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JEL Codes : D11, D43, L13

**Keywords : discrete choices, product differentiation, imperfect competition, elimination-by-aspects, Edgeworth cycles.**

# Differentiated Duopoly with ‘Elimination By Aspects’

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## Abstract

“Elimination by aspects” (EBA) is a discrete model of probabilistic choice worked out by Tversky in 1972 which supposes that decision makers follow a particular heuristic during a process of sequential choice. Options are described by their attributes and, at each decision stage, the individuals eliminate all the options not having an expected given attribute, and so until only one option remains. In this paper, probabilities resulting from the EBA model are used to construct demands of a differentiated duopoly with imperfect rationality. These demands are consistent with partial heterogeneity of tastes and may be linked with a spatial framework in which consumers have convex perception of distance. In this model, a Nash price equilibrium in pure strategies exists if the cost of the highest attributes level firm is not too low. In this case, the “differentiation by attributes” form retained here is both horizontal and vertical, which is not very frequent in the literature. When the equilibrium does not exist, the interaction of best response functions of the firms induces an Edgeworth cycle instead of an exit of the lowest attributes level firm. This result underlines the role of cost difference in the existence of such a cycle.

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# 1 Introduction

The growing number of products together with the multiplication of their attributes makes consumers' choice more and more difficult. In such an environment, the bounded cognitive capacity of consumers leads them to use *decision heuristics*, which are simple rules of reasoning reducing the complexity of a decision-making problem. This observation justifies the introduction of imperfectly rational consumers into the oligopoly theory, which rests on the assumption of perfectly rational agents. In this perspective, this paper applies to a standard differentiated duopoly the “elimination by aspects” (or EBA) heuristic proposed by Tversky (1972a, b)<sup>1</sup> and identifies the corresponding properties of the market outcome.

As McFadden (2001) underlines it, literature devoted to the analysis of individual choices among a *discrete* number of options is largely dominated by the random utility models. These models suppose that the utility of consumers confronted with various options comprises a random variable: that can come from changing state of minds (cognitive interpretation) or incapacity of the modeler to apprehend individual behaviors (econometric interpretation). The most widespread models are the logit (Luce and Suppes, 1965; McFadden, 1974), nested logit (Ben-Akiva, 1973), mixed logit (McFadden and Train, 2000) and, more generally, models with “generalized extreme values” (McFadden, 1978). Their success can be explained by their proximity with the standard approach of deterministic utility maximization, while integrating a particular form of imperfect rationality (Chen and al, 1997).

Nevertheless, among discrete choice models, the class of random decision rule models has been rarely explored by economists up to now, whereas these models are widely diffused in psychology or marketing. Models proposed by Luce (1959) or Tversky (1972a, b) belong to this class. In a context of questioning about utility maximization, which embodies the perfect rationality, these models fit clearly in cognitive interpretation. They postulate that the utility assigned to the choice options is deterministic, but that *the decision rule used by consumers is intrinsically probabilistic*. As Tversky notices it, “when faced with a choice among several alternatives, people often experience uncertainty and exhibit inconsistency. That is, people are often not sure which alternative they should select, nor do they always make the same choice under seemingly identical conditions” (1972a, p 281).

In the EBA model, choice options are represented by sets of characteristics: for instance, if the option is a good, this last is described by the set of attributes it has, an approach which recalls that of Lancaster (1966). Following such a perspective, Tversky suggests to describe the final choice as the result of a stochastic process of successive elimination of the options :

- (a) the common characteristics of the considered choice set are eliminated as any discriminating choice cannot be based on them ;
- (b) a characteristic is randomly selected and all the options not having this characteristic are eliminated. The higher the utility of characteristic is, the larger the probability of selecting this characteristic is ;
- (c) if remaining options still have specific characteristics, one turns over at the first stage. In the contrary, if the residual choices have the same characteristics, the procedure ends. If only one option remains, it is selected. In the contrary, all the remaining options have the same probability to be selected.

This heuristic particularly fits with the paradigm developed by Gigerenzer and Selten (2001). For them, decision makers use fast and frugal rules, forming a *toolbox*, to solve their decision problems, often

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<sup>1</sup>Tversky employed the term of “aspects”: for our part, we indifferently use “attributes” or “characteristics”

tackled in situation of uncertainty, limited resources and constrained time. This toolbox is *adaptive* because heuristics used depend on context, importance of decision, and so on. As Payne and Bettman (2001) showed it, the EBA heuristic belongs to this toolbox's effective heuristics because it allows decisions of "good quality" for a relatively limited cognitive effort.

The EBA model of Tversky thus makes it possible to take into account a relevant form of bounded rationality of consumers choosing on markets of differentiated products. It also has a similar degree of flexibility to the random utility models such as nested logit or probit.

In this paper, probabilities resulting from the EBA model and depicting consumer's behavior, are used to construct individual demands. Then we show that the existence of a price Nash equilibrium in a differentiated duopoly depends on conditions on the levels of attributes of goods and on the degree of unit cost asymmetry. In particular, there always exists a unique equilibrium in pure strategies if the unit cost of the firm having the highest level of attributes is not too low comparing to that of its rival.

When this equilibrium exists, our model allows a new analysis of the role of product differentiation because it integrates the existing shapes of differentiation while providing a more general approach. The literature usually distinguishes between vertical and horizontal differentiation. For the former, goods differ by their quality (all consumers prefer the highest quality good when prices are equal) as in the models of Mussa and Rosen (1978), Gabszewicz and Thisse (1979) or Shaked and Sutton (1982). For the latter, varieties of goods differ (at equal prices, each consumer chooses its own preferred basket of varieties) as in the models of Hotelling (1929) or Dixit and Stiglitz (1977). *In the EBA model, goods are differentiated by their specific attributes.* Consequently, when two goods have the same level of specific attributes, differentiation is horizontal. When only one good has all the specific attributes on the market (and the other none), differentiation is vertical. Finally, when each good has specific attributes with different levels, the two previous dimensions of differentiation are taken into account. Such models are not very frequent in the literature and often of a rather heavy formalism. It is the case, for instance, of Neven and Thisse (1990) model which use four parameters to model the differentiation (compared to two in the EBA model).

The introduction of imperfect rationality affects also modes of competition between firms. Thus, at equilibrium, the firm offering the highest level of attributes seems to be a *reference* for its rival in term of tariff choice. Formally, this means that the price of the second does not depend on its unit cost but only on that of the first. The existence of such a tariff relation was already observed on some markets (Lazer, 1957) and analyzed theoretically with the focal prices of Schelling (1960) and the *imitation equilibrium* of Ostmann and Selten (2001).

When price equilibrium does not exist, the interaction of firm's best response functions during a succession of periods leads to a trend of prices taking the form of an Edgeworth cycle. Let us recall that such a cycle, studied by Edgeworth (1925) and Maskin and Tirole (1988), comprises a long price war phase followed by an abrupt increase in prices when firms' margins become too low. Noel (2004) recently showed that such a cycle can be a Markov Perfect Equilibrium when products are horizontally differentiated. Without using this particular concept of equilibrium, our analysis reveals that such cycles can exist with less sophisticated strategies of the producers and with a more general form of differentiation.

This paper is organized in the following way. Choice probabilities of goods when consumers use the EBA heuristic are presented in section 2, while properties of the corresponding demand functions are

studied in section 3. Existence conditions of a price Nash equilibrium are analyzed in section 4. Section 5 specifies the properties of this equilibrium and compares them with the existing literature. Section 6 is devoted to the strategic interactions between firms when equilibrium does not exist and highlights the price cycle properties. Our conclusions are presented in section 7.

## 2 Choice probabilities in Tversky's EBA model

Suppose that a consumer must choose a product among a set of two goods sold at the same price. Each good  $i$  has a set of *attributes* (or characteristics) and each of them can be either *specific* to the considered good or *shared* by the two goods, as shown in the following figure :

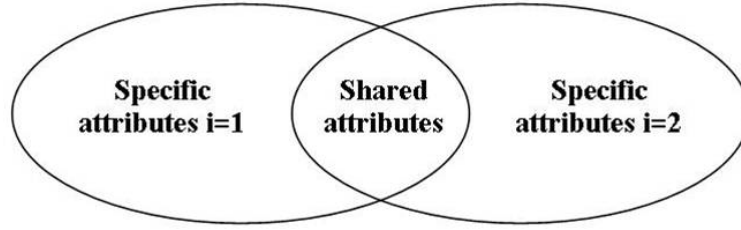


Figure 1: Specific and shared attributes

Consumers following the EBA heuristic do not use attributes shared by the two goods, since they do not allow to discriminate between goods : that's why they focus on specific attributes. Each *specific* attribute  $k$  provides to the consumer a utility  $u_{ik}$  (utilities are supposed to be additive). Consumers select initially one of them and eliminate the good not having this specific attribute (they buy the other product). When  $K_i$  is the set of specific attributes of good  $i$ , the probability of buying this good is then given by Tversky's formula (1972a) of the EBA model :

$$P_i = \sum_{k \in K_i} P_k \cdot P_i^k \quad (2.1)$$

where  $P_k$  is the probability of choosing attribute  $k$  among the set  $K = K_1 \cup K_2$  of the two goods specific attributes, and  $P_i^k$  is the probability of choosing the product  $i$  when attribute  $k$  has been selected.

