An Axiomatization of the Prekernel of Nontransferable Utility Games*

Roberto Serrano

Department of Economics, Brown University, Providence, Rhode Island 02912, USA

AND

Ken-Ichi Shimomura

OSIPP, Osaka University, 1-16 Machikaneyama, Toyonaka, Osaka 560, Japan

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Abstract:

We characterize the prekernel of NTU games by means of consistency, converse consistency, and five axioms of the Nash type on bilateral problems. The intersection of the prekernel and the core is also characterized with the same axioms over the class of games where the core is nonempty.

Keywords: prekernel, NTU games, consistency, converse consistency

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1. Introduction

Paraphrasing Nash (1951), there are two approaches to game theory: axiomatic and strategic. By giving different insights on a problem, the two approaches are complementary. This paper applies the axiomatic approach to the prekernel of games in coalitional form, thereby complementing the strategic analysis that Serrano (1995) made of this solution concept. The extension of the prekernel to games of non-transferable utility (NTU) has been perceived as a challenging problem, and the strategic approach has proven to be useful to this aim. Thus, Serrano (1995) finds that the prekernel is the set of payoffs such that every pair of players' Nash product is critical (Previous attempts to extend the prekernel to NTU games (Kalai (1975), Billera and McLean (1994)) are not satisfactory. Both papers try to extend the notion of a coalition excess to NTU games).

Harsanyi (1959) proposes "reduced games" with respect to pairs of players to analyze "consistency" and "converse consistency" on a class of multi-person pure bargaining problems. In the same spirit as Harsanyi's, we investigate the implications in the model of NTU games of a well-known internal consistency property and its converse with respect to bilateral negotiations as formulated by two-person Davis-Maschler (1965) reduced games. Consistency and its converse are the key axioms used by Peleg (1986) in order to characterize the prekernel of transferable utility (TU) games. Roughly speaking, our characterization combines the axioms of Peleg (1986) for the TU prekernel and those of Nash (1950) for the Nash solution to bilateral bargaining problems. In this sense, our work resembles Aumann's (1985) and Hart's (1985) axiomatizations of the Shapley NTU value and the Harsanyi value, respectively. These theorems combine the axioms of Nash (1950) and those of Shapley (1953). Our theorem is also a pure axiomatization, in the sense that it is not restricted to a class where a certain solution concept is nonvacuous. We regard the existence problem and the axiomatic characterization as two completely separate issues. Our main theorem says that, for the class of smooth NTU games, the prekernel is the only solution that satisfies consistency, converse consistency, and a set of five axioms of the Nash type imposed on the subclass of two-person smooth problems[1]: *nonemptiness*, *scale invariance, equal treatment for TU games*, *Pareto efficiency*, and *local independence*.

The equal treatment property for TU games is a weaker requirement than Nash's original symmetry axiom. On the other hand, local independence is stronger than the "independence of irrelevant alternatives" axiom. Local independence is introduced and studied by Nagahisa (1991) for the characterization and implementation of the

[1] In this sense, our work resembles Aumann's (1985) and Hart's axiomatizations of the Shapley NTU value and the Harsanyi value, respectively. These theorems combine the axioms of Nash (1950) and those of Shapley (1953).

Walrasian allocation rule in exchange economies (see also Dutta, Sen and Vohra (1995), Nagahisa and Suh (1995), Saijo, Tatamitani and Yamato (1993)). The basic idea can be traced back to Inada (1964), who proposes this condition to investigate the Arrow impossibility theorem in economic environments. The condition says that "if at a commodity allocation all agents have a common marginal rate of substitution under preference profiles u and u', then the allocation should be chosen as a socially optimal outcome for u' whenever it is selected for u." The version in this paper expresses essentially the same concept in the payoff space.

We also show that for the class of smooth games with the cores nonempty, the same axioms as in our main theorem characterize the intersection of the core and the prekernel. This is related to Moldovanu's (1990) partial axiomatization of this intersection for convex assignment problems.

The paper is organized as follows. Section 2 presents the model, while Section 3 is devoted to the consistency properties of the prekernel. Section 4 contains our main result, as well as examples to show that the axioms are independent.

2. The Basic Model

Denote by R the set of the real numbers. If N is a nonempty finite set, denote by $|N|$ the cardinality of N, and by R^N the set of all functions from N to R. We identify an element x of R^N with an $|N|$-dimensional vector whose components are indexed by members of N; thus we write x_i for $x(i)$. If $x \in R^N$ and $S \subset N$, we write x_S the restriction of x to S, which is the element of R^S that associates x_i with each $i \in S$. Let $S \subset N$, and $Y \subset R^S$. We define $\_Y = \{y \in Y \mid \text{There is no } x \in Y \text{ such that } x_i > y_i \text{ for all } i \in S\}$, and $\mathrm{int}Y$ as the interior of Y. A *representation for* Y is a function g from R^S to R such that $Y = \{x \in R^S \mid g(x) ² 0\}$ and $\mathrm{int}Y = \{x \in R^S \mid g(x) < 0\}$. We also write $g_i(x)$ for the partial derivative of g at x Œ R^S with respect to component $i \in S$, and $\nabla g(x)$ for the gradient vector of g at $x \in R^S$.

