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THE ANIMAL SPIRITS HYPOTHESIS AND THE BENHABIB-FARMER CONDITION FOR INDETERMINACY[*]

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Abstract

This paper provides a self-contained review of the introduction of the animal spirits hypothesis into the infinite horizon optimal growth model. The analysis begins with an economic discussion of Pontryagin's maximum principles. Thereafter, I develop a version of the increasing-returns Benhabib-Farmer model by showing the possible sub-optimality of the central planner solution and deriving the bifurcation condition for indeterminacy. Moreover, I give some insights on how to model intrinsic and extrinsic uncertainty. Finally, analysing the equilibrium condition of the labour market, I provide an intuitive rationale for the mechanism that in this model might lead prophecies to be self-fulfilling.

Keywords: Maximum problems in continuous time; indeterminate equilibrium paths; self-fulfilling prophecies.

JEL Classification: C1; E21; E32.

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1. Introduction

The importance of beliefs and expectations in economics – together with the possibility of an involuntary unemployment equilibrium – is probably one of the most important elements of the Keynesian legacy. After the publication of the *General Theory* (1936), the term 'animal spirits' used by Keynes as a synonymous for the entrepreneurial urge to action, has been also widely exploited to describe situations in which self-validating expectations or beliefs are the main sources of expansions and/or depressions.[1]

In spite of the old and ample evidence about the influence of subjective factors on economic outcomes, early writers – including Keynes – did not develop dynamic models in which realized outcomes were related to agents' expectations.[2] Only the rational expectations (RE) revolution of the 70s provided a straightforward way to endogenize beliefs. Specifically, macroeconomists of RE assumed that agents' expectations were essentially the same as the predictions of the relevant economic theory (e.g. Muth 1961).

Dynamic RE models were developed by using two distinct frameworks, i.e. the overlapping generations (OLG) model and the infinite horizon (IH) model. The former assumes that there is an infinite set of agents each of whom lives for a finite number of periods (usually two or three, e.g. Samuelson 1958). By contrast, the latter assumes a finite number of agents (usually one) that live forever (e.g. Koopmans 1965). In both kinds of framework, in spite of the respective distinctions, it is possible to show that if the RE equilibrium path is unique, then expectations must be univocally determined by technology, preferences and endowments. As a consequence, dynamic RE models with a unique equilibrium path do not allow for self-fulfilling prophecies.

At the beginning of the 80s, relaxing the hypothesis of RE equilibrium path uniqueness, both OLG and IH models have also been used to develop the idea that animal spirits might exert independent influence on economic activity. However, taking into account the difficulties to synchronize the OLG framework with the average period of business cycles, IH models are now considered the favoured candidates to explain how 'extrinsic uncertainty', i.e. random phenomena that do not affect fundamentals sometimes referred as 'sunspot activity' (e.g. Woodford 1994),

[1] However, Keynes (1936) did not use animal spirits to mean self-fulfilling beliefs; instead his view of uncertainty was closer to Knight's (1921) concept of an event for which there is too little information to make a frequentist statement about probabilities.

[2] Azariadis (1981) recalls the Dutch 'tulip mania', the South Sea bubble in England and the collapse of the Mississippi Company as three well-documented cases of speculative price movements which historians consider unwarranted by 'objective' conditions. More recently, it is worth reminding the bubble of dot-com financial assets and the bubble of US house prices (e.g. Shiller 2005).

might spark fluctuations in which prices or quantities change simply because are they expected to and price signals convey no structural information.

This paper provides a self-contained review of the introduction of the animal spirits hypothesis into the IH optimal growth model by updating the 'classical' contributions by Dorfman (1969) and Shell (1969) that do not consider the issue of extrinsic uncertainty. My analysis begins with an economic discussion of Pontryagin's maximum principles aimed at stressing the importance of the Hamiltonian function concavity and the role of the transversality conditions in defining the sufficient first-order conditions (FOCs) for a maximum problem in discrete and continuous time. Thereafter, I develop a version of the Benhabib-Farmer (1994) increasing returns model – which is a prime example inside the literature of IH models with self-fulfilling prophecies – by showing the problematic nature of the social planner solution and the optimality of the decentralized 'symmetric' market-clearing equilibrium. To the best of my knowledge, formal proofs for those results are missing in the literature of indeterminacy. Moreover, after the derivation of the bifurcation condition for indeterminacy, I give some theoretical insights on how to model intrinsic (or fundamental) and extrinsic uncertainty. Finally, analyzing the spot equilibrium condition of the labour market, I provide an intuitive rationale for the mechanism that in this model economy might lead prophecies to be self-validating.

The paper is arranged as follows. Section 2 derives some basic principles for the solution of maximum problems in discrete and continuous time. Section 3 develops a version of the Benhabib-Farmer (1994) model. Section 4 provides a rationale for the mechanism that allows prophecies to be self-fulfilling. Finally, section 5 concludes.