Since the discriminating attributes are specific, it results necessarily that  $P_i^k = 1, \forall i, k$ . Moreover, Tversky supposes that the probability of choosing a discriminating attribute  $k$  equals the ratio between the utility of this attribute and the sum of utilities of all specific attributes. When the other firm is  $j, j \neq i$ , this probability is given by the formula:

$$P_k = \frac{u_{ik}}{\sum_{k \in K_1} u_{1k} + \sum_{k \in K_2} u_{2k}} \quad (2.2)$$

Denote  $u_i = \sum_{k \in K_i} u_{ik}$  and then the probability of choosing good  $i$  is given by :

$$P_i = \frac{u_i}{u_i + u_j} \quad (2.3)$$

This probability is thus equal to the ratio of specific utilities of attributes for the various goods. If a good does not have any specific attribute, it will never be chosen by the consumers.

These results call several remarks. In the case of two goods, the EBA model of Tversky corresponds to the structure developed by Restle (1961). Moreover, this expression is formally equivalent to the Luce model (1959) if one interprets the parameters  $u_i$  as *the utility of the goods* and not as the *utility of specific attributes* of the goods. Thus, by replacing “ $u_i$ ” by “ $\exp(u_i/\mu)$ ”, probabilities correspond to those of the binomial logit. However, this equivalence ceases to be true when prices of goods differ. Indeed, in the logit model, price is a component of options utility. At the contrary, Rotondo (1986) showed that price differences between goods should be considered as an attribute in the EBA model.

In a set of two goods, let us now distinguish between “price” attribute, having the index  $k = 0$ , and “non-prices” attributes indicated with  $k \geq 1$ . Also, suppose that  $p_i > p_j$ . Then, according to Rotondo (1986), good  $j$  has an additional “price” attribute compared to good  $i$  such that :

$$u_{j0} = \int_{p_j}^{p_i} w(\lambda) d\lambda \quad (2.4)$$

In this case, the shape of price difference depends on the function  $w$  retained: tests carried out by Rotondo show that a linear price difference seems to be a good approximation of individual behaviors, which corresponds to  $w(\lambda) = 1$ . The price attribute takes the following form :

$$u_{j0} = \theta(p_i - p_j) \quad (2.5)$$

Thus, consumers can eliminate a good because it does not have a particular non-price attribute or because its price is too high compared to that of the rival good. For instance, a consumer who chooses between two cars can eliminate one of them because it is too expensive (if the discriminating attribute he selects is the price) or because it does not have a particular equipment, as the airbag (if this last corresponds to the discriminating attribute). In a set of homogeneous goods ( $u_i = u_j = 0$ ), the least expensive good is always selected with unitary probability.

Let us note that the choice probabilities differ from those of the logit model, and in particular do not have the property of “independence of irrelevant alternative” (IIA). The existence of this empirically irrelevant property (as shown in the red bus-blue bus paradox revealed by Debreu, 1960), induced the development of more sophisticated random utility models than the logit, like the nested logit or the mixed logit. The EBA model provides here an alternative solution among random decision rule models to avoid this property.<sup>2</sup>

### 3 Properties of the EBA demands

In this section, we initially study the shape of demands when price varies. Then, we show that a partial heterogeneity of preferences of the agents can be integrated in the EBA model, which belongs in theory to the representative consumer paradigm. Finally, we show that this model may also be linked with a spatial approach of differentiation.

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<sup>2</sup>A more complete presentation of the EBA model and its links with various discrete choice models is provided by Laurent (2006a)

### 3.1 Relation between demand and price

Consider now a market with  $N$  representative consumers similar to those of the previous section and buying exactly one unit of one product.<sup>3</sup> In this duopoly, each firm produces a differentiated good  $i$ <sup>4</sup> sold at the price  $p_i$ , for  $i = \{1, 2\}$ . The EBA model presented in the previous section is used to construct the demand for good  $i$ :  $X_i = NP_i$  where  $P_i$  corresponds to the probability of choosing  $i$ , given by equation (2.1). This demand for good  $i$  depends on the price hierarchy retained. It is a continuous function that can be described as :

- if  $p_i \geq p_j$ ,

$$X_i = \frac{Nu_i}{u_i + u_j + \theta(p_i - p_j)} \quad (3.1)$$

- if  $p_j \geq p_i$ ,

$$X_i = \frac{N(u_i + \theta(p_j - p_i))}{u_i + u_j + \theta(p_j - p_i)} \quad (3.2)$$

The parameter  $\theta$  can be interpreted as the relative importance of price attribute compared to non-price attributes. Each demand for a product is clearly decreasing with its price but the study of concavity gives the following result :

$$\left. \frac{\partial^2 X_i}{\partial p_i^2} \right|_{p_i \geq p_j} = \frac{2N\theta^2 u_i}{(u_i + u_j + \theta(p_i - p_j))^3} > 0 \quad (3.3)$$

$$\left. \frac{\partial^2 X_i}{\partial p_i^2} \right|_{p_j \geq p_i} = \frac{-2N\theta^2 u_j}{(u_i + u_j + \theta(p_j - p_i))^3} < 0 \quad (3.4)$$

Demand is strictly concave as long as  $p_j > p_i$  but becomes strictly convex as soon as  $p_i > p_j$ . This demand can be represented graphically when price varies. Here an example with  $N = 1$ ,  $p_2 = 4$ ,  $\theta = 1$ ,  $u_1 = 2$  and  $u_2 = 1$ .

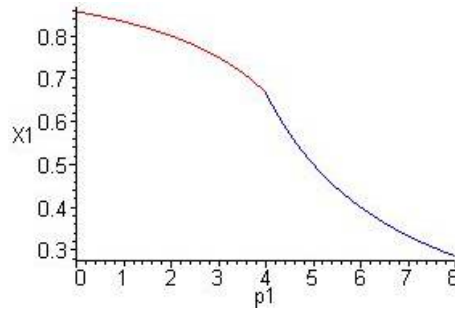


Figure 2: Evolution of demand according to the price in the EBA model

Thus, demand has a kink implying its non-concavity. This type of kink is not very widespread in the literature. Among differentiated products models, Gabszewicz and Thisse (1986) obtain a demand with convex and concave kinks in an Hotelling model with particular transportation costs function. However, this type of kink appears especially in models with “market inertia” or “switching costs” (Klemperer, 1984,

<sup>3</sup>This assumption will be discussed in section 5

<sup>4</sup>The firm selling good  $i$  will be described as “firm  $i$ ”

Scotchmer, 1986, Farrel and Shapiro, 1987). These models also use price differences and have sometimes the same properties than the framework we develop here, as it will be specified later.

### 3.2 Representative consumer or heterogeneous preferences ?

The strong assumption according to which all consumers have the same preferences for goods allows to interpret the parameter  $u_i$  as the perceived utility of specific attributes for the product  $i$  by a representative consumer. However, the EBA model is not inconsistent with a partial heterogeneity of preferences.

Consider a population of size  $N$  divided in  $m = \{1, \dots, M\}$  groups of consumers with  $\sum_{m=1}^M N_m = N$  where  $N_m$  denotes the number of individuals in group  $m$ . The taste for specific attributes varies between groups but not within groups: each agent of group  $m$  receives a utility  $u_{im}$  during the consumption of good  $i$ . Nevertheless, all consumers have the same coefficient of preference  $\theta$ . The firm cannot discriminate between the groups.

In this case, if  $p_1 \geq p_2$ , demands are given by :

$$X_1 = \sum_{m=1}^M \frac{N_m u_{1m}}{u_{1m} + u_{2m} + \theta(p_1 - p_2)} \quad (3.5)$$

and

$$X_2 = \sum_{m=1}^M \frac{N_m (u_{2m} + \theta(p_1 - p_2))}{u_{1m} + u_{2m} + \theta(p_1 - p_2)} \quad (3.6)$$

Such expressions can be simplified if we consider the following assumption :

**Assumption 1** : *The sum of specific non-price attributes utilities for the set of goods is constant between groups of individuals* :  $\sum_{i=1}^n u_{im} = \sum_{i=1}^n u_{il} \quad \forall l \neq m$

This assumption means that each consumer values identically the total set of attributes. For instance, if those attributes are accessories of a car, this means that each individual associates the same utility to the set of all possible existing accessories. This assumption seems reasonable when the number of attributes is sufficiently high: in this case, the global utility is almost equivalent for all the people.

By fixing  $u = u_{1m} + u_{2m} \quad \forall m$ , it becomes possible to reformulate demands as such :

$$X_1 = \frac{\left( \sum_{m=1}^M N_m u_{1m} \right)}{u + \theta(p_1 - p_2)} = \frac{N \left( \sum_{m=1}^M \frac{N_m}{N} u_{1m} \right)}{u + \theta(p_1 - p_2)} \quad (3.7)$$

and

$$X_2 = \frac{\left( \sum_{m=1}^M N_m u_{2m} + N \theta(p_1 - p_2) \right)}{u + \theta(p_1 - p_2)} = \frac{N \left( \sum_{m=1}^M \frac{N_m}{N} u_{2m} + \theta(p_1 - p_2) \right)}{u + \theta(p_1 - p_2)} \quad (3.8)$$

Thereafter, if  $u_i = \sum_{m=1}^M \frac{N_m}{N} u_{im}$  then one finds the traditional formulation of demand. In this case, the parameter  $u_i$  can be interpreted differently: it is the *weighted average utility in the population* of good  $i$  specific attributes set. We thus show that it is possible to introduce a certain degree of preference heterogeneity if assumption 1 holds.