The pair (N, V) is a *coalitional game*, or simply a *game* if N is a nonempty finite set, and V is a correspondence that associates with every $S \subset N$ a nonempty subset $V(S)$ of R^S such that

(1) $V(S)$ is closed, and comprehensive;
(2) for each $x_S \in R^S$, $\_V(S) \cap (\{x_S\}+R_+^S)$ and $\_V(S) \cap (\{x_S\}-R_+^S)$ are compact; and
(3) for each $(x_S, y_S) \in V(S) \times \_V(S)$, $x_S = y_S$ if x_S ³ y_S (nonlevelness)[2].

Let $V(N)$ be the class of correspondences V such that all (N, V) are games. A member of N is a *player*, and a nonempty subset of N is a *coalition* in the game (N, V). A *payoff to player* i is a point of $R^{\{i\}}$, and a *payoff profile on coalition* S is a point of R^S.

The game (N, V) is *smooth* if there is a differentiable representation g for V(N) with positive gradients on _V(N); namely for each $i \in N$, $g_i(x) > 0$ at any $x \in$ _V(N). A class Γ of games is *rich* if for every $(N, V) \in \Gamma$, Γ contains all two-person games in which the players are members of N.

A *transferable utility game*, or simply a *TU game*, is a smooth game (N, V) which is defined by a function v that associates with every coalition S a real number v(S) such that $V(S) = \{x_S \in R^S \mid$ _$_{i \in S} x_i$ ² $v(S)\}$ for every coalition S. We abuse the notation, and use (N, v) to denote the associated coalitional game.

Let Γ be a nonempty class of games. A *solution on* Γ is a relation σ which associates with every $(N, V) \in \Gamma$ a subset $\sigma(N, V)$ of V(N) (could be empty).

Definition. Let ({i, j}, V) be a two-person smooth game. The ***prekernel of* ({i, j}, V)** is:

Prk({i, j}, V) = {x ∈ _V({i, j}) | $g_i(x)(x_i - v_i) = g_j(x)(x_j - v_j)$},

where g is a representation for V({i, j}), and (v_i, v_j) = (maxV({i}), maxV({j})).

Remark 2.1. Solution Prk reduces to the Nash bargaining solution on the class of two-person smooth games ({i, j}, V) such that V({i, j}) is a convex set containing (v_i, v_j).

Definition. Let Γ be a nonempty class of games. Then a solution σ on Γ satisfies

nonempty-valuedness (***NEV***) if $\sigma(N, V)$ _ Ø for each $(N, V) \in \Gamma$; and

Pareto efficiency (***PE***) if $\sigma(N, V) \subset$ _V(N) for each $(N, V) \in \Gamma$.

Remark 2.2. On the class of two-person smooth games, Prk satisfies NEV and PE.

Let (N, v) be a TU game, and i, j be two distinct players in N. Then i and j are *substitutes in* (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N \backslash \{i, j\}$.

Definition. Let Γ be a class of games. A solution σ on Γ satisfies the ***equal treatment property*** (***ETP***) ***for TU games*** if for each $x \in \sigma(N, v)$, $x_i = x_j$ whenever (N, v) is a TU game in Γ, and i and j are substitutes in (N, v).

Remark 2.3. On the class of two-person smooth games, Prk satisfies ETP for TU games.

Let (N, V) be a game, $\alpha \in R_{++}^N$, and $\beta \in R^N$. For each coalition S, we define the function $\lambda_S^{\alpha\beta}$ from R^S to itself by $\lambda_S^{\alpha\beta}(x_S) = (\alpha_i x_i + \beta_i)_{i \in S}$; for each x_S

$\in R^S$. We then consider $\lambda^{\alpha\beta}(V)$ as the correspondence that associates with every coalition S a set $\lambda^{\alpha\beta}(V)(S) = \{y_S \in R^S \mid y_S = \lambda_S^{\alpha\beta}(x_S)$ for some $x_S \in V(S)\}$.

Definition. Let Γ be a class of games. A solution σ on Γ satisfies ***scale invariance*** (***SIV***) if for each $(N, V) \in \Gamma$, each $\alpha \in R_{++}^N$, and each $\beta \in R^N$, $\sigma(N, \lambda^{\alpha\beta}(V)) = \lambda_N^{\alpha\beta}(\sigma(N, V))$ whenever $\sigma(N, V)$ _ Ø.

Remark 2.4. On the class of two-person smooth games, Prk satisfies SIV.

Definition. Let Γ be a nonempty class of two-person smooth games. A solution σ on Γ satisfies ***local independence*** (***LID***) if for each $(\{i, j\}, V) \in \Gamma$, each $x \in \sigma(\{i, j\}, V)$, and each $V' \in V(\{i, j\})$,

$$x \in \partial V(\{i, j\}) \cap \partial V'(\{i, j\});$$
$$(\max V(\{i\}), \max V(\{j\})) = (\max V'(\{i\}), \max V'(\{j\}));$$
$$\nabla g(x) \mathbin{//} \nabla g'(x)^2$$
$$\Rightarrow \qquad x \in \sigma(\{i, j\}, V'),$$

where g and g' are respectively representations for $V(\{i, j\})$ and $V'(\{i, j\})$.

Remark 2.5. On the class of two-person smooth games, Prk satisfies LID.

