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REDUCE COMPUTATION IN PROFILE EMPIRICAL LIKELIHOOD METHOD*

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Since its introduction by Owen in [29, 30], the empirical likelihood method has been extensively investigated and widely used to construct confidence regions and to test hypotheses in the literature. For a large class of statistics that can be obtained via solving estimating equations, the empirical likelihood function can be formulated from these estimating equations as proposed by [35]. If only a small part of parameters is of interest, a profile empirical likelihood method has to be employed to construct confidence regions, which could be computationally costly. In this paper we propose a jackknife empirical likelihood method to overcome this computational burden. This proposed method is easy to implement and works well in practice.

1. Introduction.   Empirical likelihood method was introduced by Owen ([29, 30]) to construct confidence regions for the mean of a random vector. Like the bootstrap and jackknife methods, the empirical likelihood method is a nonparametric one. Without assuming a family of distributions for the

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data, the empirical likelihood ratio statistics can be defined to share similar properties as the likelihood ratio for parametric distributions. For instance, the empirical likelihood method produces confidence regions whose shape and orientation are determined entirely by the data. In comparison with the normal approximation method and the bootstrap method for constructing confidence intervals, the empirical likelihood method does not require a pivotal quantity, and it has better small sample performance (see [15]).

As an effective way for interval estimation and goodness-of-fit test, the empirical likelihood method has been extended and applied in many different fields such as regression models ([11]), quantile estimation ([9]), additive risk models ([22]), two-sample problems ([36], [19], [3], [43]), time series models ([16], [14], [8], [28], [27], [4], [5]), heavy-tailed models ([21], [32], [33, 34]), high dimensional data ([10]) and copulas ([6]).

A common feature in employing the empirical likelihood method is to work with linear constraints such as linear functionals, and a powerful way in formulating the empirical likelihood ratio statistic is via estimating equations as in [35]. When the constraints involve nonlinear equations, an important trick is to transform these nonlinear constraints to some linear constraints by introducing some link variables. After this transformation, a profile empirical likelihood method is employed. For example, if one wants to construct an empirical likelihood confidence interval for the variance $\theta = E\{X - EX\}^2$, one can introduce a link variable $\beta = E(X)$ and then formulate the empirical likelihood by using equations $E(X) = \beta$ and $E(X^2) = \theta + \beta^2$. Some other examples include the study of ROC curves ([12]), copulas ([6]) and difference of quantiles ([44], [41]). Apparently these extra constraints add

more computational burden to the empirical likelihood method. Another computational difficulty arises when the profile empirical likelihood method involves a large number of nuisance parameters.

Suppose $X_1, \cdots, X_n$ are independent random vectors with common distribution function F and there is a q-dimensional parameter θ associated with F. Let y^T denote the transpose of the vector y and

$$G(x; \theta) = (g_1(x; \theta), \cdots, g_s(x; \theta))^T$$

denote $s(\geq q)$ functionally independent functions, which connect F and θ through the equations $EG(X_1; \theta) = 0$. Write $\theta = (\alpha^T, \beta^T)^T$, where α and β are q_1-dimensional and q_2-dimensional parameters, respectively, and $q_1 + q_2 = q$. In order to construct confidence regions for α, [35] proposed to use the following profile empirical likelihood ratio

$$(1.1) \qquad l(\alpha) = 2l_E((\alpha^T, \hat{\beta}^T(\alpha))^T) - 2l_E(\tilde{\theta}),$$

where $l_E(\theta) = \sum_{i=1}^{n} \log\{1 + t^T(\theta)G(X_i; \theta)\}$, $t = t(\theta)$ is the solution of the following equation

$$(1.2) \qquad 0 = \frac{1}{n} \sum_{i=1}^{n} \frac{G(X_i; \theta)}{1 + t^T G(X_i; \theta)},$$

$\tilde{\theta} = (\tilde{\alpha}^T, \tilde{\beta}^T)^T$ minimizes $l_E(\theta)$ with respect to θ, and $\hat{\beta}(\alpha)$ minimizes $l_E((\alpha^T, \beta^T)^T)$ with respect to β for fixed α. It has been shown that $l(\alpha_0)$ converges in distribution to $\chi^2_{q_1}$ under some regularity conditions, where α_0 denotes the true value of α. For the second order properties of the empirical likelihood method based on estimating equations, we refer to [7].

The computational complexity in using the profile empirical likelihood method comes from computing $l_E((\alpha^T, \hat{\beta}^T(\alpha))^T)$. When the nuisance parameter β is known, one can simply replace $\hat{\beta}^T(\alpha)$ and $\tilde{\beta}$ in (1.1) by the true value

of β so that the computation is reduced significantly. In order to avoid computing $l_E((\alpha^T, \hat{\beta}^T(\alpha))^T)$, one may choose to replace $\hat{\beta}^T(\alpha)$ by some other different estimator, for example, solving q_2 equations of $n^{-1} \sum_{i=1}^n G(X_i; \theta) = 0$. Although this reduces computation especially when one can find an explicit estimator of β in terms of the sample and α, Wilks' theorem doesn't hold, that is, $l(\alpha_0)$ does not converge in distribution to $\chi_{q_1}^2$. Instead, $l(\alpha_0)$ generally converges in distribution to a weighted sum of independent chi-square random variables; see [17]. Since the weights in the limit have to be estimated, this empirical likelihood method does not preserve the important properties of the standard empirical likelihood method: self-studentization, automatically determined shape of confidence region and Bartlett correction. Therefore, it is of importance to develop an empirical likelihood method which has a chi-square limit and is computationally efficient than the profile empirical likelihood method especially when the number of nuisance parameters is large. Moreover, when some estimating equations involve U-statistics, the profile empirical likelihood method is extremely complicated.

Motivated by the recent study on using jackknife empirical likelihood method to deal with nonlinear constraints in U-statistics ([18]), we propose a jackknife empirical likelihood method to construct confidence regions for the interesting parameter α with the nuisance parameter β being simply replaced by some estimator. The jackknife empirical likelihood ratio statistic is obtained by applying the standard empirical likelihood method to the jackknife pseudo sample. The jackknife method was originally used to estimate the variance of a statistic and to construct bias-corrected estimators of parameters. See, e.g., [40] for details. The proposed jackknife empirical

likelihood method allows us to compute the nuisance parameters simply through a subset of estimating equations and yet still retains the attractive chi-square limiting distribution for the empirical likelihood ratio.

We organize this paper as follows. In Section 2, the new methodology and main results are given. Section 3 presents a simulation study. Section 4 presents two case studies of financial applications. All proofs are given in Section 5.

2. Methodology and main results. As in the introduction, let $G(x; \theta) = (g_1(x; \theta), \cdots, g_s(x; \theta))^T$ denote $s(\geq q)$ functionally independent functions with $EG(X_1; \theta_0) = 0$, where $\theta_0 = (\alpha_0^T, \beta_0^T)^T$ denotes the true value of $\theta = (\alpha^T, \beta^T)^T$, and α and β are q_1-dimensional and q_2-dimensional parameters, respectively. Note that only the parameter α is of interest under consideration. To remove the nuisance parameter β, we propose to first estimate it from some q_2 estimating equations, and then work with the remaining $s - q_2$ equations, where β is replaced by the obtained estimator. The details are as follows.

Define

$$G_a(x; \alpha, \beta) = \left(g_1(x; \alpha, \beta), \cdots, g_{s-q_2}(x; \alpha, \beta)\right)^T$$

and

$$G_b(x; \alpha, \beta) = \left(g_{s-q_2+1}(x; \alpha, \beta), \cdots, g_s(x; \alpha, \beta)\right)^T.$$

Without loss of generality, we solve the last q_2 equations of $n^{-1} \sum_{i=1}^n G(X_i; \theta)$ to get an estimator for β. That is, $\tilde{\beta}(\alpha; \mathbf{X})$ is the solution to

$$(2.1) \qquad \frac{1}{n} \sum_{i=1}^n G_b(X_i; \alpha, \beta) = 0$$

with respect to β for each fixed α, where $\mathbf{X} = (X_1, \cdots, X_n)^T$. Obviously, the best choice of the q_2 equations is to have explicit formula for $\tilde{\beta}(\alpha; \mathbf{X})$, if possible. Set

$$T_n(\alpha) = \frac{1}{n} \sum_{i=1}^{n} G_a(X_i; \alpha, \tilde{\beta}(\alpha, \mathbf{X}))$$

and let $\tilde{\beta}(\alpha; \mathbf{X}_{-i})$ denote the solution to the equations

(2.2)
$$\frac{1}{n-1} \sum_{j=1, j \neq i}^{n} G_b(X_j; \alpha, \beta) = 0$$

with respect to β for each fixed α, where $\mathbf{X}_{-i} = (X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)^T$. Similarly, define

$$T_{n,-i}(\alpha) = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} G_a(X_j; \alpha, \tilde{\beta}(\alpha; \mathbf{X}_{-i})).$$

Then the jackknife pseudo sample is defined as

$$Y_i(\alpha) = (Y_{i,1}(\alpha), \cdots, Y_{i,s-q_2}(\alpha))^T = nT_n(\alpha) - (n-1)T_{n,-i}(\alpha) \quad \text{for} \quad i = 1, \cdots, n.$$

As in [42], one expects that $Y_i's$ are approximately independent. This motivates us to apply the standard empirical likelihood method to the jackknife sample $Y_1(\alpha), \cdots, Y_n(\alpha)$ for constructing empirical likelihood confidence regions for α. Hence we define the jackknife empirical likelihood function as

$$L^J(\alpha) = \sup \left\{ \Pi_{i=1}^{n}(np_i) : \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i Y_i(\alpha) = 0, \ p_1 \geq 0, \cdots, p_n \geq 0 \right\}.$$

It follows from the Lagrange multiplier technique that the above maximization is achieved at $p_i = n^{-1}\{1 + \lambda^T Y_i(\alpha)\}^{-1}$ and the log empirical likelihood ratio $\ell^J(\alpha) = -\log L^J(\alpha)$ is given by

(2.3)
$$\ell^J(\alpha) = \sum_{i=1}^{n} \log\{1 + \lambda^T Y_i(\alpha)\},$$

where $\lambda = \lambda(\alpha)$ satisfies

$$(2.4) \qquad \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i(\alpha)}{1 + \lambda^T Y_i(\alpha)} = 0.$$