### 3.3 Demands and spatial differentiation

Let us see now how the EBA model demands can be connected to the main two approaches of differentiation: Kaldorian approach of localized competition and Chamberlinian approach of non-localized competition. Kaldorian models suppose that the consumers differ by their tastes, these preferences being described in a space of characteristics. They are thus known as “spatial” models, like Hotelling (1929) and Salop (1979) models. The intensity of competition is stronger between firms localized closer because their products are closer substitutes. Chamberlinian models consider representative consumers having a taste for difference. General random utility models were developed by Perloff and Salop (1985) and Sattinger (1984). Competition is not localized: if a firm modifies its price, the resulting effect is also distributed on all the other firms.

Whereas the literature initially underlined the differences in asymptotic properties between the models belonging to these two categories, some recent works sought to integrate these two approaches (like Deneckere and Rotschild, 1992). Anderson, De Palma and Thisse (1992) identified conditions under which an equivalence may be established between these structures. For instance, logit model demands with representative consumers can be modelled as demands with heterogeneous preferences for a particular density function (op. cit. p 118). Formulation of Tversky’s model demand is consistent with representative consumer approach since all consumers have the same probabilities of choice. Nevertheless, they do not maximize their utility, as it is usually the case, but use the EBA heuristic. We show now that the EBA model demand is also compatible with localized competition under some assumptions.

Suppose that two firms are localized at the bounds of a segment (firm 1 on “left”) whose length is  $l$ . The  $N$  consumers are uniformly distributed with density  $N/l$  along the segment and buy a single unit of good at one of the firms. Each consumer is supposed to bear a linear cost per unit of covered distance : thus, a consumer located at point  $x$  on the segment bears  $C_1(x) = tx$  to go to firm 1 and a cost  $C_2(x) = t(l - x)$  to go to firm 2. These assumptions are rather common in the literature: this cost can be interpreted as a physical cost of transportation or as a cost related to the distance of the preferred product variety.

Products sold by firms get a particular specific utility to consumers. For the firm proposing the highest price good, this utility equals the utility of product’s specific attributes. For the firm selling the lowest price good, the utility of specific attributes is increased by the price difference, which corresponds to the saving made by consumers choosing this good. Suppose that  $p_1 > p_2$ . In this case,  $U_1^S = u_1$  and  $U_2^S = u_2 + \theta(p_1 - p_2)$ .

Finally, suppose that the consumers choose for their purchase the firm which gets the highest specific utility per unit of distance. Such a decision rule is not usual in models of spatial differentiation in which consumers maximize their mill utility, the converse of the mill prices (which comprises the price of the product and a weighted distance by transport cost). In the Hotelling model, this utility decreases in a linear way with the distance. In the quadratic Euclidean distance model (D’ Aspremont, Gabszewicz and Thisse, 1979), consumers have a decreasing and concave utility with the distance. However, when consumers maximize their utility per unit of distance, as in the EBA model, *their utility is decreasing and convex with the distance.*

Let us now establish the position of the indifferent consumer between buying at firm 1 and at firm 2.

This last verifies  $\frac{U_1^S}{C_1(x)} = \frac{U_2^S}{C_2(x)}$ . As parameter  $t$  plays no role here, this equality leads to :

$$(l-x)u_1 = x(u_2 + \theta(p_1 - p_2)) \Leftrightarrow x = \frac{lu_1}{u_1 + u_2 + \theta(p_1 - p_2)} \quad (3.9)$$

Demands are then given by  $D_1 = Nx = \frac{lNu_1}{u_1 + u_2 + \theta(p_1 - p_2)}$  and  $D_2 = N(l-x) = \frac{lN(u_2 + \theta(p_1 - p_2))}{u_1 + u_2 + \theta(p_1 - p_2)}$ .

For  $l = 1$ , we find demand functions (3.1) and (3.2). As in the logit model, the EBA model demands are consistent both with the representative consumer approach and with an “address” model in which decision depends on the utility of relative advantages per unit of distance.

Note that establishing a link between representative consumer approach and address approach implies, for random utility models (like the logit model), a modification of the density function which affects the product utility compared to the basic spatial model. But for a random decision rule model (like the EBA model), establishing such a link is made possible by the modification of the decision rule, compared to the spatial model.

## 4 Existence and uniqueness of the price equilibrium

After having studied the EBA duopoly demands, we show now that a price equilibrium exists in such a framework. Each firm plays a non-cooperative game with its rival in which the strategy consists in price determination. Let  $S_i \subseteq \mathbb{R}_+$  be the set of strategies for good  $i$  (the set of acceptable prices). Suppose that each firm bears a unit cost  $c_i$  and a fixed cost  $F_i$ , this last being sufficiently weak to guarantee the positivity of profits, of which here expressions:

- if  $p_i \geq p_j$ ,

$$\Pi_i = \frac{Nu_i(p_i - c_i)}{u_i + u_j + \theta(p_i - p_j)} - F_i \quad (4.1)$$

- if  $p_j \geq p_i$ ,

$$\Pi_i = \frac{N(u_i + \theta(p_j - p_i))(p_i - c_i)}{u_i + u_j + \theta(p_j - p_i)} - F_i \quad (4.2)$$

With such an expression, firms are implicitly supposed to be risk neutral. We use the concept of Nash equilibrium in pure strategies. In a two goods market, a “price equilibrium” is a price vector  $(p_1^*, p_2^*)$  such that each firm  $i$  ( $i = \{1, 2\}$ ) maximizes its profit for the value  $p_i^*$  of  $p_i$  conditionally to the price  $p_j$  ( $j \neq i$ ) set by the other firm  $j$ . Formally, this means that :

$$\Pi_i(p_i^*, p_j^*) \geq \Pi_i(p_i, p_j^*) \quad \forall p_i \in S_i, \quad \forall i, j \in \{1, 2\}, i \neq j$$

When demand  $X_i$  is concave with prices, it is not necessary to prove the existence of such an equilibrium: Caplin and Nalebuf (1991) showed that if  $1/X_i$  is increasing and convex with prices, then profit is concave, which guarantees the existence of price equilibrium. However, the EBA model demands have a kink which does not allow to certify the concavity: theorem of Caplin and Nalebuf (1991) cannot be mobilized here. For which values of unit costs does such an equilibrium exist there?

**PROPOSITION 1** *Necessary and sufficient conditions of Nash equilibrium existence in  $p_i \geq p_j$ , with  $i, j \in \{1, 2\}$  and  $i \neq j$ , are*

$$u_i \geq u_j \quad (4.3)$$

and

$$c_i - c_j \geq \frac{\sqrt{u_i u_j} - u_i}{\theta} \quad (4.4)$$

*If this equilibrium exists, then it is unique.*

**Preuve :** Proof of this proposition is presented in Appendix.

This proof of equilibrium existence will adopt the following method. In first time, we establish the best response function of firm i for a price  $p_j$  chosen by its rival, according to whether i chooses  $p_i \leq p_j$  or  $p_j \leq p_i$ . It follows from this analysis that the best response functions *on all the interval of definition*, established in a second time, depend on the hierarchy between  $u_i$  and  $u_j$ . Finally, symmetrical and asymmetrical price equilibrium and their conditions of existence are identified.<sup>5</sup>

The couple of equilibrium price in  $p_i > p_j$  is given by<sup>6</sup>:

$$p_i^* = \frac{u_i + \sqrt{\Delta}}{2\theta} + c_i \quad (4.5)$$

$$p_j^* = \frac{u_i + u_j}{\theta} + c_i \quad (4.6)$$

where  $\Delta = u_i^2 + 4u_i(u_i + u_j + \theta(c_i - c_j))$ . At this equilibrium, firm i, whose product is the most appreciated by consumers, sets a higher price than its rival. This equilibrium exists when the unit cost of i is higher than that of j, which seems intuitive. However, firm j may also have a higher unit cost than i since  $\sqrt{u_i u_j} - u_i < 0$  but the gap of unit costs should be necessarily weak. Properties of this equilibrium are detailed in section 5. Finally, note that the two necessary and sufficient conditions do not guarantee the existence of equilibrium for all values of cost or utility parameters. The case with no equilibrium will be studied in section 6.

## 5 Economic properties of price equilibrium

This section study equilibrium configurations when attributes verify the hierarchy  $u_1 \geq u_2$ .<sup>7</sup> First of all, we see that this model covers many forms of differentiation depending on the values of the parameters. The analysis of the equilibrium prices will also show that firm 1 is used as reference point by its rival for its tariff choices. Moreover, profits and market shares are compared. The last section shows that the parameter  $\theta$  plays a role similar to a participation constraint in this model.

<sup>5</sup>An other method consists in determining local equilibrium existing on an interval and, if necessary, to check if this local equilibrium is global. This method leads obviously to the same result, but at the cost of more tiresome calculations...

<sup>6</sup>Let us note that the Nash equilibrium is weak because firm i could obtain the same profit by choosing another price belonging to an interval  $[p_j; +\infty[$

<sup>7</sup>This assumption aims at reducing notations with no loss of generality

## 5.1 What kind of differentiation ?

In this section, we show that various equilibrium configurations can be interpreted according to the values of parameters  $u_1$  and  $u_2$ .