Remark 2.6. Let (N, V) be a smooth game. Suppose that there is a *smooth economy* (N, Z, A, u) which generates the outcomes of V as the utility possibility sets of coalitions: Z is a common consumption set for all the agents in N, A is a correspondence that associates with every coalition S a nonempty subset A(S) of Z, which denotes the set of feasible allocations for coalition S, and $u = (u_i)_{i\in N}$ is a profile of agent's utility functions which are defined on Z and differentiable in its interior.[2]

For each coalition S, define the function u_S by the Cartesian product of u_i, $i \in S$. Then V(S) is derived as the image $u_S(A(S))$ of A(S) under u_S. For each utility profile x on $\partial V(N)$, the gradient vector of $\partial V(N)$ at x can be shown to be proportional to the vector of marginal utilities of all the agents with respect to any commodity. Its direction is then unique up to any transformations of utility functions such that the marginal utility vectors of all the agents change proportionally.

Let E be a nonempty class of smooth economies. For each $(N, Z, u, A) \in E$, denote by U(N, Z, A) the class of utility functions u' such that $(N, Z, u', A) \in E$. An *allocation rule on* E is a relation φ that associates with every $(N, Z, u, A) \in E$ a subset $\varphi(N, Z, u, A)$ of A(N). The Pareto rule on E is the allocation rule P that assigns to every $(N, Z, u, A) \in E$ the set P(N, Z, u, A) of Pareto efficient allocations in $A(N)$. We may thus translate the above version of local independence for the payoff space to that for the commodity space as follows:

An allocation rule φ on E satisfies *local independence* if for each $(N, Z, u, A) \in E$, each $z \in \varphi(N, Z, u, A) \cap \mathrm{int}Z$, and each $u' \in U(N, Z, A)$,

$$z \in P(N, Z, u, A) \cap P(N, Z, u', A);$$
$$\sup u_i(A(i)) = \sup u_i'(A(i)),\ u_{i,}(z) = u_{i,}'(z) \text{ for every } i \in N;$$
$$\nabla u(z) \,//\, \nabla u'(z)$$
$$\Rightarrow \qquad z \in \varphi(N, Z, u', A),$$

were $\nabla u(z) = (\nabla u_i(z))_{i\in N}$, and $\nabla u'(z) = (\nabla u_i'(z))_{i\in N}$.

This is not identical with the original one by Nagahisa (1991). He does not put on the conditional part either the restrictions of Pareto efficiency or the invariance of utilities. Further the proportionality implies more than the invariance of marginal rates of substitutions. The above requirement is thus weaker than his one.

Proposition 1. *Let* $\Gamma^{\{i, j\}}$ *be the class of two-person smooth games* $(\{i, j\}, V)$. *Then a solution on* $\Gamma^{\{i, j\}}$ *satisfies NEV, PE, ETP for TU games, SIV, and LID if and only if it is* Prk.

Proof. The solution Prk on $\Gamma^{\{i, j\}}$ satisfies NEV, PE, ETP for TU games, SIV, and LID. Now we prove the uniqueness. Say $i = 1$, and $j = 2$. Let $(\{1, 2\}, V)$ be a two-person smooth game, and σ a solution on $\Gamma^{\{i, j\}}$ which satisfies NEV, PE, ETP for TU games, SIV, and LID. We prove that $\sigma(\{1, 2\}, V) = \mathrm{Prk}(\{1, 2\}, V)$.

By NEV, there exists $x = (x_1, x_2) \in \mathrm{Prk}(\{1, 2\}, V)$. By PE, $x \in \partial V(\{1, 2\})$. By differentiability, there is a unique tangent line of the curve _V({1, 2}) at x: $\nabla g(x)\cdot(z - x) = g_1(x)(z_1 - x_1) + g_2(x)(z_2 - x_2) = 0$. Define the two-person smooth game $(\{1, 2\}, V')$ by $V'(\{1\}) = V(\{1\})$, $V'(\{2\}) = V(\{2\})$, and $V'(\{1, 2\}) = \{z \in R^{\{1, 2\}} \mid \nabla g(x)\cdot(z - x)$ ^{2} $0\}$. Then, by the LID of Prk, $x \in \mathrm{Prk}(\{1, 2\}, V')$. Note that

$$\mathrm{Prk}(\{1, 2\}, V')=\{([(g_2(x)/g_1(x)(x_2-v_2')+x_1+v_1']/2,\ [(g_1(x)/g_2(x)(x_1-v_1')+x_2+v_2']/2)\},$$

which is the midpoint of the segment on the line _V'({1, 2}) truncated by $(v_1', v_2') = (\max V'(\{1\}), \max V'(\{2\}))$. Hence, $\{x\} = \mathrm{Prk}(\{1, 2\}, V')$ (see Figure 1).[2] Define the TU game $(\{1, 2\}, w)$ by $w(\{1\}) = 0 = w(\{2\})$, and $w(\{1, 2\}) = g_1(x)(x_1 - v_1') + g_2(x)(x_2 - v_2')$. By NEV, PE and ETP for TU games, $\sigma(\{1, 2\}, w) = \{(1/2)w(\{1, 2\}), (1/2)w(\{1, 2\})\}$. Let $\alpha = (1/g_1(x), 1/g_2(x))$, and $\beta = (v_1', v_2')$. By SIV,

$$\sigma(\{1, 2\}, \lambda^{\alpha\beta}(w)) \qquad = \lambda_N^{\alpha\beta}(\sigma(\{1, 2\}, w))$$
$$= \{((1/2g_1(x))w(\{1, 2\}) + v_1',\ (1/2g_2(x))w(\{1, 2\}) + v_2'))\}$$
$$= \mathrm{Prk}(\{1, 2\}, V') \qquad = \{x\}.$$