Before we present our main results, we first list the assumptions we need. For this purpose, denote $\frac{\partial y}{\partial x} = (\frac{\partial y_i}{\partial x_j})_{1 \le i \le m, 1 \le j \le n}$ for $y = (y_1, \cdots, y_m)^T$ and $x = (x_1, \cdots, x_n)^T$, and define

$$\hat{\alpha} = \arg\min \ell^J(\alpha), \quad \Sigma = (E\{g_k(X_1; \alpha_0, \beta_0)g_l(X_1; \alpha_0, \beta_0)\})_{1 \le k,l \le s} = (\sigma_{k,l})_{1 \le k,l \le s},$$

$$\Sigma_1 = E\{\frac{\partial}{\partial\beta}G_b(X_1; \alpha_0, \beta_0)\}, \quad \Sigma_2 = E\{\frac{\partial}{\partial\alpha}G_b(X_1; \alpha_0, \beta_0)\},$$

$$\Sigma_3 = E\{\frac{\partial}{\partial\alpha}G_a(X_1; \alpha_0, \beta_0)\} - E\{\frac{\partial}{\partial\beta}G_a(X_1; \alpha_0, \beta_0)\}\Sigma_1^{-1}\Sigma_2$$

and $\Sigma^* = (\sigma^*_{k,l})_{1 \le k,l \le s-q_2}$, where

$$\begin{aligned}
\sigma^*_{k,l} &= \sigma_{k,l} - E\{\tfrac{\partial}{\partial\beta}g_k(X_1; \alpha_0, \beta_0)\}\Sigma_1^{-1}(\sigma_{s-q_2+1,l}, \cdots, \sigma_{s,l})^T \\
&\quad - E\{\tfrac{\partial}{\partial\beta}g_l(X_1; \alpha_0, \beta_0)\}\Sigma_1^{-1}(\sigma_{s-q_2+1,k}, \cdots, \sigma_{s,k})^T \\
&\quad + E\{\tfrac{\partial}{\partial\beta}g_k(X_1; \alpha_0, \beta_0)\}\Sigma_1^{-1}(\sigma_{ij})_{s-q_2+1 \le i,j \le s}\Sigma_1^{-1}E\{\tfrac{\partial}{\partial\beta^T}g_l(X_1; \alpha_0, \beta_0)\}.
\end{aligned}$$

Some regularity conditions are as follows:

- A1) There is a neighborhood of α_0 and β_0, say $\Omega_{\alpha_0} \times \Omega_{\beta_0}$, such that $G_b(x; \alpha, \beta)$ are continuous function of $\alpha \in \Omega_{\alpha_0}$ and $\beta \in \Omega_{\beta_0}$ for all x, and $\sup_{\alpha \in \Omega_{\alpha_0}, \beta \in \Omega_{\beta_0}} ||G_b(x; \alpha, \beta)||^3 \le K(x)$ for some function K satisfying that $EK(X_1) < \infty$;

- A2) For each $\alpha \in \Omega_{\alpha_0}$, there is a function $\beta(\alpha) \in \Omega_{\beta_0}$ such that $EG_b(X_1; \alpha, \beta(\alpha)) = 0$;

- A3) $||\frac{\partial}{\partial\beta}G_b(x; \alpha, \beta)||^3$, $||\frac{\partial^2}{\partial\beta^T\partial\beta}g_l(x; \alpha, \beta)||^3$ and $|\frac{\partial^3}{\partial\beta_i\partial\beta_j\partial\beta_m}g_l(x; \alpha, \beta)|$ for $l = s - q_2 + 1, \cdots, s$, $i, j, m = 1, \cdots, q_2$ are bounded by $K(x)$ uniformly in $\alpha \in \Omega_{\alpha_0}$ and $\beta \in \Omega_{\beta_0}$;

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- A4) $\beta(\alpha)$ defined in A2) has continuous first derivatives;

- A5) $||\frac{\partial}{\partial\alpha}G_b(x;\alpha,\beta)||^3$ and $|\frac{\partial^3}{\partial\alpha_i\partial\beta_j\partial\beta_m}g_l(x;\alpha,\beta)|$ for $l = s-q_2+1,\cdots,s$, $i = 1,\cdots,q_1$, $j,m = 1,\cdots,q_2$ are bounded by $K(x)$ uniformly in $\alpha \in \Omega_{\alpha_0}$ and $\beta \in \Omega_{\beta_0}$;

- A6) Assume Σ_1 is invertible, Σ^* is positive definite and Σ_3 has rank q_1;

- A7) $||\frac{\partial}{\partial\beta}G_a(x;\alpha,\beta)||$, $||G_a(x;\alpha,\beta)||^3$, $|\frac{\partial^3}{\partial\alpha_k\partial\beta_j\partial\beta_m}g_l(x;\alpha,\beta)|$ and $|\frac{\partial^3}{\partial\beta_i\partial\beta_j\partial\beta_m}g_l(x;\alpha,\beta)|$ for $i,j,m = 1,\cdots,q_2$, $k = 1,\cdots,q_1$ and $l = 1,\cdots,s-q_2$ are bounded by $K(x)$ uniformly in $\alpha \in \Omega_{\alpha_0}$ and $\beta \in \Omega_{\beta_0}$.

The following two propositions show the existence and some properties of $\tilde{\beta}(\alpha;\mathbf{X})$, $\tilde{\beta}(\alpha_0;\mathbf{X}_{-i})$ and $\hat{\alpha}$.

Proposition 1. i) Under conditions A1) and A2), with probability one, there exist solutions $\tilde{\beta}(\alpha;\mathbf{X}) \in \Omega_{\beta_0}$ and $\tilde{\beta}(\alpha;\mathbf{X}_{-i}) \in \Omega_{\beta_0}$ to (2.1) and (2.2), respectively, such that

$$(2.5) \quad \tilde{\beta}(\alpha;\mathbf{X}) - \beta(\alpha) = o(1) \quad \text{and} \quad \max_{1\leq i\leq n} ||\tilde{\beta}(\alpha;\mathbf{X}_{-i}) - \beta(\alpha)|| = o(1)$$

almost surely for each $\alpha \in \Omega_{\alpha_0}$.

ii) Under conditions A1)–A3), we have

$$(2.6) \quad \tilde{\beta}(\alpha_0;\mathbf{X}) - \beta_0 + \Sigma_1^{-1}\frac{1}{n}\sum_{i=1}^n G_b(X_i;\alpha_0,\beta_0) = O_p(n^{-1}),$$

$$(2.7) \quad \max_{1\leq i\leq n} ||\tilde{\beta}(\alpha_0;\mathbf{X}_{-i})-\beta_0+\Sigma_1^{-1}\frac{1}{n-1}\sum_{j=1,j\neq i}^n G_b(X_j;\alpha_0,\beta_0)|| = O_p(n^{-1}),$$

$$(2.8) \quad \max_{1\leq i\leq n} ||\tilde{\beta}(\alpha_0;\mathbf{X}) - \tilde{\beta}(\alpha_0;\mathbf{X}_{-i}) + \Sigma_{n1}^{-1}D_n(i)|| = o_p(n^{-3/2}),$$

where

$$\Sigma_{n1} = \frac{1}{n}\sum_{j=1}^{n}\frac{\partial}{\partial\beta}G_b(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X})), \quad D_n(i) = (D_{n,s-q_2+1}(i),\cdots,D_{n,s}(i))^T$$

and

$$\begin{aligned}
&D_{n,l}(i)\\
=\ &\tfrac{1}{2}n^{-1}G_b^T(X_i;\alpha_0,\beta_0)\Sigma_1^{-1}E\{\tfrac{\partial^2}{\partial\beta^T\partial\beta}g_l(X_1;\alpha_0,\beta_0)\}\Sigma_1^{-1}n^{-1}G_b(X_i;\alpha_0,\beta_0)\\
&+n^{-1}g_l(X_i;\alpha_0,\beta_0) - n^{-1}\{\tfrac{\partial}{\partial\beta}g_l(X_i;\alpha_0,\beta_0)\}\Sigma_1^{-1}\tfrac{1}{n-1}\sum_{j=1}^{n}G_b(X_j;\alpha_0,\beta_0)\\
&+n^{-1}\{\tfrac{\partial}{\partial\beta}g_l(X_i;\alpha_0,\beta_0)\}\Sigma_1^{-1}\tfrac{1}{n-1}G_b(X_i;\alpha_0,\beta_0).
\end{aligned}$$

iii) Under conditions A1)-A5), we have

$$(2.9) \quad \begin{cases} \max_{1\le i\le n}\|\frac{\partial}{\partial\alpha}\tilde{\beta}(\alpha_0;\mathbf{X}_{-i}) + \Sigma_1^{-1}\Sigma_2\| = O_p(n^{-1/2}),\\[2mm] \|\frac{\partial}{\partial\alpha}\tilde{\beta}(\alpha_0;\mathbf{X}) + \Sigma_1^{-1}\Sigma_2\| = O_p(n^{-1/2}), \end{cases}$$

and

$$(2.10) \qquad \max_{1\le i\le n}\|\frac{\partial}{\partial\alpha}\tilde{\beta}(\alpha_0;\mathbf{X}) - \frac{\partial}{\partial\alpha}\tilde{\beta}(\alpha_0;\mathbf{X}_{-i}) + \Sigma_1^{-1}A_i\| = o_p(n^{-1}),$$

where $A_i = (a_{s-q_2+1}(i),\cdots,a_s(i))^T$ and

$$\begin{aligned}
a_l^T(i) =\ &-\tfrac{1}{n}G_b^T(X_i;\alpha_0,\beta_0)\Sigma_1^{-1}E\{\tfrac{\partial^2}{\partial\beta^T\partial\alpha}g_l(X_1;\alpha_0,\beta_0)\} + \tfrac{1}{n}\tfrac{\partial}{\partial\alpha}g_l(X_i;\alpha_0,\beta_0)\\
&+\tfrac{1}{n}G_b^T(X_i;\alpha_0,\beta_0)\Sigma_1^{-1}E\{\tfrac{\partial^2}{\partial\beta^T\partial\beta}g_l(X_1;\alpha_0,\beta_0)\}\Sigma_1^{-1}\Sigma_2\\
&-\tfrac{1}{n}\tfrac{\partial}{\partial\beta}g_l(X_i;\alpha_0,\beta_0)\Sigma_1^{-1}\Sigma_2.
\end{aligned}$$

Proposition 2. Under A1)–A7), with probability tending to one, $\ell^J(\alpha)$ attains its minimum value at some point $\hat{\alpha}$ in the interior of the ball $\|\alpha - \alpha_0\| \le n^{-1/3}$, and $\hat{\alpha}$ and $\hat{\lambda} = \lambda(\hat{\alpha})$ satisfy

$$Q_{1n}(\hat{\alpha},\hat{\lambda}) = 0 \quad \text{and} \quad Q_{2n}(\hat{\alpha},\hat{\lambda}) = 0,$$

where

$$Q_{1n}(\alpha,\lambda) = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i(\alpha)}{1+\lambda^T Y_i(\alpha)}$$

and

$$Q_{2n}(\alpha, \lambda) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + \lambda^T Y_i(\alpha)} \{\frac{\partial}{\partial \alpha} Y_i(\alpha)\}^T \lambda.$$

Next we show that Wilks' theorem holds for the proposed jackknife empirical likelihood method.