### 5.1.1 Pure horizontal differentiation

When  $u_1 = u_2 = u > 0$ , two conclusions can be drawn : first, goods are differentiated and, second, the specific attributes of each good are appreciated in the same way within the population of consumers. Such a configuration, in which all varieties have a positive demand when they are sold at same price, refers to a *pure horizontal differentiation*.

In this framework, there is always a Nash price equilibrium. If  $c_1 = c_2 = c$ , then equilibrium prices are given by :  $p_1 = p_2 = \frac{2u}{\theta} + c$ . Price is higher than marginal cost, which express the differentiation. This last is horizontal since prices are equal when marginal costs are equal: it is, for instance, the case in the Kaldorian model of Hotelling (1929) or in the Chamberlinian model of Dixit and Stiglitz (1977). It is also the type of differentiation in the initial version of the logit model. Let us recall that, in the case of two goods, logit price is given by  $p = 2\mu + c$  where  $\mu$  is an index of consumers taste heterogeneity. The  $u/\theta$  ratio in the EBA model thus plays a similar role to that of parameter  $\mu$  in the logit model.

If  $c_i > c_j$ , then we find an equilibrium in  $p_i > p_j$  with  $p_i = c_i + \frac{u + \sqrt{9u^2 + 4u\theta(c_i - c_j)}}{2\theta}$  and  $p_j = \frac{2u}{\theta} + c_i$ . This result, according to which the firm having the highest marginal cost fixes the highest price, is consistent with an Hotelling model in which costs are asymmetric.

### 5.1.2 Pure vertical differentiation

When  $u_1 > 0$  and  $u_2 = 0$ , only one of the goods has additional specific attributes. Thus, at equal price, all consumers would prefer having good 1 rather than good 2: existence of such a preference hierarchy is the sign of a *pure vertical differentiation*. Traditionally, this type of differentiation is interpreted by supposing that the firms sell products of different “qualities”, in a subjective sense. In the logit model, quality is defined as the « *dot product of the observable characteristics with the consumer valuations* » (Anderson, De Palma and Thisse, 1992, p 66) but, in the EBA model, quality is the consumer supplement of utility get by all *specific* attributes of a good. Indeed, this definition takes into account only the specific attributes, not all the attributes, since common characteristics do not intervene in the choice process of the EBA model.

Several models of pure vertical differentiation have been developed in the literature: in the model of Mussa and Rosen (1978), marginal willingness to pay the quality varies depending on consumers, whereas the income of individuals varies in the model of Gabszewicz and Thisse (1979). These models established the existence and properties of “quality then price” equilibrium and were extended by Shaked and Sutton (1982) who consider a three-stage game with entry, quality choice and then price competition.

In the EPA model with vertical differentiation, the only possible equilibrium when  $u_1 > u_2$  verifies  $p_1 > p_2$  and the remaining necessary and sufficient equilibrium condition (4.4) is reformulated into  $\Delta c \geq -u_1$ . If this last is verified, equilibrium prices are given by  $p_1 = c_1 + \frac{u_1 + \sqrt{u_1^2 + 4u_1(u_1 + \Delta c)}}{2\theta}$  and  $p_2 = \frac{u_1}{\theta} + c_1$ . The highest quality firm also sets the highest price, which is standard in this type of

model.

### 5.1.3 Vertical and horizontal differentiation

The general case with  $u_i > u_j > 0$  can be interpreted as *double differentiation*: differentiation is horizontal up to the level  $u_j$ , goods offering to the consumer comparable levels of services, then vertical for a level  $u_i - u_j$ , product i proposing also additional attributes. We can note that this approach allows to model in a simple way a double differentiation<sup>8</sup> coupled with a costs asymmetry.

Literature on models with double differentiation is still embryonic. Neven and Thisse (1990) first propose a duopoly model allowing the triple analysis of equilibrium in price, quality and variety competition. This model supposes null production costs, a horizontal differentiation of Hotelling type according to a variety  $y_i$  with a quadratic transport cost (à la D'Aspremont, Gabszewicz and Thisse, 1979) and a vertical differentiation à la Mussa and Rosen (1978) according to quality  $q_i$  with a uniform distribution of consumers taste for quality. Then, the authors show that, for a same segment of demand, equilibrium prices differ according to whether the gap of differentiation is more important for the vertical characteristic or the horizontal one. Suppose, on the one hand, that firm i is dominant in quality ( $q_i > q_j$ ) and, on the other hand, that  $y_i > y_j$  (other cases are symmetric) : in this case, the range of vertical differentiation is  $q_i - q_j$  and that of horizontal differentiation is  $y_i - y_j$  and the results of the model differ according to whether  $q_i - q_j \stackrel{\leq}{\geq} y_i - y_j$ .

Such a result is not highlighted in the EBA model : the range of vertical differentiation is  $u_i - u_j$  whereas the range of horizontal differentiation (on many attributes) can be noted  $u_j - (-u_j) = 2u_j$  but the condition  $u_i > 3u_j$  plays no role on the market outcome. This difference can be explained as well by the nature of differentiation in the EBA model as by the shape of demands, and thus by this imperfect rationality of probabilistic type.

#### 5.1.4 The special case of Bertrand

When  $u_1 = u_2 = 0$  or  $\theta \rightarrow +\infty$ , any form of differentiation vanishes. For  $c_1 = c_2 = c$ , one finds the price competition of a Bertrand type, with homogeneous goods, and in which  $p_1 = p_2 = c$ .

## 5.2 Equilibrium prices and the “reference firm”

Let us note initially  $\Delta c = \theta(c_1 - c_2)$ . This term represents cost difference weighted by relative importance of prices for consumers. We consider now the most general structure of differentiation with  $u_1 > u_2 > 0$  and study the equilibrium prices, whose form is sometimes original.

Firstly, the price  $p_1^* = c_1 + \frac{u_1}{2\theta} + \frac{\sqrt{\Delta}}{2\theta}$  is of a rather classical formulation : it equals to the unit cost of the firm plus a margin, since products are differentiated. This margin of firm 1 is  $m_1 = \frac{u_1 + \sqrt{u_1^2 + 4u_1(u_1 + u_2 + \Delta c)}}{2\theta}$ . It increases with  $u_1$ , which means that the differentiation is strongly valued by the consumer. But this margin is (although slightly) also positively correlated with  $u_2$  and  $\Delta c$ . The first point means that the price is increasing with the *global* degree of differentiation on the market:

<sup>8</sup>Usually, *each* dimension of differentiation is modelled by two parameters. Here, the two parameters completely apprehend the two dimensions of differentiation

an effort of differentiation from a competitor will profit to all the protagonists. The second point is more original: margin of firm 1 is increasing with its cost disadvantage compared to firm 2.

These two elements also seem at the origin of the non-conventional form of price equilibrium of firm 2  $p_2^* = \frac{u_1 + u_2}{\theta} + c_1$ . Price grows with  $u_1$  and  $u_2$ , which seems to confirm the idea according to which prices are increasing with the total effort of differentiation on the market. But, here again, it could seem surprising that the price does not depend on  $c_2$  but increases with  $c_1$ .

In order to explain these unusual relations between prices and costs, a come back to the study of equilibrium determination is required. For a given level of attributes and costs, if the price set by firm 2 is too low, any increase in  $p_1$  raises the margin of firm 1 more than proportionally that the reduction of demand, which breaks the existence of equilibrium. On the other hand, if the price set by firm 2 is too high, firm 1 is not inclined to accept a disadvantage in price, which strongly reduces the attractiveness of its product, and price equilibrium cannot be asymmetrical. Finally, *for a particular gap of levels of attributes and a particular gap of costs, a fixed and strict gap of price is required for the existence of equilibrium on the market.* This conclusion reminds practices of tariff imitation, like those described by Lazer (1957 p. 130-131), and in particular the case where the firm selling the “best quality” good sets a *reference price* on the market (or “focal price”). In this case, the other firms tariff at reference price minus a certain amount, which is function of the quality gap with the reference firm<sup>9</sup>. Such an explanation allows to explain the existence of a price equilibrium when  $c_1$  is much higher than  $c_2$ .

Introduction of imperfect rationality in consumer behavior can thus involve a modification of relationships between firms, the firm selling the more appreciated good acquiring a statute of “reference”. In a more symmetrical framework, let us note that a model of price determination based on imitation between firms in a Bertrand oligopoly was also proposed by Ostmann and Selten (2001).

### 5.3 Equilibrium demands and profits

At the equilibrium, demands takes the values  $X_1 = NP_1^* = \frac{2Nu_1}{u_1 + \sqrt{\Delta}}$  and  $X_2 = NP_2^* = \frac{N\sqrt{\Delta} - u_1}{(u_1 + \sqrt{\Delta})}$  with always  $\Delta = u_1^2 + 4u_1(u_1 + u_2 + \Delta c)$ . Let us determine under which conditions a firm can obtain the largest market share. Comparison shows that:

$$\begin{aligned} X_1^* > X_2^* &\Leftrightarrow 3u_1 > \sqrt{\Delta} \\ &\Leftrightarrow \Delta c < u_1 - u_2 \end{aligned} \tag{5.1}$$

This condition stresses the importance of cost parameter in determination of firms’ market shares : firm selling the “more appreciated” product (such that  $u_1 > u_2$ ) will make the largest market share if the gap of differentiation is sufficiently high compared to the gap of costs. Indeed, a too strong difference in costs would generate a too high price for firm 1 and a transfer on demand of firm 2. Such a relation is relatively new in the literature on differentiation in which unit costs of firms are often equal.

