=0}^{d_i} \left(\max_{s\in[a,b]} |a_{i,k}(s)|\right) |w|^k,$$

from which the first majoration is immediate.

Lemma 3: *f is C^∞ w.r.t. each of its variables $t \in [0,T[$, $s \in]0, T-t[$, $w \in \mathbb{R}$, and its derivatives are bounded.*

Proof: this comes directly from Lemma 1. In particular, there exist constants K_i and K'_i so that:

$$\left|\frac{\partial^i f}{\partial t^i}\right| \leq K_i \qquad , \qquad \left|\frac{\partial^i f}{\partial s^i}\right| \leq K'_i$$

Lemma 4: *u is C^∞ w.r.t. each of its variables $t \in [0,T[$, $s \in]0, T-t[$, $x \in \mathbb{R}$. For any compact set $C' \subset \mathbb{R}$:*

$$\begin{array}{lcl} \left|\frac{\partial^i u}{\partial t^i}\right| & \leq & K_i \\ \left|\frac{\partial^i u}{\partial x^i}\right| & \leq & K_{i,C'}, \forall x \in C' \end{array}$$

and the following differential equation is verified by u:

$$u''_{xx} = 2(u'_s - \int_{\mathbb{R}} f'_s g)$$

Proof:

- Using Lemma 3, we obtain:

  $$\left|\frac{\partial^i}{\partial t^i}(fg)\right| = \left|\frac{\partial^i f}{\partial t^i}\, g\right| \leq K_i g$$

  Then, the derivability of u w.r.t. t and the first majoration immediately come from expression (17) and the differentiation under the integral sign theorem.

- Using Lemmae 2 and 3, we obtain (for any $x \in C'$):

  $$\left|\frac{\partial^i}{\partial x^i}(fg)\right| = \left|f\, \frac{\partial^i g}{\partial x^i}\right| \leq K_0 \psi^i_{s,C'}(w)$$

  Since $\psi^i_{s,C'}$ is summable over $\mathbb{R}$, we deduce the derivability of u w.r.t. x, along with the second majoration, taking:

  $$K_{i,C'} = K_0 \int_{\mathbb{R}} \psi^i_{s,C'}(w)\, dw.$$

- Each derivative $\frac{\partial^i}{\partial s^i}(fg)$ is a sum of terms which can be written:

  $$\frac{\partial^p f}{\partial s^p} \cdot \frac{\partial^q g}{\partial s^q} \;\; (p+q=i)$$

  From Lemma 2 (taking $C=[0,T]$) and Lemma 3, we get:

  $$\left|\frac{\partial^p f}{\partial s^p} \cdot \frac{\partial^q g}{\partial s^q}\right| \leq K'_p \varphi^q_{x,[0,T]}(w),$$

  which is integrable over $\mathbb{R}$. Hence, u is C^∞ w.r.t. s.

- By deriving under the integral sign, we have:

  $$\begin{array}{rcl} u''_{xx}(t,s,x) & = & \int_{\mathbb{R}} f(t,s,w) g''_{xx}(x,s,w)\, dw \\ u'_s(t,s,x) & = & \int_{\mathbb{R}} f(t,s,w) g'_s(x,s,w)\, dw \\ & + & \int_{\mathbb{R}} f'_s(t,s,w) g(x,s,w)\, dw, \end{array}$$

  (splitting into two sums is allowed since $f'_s g$ is summable). Thanks to the heat equation verified by the gaussian kernel:

  $$g'_s = \frac{1}{2} g''_{xx}$$

  we finally get third assertion of Lemma 4:

  $$\begin{array}{rcl} u''_{xx} & = & 2 \int_{\mathbb{R}} f(t,s,w) g'_s(x,s,w)\, dw \\ & = & 2\left(u'_s - \int_{\mathbb{R}} f'_s(t,s,w) g(x,s,w)\, dw\right). \end{array}$$

Lemma 5: *v is C^∞ w.r.t. each of its variables $t \in [0,+\infty[$ and $x \in \mathbb{R}$ and, $v'_t + \frac{1}{2} v''_{xx} = -\phi(t,t,x) + \zeta v$.*

Proof: the regularity of v is a direct consequence of equation (18) and Lemma 4. The differential equation checked by v is obtained as follows:

$$\begin{array}{rcl} v'_t & = & \int_0^{T-t} u'_t\, ds - u(t, T-t, x) \\ & = & \int_0^{T-t} \int_{\mathbb{R}} f'_t\, g\, dw\, ds - u(t,T-t,x) \\ & = & \int_0^{T-t} \int_{\mathbb{R}} (f'_s\, g + \zeta f\, g)\, dw\, ds - u(t,T-t,x) \\ & = & \int_0^{T-t} \left(u'_s - \frac{1}{2} u''_{xx}\right) ds + \zeta v - u(t,T-t,x) \\ & = & u(t,T-t,x) - \lim_{s\to 0} u - \frac{1}{2} v''_{xx} + \zeta v - u(t,T-t,x). \end{array}$$