Theorem 1. Under conditions A1)–A7), we have $LR(\alpha_0) := 2\ell^J(\alpha_0) - 2\ell^J(\hat{\alpha})$ converges in distribution to $\chi^2_{q_1}$, where $\hat{\alpha}$, $\tilde{\beta}(\alpha; \mathbf{X})$ and $\tilde{\beta}(\alpha; \mathbf{X}_{-i})$ are given in Propositions 1 and 2.

Based on Theorem 1, an asymptotically accurate $100\gamma\%$ confidence region for α is given by

$$I_\gamma = \{\alpha : LR(\alpha) \le \chi^2_{q_1}(\gamma)\},$$

where $\chi^2_{q_1}(\gamma)$ is the γ quantile of a χ^2 distribution with q_1 degrees of freedom.

Remark 1. When $s = q$, we have $\ell^J(\hat{\alpha}) = 0$. Moreover, when $\tilde{\beta}(\alpha; \mathbf{X})$ has an explicit formula in terms of the sample $\mathbf{X}$ and α, the computation of the proposed jackknife empirical likelihood method is only slightly heavier than the standard empirical likelihood method. Indeed, the software R package for the empirical likelihood method can be employed as in our simulation study.

Remark 2. If one is interested in a part of $\alpha = (\tilde{\alpha}_1^T, \tilde{\alpha}_2^T)^T$, say $\tilde{\alpha}_1$, then we can show that $2 \min_{\tilde{\alpha}_2} \ell^J(\alpha) - 2\ell^J(\hat{\alpha})$ converges in distribution to a chi-square limit with the degrees of freedom being the length of $\tilde{\alpha}_1$. This method may be called jackknife profile empirical likelihood method, which is appealing when a part of nuisance parameters can be solved explicitly.

Remark 3. When equations (2.1) are independent of α, then regularity

conditions involving the partial derivatives of G_b with respect to α can be removed.

3. Simulation study. Suppose the random vector (X, Y) has marginal distribution functions F_1, F_2 and copula $C(x, y) = P(F_1(X) \leq x, F_2(Y) \leq y)$. In fitting a parametric family to the copula, a useful quantity is the Spearman's rho defined as $\rho_s = 12E\{F_1(X)F_2(Y)\} - 3$. For example, if one employs the Gaussian copula

$$C(u, v; \theta) = \int_{-\infty}^{\Phi^-(u)} \int_{-\infty}^{\Phi^-(v)} \frac{1}{2\pi\sqrt{1-\theta^2}} \exp\{-\frac{s^2 - 2\theta st + t^2}{2(1-\theta^2)}\} \, dsdt,$$

or the t copula

$$C(u, v; \theta) = \int_{-\infty}^{t_4^-(u)} \int_{-\infty}^{t_4^-(v)} \frac{1}{2\pi\sqrt{1-\theta^2}} \{1 + \frac{s^2 - 2\theta st + t^2}{4(1-\theta^2)}\}^{-3} \, dsdt,$$

where $\theta \in [-1, 1]$ and Φ^- and t_ν^- denote the inverse function of the standard normal distribution function and t distribution function with degrees of freedom ν, respectively, then $\rho_s = 6\pi^{-1}\arcsin(\theta/2)$. Hence, Spearman's rho is of importance in fitting a parametric copula. Here, we consider constructing a confidence interval for the Spearman's rho by fitting either the Gaussian copula or the t copula and modeling marginals by either a normal distribution or a t distribution. In this case, a profile empirical likelihood method can be employed to construct a confidence interval for ρ_s by considering the following estimating equations

$$\begin{cases} \rho_s = 12E\{F_1(X; \mu_1, \sigma_1)F_2(Y; \mu_2, \sigma_2)\} - 3 \\ 0 = EX - \mu_1 = EY - \mu_2 \\ 0 = EX^2 - \mu_1^2 - \sigma_1^2 = EY^2 - \mu_2^2 - \sigma_2^2 \end{cases}$$

when F_1 and F_2 have normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, and

$$
\begin{cases}
\rho_s = 12E\{F_1(X; \nu_1)F_2(Y; \nu_2)\} - 3 \\
0 = EX^2 - \nu_1/(\nu_1 - 2) = EY^2 - \nu_2/(\nu_2 - 2)
\end{cases}
$$

when F_1 and F_2 have distributions t_{ν_1} and t_{ν_2}, respectively. On the other hand, the proposed jackknife empirical likelihood method can be employed to the above estimating equations as well.

First we draw $10,000$ random samples with sample size $n = 100$ and 300 from the Gaussian copula and t copula by using the package 'copula' in R and transform the marginals to have either normal distributions or t distributions. For computing the coverage probabilities of the proposed jackknife empirical likelihood method, we employ the package 'emplik' in the software R. For computing the coverage probabilities of the profile empirical likelihood method, we use the package 'emplik' to obtain the likelihood ratio as a function of nuisance parameters and then use the package 'nlm' to find the minimum. These coverage probabilities are reported in Tables 1–4. From Tables 1–4, we observe that i) the proposed jackknife empirical likelihood method performs much better than the profile empirical likelihood method when $n = 100$ and marginal distributions follow from t distributions; and ii) both methods perform well when $n = 300$ although the jackknife empirical likelihood method is slightly better.

Second, we calculate the average interval lengths for both methods by drawing $1,000$ random samples from the above models. Tables 5-8 show that the interval length of the proposed jackknife empirical likelihood method is slightly bigger than that of the profile empirical likelihood method.

Third, we draw $10,000$ random samples from a 3-dimensional normal copula and t copula with marginal distributions t_7, t_8, t_9 and $\theta = (\theta_1, \theta_2, \theta_2)^T$. Then we apply the proposed jackknife empirical likelihood method and the profile empirical likelihood method to construct confidence regions for the three Spearman's rho. Coverage probabilities are reported in Tables 9 and 10, which show that the proposed jackknife empirical likelihood method performs better than the profile empirical likelihood.

In conclusion, the proposed jackknife empirical likelihood method provides good coverage accuracy and the computation is much simpler than the profile empirical likelihood method. Moreover, the package "emplik" in the software R is ready to use for the proposed method.

4. Case studies.

4.1. *Testing the drift parameter in the variance gamma model.* The class of variance gamma (VG) distributions was introduced by [25] as an alternative model for stock returns beyond the usual normal distribution assumption. It has so far been used extensively by financial economists especially in pricing financial derivatives, see [24] and [23] for applications, and [38] for a historical account of the development.

The VG process Z_t is a time-changed Lévy process where the subordinator is a Gamma process. It is parameterized by three parameters: the drift parameter μ, the volatility parameter σ, and the subordinator variance parameter ν. More specifically, let S_t be a gamma process subordinator with a unit mean rate and a variance rate ν where $\nu > 0$. Let W_t be a standard Brownian motion. Then the VG process is defined as $dZ_t = \mu dS_t + \sigma dW_{S_t}$.

That is, the calendar time t is now replaced with the time change S_t. Let $X \equiv Z_{t+\delta} - Z_t$ be the increment of Z_t with interval δ. The characteristic function of X is $Ee^{iuX} = e^{\Psi(u)\delta}$, where the characteristic exponent $\Psi(u)$ is given by $\Psi(u) = -\frac{1}{\nu} \log\{1 + \frac{u^2\sigma^2\nu}{2} - i\mu\nu u\}$.

Given a sample $\mathbf{X}$ of increments X with sample size n, we are interested in the hypothesis $H_0 : \mu = 0$. This amounts to asking whether it is sufficient to model the data in interest using a martingale process. For example, it is interesting to know whether one needs to introduce a drift parameter for the log change of US/Japan exchange rate process or not. The parameter of interest is thus μ, and the two nuisance parameters are (σ, ν). To employ the jackknife empirical likelihood estimation method in this paper, we need the estimating equations. One approach would be to just use the mean equation alone. However, this ignores all higher moments which are the very reason why financial economists use variance gamma process as an alternative to Brownian motion with drift. Therefore, below we use a different approach based on higher-order moments. The lowest-order moments or central moments would seem to be natural choices. However, the odd moments or central moments are all proportional to μ and thus all zero under the null $H_0 : \mu = 0$. Thus we use only the even moments with orders 2, 4 and 6.

By differentiating the characteristic function of the VG process increment, the raw even moments $m_j(\mu, \nu, \sigma) := EX^j$ can be computed easily and estimating equations are obtained by equating the empirical moments to the raw moments. Then we apply the jackknife empirical likelihood method to the last equation by solving the first two equations for ν and σ.

We use four different financial time series, namely, the S&P 500 index (SPX), the CBOE volatility index (VIX), the effective federal funds rate (FFR), and the exchange rate between the British Pound and US Dollar (GBP/USD). These four time series are very important in four distinctive large financial sectors, that is, the equity market, the financial derivative market, the fixed-income market, and the foreign exchange market. The sample periods are January 1950 to October 2009 for the SPX, January 1990 to October 2009 for the VIX, July 1954 to September 2009 for the FFR, and January 1971 to July 2009 for the GBP/USD.

Figure 1 plots these four time series on the same graph. The subplot for the SPX shows a few significant events, including the amazing run of the SPX before year 1987, the market crash in 1987, the internet boom in late 90s, the IT bubble burst in 2001, and most recently the subprime mortgage crisis. From the second subplot, we see that the VIX has been largely stable, except for the abrupt surge during the recent market turmoil. The FFR has had quite a few surges, especially during the hyper-inflation period in the late 70s and early 80s, and has recently reached unprecedented low levels due to the Fed's effort to boost the economy. The most noticeable feature on the GBP/USD subplot is the free fall and fast appreciation of the British Pound before and after the Plaza Accord in 1985. The recent global financial crisis has also weakened the British Pound due to the flight to quality by international investors.

Our samples are the weekly log change processes of the above four financial time series, which are plotted in Figure 2. This graph confirms the common notion that financial data often deviate a lot from the normal dis-

tribution assumption, see for example [2] or [20]. [26] traces awareness of this non-normality as far back as year 1915. This nonnormality is especially true for the times series of the log change of the effective federal funds rate, where we see a huge kurtosis.