A similar analysis can be made about profits which take the outcome values  $\Pi_1^* = \frac{Nu_1}{\theta} - F_1$  and  $\Pi_2^* = \frac{N(u_1 + u_2 + \Delta c)(\sqrt{\Delta} - u_1)}{\theta(\sqrt{\Delta} + u_1)} - F_2$ . Calculus of first order conditions allows to check that  $\Pi_i^*$  increases

<sup>9</sup>Recall however that the information is perfectly symmetrical in the concept of equilibrium used here

with  $u_i, \forall i$ . A comparison of profits can be realized when fixed costs are identical  $F_1 = F_2$ . While posing  $x = u_1 + u_2 + \Delta c$ , we find that :

$$\begin{aligned} \Pi_1^* \geq \Pi_2^* &\Leftrightarrow \sqrt{\Delta}(u_2 + \Delta c) < u_1(u_1 + u_2 + \Delta c) + u_1^2 \\ &\Leftrightarrow (u_1^2 + 4u_1x)(x^2 - 2u_1x + u_1^2) < u_1^2x^2 + u_1^4 + 2u_1^3x \\ &\Leftrightarrow \Delta c < u_1 - u_2 \end{aligned}$$

The condition (5.1) is also crucial in the profit analysis. This omnipresence reveals a convergence of firms' goals in this model: maximization of profit, i.e. short term objective, is consistent with maximization of market share, i.e. long term objective. Such a convergence is also present in the "switching costs" models previously mentioned, as Farrel (1986) shows it. If the valuation gap of products between firms is higher than the weighted cost gap, then the firm selling the "more appreciated" product (such that  $u_1 > u_2$ ) obtains both a larger profit and a higher market share. By referring again to the assumption  $c_1 = c_2$ , which is frequent in models of product differentiation, one finds thus the standard result : firm selling the "more appreciated" good always makes a higher profit than its rival.

## 5.4 Market participation for consumers and outside option

We explain initially how the outside option is generally integrated in models of differentiation. Then we show that the assumption of complete coverage of market can also be lifted in the EBA model.

### 5.4.1 Outside option in models of differentiation

Up to now, we have made the assumption of complete coverage of market, which means that each consumer buys exactly a unit of good. However, differentiated products models generally take into account the possibility that some consumers decide not to buy at all. Indeed, these consumers may prefer to choose an "outside option" or keep their money, which gets a certain utility to them. Thus, interest of non-consumption is represented by a parameter in the model and, according to the interval to which this parameter belongs, the market is covered or not: that defines the participation constraint of the model.

This parameter is often modelled as a utility of reservation, noted for instance  $U_0$ : it is the case in the vertical differentiation models or in the logit model of oligopoly. When this parameter varies, the properties of the equilibrium are rather similar in all these models: equilibrium prices decrease with this parameter, as well as the margins of firms and generally their profits. Indeed, the possibility that the consumers leave the market plays like a bridle on the firms price level determination, which can also affect the quality choices.

In vertical differentiation, the role played by  $U_0$  on demands is not always clear. Let us take the example of the duopoly of Gabszewicz and Thisse (1979) in which firm A sells the highest quality good and firm B a lower quality good (at a lower price)<sup>10</sup>. In this model, there exists 2 modes :

- when market is covered (mode  $D_2$ ) and for some values of parameters<sup>11</sup>, demand of firm B grows with  $U_0$  whereas that of firm A decrease with this parameter. In this mode, an increase in  $U_0$  results in a larger price disutility and thus by a transfer to firm B of the demand previously addressed to firm A.

<sup>10</sup>We will not consider here the case in which one firm is excluded from the market

<sup>11</sup>In particular when the fixed part of consumers income is not too high

- when market is not covered (mode  $D_1$ ), demands of the two firms decrease with  $U_0$  (but this decrease is larger for firm A). In this mode, the increase in  $U_0$  induces a market exit of B's consumers. However, as a part of A's demand transfers from A to B, the reduction of B's demand is lower than that of A.

#### 5.4.2 An equivalence in the EBA model

Let us see now how to model an outside option in the EBA framework. Note that it is impossible to use a utility of reservation because the choice in the EBA model is based on comparisons between specific attributes and prices. The "own" utilities of options are thus not taken into account. Nevertheless, let us show that the parameter  $\theta$  plays in the EBA model a similar role than the parameter  $U_0$  in the model of Gabszewicz and Thisse (1979).

First of all, firms' prices and margins decrease with  $\theta$ . We have previously denoted that this parameter measures the relative preference for consumers of price difference compared to the non-price attributes : it thus seems logical that  $p_1$  decreases with  $\theta$  but more paradoxical than  $p_2$  also diminishes. This apparent contradiction is solved by recalling that the firm 1 plays a role of "reference" for its rival in price determination : since  $p_1$  decrease with  $\theta$ , it is also true for  $p_2$ . Moreover, it seems plausible that the consumers leave the market when they attach to prices a great importance compared to non-price attributes.

Then, when the market is completely covered and  $\Delta c \geq 0$ , then  $\frac{\partial X_1}{\partial \theta} \leq 0$  and  $\frac{\partial X_2}{\partial \theta} \geq 0$ . These results correspond exactly to the case described by mode  $D_2$  in the model of Gabszewicz and Thisse.

Finally, the model remains to be generalized in order to replicate the mode  $D_1$  in which market is not covered. Suppose that a consumer  $k$  decides to enter on the market only if  $\theta$  is lower than a personal psychological threshold  $\bar{\theta}_k$  (the interpretation of such a relation will be specified in section 5.4.3). Also suppose that this threshold is a random variable i.i.d in the population of consumers according to a uniform law. In this case, the number of consumers deciding to enter on the market is given by  $\sum_{k=1}^N P(\bar{\theta}_k > \theta) = NP(\bar{\theta} > \theta)$  where  $\bar{\theta}$  is the average psychological threshold. When  $P(\theta < \bar{\theta}) < 1$ , such a probability makes it possible to represent a *participation constraint* in the EBA model. Demand for good  $i$  can now be represented by  $X_i = NP(\theta < \bar{\theta})P_i$ . By always supposing that  $\Delta c \geq 0$ , let us study how demands vary with  $\theta$  :

$$\left. \frac{\partial X_1}{\partial \theta} \right|_{P(\theta < \bar{\theta}) < 1} = \frac{2Nu_1[P'(\theta < \bar{\theta})(u_1 + \sqrt{\Delta})\sqrt{\Delta} + 2u_1(c_1 - c_2)P(\theta < \bar{\theta})]}{\sqrt{\Delta}(u_1 + \sqrt{\Delta})^2} < 0 \quad (5.2)$$

$$\left. \frac{\partial X_2}{\partial \theta} \right|_{P(\theta < \bar{\theta}) < 1} = \frac{4Nu_1[P'(\theta < \bar{\theta})(u_1 + u_2 + \theta(c_1 - c_2))\sqrt{\Delta} + u_1(c_1 - c_2)P(\theta < \bar{\theta})]}{\sqrt{\Delta}(u_1 + \sqrt{\Delta})^2} \quad (5.3)$$

with  $\Delta = u_1^2 + 4u_1(u_1 + u_2 + \Delta c)$ . Thus, if the demand of product 1 is still decreasing with  $\theta$ , the sign of demand's derivative for product 2 is unknown in general case, since it depends on the shape of the probability used and of the values of parameters. However, this slope may be negative: for instance, it is true when  $c_1 = c_2$ . If this property holds, one finds the characteristic relation of mode 1 in the model of Gabszewicz and Thisse: the rise of  $\theta$  profits to the outside option and demand for product 2 decreases with  $\theta$ .



### 5.4.3 Interpretation and aspiration threshold

This participation constraint is not linked directly to the prices but only to a “preference” parameter of prices, which can seem surprising at first sight. However, the decision of access to the market (first stage) must be distinguished to the choice among the set of goods proposed on this market (second stage). Up to now, our analysis was focused exclusively on the second stage, in which tariff arbitrages occur. But such arbitrages are not necessarily present during the first stage on mature markets of differentiated products.

On this type of market, the purchase motivation come rather from the direct utility of the product<sup>12</sup>: the consumer’s decision of access to market is thus the fruit of a social, personal or professional context, which determines the valuation of the product. Consequently, consumers differ in their *aspiration level* on services rendered by the good (to remind the concept introduced by Simon, 1955): they decide to access to the market if the utility of product exceeds this threshold.<sup>13</sup>

In the EBA model, if prices or specific attributes allow to choose among the set of goods, it seems that the *common attributes* of these various goods<sup>14</sup> rather explain the consumers decision of entry on the market. Let us suppose that these common attributes get a rough utility  $u_0$  and that the consumer  $k$  enters if  $u_0$  exceeds its own aspiration threshold  $\bar{U}_k$ . In this case, consumers for which  $\frac{u_0}{\theta} > \bar{U}_k$  decide to purchase on the market. But this expression can also be rewritten as  $\theta \leq \bar{\theta}$  with  $\bar{\theta}_k = \frac{u_0}{\bar{U}_k}$ , which allow us to justify the form retained for the participation constraint. This threshold  $\bar{\theta}$  depends on a non-observable psychological parameter: the number of consumers entering on the market is thus considered as exogenous by firms for their tariff choice. The use of an aspiration threshold in first stage seems besides consistent with the use of choice heuristic at the second stage.