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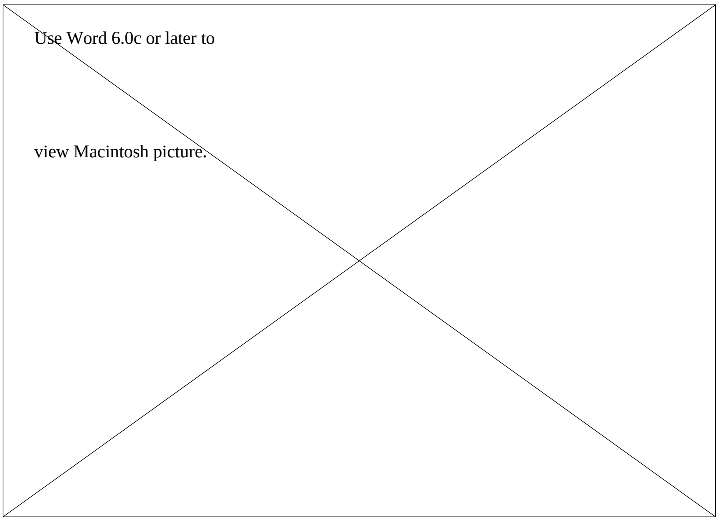
[2] We do not necessarilly assume that $(v_i, v_j) \in V(\{i, j\})$ for all $(\{i, j\}, V) \in \Gamma^{\{i, j\}}$.

Note that $(\{1, 2\}, \lambda^{\alpha\beta}(w))$ is the game $(\{1, 2\}, V')$. Hence, $\sigma(\{1, 2\}, V') = \sigma(\{1, 2\}, \lambda^{\alpha\beta}(w)) = \{x\}$, so that $x \in \sigma(\{1, 2\}, V')$. By LID, $x \in \sigma(\{1, 2\}, V)$. Thus, $Prk(\{1, 2\}, V) \subset \sigma(\{1, 2\}, V)$. In exactly the same way, we can show that $\sigma(\{1, 2\}, V) \subset Prk(\{1, 2\}, V)$. Hence, $\sigma(\{1, 2\}, V) = Prk(\{1, 2\}, V)$. `



3. Reduced Game Properties of the Prekernel

The following is a two-person "reduced game" studied by Peleg (1986, 1992). Let $\Pi^N \equiv \{P \subset N \mid |P| = 2\}$, which is the set of two-person coalitions in N.

Definition. Let (N, V) be a game, $x \in V(N)$, and $P \in \Pi^N$. The ***two-person reduced game of* (N, V) *with respect to* P *given* x** is the pair (P, V_{xP}) of P and the correspondence V_{xP} that associates with every $S \subset P$ a subset $V_{xP}(S)$ of R^S, where $V_{xP}(\{i\}) = \{y_i \in R^{\{i\}} \mid (y_i, x_Q) \in V(\{i\} \cup Q), Q \subset N \backslash P\}$ for each $i \in P$, and $V_{xP}(P) = \{y_p \in R^p \mid (y_p, x_{-p}) \in V(N)\}$.[34]

Definition. Let (N, V) be a game. The ***prekernel of* (N, V)** is:
$Prk(N, V) = \{x \in {\_}V(N) \mid g_i(x)(x_i - v_i(x_{-\{i, j\}})) = g_j(x)(x_j - v_j(x_{-\{i, j\}}))$ for each $i, j \in N\}$,
where g is a representation for V(N), and $v_i(x_{-\{i, j\}}), v_j(x_{-\{i, j\}})) = (\max V_{x\{i,j\}}(\{i\}), \max V_{x\{i,j\}}(\{j\}))$.

Definition. Let Γ be a nonempty class of smooth games. A solution σ on Γ satisfies ***local independence*** (***LID***) if for each $(N, V) \in \Gamma$, each $x \in \sigma(N, V)$, and each $V' \in V(N)$,

$x \in \partial V(N) \cap \partial V'(N)$;
$\forall P \in \Pi^N, \forall Q \subset N \backslash P, \forall i \in P, v_i(\{i\} \cup Q; x_Q) = v_i'(\{i\} \cup Q; x_Q)$;
$\nabla g(x) // \nabla g'(x)$

$\Rightarrow \qquad x \in \sigma(N, V')$,

where g and g' are respectively representations for $V(N)$ and $V'(N)$, $v_i(\{i\} \cup Q; x_Q) = \max\{y_i \in R^{\{i\}} | (y_i, x_Q) \in V(\{i\} \cup Q)\}$, and $v_i'(\{i\} \cup Q; x_Q) = \max\{y_i \in R^{\{i\}} | (y_i, x_Q) \in V'(\{i\} \cup Q)\}$ for each $P \in \Pi^N$, each $Q \subset N \backslash P$, and each $i \in P$.

Remark 3.1. To redefine the above statement for the commodity space, we need to impose more restrictions of invariance of utility levels on the translation in Remark 2.6. Thus the above version is even weaker than the previous one.

Definition. Let Γ be a nonempty class of games. Then a solution σ on Γ satisfies ***bilateral consistency*** (***BCS***) if

$$\forall (N, V) \in \Gamma, \forall x \in \sigma(N, V), \forall P \in \Pi^N, (P, V_{xP}) \in \Gamma \ \& \ x_P \in \sigma(P, V_{xP}).$$

Definition. Let Γ be a nonempty class of games. Then a solution σ on Γ satisfies ***converse consistency*** (***CCS***) if

$$\forall (N, V) \in \Gamma, \forall x \in \_V(N),[(\forall P \in \Pi^N, x_P \in \sigma(P, V_{xP})) \Rightarrow x \in \sigma(N, V)].$$

Remark 3.2. On a rich class of smooth games, Prk satisfies BCS and CCS.