Table 11 reports the test statistic $LR(0)$ for the log changes of the four financial time series considered. The statistic is asymptotically χ_1^2 distributed, and the asymptotic p-values are reported. The skewness and kurtosis for each time series are also included in the table, as well as the result of a naive t-test. As we would expect, the statistic $LR(0)$ strongly rejects $H_0: \mu = 0$ for the log changes of the S&P 500 index with a p-value of only 0.0002. However, the statistic cannot reject $H_0: \mu = 0$ for the log changes of the CBOE volatility index, and the exchange rate between British Pound and US Dollar. These are the same conclusions one gets from the naive t-test assuming that the sample is drawn from a normal distribution with unknown variance. However, the test on the log changes of FFR is interesting. A naive t-test suggests that $\mu = 0$, while the test statistic $LR(0)$ rejects $\mu = 0$ at more than 99% level. This apparent discrepancy might be explained by the fact that we have used higher-order moments in the estimating equations and that the empirical distribution of log changes of FFR is very different from the normal distribution with a very large kurtosis. It also highlights the advantage of using empirical likelihood estimation over the naive t-test which assumes a normal distribution.

4.2. *Testing whether a normal tempered stable process is normal inverse Gaussian.* A broader class of models that includes the VG process as a special case is the so-called *normal tempered stable* process. See [13]. This class

of models is obtained by time changing an independent Brownian motion with drift by a tempered stable subordinator S_t. The Lévy measure of S_t with index A and parameter ν is given by

$$\rho_S(x) = \frac{1}{\Gamma(1-A)} \left(\frac{1-A}{\nu}\right)^{1-A} \frac{e^{-(1-A)x/\nu}}{x^{A+1}} 1_{x>0},$$

where $\nu > 0$ and $0 \leq A < 1$ are two constants. The increment X of the time-changed process Y_t again has a closed-form characteristic function with characteristic exponent as follows

$$\Psi(u) = \frac{1-A}{\nu A} \left\{1 - \left(1 + \frac{\nu(u^2\sigma^2/2 - i\mu u)}{1-A}\right)^A\right\}.$$

Two important special cases are the VG process (where $A = 0$) and the normal inverse Gaussian process (where $A = 1/2$). See [1] and [37] for early studies on the normal inverse Gaussian process.

We are now interested in the null hypothesis $H_0 : A = 1/2$, that is, whether the data implies a normal inverse Gaussian process. Thus, the parameter of interest is A and the nuisance parameters are now (μ, ν, σ). To employ the jackknife empirical likelihood estimation method, we construct estimating equations from the raw moments $m_j(A, \mu, \nu, \sigma) := EX^j$. It turns out that if $\mu = 0$ in the normal tempered stable process, then the lowest four moments are not functions of A. Also, if $\mu = 0$, then all odd moments are zero. Thus, to reliably estimate A using moments when μ is small, we have to include a moment m_j with $j \geq 6$. We choose to use the moments m_1, m_2, m_4 and m_6 which can be calculated straightforwardly. Then the jackknife empirical likelihood method is applied to the last equation by using the first three equations to compute $\tilde{\mu}(A; \mathbf{X})$, $\tilde{\nu}(A; \mathbf{X})$ and $\tilde{\sigma}(A; \mathbf{X})$.

We apply the above procedure to the same four time series as before. The results are shown in Table 12. As we see, the test statistic $LR(1/2)$ cannot reject $H_0 : A = 1/2$ for the log changes of the S&P 500 index, the CBOE VIX index, and the exchange rate between the British Pound and US Dollar. However, it strongly rejects $A = 1/2$ for the log change time series of the effective federal funds rate.

5. Proofs. **Proof of Proposition 1.** Since condition A1) implies that

$$(5.1) \qquad \max_{1 \le i \le n} \sup_{\beta \in \Omega_{\beta_0}} \frac{1}{n-1} ||G_b(X_i; \alpha, \beta)|| = o(n^{-1/2})$$

almost surely for each $\alpha \in \Omega_{\alpha_0}$, we have

$$\frac{1}{n-1} \sum_{j=1, j \neq i}^{n} G_b(X_j; \alpha, \beta) \;\; = \;\; \frac{1}{n-1} \sum_{j=1}^{n} G_b(X_j; \alpha, \beta) - \frac{1}{n-1} G_b(X_i; \alpha, \beta)$$

$$\to \;\; EG_b(X_1; \alpha, \beta)$$

almost surely and uniformly in $1 \le i \le n$ for $\alpha \in \Omega_{\alpha_0}$ and $\beta \in \Omega_{\beta_0}$. Hence Part i) follows from the above equation and Theorem B in Section 7.2.1 of [39].

It follows from Taylor's expansion that

(5.2)

$$0 \;\; = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} G_b(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}))$$

$$= \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} G_b(X_j; \alpha_0, \beta_0)$$

$$+ \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} \{ \frac{\partial}{\partial \beta} G_b(X_j; \alpha_0, \gamma_i \beta_0 + (1 - \gamma_i) \tilde{\beta}(\alpha_0; \mathbf{X}_{-i})) \} \{ \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}) - \beta_0 \},$$

where $\gamma_i \in [0, 1]$ for each $i = 1, \cdots, n$. Since A1) and (2.5) imply that

(5.3)

$$\max_{1 \le i \le n} \frac{1}{n-1} ||G_b(X_i; \alpha_0, \gamma_i \beta_0 + (1 - \gamma_i) \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}))|| = o(n^{-1/2}) \quad \text{almost surely,}$$

we have

(5.4)
$$\max_{1\le i\le n}||\frac{1}{n-1}\sum_{j=1,j\ne i}^{n}\frac{\partial}{\partial\beta}G_b(X_j;\alpha_0,\gamma_i\beta_0+(1-\gamma_i)\tilde{\beta}(\alpha_0;\mathbf{X}_{-i}))-\Sigma_1||=o_p(1).$$

Hence (2.7) follows from (5.2) and (5.4). Similarly we can prove (2.6).

By (2.6), (2.7) and (5.3), we have

(5.5)
$$\begin{cases} \max_{1\le i\le n}||\tilde{\beta}(\alpha_0;\mathbf{X})-\tilde{\beta}(\alpha_0;\mathbf{X}_{-i})||=o_p(n^{-1/2}) \\ \max_{1\le i\le n}||\tilde{\beta}(\alpha_0;\mathbf{X}_{-i})-\beta_0||=O_p(n^{-1/2}). \end{cases}$$

It follows from (5.5) and Taylor's expansion that

$$\begin{aligned} 0 &= \tfrac{1}{n}\{\sum_{j=1}^{n}G_b(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X}))-\sum_{j=1,j\ne i}^{n}G_b(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X}_{-i}))\} \\ &= \tfrac{1}{n}\sum_{j=1}^{n}\{G_b(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X}))-G_b(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X}_{-i}))\} \\ &\quad +n^{-1}G_b(X_i;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X}_{-i})) \\ &= \Sigma_1\{\tilde{\beta}(\alpha_0;\mathbf{X})-\tilde{\beta}(\alpha_0;\mathbf{X}_{-i})\}+o_p(n^{-1}) \\ &\quad +n^{-1}G_b(X_i;\alpha_0,\beta_0)+o_p(n^{-1}) \end{aligned}$$

uniformly in $1\le i\le n$, which implies that

(5.6)
$$\max_{1\le i\le n}||\tilde{\beta}(\alpha_0;\mathbf{X})-\tilde{\beta}(\alpha_0;\mathbf{X}_{-i})+\Sigma_1^{-1}\frac{1}{n}G_b(X_i;\alpha_0,\beta_0)||=o_p(n^{-1}).$$

Note that A3) implies that

(5.7)
$$\max_{1\le i\le n}\sup_{\alpha\in\Omega_{\alpha_0},\beta\in\Omega_{\beta_0}}||n^{-1}\frac{\partial^2}{\partial\beta^T\partial\beta}g_l(X_i;\alpha,\beta)||=o(n^{-1/2})$$

almost surely for $l=s-q_2+1,\cdots,s.$

Using (2.7), (5.5)–(5.7) and Taylor's expansion, we have

$$
\begin{aligned}
0 \;=\;& \tfrac{1}{n}\{\sum_{j=1}^{n} g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X})) - \sum_{j=1,j\neq i}^{n} g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}_{-i}))\} \\
=\;& \tfrac{1}{n}\sum_{j=1}^{n}\{g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X})) - g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}_{-i}))\} \\
& + n^{-1}g_l(X_i;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}_{-i})) \\
=\;& \tfrac{1}{n}\sum_{j=1}^{n}\{\tfrac{\partial}{\partial\beta}g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}))\}\{\tilde\beta(\alpha_0;\mathbf{X}) - \tilde\beta(\alpha_0;\mathbf{X}_{-i})\} \\
& + \tfrac{1}{2n}\sum_{j=1}^{n}\{\tilde\beta(\alpha_0;\mathbf{X}) - \tilde\beta(\alpha_0;\mathbf{X}_{-i})\}^T\{\tfrac{\partial^2}{\partial\beta^T\partial\beta}g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}))\} \\
& \{\tilde\beta(\alpha_0;\mathbf{X}) - \tilde\beta(\alpha_0;\mathbf{X}_{-i})\} + n^{-1}g_l(X_i;\alpha_0,\beta_0) \\
& + n^{-1}\{\tfrac{\partial}{\partial\beta}g_l(X_i;\alpha_0,\beta_0)\}\{\tilde\beta(\alpha_0;\mathbf{X}_{-i}) - \beta_0\} + o_p(n^{-3/2}) \\
=\;& \tfrac{1}{n}\sum_{j=1}^{n}\{\tfrac{\partial}{\partial\beta}g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}))\}\{\tilde\beta(\alpha_0;\mathbf{X}) - \tilde\beta(\alpha_0;\mathbf{X}_{-i})\} \\
& + \tfrac{1}{2n}\{\tfrac{1}{n}G_b^T(X_i;\alpha_0,\beta_0)\Sigma_1^{-1}\}\{\sum_{j=1}^{n}\tfrac{\partial^2}{\partial\beta^T\partial\beta}g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}))\} \\
& \{\Sigma_1^{-1}\tfrac{1}{n}G_b(X_i;\alpha_0,\beta_0)\} + n^{-1}g_l(X_i;\alpha_0,\beta_0) \\
& - n^{-1}\{\tfrac{\partial}{\partial\beta}g_l(X_i;\alpha_0,\beta_0)\}\Sigma_1^{-1}\tfrac{1}{n-1}\sum_{j=1,j\neq i}^{n}G_b(X_j;\alpha_0,\beta_0) \\
& + o_p(n^{-3/2}) \\
=\;& \{\tfrac{1}{n}\sum_{j=1}^{n}\tfrac{\partial}{\partial\beta}g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}))\}\{\tilde\beta(\alpha_0;\mathbf{X}) - \tilde\beta(\alpha_0;\mathbf{X}_{-i})\} \\
& + \tfrac{1}{2}\{\tfrac{1}{n}G_b^T(X_i;\alpha_0,\beta_0)\Sigma_1^{-1}\}\{E\tfrac{\partial^2}{\partial\beta^T\partial\beta}g_l(X_1;\alpha_0,\beta_0)\} \\
& \{\Sigma_1^{-1}\tfrac{1}{n}G_b(X_i;\alpha_0,\beta_0)\} + n^{-1}g_l(X_i;\alpha_0,\beta_0) \\
& - n^{-1}\{\tfrac{\partial}{\partial\beta}g_l(X_i;\alpha_0,\beta_0)\}\Sigma_1^{-1}\tfrac{1}{n-1}\sum_{j=1}^{n}G_b(X_j;\alpha_0,\beta_0) \\
& + n^{-1}\{\tfrac{\partial}{\partial\beta}g_l(X_i;\alpha_0,\beta_0)\}\Sigma_1^{-1}\tfrac{1}{n-1}G_b(X_i;\alpha_0,\beta_0) + o_p(n^{-3/2})
\end{aligned}
$$

uniformly in $1\leq i\leq n$ for $l = s - q_2 + 1,\cdots,s$, which imply (2.8).