2. Basic Principles on Optimal Control Problems

In this section I provide a quick economic review of Pontryagin's maximum principle that will be useful to solve the Benhabib-Farmer (1994) model (e.g. Pontryagin *et al.* 1962). Specifically, starting from the easiest discrete case, I derive the necessary – and sometimes sufficient – conditions for the solution of a maximum problem defined over a finite and an infinite horizon.

2.1 *Finite-Horizon*

Suppose to be interested in the solution of the following discrete-time finite-horizon maximum problem:

$$\max_{\{u_t, x_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \left(\frac{1}{1+\rho} \right)^t J(u_t, x_t) \qquad \rho > 0 \tag{1}$$

s.to

$$x_{t+1} - x_t = f(u_t, x_t)$$
$$x_0 = \bar{x} > 0 \tag{2}$$

Interpreting ρ as a discount rate, i.e. the psychological cost of having something after one period instead of having it at once, the problem outlined above suggests that I'm interested in finding a sequence $\{u_t, x_{t+1}\}_{t=0}^{T}$ that maximizes the discounted sum of different realizations of the instantaneous objective function $J(\cdot)$ subject to the dynamic constraints implied by (2). Thereafter, it immediately follows that the elements of $\{u_t, x_{t+1}\}_{t=0}^{T}$ do not have the same degree of freedom. Specifically, while the vector u_t is free, within limits, to be chosen (control variables), the evolution of x_t (state variables) is determined by $f(\cdot)$.

The problem in (1) and (2) can be solved by writing a Lagrangian. Hence,

$$L(\cdot) \equiv J(u_0, x_0) - \lambda_0(x_1 - x_0 - f(u_0, x_0)) + \frac{1}{1+\rho}(J(u_1, x_1) - \lambda_1(x_2 - x_1 - f(u_1, x_1)))...$$
$$+\left(\frac{1}{1+\rho}\right)^T (J(u_T, x_T) - \lambda_T(x_{T+1} - x_T - f(u_T, x_T))) \tag{3}$$

where λ is a vector of Lagrange multipliers.

Standard results on duality suggest that λ can be interpreted as a system of implicit prices that defines the values of the marginal contribution to $J(\cdot)$ of a variation in x (e.g. Mas-Colell et al. 1995).

Assuming that $L(\cdot)$ is concave, the sufficient FOCs for a maximum take the following form:

$$\frac{\partial L(\cdot)}{\partial u_t^*} = 0 \Leftrightarrow \frac{\partial J(\cdot)}{\partial u_t^*} + \lambda_t^* \frac{\partial f(\cdot)}{\partial u_t^*} = 0 \qquad t = 0,...,T \tag{4}$$

$$\frac{\partial L(\cdot)}{\partial x_t^*} = 0 \Leftrightarrow \lambda_t^* - \lambda_{t-1}^* = \rho\lambda_{t-1}^* - \frac{\partial J(\cdot)}{\partial x_t^*} - \lambda_t^* \frac{\partial f(\cdot)}{\partial x_t^*} = 0 \qquad t = 0,...,T \tag{5}$$

$$\frac{\partial L(\cdot)}{\partial \lambda_t^*} = 0 \Leftrightarrow x_{t+1}^* - x_t^* = f(x_t^*, u_t^*) \qquad t = 0,...,T \tag{6}$$

where $\{u_t^*, x_{t+1}^*\}_{t=0}^{T}$ is optimal sequence that solves the dynamic problem in (1) and (2).

Now suppose to define the following auxiliary or Hamiltonian function:

$$H_t(\cdot) \equiv J(x_t, u_t) + \lambda_t f(u_t, x_t) \tag{7}$$

In each period, the Hamiltonian is defined as the sum of the instantaneous objective function and the accrual of x evaluated at its marginal worth. Therefore, $H_t(\cdot)$ measures the total contribution of the activities going on at time t, including both the direct contribution to the summation in (1) and the value of x accrued in t (e.g. Dorfman 1969).

As far as (7) is taken into account, the FOCs in (4)-(6) can be re-arranged as follows:

$$\frac{\partial H_t(\cdot)}{\partial u_t^*} = 0 \qquad t = 0,...,T \tag{8}$$

$$-\left(\lambda_t^* - \lambda_{t-1}^*\right) = \frac{\partial H_t(\cdot)}{\partial x_t^*} - \rho\lambda_{t-1}^* \qquad t = 0,...,T \tag{9}$$

$$x_{t+1}^* - x_t^* = \frac{\partial H_t(\cdot)}{\partial \lambda_t^*} = f\left(u_t^*, x_t^*\right) \qquad t = 0,...,T \tag{10}$$

The expressions in (8)-(10) are the core of the so-called 'maximum principle' and they are suited for a straightforward interpretation. First, (8) suggests to maximize the Hamiltonian with respect to control variables. Obviously, this can be done by differentiating $H_t(\cdot)$ with respect to u_t and equating the partial derivatives to zero at any time. As stated above, a necessary and sufficient condition for this procedure to detect a proper maximum requires the Hamiltonian concavity, i.e. the Hessian matrix of $H_t(\cdot)$ evaluated along the optimal sequence has to be semidefinite (or definite for strictly concavity) negative. Second, (9) states that along the optimal sequence, the decrease in the marginal value of a variation in x is given by the marginal contribution of the control variables to the Hamiltonian net of the yield on the marginal value of the accrual in x. In other words, at any time, the marginal value of the accrual in the control variables flows into the marginal contribution of the control variables net of their psychological cost. Such a reading of the FOC on x_t in terms of an asset equation is provided by Dorfman (1969). Finally, (10) replicates the original set of difference equations for the state variables.