$f(t,.,w)$ being continuous at point $s = 0$, we have $\lim_{s\to 0} u(t,s,x) = \int_{\mathbb{R}} \phi(t,t,w) g(x,0,w)\, dw$ where the term $g(x,0,w)$ has to be understood as the Dirac distribution at x. Therefore:

$$v'_t = -\phi(t,t,x) - \frac{1}{2} v''_{xx} + \zeta v.$$

Lemma 6: *$\frac{\partial v}{\partial x}$(t,x) is null only on a curve $x = x(t)$.*

Proof: this lemma will not be rigorously proved, but instead inferred from numerical simulation of the surface $\frac{\partial v}{\partial x}(t,x),\ t \in [0,T[,\ x \in \mathbb{R}$. Fig. 7 shows this surface. Clearly, one can observe that on the left side of the red line, $\frac{\partial v}{\partial x}$ goes to zero only on a curve $x(t)$. What can be proved analytically is that $\frac{\partial v}{\partial x}(t,x) < 0$ on the right side of the line, where paradoxically the surface is very close to zero. Let us write $\frac{\partial v}{\partial x}$ in a new way:

$$\begin{aligned} \frac{\partial v}{\partial x}(t,x) &= \int_0^{T-t} \int_{\mathbb{R}} f(t,s,w) \frac{\partial g}{\partial x}(x,s,w)\, dw\, ds \\ &= \int_0^{T-t} \int_{\mathbb{R}} f(t,s,w) \frac{w-x}{s} g(x,s,w)\, dw\, ds \\ &= \int_0^{T-t} \mathbb{E}\left[\phi(t,t+s,x+\overline{W}_s)\frac{\overline{W}_s}{s}\right] ds, \end{aligned}$$

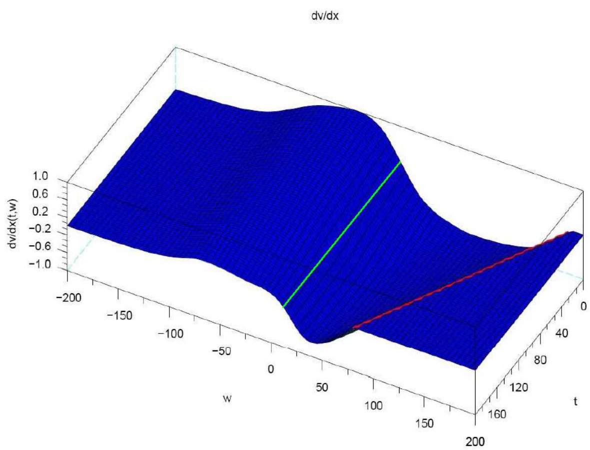

Fig. 7. The surface $\partial v/\partial x$ (we kept the previous parameters). On the right of the red line, we show in Lemma 6 that it is nonnull. On the left, we see that it is null only on the green line.

where $\overline{W}_s$ is a standard brownian motion. $\phi(t, t+s, x+\overline{W}_s)$ is always positive, and null *iff* $t+s \geq a(x+\overline{W}_s)+b$, where we have introduced two coefficients:

$$a = -\frac{\sigma}{\alpha - \sigma^2/2} \qquad , \qquad b = \frac{\log(X'_{max}/x_0)}{\alpha - \sigma^2/2}.$$

Hence:

$$\begin{aligned} \frac{\partial v}{\partial x}(t,x) &= \int_0^{T-t} \mathbb{E}\left[\phi(t,t+s,x+\overline{W}_s)\frac{\overline{W}_s}{s}\, \mathbb{1}_{\{t+s<a(x+\overline{W}_s)+b\}}\right] ds \\ &= \int_0^{T-t} \mathbb{E}\left[\phi(t,t+s,x+\overline{W}_s)\frac{\overline{W}_s}{s}\, \middle|\, \overline{W}_s < \frac{t+s-b}{a} - x\right] \\ &\quad \times \mathbb{P}\left(t+s < a(x+\overline{W}_s)+b\right)\, ds \end{aligned}$$

If $\frac{t-b}{a} - x \leq 0$, it is immediate to obtain $\frac{\partial v}{\partial x}(t,x) < 0$. The line $\frac{t-b}{a} - x = 0$ being precisely the red line of Fig. 7, we have the result.