Since $0 = \tfrac{1}{n-1}\sum_{j=1,j\neq i}^{n}G_b(X_j;\alpha,\tilde\beta(\alpha;\mathbf{X}_{-i}))$, we have

$$
\begin{aligned}
0 \;=\;& \tfrac{1}{n-1}\sum_{j=1,j\neq i}^{n}\tfrac{\partial}{\partial\alpha}G_b(X_j;\alpha,\tilde\beta(\alpha;\mathbf{X}_{-i})) \\
& + \tfrac{1}{n-1}\sum_{j=1,j\neq i}^{n}\{\tfrac{\partial}{\partial\beta}G_b(X_j;\alpha,\tilde\beta(\alpha;\mathbf{X}_{-i}))\}\tfrac{\partial}{\partial\alpha}\tilde\beta(\alpha;\mathbf{X}_{-i}).
\end{aligned}
$$

Note that A5) implies that

$$
\max_{1\leq i\leq n}\frac{1}{n-1}\|\frac{\partial}{\partial\alpha}G_b(X_i;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}_{-i}))\| = o(n^{-1/2})
$$

almost surely. Hence

$$\frac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}) = -\Sigma_1^{-1} \Sigma_2 + O_p(n^{-1/2})$$

uniformly in $1 \le i \le n$. Similarly, we have

$$\frac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}) = -\Sigma_1^{-1} \Sigma_2 + O_p(n^{-1/2}),$$

i.e., (2.9) holds.

It follows from (5.5), (5.6), A3) and A5) that

$$\frac{1}{n} \sum_{j=1}^{n} \{ \frac{\partial}{\partial \alpha} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X})) - \frac{\partial}{\partial \alpha} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i})) \}$$

$$= \frac{1}{n} \sum_{j=1}^{n} \{ \tilde{\beta}(\alpha_0; \mathbf{X}) - \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}) \} \frac{\partial^2}{\partial \beta^T \partial \alpha} g_l(X_j; \alpha_0, \beta_0) + o_p(n^{-1})$$

$$= -\frac{1}{n} G_b^T(X_i; \alpha_0, \beta_0) \Sigma_1^{-1} E\{ \frac{\partial^2}{\partial \beta^T \partial \alpha} g_l(X_1; \alpha_0, \beta_0) \} + o_p(n^{-1})$$

and

$$\frac{1}{n} \sum_{j=1}^{n} \{ \frac{\partial}{\partial \beta} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X})) - \frac{\partial}{\partial \beta} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i})) \}$$

$$= \frac{1}{n} \sum_{j=1}^{n} \{ \tilde{\beta}(\alpha_0; \mathbf{X}) - \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}) \} \frac{\partial^2}{\partial \beta^T \partial \beta} g_l(X_j; \alpha_0, \beta_0) + o_p(n^{-1})$$

$$= -\frac{1}{n} G_b^T(X_i; \alpha_0, \beta_0) \Sigma_1^{-1} E\{ \frac{\partial^2}{\partial \beta^T \partial \beta} g_l(X_1; \alpha_0, \beta_0) \} + o_p(n^{-1})$$

uniformly in $1 \leq i \leq n$ for $l = s - q_2 + 1, \cdots, s$. Hence

$$
\begin{aligned}
0 \; =\; & \tfrac{1}{n}\{\textstyle\sum_{j=1}^{n} \tfrac{\partial}{\partial \alpha} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X})) - \sum_{j=1, j \neq i}^{n} \tfrac{\partial}{\partial \alpha} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}))\} \\
& + \tfrac{1}{n}\{\textstyle\sum_{j=1}^{n} \big(\tfrac{\partial}{\partial \beta} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}))\big) \tfrac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}) \\
& - \textstyle\sum_{j=1, j \neq i}^{n} \big(\tfrac{\partial}{\partial \beta} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}))\big) \tfrac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}_{-i})\} \\
=\; & \tfrac{1}{n}\textstyle\sum_{j=1}^{n}\{\tfrac{\partial}{\partial \alpha} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X})) - \tfrac{\partial}{\partial \alpha} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}))\} \\
& + \tfrac{1}{n} \tfrac{\partial}{\partial \alpha} g_l(X_i; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i})) \\
& + \tfrac{1}{n}\textstyle\sum_{j=1}^{n}\{\tfrac{\partial}{\partial \beta} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X})) - \tfrac{\partial}{\partial \beta} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}))\} \tfrac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}) \\
& + \tfrac{1}{n}\textstyle\sum_{j=1}^{n}\{\tfrac{\partial}{\partial \beta} g_l(X_j; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}))\}\{\tfrac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}) - \tfrac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}_{-i})\} \\
& + \tfrac{1}{n}\{\tfrac{\partial}{\partial \beta} g_l(X_i; \alpha_0, \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}))\} \tfrac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}_{-i}) \\
=\; & -\tfrac{1}{n} G_b^T(X_i; \alpha_0, \beta_0) \Sigma_1^{-1} E\{\tfrac{\partial^2}{\partial \beta^T \partial \alpha} g_l(X_1; \alpha_0, \beta_0)\} \\
& + \tfrac{1}{n} \tfrac{\partial}{\partial \alpha} g_l(X_i; \alpha_0, \beta_0) \\
& + \tfrac{1}{n} G_b^T(X_i; \alpha_0, \beta_0) \Sigma_1^{-1} E\{\tfrac{\partial^2}{\partial \beta^T \partial \beta} g_l(X_1; \alpha_0, \beta_0)\} \Sigma_1^{-1} \Sigma_2 \\
& + \{\Sigma_1 + o_p(1)\}\{\tfrac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}) - \tfrac{\partial}{\partial \alpha} \tilde{\beta}(\alpha_0; \mathbf{X}_{-i})\} \\
& - \tfrac{1}{n}\{\tfrac{\partial}{\partial \beta} g_l(X_i; \alpha_0, \beta_0)\} \Sigma_1^{-1} \Sigma_2 + o_p(n^{-1})
\end{aligned}
$$

for $l = s - q_2 + 1, \cdots, s$, which imply (2.10).

Before we prove Proposition 2, we need two lemmas.

Lemma 1. Under conditions A1)–A7), we have

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i(\alpha_0) \xrightarrow{d} (W_1, \cdots, W_{s-q_2})^T
$$

where for $l = 1, \cdots, s - q_2$,

$$
W_l = Z_l - E\{\frac{\partial}{\partial \beta} g_l(X_1; \alpha_0, \beta_0)\} \Sigma_1^{-1} (Z_{s-q_2+1}, \cdots, Z_s)^T
$$

and $(Z_1, \cdots, Z_s)^T \sim N(0, \Sigma)$.

Proof. Write

$$\begin{aligned}
Y_{i,l}(\alpha_0) & = \sum_{j=1}^{n}\{g_l(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X})) - g_l(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X}_{-i}))\} \\
& \quad + g_l(X_i;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X}_{-i})) \\
& = I_1(i,l) + I_2(i,l)
\end{aligned}$$

for $i = 1,\cdots,n$ and $l = 1,\cdots,s - q_2$.

By Proposition 1, (5.5) and A7), we have

$$\begin{aligned}
& I_1(i,l) \\
= \ & \sum_{j=1}^{n}\{\tfrac{\partial}{\partial\beta}g_l(X_j;\alpha_0,\tilde{\beta}(\alpha_0,\mathbf{X}))\}\{\tilde{\beta}(\alpha_0;\mathbf{X}) - \tilde{\beta}(\alpha_0;\mathbf{X}_{-i})\} \\
& + \tfrac{1}{2}\sum_{j=1}^{n}\{\tilde{\beta}(\alpha_0;\mathbf{X}) - \tilde{\beta}(\alpha_0;\mathbf{X}_{-i})\}^T\tfrac{\partial^2}{\partial\beta\partial\beta^T}g_l(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X})) \\
& \{\tilde{\beta}(\alpha_0;\mathbf{X}) - \tilde{\beta}(\alpha_0;\mathbf{X}_{-i})\} + o_p(n^{-1/2}) \\
= \ & -\sum_{j=1}^{n}\{\tfrac{\partial}{\partial\beta}g_l(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X}))\}\Sigma_{n1}^{-1}D_n(i) \\
& + \tfrac{1}{2}\sum_{j=1}^{n}D_n^T(i)\Sigma_{n1}^{-1}\{\tfrac{\partial^2}{\partial\beta\partial\beta^T}g_l(X_j;\alpha_0,\tilde{\beta}(\alpha_0;\mathbf{X}))\}\Sigma_{n1}^{-1}D_n(i) + o_p(n^{-1/2})
\end{aligned}$$

where $o_p(n^{-1/2})$ holds uniformly in $i = 1,\cdots,n$ for $l = 1,\cdots,s - q_2$. It follows from (5.1) that

$$\begin{aligned}
& n\sum_{i=1}^{n}D_n(i) \\
= \ & \sum_{i=1}^{n}G_b(X_i;\alpha_0,\beta_0) \\
& -\{\sum_{i=1}^{n}\tfrac{\partial}{\partial\beta}G_b(X_i;\alpha_0,\beta_0)\}\Sigma_1^{-1}\tfrac{1}{n-1}\sum_{j=1}^{n}G_b(X_j;\alpha_0,\beta_0) + o_p(n^{1/2}) \\
= \ & n\{-\tfrac{1}{n}\Sigma_1 + \Sigma_1 - \tfrac{1}{n}\sum_{i=1}^{n}\tfrac{\partial}{\partial\beta}G_b(X_i;\alpha_0,\beta_0)\}\Sigma_1^{-1}\tfrac{1}{n-1}\sum_{j=1}^{n}G_b(X_j;\alpha_0,\beta_0) \\
& + o_p(n^{1/2}) \\
= \ & o_p(n^{1/2}).
\end{aligned}$$