Remark 3.3. Let $\Gamma^{\{i, j\}}$ be the class of two-person smooth games $(\{i, j\}, V)$. Then a solution on $\Gamma^{\{i, j\}}$ satisfies NEV, PE, ETP for TU games, SIV, LID, BCS, and CCS if and only if it is Prk.

4. A Characterization of the Prekernel

We have checked that on a class of smooth games, the solution Prk is nonempty-valued, and satisfies PE, ETP for TU games, SIV, and LID. We next show that it is uniquely characterized by two-person versions of all the axioms together with bilateral consistency and its converse.

Theorem 1. *Let Γ_0 be a rich class of smooth games. A solution on Γ_0 satisfies NEV for two-person games, PE for two-person games, ETP for two-person TU games, SIV for two-person games, and LID for two-person games, BCS and CCS if and only if it is* Prk.

Proof. The solution Prk on Γ_0 satisfies NEV for two-person game, PE for two-person game, ETP for two-person TU games, SIV, LID, BCS, and CCS. Now we prove the uniqueness. Let $(N, V) \in \Gamma_0$, and σ a solution on Γ_0 that satisfies NEV, PE, ETP for TU games, SIV and LID for two-person games, and satisfies BCS and CCS on Γ_0. We prove that $\sigma(N, V) = Prk(N, V)$. The proof for $|N| = 1$ is trivial. We have already proven the case of $|N| = 2$. Then consider the case of $|N| \geq 3$.

Suppose that Prk(N, V) _ Ø. Let $x \in Prk(N, V)$. By the BCS of Prk, $x_P \in Prk(P, V_{xP})$ for every $P \in \Pi^N$. Hence, $x_P \in \sigma(P, V_{xP})$ for every $P \in \Pi^N$. By the CCS of σ, $x \in \sigma(N, V)$. Hence, $Prk(N, V) \subset \sigma(N, V)$. Note that σ(N, V) _ Ø. Then we can similarly show that $\sigma(N, V) \subset Prk(N, V)$. Thus, $\sigma(N, V) = Prk(N, V)$.

Suppose that $Prk(N, V) = \emptyset$. Let x ∈ _V(N). By the CCS of Prk, there exist at least one pair Q of players in N such that $x_Q \notin Prk(Q, V_{xQ})$. Since (Q, V_{xQ}) is a two-person game, we have $Prk(Q, V_{xQ}) = \sigma(Q, V_{xQ})$, so that $x_Q \notin \sigma(Q, V_{xQ})$. By the BCS of σ, $x \notin \sigma(N, V)$. Hence, there is no payoff profile in $\sigma(N, V)$, so that $\sigma(N, V) = \emptyset$. Thus, $\sigma(N, V) = Prk(N, V)$. `

In Theorem 1, we have used seven axioms. It does not look so elegant, and Lensberg (1988) actually proves that under consistency, we do not need any such conditions as the "independence of irrelevant alternatives" to axiomatize the Nash solution on the class of pure bargaining problems with valuable populations. The following examples show that we are not able to drop any of the seven axioms to characterize Prk on a class of smooth games even with valuable populations.

Example 4.1: For every $(N, V) \in \Gamma_0$, let $\sigma(N, V) = \emptyset$**.** Then σ vacuously satisfies all the conditions except NEV for two-person games.

Example 4.2: For every two-person game (P, V), define

$$b(P, V) = \begin{cases} (v_i)_{i\in P} & \text{if } (v_i)_{i\in P} \in \mathrm{int}V(P); \\ Prk(P, V) & \text{if } (v_i)_{i\in P} \notin \mathrm{int}V(P). \end{cases}$$

For every $(N, V) \in \Gamma_0$, let $\sigma(N, V) = \{x \in V(N) | \ x_P \in b(P, V_{xP}) \text{ for all } P \in \Pi^N\}$. Then σ satisfies all the conditions except PE for two-person games.

Example 4.3: For every $(N, V) \in \Gamma_0$, let σ(N, V) = _V(N). Then σ satisfies all the conditions except ETP for two-person TU games.

Example 4.4: For every $(N, V) \in \Gamma_0$, let σ(N, V) = {x ∈ _V(N)| $x_i - v_i(x_{-\{i, j\}}) = x_j - v_j(x_{-\{i, j\}})$ for each $i, j \in N$}. Then σ satisfies all the conditions except SIV for two-person games.

Example 4.5: For every two-person game (P, V), define $a(P, V) = (a_i(P, V))_{i\in P}$ by $a_i(P, V) = \max\{x_i \in R^{\{i\}} | \ (x_i, v_{-i}) \in V(P)\}$ for all $i \in P$. For every $(N, V) \in \Gamma_0$, let $\sigma(N, V) = \{x \in \partial V(N) | \ x_P \in [(v_i(x_{-P}))_{i\in P}, a(P, V_{xP})] \text{ for each } P \in \Pi^N\}$, where [c, d]

= $\{(1 - t)c + td$ | 0 ² t ² 1} for each $c, d \in R^P$. That is, for every pair P of players in N, x_P is the maximal point of the feasible set V_{xP} on the segment connecting $(v_i(x_{-P}))_{i \in P}$ to $a(P, V_{xP})$ if $x \in \sigma(N, V)$. Note that σ is a modification of the Kalai-Smorodinsky bargaining solution, and σ satisfies all the conditions except LID for two-person games.