C. *Expression of the risk premium θ_t*

In this section, we justify the existence of the market price of risk θ_t and deduce its expression from Itō's lemma and the differential equation checked by v. Itō's lemma holds since v is regular. It gives:

$$dS_t = \left[\frac{\partial v}{\partial t}(t,W_t) + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,W_t)\right] dt + \frac{\partial v}{\partial x}(t,W_t)\, dW_t$$

Provided $X'_{max} > X_{max}$, we have $v(t,x) > 0$ for any (t,x). Hence $S_t > 0$ and we can write:

$$\frac{dS_t}{S_t} = \frac{1}{v}\left(\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\right) dt + \frac{1}{v}\frac{\partial v}{\partial x}\, dW_t.$$

By identifying this equation with the dynamics of the underlying asset (8), we get the expression of the expected total return on the asset μ_t and the volatility κ_t:

$$\begin{cases} \mu_t = \frac{1}{v}\left(\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\right) \\ \kappa_t = \frac{1}{v}\frac{\partial v}{\partial x} \end{cases}.$$

Lemma 6 ensures that $\kappa_t \neq 0$ *a.s.*, therefore the market price of risk θ_t is well defined, and:

$$\theta_t = \left(\frac{\partial v}{\partial x}\right)^{-1}\left(\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} - \zeta v\right).$$

The final expression for θ_t is a consequence of Lemma 5:

$$\theta_t = -\frac{\phi(t,t,W_t)}{\frac{\partial v}{\partial x}(t,W_t)} = -\frac{\pi^2(t,W_t) - \pi^1(t,W_t)}{\frac{\partial v}{\partial x}(t,W_t)}.$$

D. *Novikov's condition*

Now we have to prove that:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2\, dt\right)\right] < +\infty.$$

Actually, is $\int_0^T \theta_t^2\, dt$ even finite ? The question is relevant, because Lemma 6 shows that on a certain line, $\frac{\partial v}{\partial x}$ is null, and so $\theta(t,w)$ is infinite (see Fig. 8).

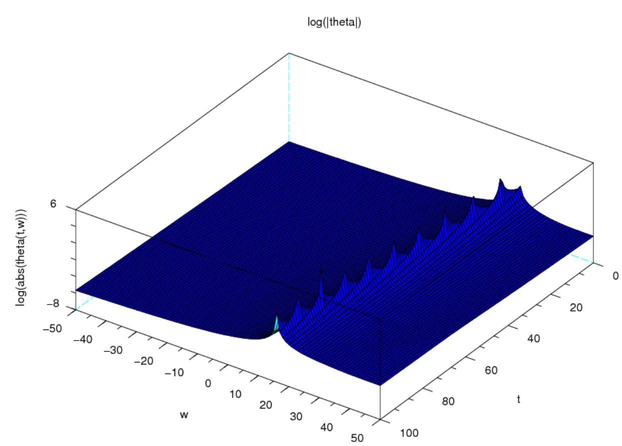

Fig. 8. Repartition of the peaks of $\theta(t,w)$. They all lie on the green line represented on Fig. 7. Normally, they should form a continuous crest, but due to discretization they show an uneven behavior.

Now, could a trajectory (W_t) come close to this line during a time long enough so that:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2\, dt\right)\right] = +\infty\ ?$$

Here we use a result of El Karoui and Gobet (see [6], Proposition 1.3.8.):

$$\mathbb{P}\left(\sup_{t\leq T}|W_t|\geq c\right)\leq 2\ \mathbb{P}(|W_T|\geq c),$$

which tends to 0 extremely rapidly when $c\to\infty$. Hence, if we choose correctly our parameters so that the critic line lies far enough from the line $w=0$, the probability to reach it during the experiment will be extremely low. Then, in practice, we will consider that θ_t remains almost surely bounded by a constant $\hat{\theta}$; and *a fortiori*, Novikov's condition will be verified, since we will have:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T\theta_t^2\,dt\right)\right]\leq \mathrm{e}^{T\hat{\theta}^2/2}.$$

E. Regressing a set of points on a 2-degree polynom

Given a set of points $(x_i,y_i)_{1\leq i\leq n}$ in $\mathbb{R}^2$, the aim of the section is to find three real numbers a, b, c, such that

$$\sum_{i=1}^n|y_i-P_{a,b,c}(x_i)|^2$$

is minimal, where:

$$P_{a,b,c}(x)=ax^2+bx+c.$$

After deriving w.r.t. a, b and c, we obtain respectively:

$$\begin{cases}a\sum x_i^4+b\sum x_i^3+c\sum x_i^2 &=\sum x_i^2y_i\\ a\sum x_i^3+b\sum x_i^2+c\sum x_i &=\sum x_iy_i\\ a\sum x_i^2+b\sum x_i+c &=\sum y_i\end{cases}\qquad(21)$$