Similarly we can verify that $n\sum_{i=1}^{n}D_n^T(i)\Delta D_n(i) = o_p(n^{1/2})$ for any $q_2 \times q_2$ matrix Δ, which is independent of i. Hence,

$$\begin{equation}
\sum_{i=1}^{n}I_1(i,l) = o_p(n^{1/2}) \quad \text{for} \quad l = 1,\cdots,s - q_2. \tag{5.8}
\end{equation}$$

It follows from Proposition 1 that

$$\sum_{i=1}^{n} I_2(i,l)$$

$$= \sum_{i=1}^{n} g_l(X_i; \alpha_0, \beta_0)$$

$$+ \sum_{i=1}^{n} \{\frac{\partial}{\partial \beta} g_l(X_i; \alpha_0, \beta_0)\}\{-\Sigma_1^{-1}\}\frac{1}{n-1}\sum_{k \neq i} G_b(X_k; \alpha_0, \beta_0) + o_p(n^{1/2})$$

$$= \sum_{i=1}^{n} g_l(X_i; \alpha_0, \beta_0)$$

$$+ \frac{1}{n-1}\sum_{i=1}^{n}\{\frac{\partial}{\partial \beta} g_l(X_i; \alpha_0, \beta_0)\}\{-\Sigma_1^{-1}\}\sum_{k=1}^{n} G_b(X_k; \alpha_0, \beta_0)$$

$$- \frac{1}{n-1}\sum_{i=1}^{n}\{\frac{\partial}{\partial \beta} g_l(X_i; \alpha_0, \beta_0)\}\{-\Sigma_1^{-1}\}G_b(X_i; \alpha_0, \beta_0) + o_p(n^{1/2})$$

$$= II_1(l) + II_2(l) - II_3(l) + o_p(n^{1/2}).$$

By (5.1) and A7), we can show that

$$II_3(l) = O_p(1) \quad \text{for} \quad l = 1, \cdots, s - q_2,$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(G_a^T(X_i; \alpha_0, \beta_0), G_b^T(X_i; \alpha_0, \beta_0))^T \xrightarrow{d} (Z_1, \cdots, Z_s)^T \sim N(0, \Sigma),$$

which imply that

(5.9)
$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} I_2(i,l) \xrightarrow{d} Z_l - E\{\frac{\partial}{\partial \beta} g_l(X_1; (\alpha_0^T, \beta_0^T)^T)\}\Sigma_1^{-1}(Z_{s-q_2+1}, \cdots, Z_s)^T.$$

Hence the lemma follows from (5.8) and (5.9).

Lemma 2. Under conditions A1)–A7), we have

$$\frac{1}{n}\sum_{i=1}^{n} Y_i(\alpha_0)Y_i^T(\alpha_0) \xrightarrow{p} (E\{W_l W_k\})_{1 \leq k,l \leq s-q_2},$$

where $W_k's$ are given in Lemma 1.

Proof. Using the same notation as in the proof of Lemma 1, it follows from Proposition 1, (5.6) and Taylor's expansion that

$$\frac{1}{n}\sum_{i=1}^{n} Y_{i,l}(\alpha_0)Y_{i,k}(\alpha_0)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\{I_1(i,l)I_1(i,k) + I_1(i,l)I_2(i,k) + I_2(i,l)I_1(i,k) + I_2(i,l)I_2(i,k)\},$$

$$\frac{1}{n}\sum_{i=1}^{n} I_1(i,l)I_1(i,k)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\{\sum_{j_1=1}^{n}\frac{\partial}{\partial\beta}g_l(X_{j_1};\alpha_0,\beta_0)\}\{-\Sigma_1^{-1}\}\{\frac{1}{n}G_b(X_i;\alpha_0,\beta_0)\}$$

$$\times\{\frac{1}{n}G_b^T(X_i;\alpha_0,\beta_0)\}\{-\Sigma_1^{-1}\}\{\sum_{j_2=1}^{n}\frac{\partial}{\partial\beta}g_k(X_{j_2};\alpha_0,\beta_0)\}^T + o_p(1)$$

$$\xrightarrow{p} \{E\frac{\partial}{\partial\beta}g_l(X_1;\alpha_0,\beta_0)\}\{-\Sigma_1^{-1}\}E\{G_b(X_1;\alpha_0,\beta_0)G_b^T(X_1;\alpha_0,\beta_0)\}$$

$$\times\{-\Sigma_1^{-1}\}\{E\frac{\partial}{\partial\beta}g_k(X_1;\alpha_0,\beta_0)\},$$

$$\frac{1}{n}\sum_{i=1}^{n} I_1(i,l)I_2(i,k)$$

$$\xrightarrow{p} \{E\frac{\partial}{\partial\beta}g_l(X_1;\alpha_0,\beta_0)\{-\Sigma_1^{-1}\}E\{G_b(X_1;\alpha_0,\beta_0)g_k(X_1;\alpha_0,\beta_0)\},$$

$$\frac{1}{n}\sum_{i=1}^{n} I_2(i,l)I_1(i,k)$$

$$\xrightarrow{p} \{E\frac{\partial}{\partial\beta}g_k(X_1;\alpha_0,\beta_0)\{-\Sigma_1^{-1}\}E\{G_b(X_1;\alpha_0,\beta_0)g_l(X_1;\alpha_0,\beta_0)\},$$

and

$$\frac{1}{n}\sum_{i=1}^{n} I_2(i,l)I_2(i,k) \xrightarrow{p} E\{g_l(X_1;\alpha_0,\beta_0)g_k(X_1;\alpha_0,\beta_0)\}.$$

Hence

$$\frac{1}{n}\sum_{i=1}^{n} Y_{i,l}(\alpha_0)Y_{i,k}(\alpha_0) \xrightarrow{p} \sigma_{l,k}^*$$

for $1 \le l, k \le s - q_2$, i.e., the lemma holds.

Proof of Proposition 2. By (2.5), A1)–A4) and Taylor's expansion, we have

(5.10)
$$\begin{cases} \tilde{\beta}(\alpha;\mathbf{X}) - \beta_0 + \Sigma_1^{-1}\frac{1}{n}\sum_{i=1}^{n} G_b(X_i;\alpha,\beta_0) = O_p(n^{-2/3}) \\ \sup_{1\le i\le n}|\tilde{\beta}(\alpha;\mathbf{X}_{-i}) - \beta_0 + \Sigma_1^{-1}\frac{1}{n-1}\sum_{j\ne i} G_b(X_j;\alpha,\beta_0)| = O_p(n^{-2/3}) \end{cases}$$

uniformly in $||\alpha - \alpha_0|| \le n^{-1/3}$. By (5.10), we have

(5.11)
$$\sup_{1\le i\le n}|\tilde{\beta}(\alpha;\mathbf{X}) - \tilde{\beta}(\alpha;\mathbf{X}_{-i})|$$

$$= \sup_{1\le i\le n}|-\Sigma_1^{-1}\{-\frac{1}{n(n-1)}\sum_{k=1}^{n} G_b(X_k;\alpha,\beta_0) + \frac{1}{n-1}G_b(X_i;\alpha,\beta_0)\}|$$

$$+O_p(n^{-2/3})$$

$$= o_p(n^{-1/2})$$

uniformly in $||\alpha - \alpha_0|| \leq n^{-1/3}$. For $i = 1, \cdots, n$ and $l = 1, \cdots, s - q_2$, define

$$II_1(i, l) = \sum_{j=1}^{n} \{g_l(X_j; \alpha, \tilde{\beta}(\alpha; \mathbf{X})) - g_l(X_j; \alpha, \tilde{\beta}(\alpha; \mathbf{X}_{-i}))\}.$$

Then, it follows from (5.11), A7) and Taylor's expansion that

$$\max_{1 \leq i \leq n} |II_1(i, l)| = o_p(n^{1/2}) \quad \text{uniformly in } ||\alpha - \alpha_0|| \leq n^{-1/3}$$

for $l = 1, \cdots, s - q_2$. Hence

(5.12)

$$\max_{1 \leq i \leq n} |Y_{i,l}(\alpha)| = \max_{1 \leq i \leq n} |II_1(i, l) + g_l(X_i; (\alpha^T, \tilde{\beta}^T(\alpha; \mathbf{X}_{-i}))^T)| = o_p(n^{1/2})$$

uniformly in $||\alpha - \alpha_0|| \leq n^{-1/3}$ for $l = 1, \cdots, s - q_2$. By (5.12), Lemmas 1-2 and similar arguments to the proof of [30], we have

$$(5.13) \quad \lambda = \{\frac{1}{n} \sum_{i=1}^{n} Y_i(\alpha) Y_i^T(\alpha)\}^{-1} \{\frac{1}{n} \sum_{i=1}^{n} Y_i(\alpha)\} + o_p(n^{-1/3}) = O_p(n^{-1/3})$$

uniformly in $||\alpha - \alpha_0|| \leq n^{-1/3}$. The rest is similar to the proof of Lemma 1 in [35].

Proof of Theorem 1. Note that

$$\sum_{i=1}^{n} A_i \xrightarrow{p} E\{\frac{\partial}{\partial \alpha} G_b(X_1; \alpha_0, \beta_0)\} - E\{\frac{\partial}{\partial \beta} G_b(X_1; \alpha_0, \beta_0)\} \Sigma_1^{-1} \Sigma_2.$$