Example 4.6: Let $(N, V) \in \Gamma_0$, and $i, j \in N$. Then i *and* j *are equivalent* if $g_i(x)(x)(v_i(\{i\} \cup S; x_S) - v_i) = g_j(x)(v_j(\{j\} \cup S; x_S) - v_j)$ for every $x \in$ _V(N), and for each $S \subset N \backslash \{i, j\}$, where $v_i(\{i\} \cup S; x_S) = \max\{y_i \in R^{\{i\}} | (y_i, x_S) \in V(\{i\} \cup S)\}$ for each $i \in N$, and each $S \subset N \backslash \{i\}$. Note that $g_i(x)(v_i(x_{-\{i, j\}}) - v_i) = g_j(x)(v_j(x_{-\{i, j\}})) - v_j)$ for every $x \in$ _V(N) if players i and j are equivalent. Now let $\sigma(N, V) = \{x \in$ _V(N) | $g_i(x)(x_i - v_i) = g_j(x)(x_j - v_j)$ if $i, j \in N$ are equivalent}. We can verify that σ satisfies the five axioms imposed on two-person games: NEV, PE, ETP for TU games, SIV, and LID. If $|N| = 2$, then the two players in N are equivalent, and $\sigma(N, V) = Prk(N, V)$. To prove that σ satisfies CCS, suppose that $x \in$ _V(N) is such that $x_P \in \sigma(P, V_{xP}) = Prk(P, V_{xP})$ for each $P \in \Pi^N$. Since Prk satisfies CCS, $x \in Prk(N, V)$. We show that $Prk(N, V) \subset \sigma(N, V)$ if |N| ³ 3. Let $x \in Prk(N, V)$. Then $x \in$ _V(N). Assume that $i, j \in N$ are equivalent. Since $x \in$ _V(N), we have $g_i(x)(v_i(x_{-\{i, j\}}) - v_i) = g_j(x)(v_j(x_{-\{i, j\}})) - v_j)$, so that $g_i(x)(x_i - v_i) - g_j(x)(x_j - v_j) = g_i(x)(x_i - v_i(x_{-\{i, j\}})) - g_j(x)(x_j - v_j(x_{-\{i, j\}}))$. Since $x \in Prk(N, V)$, we have $g_i(x)(x_i - v_i(x_{-\{i, j\}})) = g_j(x)(x_j - v_j(x_{-\{i, j\}}))$, so that $g_i(x)(x_i - v_i) - g_j(x)(x_j - v_j) = 0$. Hence, $x \in \sigma(N, V)$, i.e., $Prk(N, V) \subset \sigma(N, V)$. Thus, σ satisfies CCS. Suppose that $N = \{1, 2, 3\}$, and V is a TU game defined by $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 3$, $v(\{2, 3\}) = 2$, and $v(\{1, 2, 3\}) = 6$. Then (N, V) has no pair of equivalent players, so that $\sigma(N, V) =$ _V(N). Let $x = (2, 2, 2) \in \sigma(N, V)$, then $\sigma(\{1, 2\}, V(x_{-\{1, 2\}})) = \{(2.5, 1.5)\}$. Hence, σ does not satisfy BCS.

Example 4.7: For every $(N, V) \in \Gamma_0$, let $\sigma(N, V) = \{x \in Prk(N, V) | v_i(x_{-\{i, j\}}) = v_i(x_{-\{i, k\}})$ for each $i, j, k \in N$ with j _ i _ k}. Note that $\sigma(N, V) = Prk(N, V)$ if $|N| = 2$. Then σ satisfies all the conditions except CCS.

Remark 4. 1. The solution Prk satisfies PE, ETP for TU games, SIV, and LID over the class of n-person smooth games, n ³ 2. It does not satisfy, however, NEV over the same class (see Serrano (1995, Example 2), and Moldovanu (1990, p.188)).

We next study the intersection of the core and the prekernel on the class of games with nonempty core.

Definition. Let (N, V) be a game, $S \subset N$, and $x \in R^N$. Then **S** ***can improve upon*** **x** if there is $y \in V(S)$ such that $y_i > x_i$ for all $i \in S$. The ***core of*** **(N, V)** is:

$$C(N, V) = \{x \in V(N) | \text{ There is no coalition that can improve upon } x\}.$$

Definition. Let $(\{i, j\}, V)$ be a two-person smooth game. The ***core-kernel of*** **({i, j}, V)** is:

$$CK(\{i, j\}, V) = \{x \in C(\{i, j\}, V) \mid g_i(x)(x_i - v_i) = g_j(x)(x_j - v_j)\},$$

where g is a representation for $V(\{i, j\})$, and $(v_i, v_j) = (\max V(\{i\}), \max V(\{j\}))$.

Remark 4.2. The solution CK reduces to the Nash bargaining solution on the class of two-person smooth games $(\{i, j\}, V)$ such that $V(\{i, j\})$ is a convex set.

Proposition 2. *Let* $\Gamma_C^{\{i, j\}}$ *be the class of two-person smooth games* $(\{i, j\}, V)$ *with nonempty core. Then a solution on* $\Gamma_C^{\{i, j\}}$ *satisfies NEV, PE, ETP for TU games, SIV, and LID if and only if it is* CK.