## 6 Non-existence of an equilibrium and Edgeworth cycle

Let us now analyse the EBA duopoly when one condition of existence of price equilibrium holds but not the other.<sup>15</sup> After having highlighted the existence of an Edgeworth cycle in this framework, we put this result in prospect compared to the existing literature.

### 6.1 Edgeworth cycles in the EBA model

Let us consider a sequential infinite-horizon game with alternate moves of firms in which these firms make a choice in the space of pure price strategies. Suppose that each firm is unaware of the reaction function of its rival, which limits its temporal horizon of profit maximization to one period. Thus, each time  $t$  its “turn” to play occurs, the firm  $i$  observes  $p_j^{t-1}$  and chooses the price  $p_i^t$  maximizing its profit for the current period. Prices evolution is thus described by successive and “naive” interactions of the firms’ reaction functions.

In this framework, we show that the strategic interaction of firms generates a shape of cycle already

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<sup>12</sup>For instance, a car may appear as an indispensable means of transport at some places, a television is an essential vector of socialization for some people...

<sup>13</sup>Typically, the purchase of a car can be less essential in urban environment than in rural one, because of the presence of public transport

<sup>14</sup>Which are not used in the choice heuristic

<sup>15</sup>If the two conditions were simultaneously violated, another equilibrium configuration would be obtained

studied in the literature. Let us consider the interaction of reaction functions established in appendix 9.2.2 when (4.3) is verified but (4.4) is violated, which implies the inequality  $c_i - c_j \leq \frac{\sqrt{u_i u_j} - u_i}{\theta}$ . Properties of the resulting cycle in downward and ascending phases are studied now.

**LEMMA 2** *In this mode of EBA model, the downward phase of the cycle has several steps .*

**Proof:** Let us take again the reaction functions of the appendix 9.2.2 when  $u_i > u_j$  and suppose that firm j chooses a very high price (because its profit is strictly increasing),  $p_j \rightarrow +\infty$ . In this case, the best response of i is to choose  $p_i^*(p_j) = \frac{u_i + u_j + \theta p_j - \sqrt{u_j(u_i + u_j + \theta(p_j - c_i))}}{\theta} < p_j$  but verifying  $\lim_{p_j \rightarrow +\infty} p_i^*(p_j) \rightarrow +\infty$ .

Consequently, firm j is also incited to set  $p_j^*(p_i) = \frac{u_i + u_j + \theta p_i - \sqrt{u_i(u_i + u_j + \theta(p_i - c_j))}}{\theta} < p_i$  which is also very high, and so on. The decreasing period thus comprises several stages.

CQFD

Let us stress that during the downward phase,  $\frac{\partial(p_i^*(p_j) - p_j)}{\partial p_j} < 0$  et  $\frac{\partial(p_j^*(p_i) - p_i)}{\partial p_i} < 0$ . Thus, in downward phase, price reduction is more and more progressive. Moreover, for a particular level of price  $\bar{p}$ ,  $\frac{\partial(p_i^*(\bar{p}) - \bar{p})}{\partial \bar{p}} < \frac{\partial(p_j^*(\bar{p}) - \bar{p})}{\partial \bar{p}}$  (as condition (4.3) holds and the violation of condition (4.4) implies that  $c_j > c_i$ ). This means that the price decrease is larger when it comes from the firm j than from the firm i.

However, this downward phase cannot continue infinitely. When its rival price falls under a particular threshold, a firm may have an interest to increase its price brutally what initiates the ascending phase of the cycle, characterized by the following lemma:

**LEMMA 3** *When condition (4.3) is verified and condition (4.4) violated, firm j is the first to increase its price. This raise, which characterizes the ascending phase of cycle, is unique.*

**Proof:** The first part of this lemma will be proven by contradiction : we show that prices cannot decrease in the area in which  $\Pi_i$  is increasing, which proves that  $\Pi_j$  is increasing “before”  $\Pi_i$ . Let us suppose that we are in the area in which  $\Pi_i$  is increasing for a particular period t. In this case,  $p_j^{t-1}$  necessarily verifies  $p_j^{t-1} < \frac{(u_i + u_j)}{\theta} + c_i$ . By using the proof of asymmetric equilibrium existence, realized in section 9.2.2, we deduced from it that  $p_i^{t-1}$  should verify both  $p_i^{t-1} < \frac{u_i + \sqrt{u_1^2 + 4u_1(u_1 + u_2 + \theta(c_1 - c_2))}}{2\theta} + c_i$  and  $p_i^{t-1} > \frac{2\sqrt{u_i u_j}}{\theta} + c_j$ . However that is impossible since the condition (4.4) is violated.

Consequently, at the end of the downward phase, firm i will end up choosing a price such that  $\Pi_j$  increases and j will choose  $p_j^* \rightarrow +\infty$  at the following period. For this price, the best response of i will be to propose a lower price which will initiate a new downward phase. The ascending phase can thus comprise only one stage.

CQFD

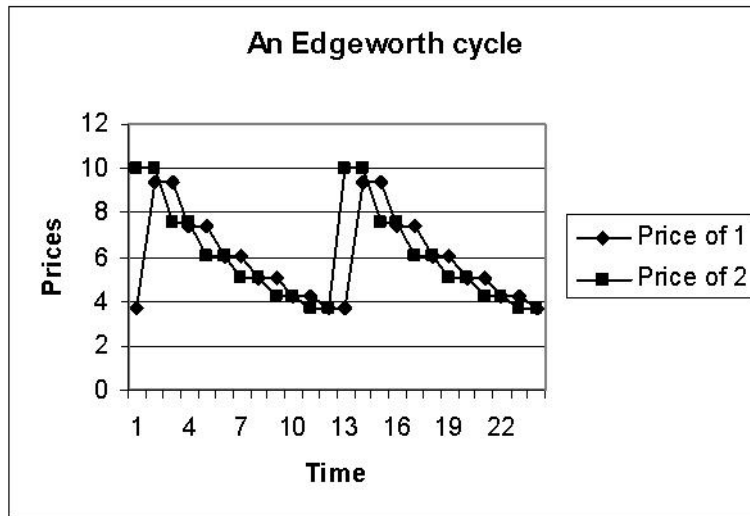
Note that this lemma just guarantees that  $\Pi_i(p_j^*)$  is not strictly increasing when  $\Pi_j(p_i^*)$  increases. Thus, just after the ascending phase, best response of firm i can consist in fixing a price either inferior or identical to that of its rival.

By combining the two previous lemmas, this shape of cycle comprising several stages in its downward phase and a single stage in ascending phase is well-known:

**PROPOSITION 4** *When equilibrium conditions are not verified simultaneously, firms' pure strategies interactions in an alternate move game leads to an Edgeworth price cycle.*

Thus, in the EBA model, when a firm cumulates disadvantages in term of unit cost and of specific attributes, this firm chooses to strongly increase its price and the form of differentiation on the market allows it to preserve a positive profit in spite of a negligible market share. This solution is better than market exit which gets a null profit.

In the absence of reservation price or outside option, maximum of profit during the ascending phase is reached for  $p \rightarrow +\infty$ , which characterizes an extremely large raise of prices and implies an extremely long downward phase. Such a width is obviously not observable in reality. But the aim of this analysis is only to show that the fundamental properties of Edgeworth cycles are verified in the absence of equilibrium in the EBA model, not to confront the cycle obtained with a particular market. A marginal modification of the EBA model reducing the width of fluctuations could however make it possible to obtain a more representative cycle. An example is provided here when maximum price has a finite bound  $p_{max} \rightarrow 10$  with  $u_1 = 2$ ,  $u_2 = 1$ ,  $c_1 = 0$ , and  $c_2 = 1$  (no equilibrium for these values):



The ascending phase of the cycle indeed comprises a single stage whereas the downward phase comprises 11 successive price reductions in this case. When  $p_{max}$  increases, it seems that this number of stages varies in a logarithmic way.

## 6.2 A setting in prospect of the results

After a recall of principal theoretical developments on Edgeworth cycles, existence conditions and characteristics of the price cycle obtained in the EBA model are connected to the existing literature.

### 6.2.1 Price cycle theories

The existence of cycles characterized by a price war during a certain period followed by a brutal increase in prices, was firstly established by Edgeworth (1925) in the case of homogeneous goods. For this author, each firm sets initially a lower price than its rival to increase its market share but did not recover the totality of demand because of capacity constraints. Thus, firms decrease their price until the war becomes too expensive: then a firm increases its price, immediately followed by its rivals. Finally, price war starts again.

Such cycles were then studied by Maskin and Tirole (1988) in a framework of dynamic and symmetrical duopoly with price competition, identical costs and homogeneous goods. These authors also show that Edgeworth cycles can be Markov Perfect Equilibrium for particular mixed strategies maximizing the firms' discounted profits <sup>16</sup>, if the discount factor is sufficiently high. Such equilibria are obtained in games with simultaneous moves or with alternate ones. This result was extended by Eckert (2003) to the case of asymmetric market share duopolies (but identical prices) and it proves the existence of price cycles at the equilibrium.