Hence, it follows from Proposition 1 that

$$
\begin{aligned}
& \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\tfrac{\partial}{\partial\alpha}Y_{i,l}(\alpha_0) \\
=\ & \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\{\sum_{j=1}^{n}\tfrac{\partial}{\partial\alpha}g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X})) \\
& -\textstyle\sum_{j=1,j\neq i}^{n}\tfrac{\partial}{\partial\alpha}g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}_{-i})) \\
& +\textstyle\sum_{j=1}^{n}\{\tfrac{\partial}{\partial\beta}g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}))\}\tfrac{\partial\tilde\beta(\alpha_0;\mathbf{X})}{\partial\alpha} \\
& -\textstyle\sum_{j=1,j\neq i}^{n}\{\tfrac{\partial}{\partial\beta}g_l(X_j;\alpha_0,\tilde\beta(\alpha_0;\mathbf{X}_{-i}))\}\tfrac{\partial\tilde\beta(\alpha_0;\mathbf{X}_{-i})}{\partial\alpha}\} \\
=\ & \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\{-\sum_{j=1}^{n}\tfrac{1}{n}G_b^T(X_i;\alpha_0,\beta_0)\Sigma_1^{-1}\tfrac{\partial^2}{\partial\beta^T\partial\alpha}g_l(X_j;\alpha_0,\beta_0) \\
& +\tfrac{\partial}{\partial\alpha}g_l(X_i;\alpha_0,\beta_0) \\
& +\textstyle\sum_{j=1}^{n}\tfrac{1}{n}G_b^T(X_i;\alpha_0,\beta_0)\Sigma_1^{-1}\tfrac{\partial^2}{\partial\beta\partial\beta^T}g_l(X_j;\alpha_0,\beta_0)\Sigma_1^{-1}\Sigma_2 \\
& -\textstyle\sum_{j=1}^{n}\tfrac{\partial}{\partial\beta}g_l(X_j;\alpha_0,\beta_0)\Sigma_1^{-1}A_i \\
& -\tfrac{\partial}{\partial\beta}g_l(X_i;\alpha_0,\beta_0)\Sigma_1^{-1}\Sigma_2\}+o_p(1) \\
\xrightarrow{p}\ & E\{\tfrac{\partial}{\partial\alpha}g_l(X_1;\alpha_0,\beta_0)\} \\
& -E\{\tfrac{\partial}{\partial\beta}g_l(X_1;\alpha_0,\beta_0)\}\Sigma_1^{-1}\Big(E\{\tfrac{\partial}{\partial\alpha}G_b(X_1;\alpha_0,\beta_0)\} \\
& -E\{\tfrac{\partial}{\partial\beta}G_b(X_1;\alpha_0,\beta_0)\}\Sigma_1^{-1}\Sigma_2\Big) \\
& -E\{\tfrac{\partial}{\partial\beta}g_l(X_1;\alpha_0,\beta_0)\}\Sigma_1^{-1}\Sigma_2 \\
=\ & E\{\tfrac{\partial}{\partial\alpha}g_l(X_1;\alpha_0,\beta_0)\}-E\{\tfrac{\partial}{\partial\beta}g_l(X_1;\alpha_0,\beta_0)\}\Sigma_1^{-1}\Sigma_2
\end{aligned}
$$

for $l=1,\cdots,s-q_2$, which imply that

$$
(5.14)\qquad\qquad \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\alpha}Y_i(\alpha_0)\xrightarrow{p}\Sigma_3.
$$

Put $V=\{\Sigma_3^T(\Sigma^*)^{-1}\Sigma_3\}^{-1}$. Similar to the proof of Theorem 1 of [35], we can show that

$$
\sqrt{n}\{\hat\alpha-\alpha_0\}=-V\Sigma_3^T(\Sigma^*)^{-1}\sqrt{n}Q_{1n}(\alpha_0,0)+o_p(1)
$$

and

$$
\sqrt{n}\hat\lambda=(\Sigma^*)^{-1}\{I-\Sigma_3 V\Sigma_3^T(\Sigma^*)^{-1}\}\sqrt{n}Q_{1n}(\alpha_0,0)+o_p(1).
$$

So,

$$
\begin{aligned}
\ell^J(\hat{\alpha}) \quad &= \textstyle\sum_{i=1}^n \hat{\lambda}^T Y_i(\hat{\alpha}) + \frac{1}{2}\sum_{i=1}^n \hat{\lambda}^T Y_i(\hat{\alpha})Y_i^T(\hat{\alpha})\hat{\lambda} + o_p(1) \\
&= \textstyle\sum_{i=1}^n \hat{\lambda}^T Y_i(\alpha_0) + \sum_{i=1}^n \hat{\lambda}^T \frac{\partial}{\partial \alpha} Y_i(\alpha_0)(\hat{\alpha}-\alpha_0) \\
&\quad + \frac{1}{2}\textstyle\sum_{i=1}^n \hat{\lambda}^T Y_i(\alpha_0)Y_i^T(\alpha_0)\hat{\lambda} + o_p(1) \\
&= \frac{n}{2}Q_{1n}^T(\alpha_0,0)(\Sigma^*)^{-1}\{I - \Sigma_3 V \Sigma_3^T(\Sigma^*)^{-1}\}Q_{1n}(\alpha_0,0) + o_p(1).
\end{aligned}
$$

Similarly

$$
\ell^J(\alpha_0) = \frac{n}{2}Q_{1n}^T(\alpha_0,0)(\Sigma^*)^{-1}Q_{1n}(\alpha_0,0) + o_p(1).
$$

Thus

$$
\begin{aligned}
&LR(\alpha_0) \\
=\quad & \{(\Sigma^*)^{-1/2}\tfrac{1}{\sqrt{n}}\textstyle\sum_{i=1}^n Y_i(\alpha_0)\}^T \{(\Sigma^*)^{-1/2}\Sigma_3 V \Sigma_3^T(\Sigma^*)^{-1/2}\} \\
& \{(\Sigma^*)^{-1/2}\tfrac{1}{\sqrt{n}}\textstyle\sum_{i=1}^n Y_i(\alpha_0)\} + o_p(1) \\
\xrightarrow{d}\quad & \chi^2_{q_1}.
\end{aligned}
$$

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TABLE 1

Empirical coverage probabilities for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 *and* 0.95 *for the 2-dimensional Gaussian copula and marginal distributions* $N(-1,1)$ *and* $N(1,1)$.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n,\theta) = (100, 0.25)$	0.8960	0.8954	0.9464	0.9461
$(n,\theta) = (100, 0.5)$	0.8960	0.8956	0.9456	0.9479
$(n,\theta) = (100, 0.75)$	0.8916	0.9021	0.9429	0.9552
$(n,\theta) = (300, 0.25)$	0.8987	0.8979	0.9491	0.9481
$(n,\theta) = (300, 0.5)$	0.8963	0.8962	0.9497	0.9492
$(n,\theta) = (300, 0.75)$	0.8972	0.8988	0.9499	0.9515

TABLE 2

Empirical coverage probabilities for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 *and* 0.95 *for the 2-dimensional Gaussian copula and marginal distributions* t_7 *and* t_8.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n,\theta) = (100, 0.25)$	0.8683	0.8001	0.9184	0.8438
$(n,\theta) = (100, 0.5)$	0.8573	0.7970	0.9114	0.8416
$(n,\theta) = (100, 0.75)$	0.8563	0.8066	0.9076	0.8505
$(n,\theta) = (300, 0.25)$	0.8961	0.8950	0.9455	0.9436
$(n,\theta) = (300, 0.5)$	0.8969	0.8960	0.9451	0.9437
$(n,\theta) = (300, 0.75)$	0.8961	0.8945	0.9453	0.9433

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TABLE 3

Empirical coverage probabilities for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 and 0.95 for the 2-dimensional t copula and marginal distributions $N(-1,1)$ and $N(1,1)$.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n,\theta) = (100, 0.25)$	0.9031	0.9015	0.9512	0.9499
$(n,\theta) = (100, 0.5)$	0.9011	0.9011	0.9475	0.9477
$(n,\theta) = (100, 0.75)$	0.8969	0.9042	0.9467	0.9540
$(n,\theta) = (300, 0.25)$	0.9011	0.8990	0.9491	0.9480
$(n,\theta) = (300, 0.5)$	0.8947	0.8922	0.9469	0.9448
$(n,\theta) = (300, 0.75)$	0.8893	0.8874	0.9449	0.9436

TABLE 4

Empirical coverage probabilities for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 and 0.95 for the 2-dimensional t copula and marginal distributions t_7 and t_8.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n,\theta) = (100, 0.25)$	0.8679	0.8054	0.9221	0.8518
$(n,\theta) = (100, 0.5)$	0.8605	0.8055	0.9149	0.8531
$(n,\theta) = (100, 0.75)$	0.8575	0.8123	0.9109	0.8612
$(n,\theta) = (300, 0.25)$	0.8948	0.8923	0.9439	0.9420
$(n,\theta) = (300, 0.5)$	0.8963	0.8937	0.9464	0.9439
$(n,\theta) = (300, 0.75)$	0.8935	0.8928	0.9447	0.9434

TABLE 5

Empirical interval lengths for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 and 0.95 for the 2-dimensional Gaussian copula and marginal distributions $N(-1,1)$ and $N(1,1)$.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n,\theta) = (100, 0.25)$	0.361	0.354	0.397	0.395
$(n,\theta) = (100, 0.5)$	0.328	0.321	0.382	0.376
$(n,\theta) = (100, 0.75)$	0.286	0.285	0.340	0.334
$(n,\theta) = (300, 0.25)$	0.207	0.205	0.247	0.245
$(n,\theta) = (300, 0.5)$	0.187	0.185	0.224	0.222
$(n,\theta) = (300, 0.75)$	0.163	0.165	0.196	0.203

TABLE 6

Empirical interval lengths for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 and 0.95 for the 2-dimensional Gaussian copula and marginal distributions t_7 and t_8.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n, \theta) = (100, 0.25)$	0.949	0.938	1.131	1.117
$(n, \theta) = (100, 0.5)$	1.036	1.011	1.233	1.201
$(n, \theta) = (100, 0.75)$	1.130	0.986	1.348	1.174
$(n, \theta) = (300, 0.25)$	0.548	0.547	0.654	0.652
$(n, \theta) = (300, 0.5)$	0.593	0.590	0.706	0.703
$(n, \theta) = (300, 0.75)$	0.634	0.632	0.755	0.752

TABLE 7

Empirical interval lengths for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 and 0.95 for the 2-dimensional t copula and marginal distributions $N(-1, 1)$ and $N(1, 1)$.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n, \theta) = (100, 0.25)$	0.381	0.376	0.399	0.399
$(n, \theta) = (100, 0.5)$	0.359	0.351	0.394	0.391
$(n, \theta) = (100, 0.75)$	0.314	0.309	0.366	0.359
$(n, \theta) = (300, 0.25)$	0.223	0.220	0.266	0.263
$(n, \theta) = (300, 0.5)$	0.206	0.203	0.246	0.243
$(n, \theta) = (300, 0.75)$	0.178	0.177	0.214	0.215

TABLE 8

Empirical interval lengths for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 and 0.95 for the 2-dimensional t copula and marginal distributions t_7 and t_8.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n, \theta) = (100, 0.25)$	0.956	0.941	1.140	1.121
$(n, \theta) = (100, 0.5)$	1.038	1.015	1.236	1.207
$(n, \theta) = (100, 0.75)$	1.118	0.976	1.334	1.164
$(n, \theta) = (300, 0.25)$	0.553	0.551	0.659	0.657
$(n, \theta) = (300, 0.5)$	0.593	0.590	0.707	0.703
$(n, \theta) = (300, 0.75)$	0.631	0.629	0.752	0.749

TABLE 9

Empirical coverage probabilities for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 *and* 0.95 *for the 3-dimensional Gaussian copula and marginal distributions* t_7, t_8 *and* t_9.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n, \theta_1, \theta_2, \theta_3) = (100, 0.25, 0.5, 0.25)$	0.8216	0.7306	0.8799	0.7746
$(n, \theta_1, \theta_2, \theta_3) = (100, 0.5, 0.5, 0.5)$	0.8164	0.7363	0.8740	0.7794
$(n, \theta_1, \theta_2, \theta_3) = (100, 0.75, 0.5, 0.75)$	0.7908	0.7524	0.8497	0.7958
$(n, \theta_1, \theta_2, \theta_3) = (300, 0.25, 0.5, 0.25)$	0.8934	0.8907	0.9434	0.9384
$(n, \theta_1, \theta_2, \theta_3) = (300, 0.5, 0.5, 0.5)$	0.8893	0.8872	0.9429	0.9388
$(n, \theta_1, \theta_2, \theta_3) = (300, 0.75, 0.5, 0.75)$	0.8867	0.8859	0.9406	0.9390