Proof. The solution CK on $\Gamma_C^{\{i, j\}}$ satisfies NEV, PE, ETP for TU games, SIV, and LID. Now we prove the uniqueness. Say $i = 1$, and $j = 2$. Let $(\{1, 2\}, V)$ be a two-person smooth game, and σ a solution on $\Gamma_C^{\{i, j\}}$ which satisfies NEV, PE, ETP for TU games, SIV, and LID. We prove that $\sigma(\{1, 2\}, V) = CK(\{1, 2\}, V)$.

By NEV, there exists $x = (x_1, x_2) \in CK(\{1, 2\}, V)$. By PE, $x \in$ _V({1, 2}). By differentiability, there is a unique tangent line of the curve _V({1, 2}) at x: $\nabla g(x)\cdot(z - x) = g_1(x)(z_1 - x_1) + g_2(x)(z_2 - x_2) = 0$. Define the two-person smooth game $(\{1, 2\}, V')$ by $V'(\{1\}) = V(\{1\})$, $V'(\{2\}) = V(\{2\})$, and $V'(\{1, 2\}) = \{z \in R^{\{1, 2\}} \mid \nabla g(x)\cdot(z - x) \,^2\, 0\}$. Then, by the LID of CK, $x \in CK(\{1, 2\}, V')$. Note that

$$CK(\{1, 2\}, V')=\{([(g_2(x)/g_1(x)(x_2-v_2')+x_1+v_1']/2, [(g_1(x)/g_2(x)(x_1 -v_1')+ x_2+v_2']/2)\},$$

which is the midpoint of the segment on the line _V'({1, 2}) truncated by $(v_1', v_2') = (\max V'(\{1\}), \max V'(\{2\}))$. Hence, $\{x\} = CK(\{1, 2\}, V')$ (see Figure 2).[3] Define the TU game $(\{1, 2\}, w)$ by $w(\{1\}) = 0 = w(\{2\})$, and $w(\{1, 2\}) = g_1(x)(x_1 - v_1') + g_2(x)(x_2 - v_2')$. By NEV, PE and ETP for TU games, $\sigma(\{1, 2\}, w) = \{(1/2)w(\{1, 2\}), (1/2)w(\{1, 2\})\}$. Let $\alpha = (1/g_1(x), 1/g_2(x))$, and $\beta = (v_1', v_2')$. By SIV,

$$\begin{aligned} \sigma(\{1, 2\}, \lambda^{\alpha\beta}(w)) &= \lambda_N^{\alpha\beta} (\sigma(\{1, 2\}, w)) \\ &= \{((1/2g_1(x))w(\{1, 2\}) + v_1', (1/2g_2(x))w(\{1, 2\}) + v_2'))\} \\ &= CK(\{1, 2\}, V') = \{x\}. \end{aligned}$$

Note that $(\{1, 2\}, \lambda^{\alpha\beta}(w))$ is the game $(\{1, 2\}, V')$. Hence, $\sigma(\{1, 2\}, V') = \sigma(\{1, 2\}, \lambda^{\alpha\beta}(w)) = \{x\}$, so that $x \in \sigma(\{1, 2\}, V')$. By LID, $x \in \sigma(\{1, 2\}, V)$. Thus, CK({1, 2},

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[3] For every $(\{i, j\}, V) \in \Gamma_C^{\{i, j\}}$, $C(\{i, j\}, V)$ _ Ø, i.e., $(v_i, v_j) \in V(\{i, j\})$.

V) ⊂ σ({1, 2}, V). In exactly the same way, we can show σ({1, 2}, V) ⊂ CK({1, 2}, V). Hence, σ({1, 2}, V) = CK({1, 2}, V). `

Definition. Let (N, V) be a game. The ***core-kernel of* (N, V)** is:

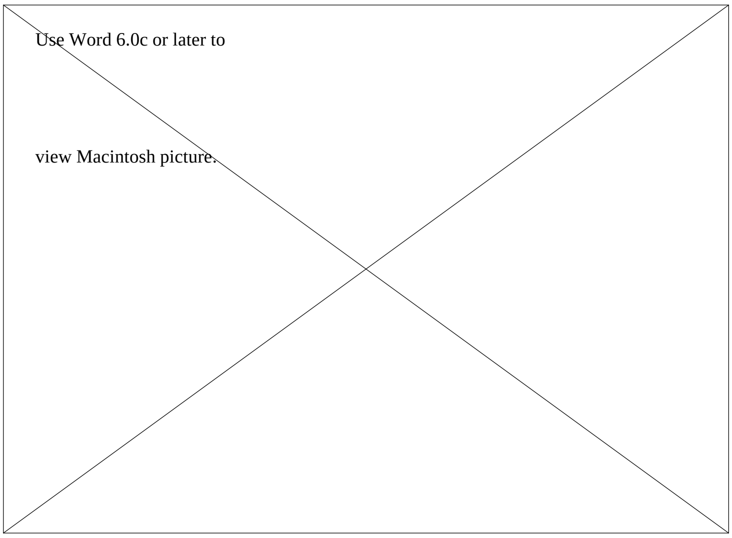
$CK(N, V) = \{x \in C(N,V) \mid g_i(x)(x_i - v_i(x_{-\{i,j\}})) = g_j(x)(x_j - v_j(x_{-\{i,j\}}))$ for each $i, j \in N\}$, where g is a representation for $V(N)$, and $(v_i(x_{-\{i,j\}}), v_j(x_{-\{i,j\}})) = (\max V_{x\{i,j\}}(\{i\}), \max V_{x\{i,j\}}(\{j\}))$.