Has (21) a solution ? Let us consider the four vectors of $\mathbb{R}^n$:

$$x^2=\begin{pmatrix}x_1^2\\ \vdots\\ x_n^2\end{pmatrix}x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\mathbf{1}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix},$$

then we can re-write the system as:

$$\begin{pmatrix}(x^2|x^2) & (x^2|x) & (x^2|\mathbf{1})\\ (x|x^2) & (x|x) & (x|\mathbf{1})\\ (\mathbf{1}|x^2) & (\mathbf{1}|x) & (\mathbf{1}|\mathbf{1})\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}(x^2|y)\\ (x|y)\\ (\mathbf{1}|y)\end{pmatrix}.$$

It is equivalent to say that the vector $y-ax^2-bx-c$ is orthogonal to $\mathbf{1}$, x and x^2. In other words, ax^2+bx+c is the orthogonal projection of y on Vect($\mathbf{1}$,x,x^2), and thus we are sure that (21) has a solution.

Is this solution unique ? If $\mathbf{1}$, x and x^2 were not independent, there would be three real numbers u, v and w such that:

$$\forall i,\quad u+vx_i+wx_i^2=0.$$

- as soon as there are more than two different values of x_i, this is impossible, and so there is a unique solution.
- if x_i takes only two different values, then the solution is not unique any more, but ($\mathbf{1}$,x,x^2) has rank 2. So we choose to regress y on $\mathbf{1}$ and x for example (and we find a line which intersects the centroids of the two corresponding subsets).
- if x_i takes only one value, then ($\mathbf{1}$,x,x^2) has rank 1. So we choose to regress y on $\mathbf{1}$ (and we find the mean of the y_is).

F. Interpolating a surface

Given a surface of equation: $z=v(t,x)$, suppose we only know a discrete set of values of z: $z_{i,j}=v(t_i,x_j)$. We want to compute $v(t,x)$ for any value of t and x. The technique is very similar to a linear interpolation in 1 dimension:

- first find the intervals $[t_i,t_{i+1}[$ and $[x_j,x_{j+1}[$ in which t and x lie
- then interpolate first w.r.t. t. Denoting $\frac{t-t_i}{t_{i+1}-t_i}$ by α_t, we obtain:

$$\begin{cases}z^1=\alpha_tz_{i,j} & +(1-\alpha_t)z_{i+1,j}\\ z^2=\alpha_tz_{i,j+1} & +(1-\alpha_t)z_{i+1,j+1}\end{cases}$$

- at least interpolate w.r.t. x. Denoting $\frac{x-x_i}{x_{i+1}-x_i}$ by α_x, we obtain:

$$z=\alpha_xz^1+(1-\alpha_x)z^2.$$

<u>*Remark*</u>
We have:

$$\begin{aligned}z=\ &\alpha_t\alpha_xz_{i,j}+(1-\alpha_t)\alpha_xz_{i+1,j}\\ &+\alpha_t(1-\alpha_x)z_{i,j+1}+(1-\alpha_t)(1-\alpha_x)z_{i+1,j+1},\end{aligned}$$

which is symmetrical in (t,i) and (x,j). Hence, we would have found the same result by interpolating first w.r.t. x, and then w.r.t. t (see Fig. 9).

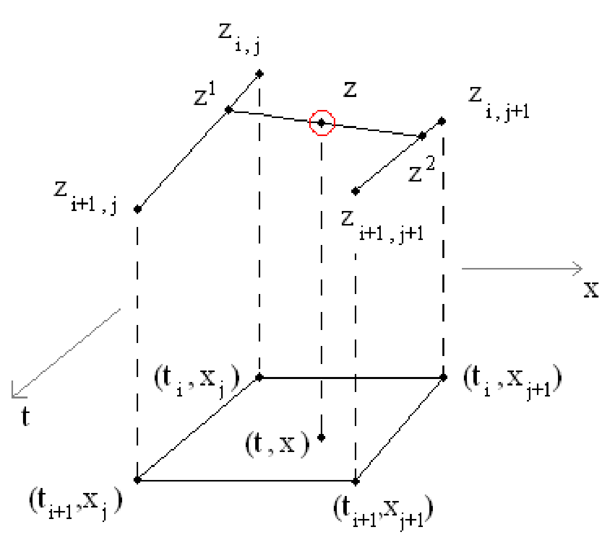

Fig. 9. Interpolation of the surface.

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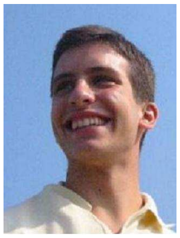

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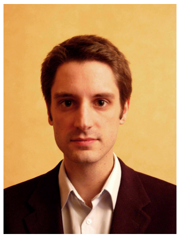

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