TABLE 10

Empirical coverage probabilities for the proposed jackknife empirical likelihood confidence interval(JELCI) and the profile empirical likelihood confidence interval(PELCI) with nominal levels 0.9 *and* 0.95 *for the 3-dimensional t copula and marginal distributions* t_7, t_8 *and* t_9.

	level 0.9 JELCI	level 0.9 PELCI	level 0.95 JELCI	levle 0.95 PELCI
$(n, \theta_1, \theta_2, \theta_3) = (100, 0.25, 0.5, 0.25)$	0.8160	0.7341	0.8733	0.7784
$(n, \theta_1, \theta_2, \theta_3) = (100, 0.5, 0.5, 0.5)$	0.8025	0.7348	0.8648	0.7835
$(n, \theta_1, \theta_2, \theta_3) = (100, 0.75, 0.5, 0.75)$	0.7860	0.7443	0.8475	0.7920
$(n, \theta_1, \theta_2, \theta_3) = (300, 0.25, 0.5, 0.25)$	0.8867	0.8819	0.9390	0.9341
$(n, \theta_1, \theta_2, \theta_3) = (300, 0.5, 0.5, 0.5)$	0.8858	0.8834	0.9372	0.9340
$(n, \theta_1, \theta_2, \theta_3) = (300, 0.75, 0.5, 0.75)$	0.8842	0.8857	0.9334	0.9352

TABLE 11

Jackknife empirical likelihood ratio test for $H_0: \ \mu = 0$ in the variance gamma model for financial time series. The samples are the weekly log changes of the SPX, VIX, FFR, and GBP/USD time series. The skewness and kurtosis of the samples are also reported as well as the naive t-statistic assuming that the samples are drawn from a normal distribution with unknown variances.

log change of	n	skewness	kurtosis	t-test	$LR(0)$	p-value
SPX	3059	-0.54	7.77	3.5451	14.3046	0.0002
VIX	1018	0.39	4.82	0.1005	1.1303	0.2877
FFR	2881	-0.32	55.83	-0.2819	7.8141	0.0052
GBP/USD	1987	-0.43	7.10	-0.6286	0.8917	0.3450

TABLE 12

Jackknife empirical likelihood ratio test for $H_0 : A = 1/2$ in a normal tempered stable process for financial time series. The weekly log changes of the SPX, VIX, FFR, and GBP/USD are studied.

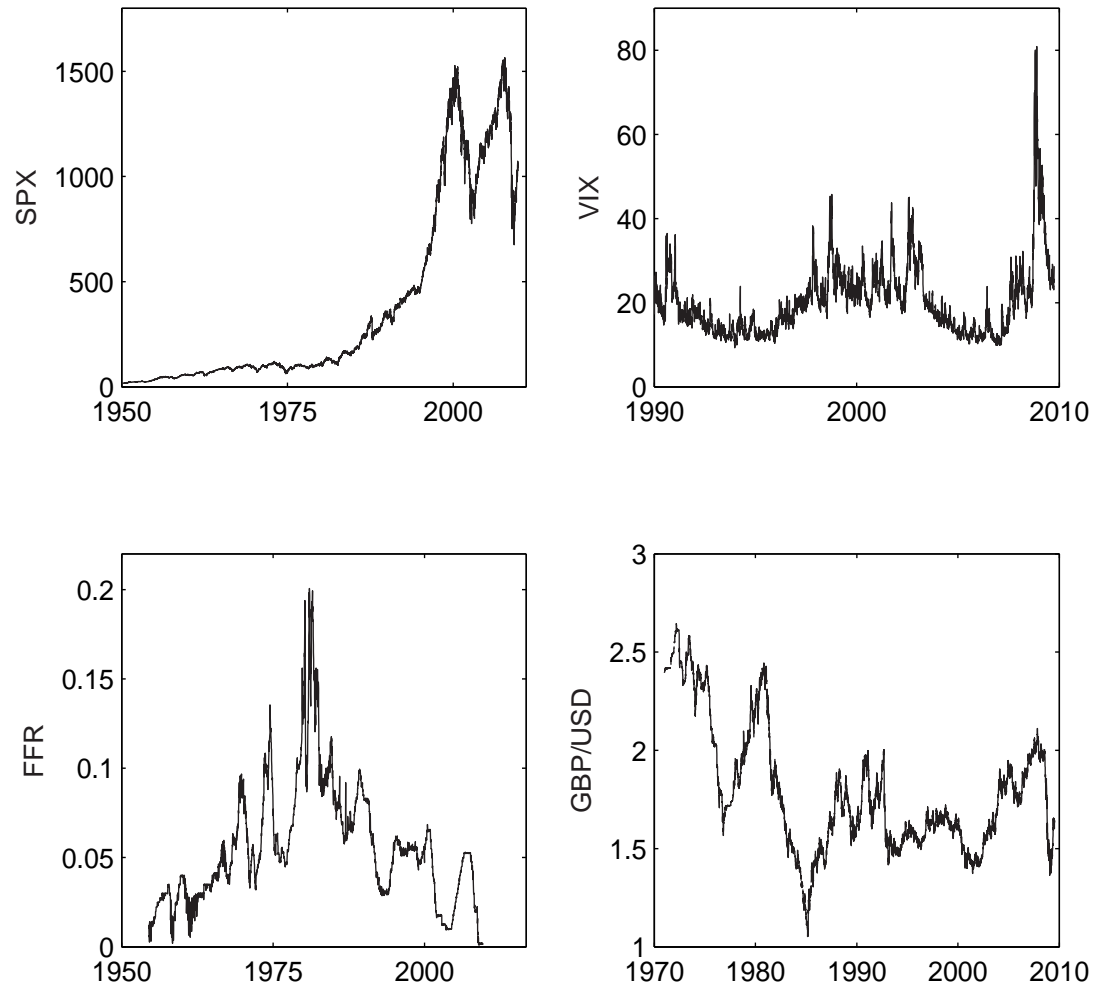| log change of	n	$LR(1/2)$	p-value
SPX	3059	1.0944	0.2955
VIX	1018	1.2903	0.2560
FFR	2881	10.1324	0.0015
GBP/USD	1987	0.2629	0.6082

FIG 1. **The historical times series of SPX, VIX, FFR and GBP/USD.** *Here SPX, VIX, FFR and GBP/USD refer to the S&P 500 index, the CBOE volatility index, the effective federal funds rate, and the exchange rate between the British Pound and US Dollar, respectively.*
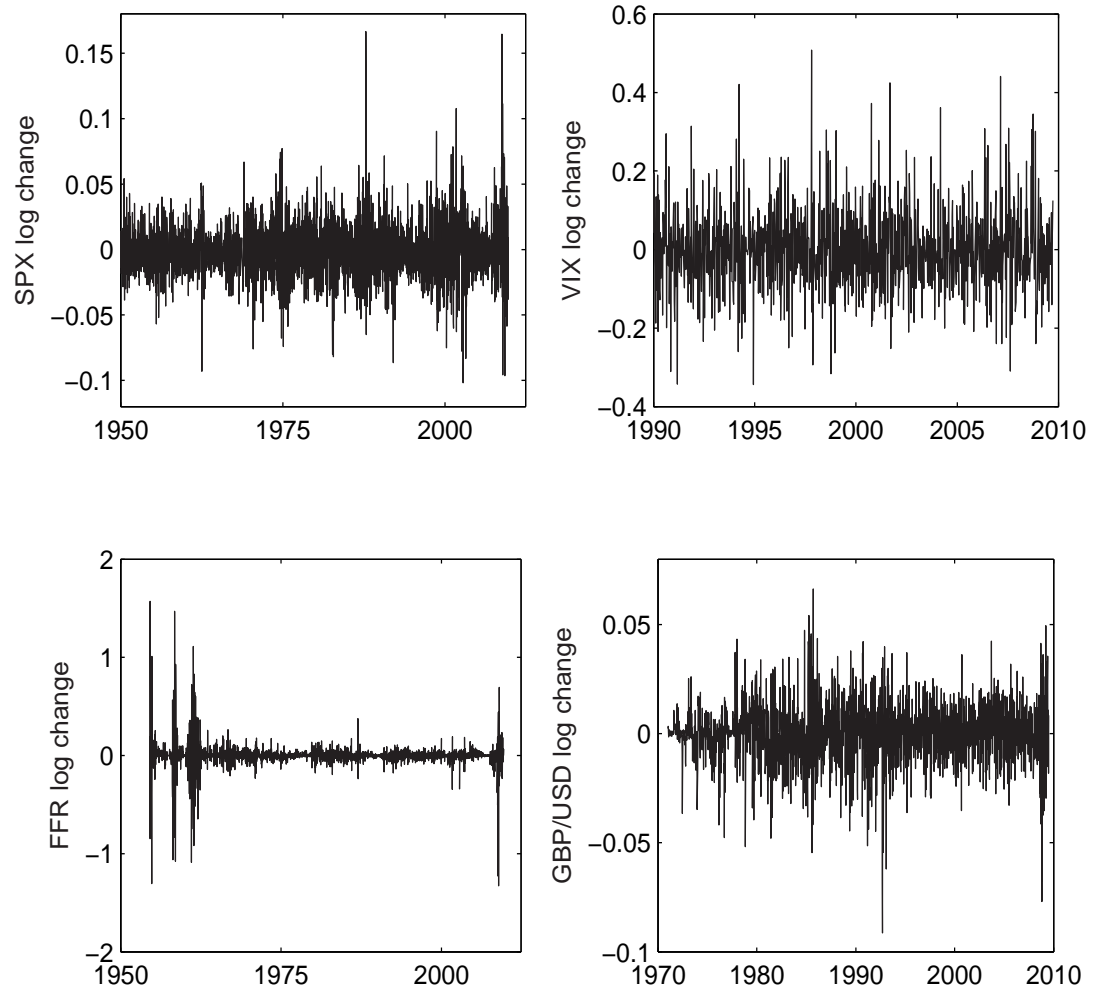
Fig 2. **The historical times series of the log changes of SPX, VIX, FFR and GBP/USD.** *Here SPX, VIX, FFR and GBP/USD refer to the S&P 500 index, the CBOE volatility index, the effective federal funds rate, and the exchange rate between the British Pound and US Dollar, respectively.*