Theorem 2. *Let* Γ_C *be a rich class of smooth games with nonempty core. A solution on* Γ_C *satisfies NEV for two-person games, PE for two-person games, ETP for two-person TU games, SIV two-person games, and LID for two-person games, BCS and CCS if and only if it is* CK.

Proof. The solution CK on Γ_C satisfies NEV for two-person game, PE for two-person game, ETP for two-person TU games, SIV, LID, BCS, and CCS. Now we prove the uniqueness. Let $(N, V) \in \Gamma_C$, and σ a solution on Γ_C that satisfies NEV, PE, ETP for TU games, SIV and LID for two-person games, and satisfies BCS and CCS on Γ_C. We prove that σ(N, V) = CK(N, V). The proof for |N| = 1 is trivial. We have already proven the case of |N| = 2. Then consider the case of |N| ≥ 3.

Suppose that CK(N, V) _ Ø. Let x ∈ CK(N, V). By the BCS of CK, $x_p \in CK(P, V(x_{-p}))$ for every $P \in \Pi^N$. Hence, $x_p \in \sigma(P, V(x_{-p}))$ for every $P \in \Pi^N$. By the CCS of σ, x ∈ σ(N, V). Hence, CK(N, V) ⊂ σ(N, V). Note that σ(N, V) _ Ø. Then we can similarly show that σ(N, V) ⊂ CK(N, V). Thus, σ(N, V) = CK(N, V).

Suppose that CK(N, V) = Ø. Let x ∈ _V(N). Then by the CCS of CK, there exist at least one pair Q of players in N such that $x_Q \notin CK(Q, V(x_{-Q}))$. Since $(Q, V(x_{-Q}))$ is a two-person game, we have $CK(Q, V(x_{-Q})) = \sigma(Q, V(x_{-Q}))$, so that $x_Q \notin \sigma(Q, V(x_{-Q}))$. By the BCS of σ, x ∉ σ(N, V). Hence, there is no payoff profile in σ(N, V), so that σ(N, V) = Ø. Thus, σ(N, V) = CK(N, V). `



Moldovanu (1990, Theorem 5.2) considers the same solution on the class of NTU assignment games. For characterization, he implicitly assumes that the solution under investigation is single-valued for two-person games of the domain. He thus uses the "independence of irrelevant alternatives" as one of the axioms instead of local independence. If we do not assume either such a single-valuedness condition or local independence, we are not able to show more than that the solution contains the intersection of the core and the prekernel, that is, $\sigma(N, V) \subset CK(N, V)$ for every game (N, V) of the domain.

The following examples show the independence of the axioms in Theorem 2.

Example 4.1': For every $(N, V) \in \Gamma_C$, let $\sigma(N, V) = \emptyset$**.** Then σ vacuously satisfies all the conditions except NEV for two-person games in Γ_C.

Example 4.2': For every two-person game (P, V), define $d(P, V) = (v_i)_{i \in P}$. For every $(N, V) \in \Gamma_C$, let $\sigma(N, V) = \{x \in V(N) | x_P \in d(P, V_{xP})$ for all $P \in \Pi^N\}$. Then σ satisfies all the conditions except PE for two-person games in Γ_C.

Example 4.3': For every $(N, V) \in \Gamma_C$, let $\sigma(N, V) = C(N, V)$. Then σ satisfies all the conditions except ETP for two-person TU games in Γ_C.

Example 4.4': For every $(N, V) \in \Gamma_C$, let $\sigma(N, V) = \{x \in \_V(N) | x_i - v_i(x_{-\{i, j\}}) = x_j - v_j(x_{-\{i, j\}})$ for each $i, j \in N\}$. Then σ satisfies all the conditions except SIV for two-person games in Γ_C.

Example 4.5': For every two-person game (P, V), define $a(P, V) \equiv (a_i(P, V))_{i \in P}$ by $a_i(P, V) = \max\{x_i \in R^{\{i\}} | (x_i , v_{-i}) \in V(P)\}$ for all $i \in P$. For every $(N, V) \in \Gamma_C$, let $\sigma(N, V) = \{x \in \_V(N) | x_P \in [(v_i(x_{-p}))_{i \in P}, a(P, V_{xP})]$ for each $P \in \Pi^N\}$. Then σ satisfies all the conditions except LID for two-person games in Γ_C.

Example 4.6': For every $(N, V) \in \Gamma_C$, let $\sigma(N, V) = Prk(N, V)$. Then σ satisfies all the conditions except ETP for two-person TU games in Γ_C.

Example 4.7': For every $(N, V) \in \Gamma_C$, let $\sigma(N, V) = \{x \in CK(N, V) \mid v_i(x_{-\{i, j\}}) = v_i(x_{-\{i, k\}})$ for each $i, j, k \in N$ with j _ i _ k$\}$. Note that $\sigma(N, V) = CK(N, V)$ if $|N| = 2$. Then σ satisfies all the conditions except CCS.

Remark 4. 3. The solution CK satisfies PE, ETP for TU games, SIV, and LID over the class of n-person smooth games with nonempty core, n ³ 2. It does not satisfy, however, NEV over the same class (see Moldovanu (1990, p.188)).

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