A more systematic extension in various directions was realized by Noel (2004): the author generalizes the duopoly demand functions used in previous papers and shows that Edgeworth cycles can occur in the case of differentiation, which interests us particularly here. In order to identify Markov Perfect Equilibria, the author uses an algorithm with unit costs variations between 0 and 1 according to periods. It allows considering only pure strategies, contrary to Maskin-Tirole: indeed, it is formally equivalent to suppose that a firm chooses various prices with a probability distribution or to consider that a firm always chooses a price of identical shape but that its unit cost fluctuates. This analysis allows to determine rather general properties on the shape of cycles. In the case of differentiation, Noel studies an alternate move game in a Hotelling framework. He shows that Edgeworth cycles occur if differentiation is not too strong. Indeed, according to the author, differentiation should decrease the probability of Edgeworth cycles existence, since it gets to the highest price firm a positive market share (and then a positive profit), which can destroy the cycle.

Results obtained in the EBA model can now be connected to the economic literature. We study both the existence of Edgeworth cycles and their structure.

### 6.2.2 Existence conditions of price cycles

Let us stress first that cycles in the EBA model are obtained by simple interaction of best response functions in one period scale and in pure strategies : cycles require neither mixed strategy nor unit costs fluctuation but, on the other hand, do not constitute equilibria. Besides, Markov Perfect Equilibria cannot be identified within this complex framework, with asymmetric reaction functions and price choice on a continuous interval. Let us note that these interactions require only short term opportunism from firms whereas Markov Perfect Equilibria require a planning by firms on a potentially high number of periods, a capacity to follow complex strategies and perfect information on past actions.

In the above mentioned papers, main decisive factors for price cycles existence seem to be the nature of reaction functions (Maskin and Tirole, 1988) and the degree of differentiation (Noel, 2004) but cost

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<sup>16</sup>For other strategies, it is also possible to obtain a kinked demand equilibrium a la Sweezy (1939)

asymmetry is rarely mentioned as a deciding factor, whereas it is the case in the EBA model.

Moreover, by supposing that the condition (4.3) holds, the probability of Edgeworth cycle existence corresponds to the probability of violation of condition (4.4). However this latter increases when the degree ( $u_i - u_j$ ) of vertical differentiation decreases and pure horizontal differentiation supposes necessarily absence of vertical differentiation. Thus, horizontal differentiation seems to induce Edgeworth cycles more frequently than vertical differentiation. This result is consistent with the conclusion of Gabszewicz and Thisse (1986) according to which a stable equilibrium exists more frequently in vertical differentiation models than in horizontal differentiation models. On the other hand, the width of horizontal differentiation does not play here any role in the existence of cycle, which differs from the conclusion of Noel. In the EBA model, cycle existence is related to the capability of market structure to support a cost asymmetry at the advantage of the more appreciated product firm.

### 6.2.3 Price cycles structure

Let us analyze now the properties of cycles obtained in the EBA model. Firstly, prices may decrease up to the unit cost in the literature, even with a Hotelling differentiation (look for instance at Noel, op cit.). It is not true in the EBA model, because of the more complex nature of differentiation supposed here.

Then, Noel (2004, p 8) identifies some general properties of the Edgeworth cycles obtained :

**P1** : at the end of the downward phase, firms can share the market during a little period. It is also true in the EBA model : in particular, after a price fall of firm  $j$ , the best response of firm  $i$  can consist in fixing the same price than  $j$ . In this context, firms have the same price during two periods but asymmetrical market shares.

**P2** : the increase is fast and sudden, and prices reached at the end of ascending phase are very high. This property also holds in the EBA model cycles.

**P3** : during the downward phase, prices fall accelerates until reaching the “floor” of the cycle. Such an acceleration of price war seems relatively difficult to explain by firms’ behavior. However this property is not verified in the EBA model since prices fall is increasingly progressive.

## 7 Conclusion

In this paper, it was established that the demand functions are kinked when consumers follow the EBA heuristic. Moreover, these demands are consistent both with a representative consumer approach and with a heterogeneous preferences approach. In particular, these demands may be obtained within a spatial framework in which consumers have convex perception of distance (whereas it is generally concave in the literature). By using these demands, we develop a discrete choice oligopoly of product differentiation. We showed that a price equilibrium in the EBA model can exist when the unit costs are asymmetrical. The firm having the more appreciated product can sell it at a higher price than its rival. Existence of equilibrium requires however that its cost is not too low compared to that of its rival.

Properties of this outcome are interesting in more than one way. First of all, this model integrates in the general case a double differentiation, vertical and horizontal, while keeping a rather light formalism. Then, it seems that the introduction of bounded rationality in consumer behavior modifies the relations between firms: thus, the firm selling the more appreciated good seems to act as “reference” for its rival

during the price determination. Moreover, the hierarchy of profits and demands seems to depend on the same criterion: comparison between the gap of specific attributes utilities and the gap of unit costs. Finally, we showed that the parameter  $\theta$  played in the EBA model a similar role to participation constraint in other models: the assumption of complete coverage of the market cannot thus constitute a limit of the structure we propose.

When no price equilibrium exists, price strategic interaction of firms leads to an Edgeworth cycle. Economic theory generally explained the existence of these cycles, frequently observed, by complex strategies of agents. This model shows, on the one hand, that this assumption is not always necessary and, on the other hand, that the costs asymmetries can play a role of trigger for such cycles.

Since these first results are encouraging, an analysis of firms' attributes choice in the EBA model framework is promising. In Laurent (2006b), we show that a subgame perfect equilibrium in "attributes then prices choice" exists when unit costs increase with levels of attributes. Imitation and endogenous outside option imply a lower level of attributes for the firm having the more appreciated product but are not inconsistent with equilibrium. The analysis of attributes choice with endogenous fixed costs is also carried out.

## 8 References

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## 9 Appendix

### 9.1 Best response function conditional to price hierarchy

Initially, let us establish the firms' best response functions in price conditionally to the price hierarchy retained. Which price  $p_i$  constitutes the best response of firm i knowing the price  $p_j$  of its rival?

#### 9.1.1 Best response in $p_i \geq p_j$

When  $p_i \geq p_j$ , the profit of i is given by :  $\Pi_i = \frac{Nu_i(p_i - c_i)}{(u_i + u_j + \theta(p_i - p_j))} - F_i$

For firm i, the first derivative is :

$$\frac{\partial \Pi_i}{\partial p_i} = \frac{Nu_i(u_i + u_j + \theta c_i - \theta p_j)}{(u_i + u_j + \theta(p_i - p_j))^2} \quad (9.1)$$

The sign of this derivative does not depend on the value of  $p_i$  and there are three cases :

- If  $p_j < \frac{u_i + u_j}{\theta} + c_i$ , then  $\frac{\partial \Pi_i}{\partial p_i} > 0$  and best response is  $p_i^*(p_j) \rightarrow +\infty$ .
- If  $p_j = \frac{u_i + u_j}{\theta} + c_i$ , then  $\frac{\partial \Pi_i}{\partial p_i} = 0$  and best response belongs to an interval  $p_i^*(p_j) \in [p_j; +\infty[$ .
- If  $p_j > \frac{u_i + u_j}{\theta} + c_i$ , then  $\frac{\partial \Pi_i}{\partial p_i} < 0$  and best response is  $p_i^*(p_j) = p_j$ .

#### 9.1.2 Best response in $p_j \geq p_i$

When  $p_j \geq p_i$ , profit of firm i is given by  $\Pi_i = \frac{N(u_i + \theta(p_j - p_i))(p_i - c_i)}{u_i + u_j + \theta(p_j - p_i)} - F_i$

The first derivative is :



$$\frac{\partial \Pi_i}{\partial p_i} = \frac{N[(u_i + u_j + \theta(p_j - p_i))(u_i + \theta(p_j - p_i)) - \theta u_j(p_i - c_i)]}{(u_i + u_j + \theta(p_j - p_i))^2} \quad (9.2)$$

This function has a single extremum whose expression is given by :

$$\frac{\partial \Pi_i}{\partial p_i} = 0 \Leftrightarrow \theta^2 p_i^2 - 2\theta p_i(u_i + u_j + \theta p_j) + (u_i + \theta p_j)(u_i + u_j + \theta p_j) + \theta u_j c_i = 0 \quad (9.3)$$

Only one root of this polynomial can verify  $p_j \geq p_i$  (the other never respects this hierarchy) :

$$p_i^c(p_j) = \frac{u_i + u_j + \theta p_j - \sqrt{u_j(u_i + u_j + \theta(p_j - c_i))}}{\theta} \quad (9.4)$$

Let us verify that this extremum is really a maximum :

$$\left. \frac{\partial^2 \Pi_i}{\partial p_i^2} \right|_{\frac{\partial \Pi_i}{\partial p_i} = 0} = \frac{2N(\theta p_i - u_i - u_j - \theta p_j)}{(u_i + u_j + \theta(p_j - p_i))^2} < 0 \quad (9.5)$$

This function is always negative in the interval  $p_j \geq p_i$  and profit is thus concave. Moreover,  $p_i^c(p_j)$  is in the interval of definition if :

$$p_j \geq \frac{u_i(u_i + u_j)}{u_j} + c_i \quad (9.6)$$

Best response function can thus take two possible forms :

- if  $p_j \geq \frac{u_i(u_i + u_j)}{u_j} + c_i$ , then profit is concave in the interval and best response is

$$p_i^*(p_j) = \frac{u_i + u_j + \theta p_j - \sqrt{u_j(u_i + u_j + \theta(p_j - c_i))}}{\theta}$$

- if  $p_j < \frac{u_i(u_i + u_j)}{u_j} + c_i$ , then profit is increasing in the interval and best response is  $p_i^*(p_j) = p_j$ .

## 9.2 Non-conditional best response functions

### 9.2.1 Asymmetry of best response functions

From the conditional reaction functions, it is possible to establish each firm best response function whatever the price hierarchy. However, the condition (9.6) indicates that the shape of reaction function depends on the hierarchy of specific attributes utilities.

Thus, when  $u_i \geq u_j$ , if profit is concave on  $[0; p_j]$ , it decreases necessarily on  $[p_j; +\infty[$  because  $\frac{u_i(u_i + u_j)}{u_j} + c_i \geq \frac{(u_i + u_j)}{\theta} + c_i$ . However, since demand is continuous, profit is also continuous and global maximum is necessarily in the interval  $[0; p_j]$ . On the other hand, when  $u_j \geq u_i$ , profit may be simultaneously concave in  $[0; p_j]$  and increasing in  $[p_j; +\infty[$  in which case determination of global maximum requires to compare profits on the two intervals. In all the other cases, there are only one local maximum.

Best response functions are thus asymmetric according to the hierarchy between  $u_i$  and  $u_j$ . *Let us suppose now that  $u_j \geq u_i$  and let us solve this indetermination on firm j's best response by comparing maximum profits on each of the two intervals  $[0; p_i]$  et  $[p_i; +\infty[$ .*

When  $p_j = \frac{u_i + u_j + \theta p_i - \sqrt{u_i(u_i + u_j + \theta(p_i - c_j))}}{\theta}$ , profit equals :

$$\Pi_j^c = \frac{N(\sqrt{u_i + u_j + \theta(p_i - c_j)} - u_i)^2}{\theta u_i} \quad (9.7)$$

When  $p_j \rightarrow +\infty$ , profit is

$$\Pi_j^{cc} = \frac{Nu_j}{\theta} \quad (9.8)$$

Comparison of profits leads to the following condition :

$$\Pi_j^c > \Pi_j^{cc} \Leftrightarrow p_i > \frac{2\sqrt{u_i u_j}}{\theta} + c_j \quad (9.9)$$

Moreover, it is possible to show that  $\frac{u_j}{u_i}(u_i + u_j) \leq 2\sqrt{u_i u_j} \leq u_i + u_j$ . This result allows us to establish prices best response functions of the firms, independently of the initial hierarchy of prices.

### 9.2.2 Expression of best response functions

*Best response function of i is :*

- if  $p_j > \frac{u_i(u_i + u_j)}{u_j \theta} + c_i$ , or  $p_j = \frac{u_i(u_i + u_j)}{u_j \theta} + c_i$  and  $u_i > u_j$ , then profit is concave in  $[0; p_j]$  and strictly decreasing in  $[p_j; +\infty[$ . In this case, best response function of i is

$$p_i^*(p_j) = \frac{u_i + u_j + \theta p_j - \sqrt{u_j(u_i + u_j + \theta(p_j - c_i))}}{\theta} \text{ which verifies } p_i \leq p_j. \text{ (case I1)}$$

- if  $\frac{u_i + u_j}{\theta} + c_i < p_j < \frac{u_i(u_i + u_j)}{u_j \theta} + c_i$  (which supposes that  $u_i > u_j$ ), then profit is strictly increasing in  $[0; p_j]$  and strictly decreasing in  $[p_j; +\infty[$ . In this case, best response of i is "on the kink" :  $p_i^*(p_j) = p_j$ . (Case I2)

- if  $p_j = \frac{u_i + u_j}{\theta} + c_i$  then profit is strictly increasing in  $[0; p_j]$  and constant in  $[p_j; +\infty[$ . Best response belongs to the interval  $p_i^*(p_j) \in [p_j; +\infty[$ . (Case I3)

- if  $p_j < \frac{u_i + u_j}{\theta} + c_i$ , then profit of i is strictly increasing in all the interval of definition and best response is  $p_i^*(p_j) \rightarrow +\infty$ . (Case I4)

*Best response function of j is :*

- if  $p_i > \frac{2\sqrt{u_i u_j}}{\theta} + c_j$ , or  $p_i = \frac{2\sqrt{u_i u_j}}{\theta} + c_j$  and  $u_i > u_j$ , then global maximum is in the interval  $[0; p_i]$  in which profit is concave and best response of j is

$$p_j^*(p_i) = \frac{u_i + u_j + \theta p_i - \sqrt{u_i(u_i + u_j + \theta(p_i - c_j))}}{\theta} \text{ which verifies } p_j \leq p_i. \text{ (case J1)}$$

- if  $p_i = \frac{2\sqrt{u_i u_j}}{\theta} + c_j$  and  $u_i = u_j$ , then profit is strictly increasing in  $[0; p_i]$  and constant in  $[p_i; +\infty[$  and the best response belongs to the interval  $p_j^*(p_i) \in [p_i; +\infty[$ . (case J2)

- if  $p_i < \frac{2\sqrt{u_i u_j}}{\theta} + c_j$ , then global maximum is reached in the interval  $[p_i; +\infty[$  in which profit is increasing and best response of j is  $p_j^*(p_i) \rightarrow +\infty$ . (Case J3)

### 9.3 Identification of equilibria

Nash price equilibrium is given by intersection of best response functions. Let us prove now that the equilibrium in  $p_i \geq p_j$  exists when conditions (4.3) and (4.4) hold.

A symmetric Nash equilibrium exists when  $p_i^*(p_j) = p_j$  and  $p_j^*(p_i) = p_i$ , what restricts the analysis to the cases I2 and I3 for i and J2 for j. Moreover, reaction function of j shows that the second condition may be verified if and only if  $u_i = u_j = u$  and  $p_i = \frac{2u}{\theta} + c_j$ . But when  $u_i = u_j$ , firm i will choose an identical price only if  $p_j = \frac{2u}{\theta} + c_i$ . The condition  $c_i = c_j = c$  is also necessary for the symmetric

equilibrium existence. But in this case,  $p_i^*(p_j) = \frac{2u}{\theta} + c$  is one of the bests responses of firm i. We have thus a Nash price equilibrium.

Let us now study the possibility of asymmetric price equilibria of the form  $p_i \neq p_j$ .

First, cases in which  $p_i \rightarrow +\infty$  and  $p_j \rightarrow +\infty$  cannot constitute equilibria. Indeed, if firm i chooses  $p_i \rightarrow +\infty$  then best response of j is given by:  $p_j^*(p_i) = \bar{p} = \frac{u_i + u_j + \theta p_i - \sqrt{u_i(u_i + u_j + \theta(p_i - c_j))}}{\theta}$ . Moreover  $\lim_{p_i \rightarrow +\infty} \bar{p} \rightarrow +\infty$ , what violates condition  $p_j < \frac{(u_i + u_j)}{\theta} + c_i$  under which firm i would choose this price  $p_i \rightarrow +\infty$  (demonstration is similar for  $p_j \rightarrow +\infty$ ). Cases I4 and J3 are thus eliminated from the analysis. Finally, the only possible equilibrium corresponding to the case J2 has already been revealed, which also exclude it of this analysis.

Consequently, if an asymmetric equilibrium exists, it verifies necessarily  $p_i > p_j$  and corresponds to the case J1 coupled with the case I3. And this last case implies that:  $p_j^c = \frac{(u_i + u_j)}{\theta} + c_i$ . For which value of  $p_i$  leads the reaction function of j to the choice of such a  $p_j$  ?

By using this reaction function, we find that  $p_i$  must verify  $\theta(p_i - c_i) = \sqrt{u_i(u_i + u_j + \theta(p_i - c_j))}$ . This equation may be rewritten as the following second degree polynomial :  $\theta^2 p_i^2 - \theta p_i(u_i + 2\theta c_i) - u_i(u_i + u_j - \theta c_j) + \theta^2 c_i^2$ . Only one root may verify  $p_i > p_j$  :  $p_i^c = \frac{u_i + \sqrt{\Delta}}{2\theta} + c_i$  with  $\Delta = u_i^2 + 4u_i(u_i + u_j + \theta(c_1 - c_2))$ .

But this price should also verify the inequality  $p_i \geq \frac{2\sqrt{u_i u_j}}{\theta} + c_j$  so that firm j really chooses  $p_j^c$ . And this condition is verified if and only if  $c_i - c_j \geq \frac{\sqrt{u_i u_j} - u_i}{\theta}$ . Prices are within intervals of definition of cases I3 and J1 and thus constitute a Nash equilibrium.

Proof of unicity may now be realized. Let us consider the first condition between  $u_1$  and  $u_2$  : there are three possible senses of hierarchy between these parameters.

- if  $u_1 > u_2$  then the only possible price equilibrium verifies  $p_1 > p_2$
- if  $u_2 > u_1$  then the only possible price equilibrium verifies  $p_2 > p_1$
- if  $u_2 = u_1$  then nature of equilibrium depends on the second necessary condition. When  $c_1 > c_2$ , the only possible price equilibrium verifies  $p_1 > p_2$ . Conversely, if  $c_2 > c_1$ , this equilibrium takes the form  $p_2 > p_1$ . Finally, if  $c_1 = c_2$ , an equilibrium with  $p_1 = p_2$  is reached.

In all these various cases, if equilibrium exists, then it is unique.