A special importance is attached to the FOC for the choice of the final value of x, i.e. x_{T+1}. Its expression can be conveyed as

$$\frac{\partial L(\cdot)}{\partial x_{T+1}} = 0 \Leftrightarrow \left(\frac{1}{1+\rho}\right)^T \lambda_T = 0 \text{ or } \lambda_T = 0 \tag{11}$$

The expression in (11) is known as 'transversality condition' and states that the actual value of the marginal evaluation of the control variables accrual at the end of the period has to be zero. This simple closing condition avoids the dynamic inefficiency of having some 'left over' value at the end of the period. Implicitly, being finite the time horizon, this means that there could be a post-planning period to worry about (e.g. Shell 1969, 1971).

The arguments put forward above suggest that in order to find the trajectory that solves the continuous-time finite-horizon maximum problem given by

$$\max_{\{u(t),x(t)\}_{t=0}^T} \int_0^T \exp(-\rho t) J\left(u(t), x(t)\right) dt \qquad \rho > 0 \tag{12}$$

s.to

$$\dot{x}(t) = f(u(t), x(t))$$
$$x(0) = x_0 > 0 \tag{13}$$

one has to follow the following steps. First, define the Hamiltonian:

$$H(u, x, \lambda) \equiv J(u, x) + \lambda f(u, x) \tag{14}$$

Second, implement the continuous-counterpart of the FOCs in (8)-(10):

$$\frac{\partial H(x(t), u(t), \lambda(t))}{\partial u} = 0 \tag{15}$$

$$-\dot{\lambda}(t) = \frac{\partial H(x(t), u(t).\lambda(t))}{\partial x} - \rho\lambda(t) \tag{16}$$

$$\dot{x}(t) = \frac{\partial H(x(t), u(t), \lambda(t))}{\partial \lambda(t)} = f(u(t), x(t)) \tag{17}$$

Finally, supplement the FOCs in (15)-(17) with the following transversality condition:

$$\exp(-\rho T)\lambda(T) = 0 \text{ or } \lambda(T) = 0 \tag{18}$$

It is worth noting that the FOCs in (8)-(10) ((15)-(17)) can be re-arranged to derive a system of difference (differential) equations for control and state variables. In the continuous case, this can be easily done by differentiating (15) with respect to time and then using (16) to eliminate $\dot{\lambda}(t)$. In the discrete case, the derivation of the difference equations for u is a simple re-statement of the so-called Euler equation obtained by combining (4) and (5). Usually, there are infinite trajectories that satisfy this system of difference (differential) equations known as Pontryagin paths. Among this infinity, the trajectories that also satisfy the transversality condition in (11) ((18)) exactly define the solution of the maximum problem in (1) and (2) ((12) and (13)). Obviously, whenever there are many trajectories that fulfil those requirements, the problem admits multiple solutions. As it will be shown in section 3, multiple solutions might even exist whenever the dynamic system arising from the FOCs has a unique stationary solution.

2.2 *Infinite-Horizon*

Consider the discrete problem in (1) and (2) extended over an infinite horizon, i.e. $T \to +\infty$. The additional issues that might arise in an infinite-horizon problem concern the fact that the discounted sum of the realizations of $J(\cdot)$ can be unbounded. In this case, no optimum exists. However, for a number of problems developed in the optimal growth literature, a maximum problem extended over an infinite horizon can be solved by applying the FOCs from the finite case supplemented by a different transversality condition. Let's see how and why.

Suppose to define the following Lagrangian:

$$L(\cdot) \equiv \sum_{t=0}^{+\infty} \left(\frac{1}{1+\rho} \right)^t \left(J(u_t, x_t) - \lambda_t \left(x_{t+1} - x_t - f(u_t, x_t) \right) \right) \tag{19}$$

Let $\{\hat{u}_t, \hat{x}_t\}_{t=0}^{+\infty}$ be a candidate sequence for the optimal solution. Thereafter, evaluate the first-order Taylor expansion of $L(u_t, x_t)$ in the neighbourhood of $L(\hat{u}_t, \hat{x}_t)$, where $\{u_t, x_t\}_{t=0}^{+\infty}$ is an arbitrary sequence. Hence,

$$L(u_t, x_t) = L(\hat{u}_t, \hat{x}_t) + L_x(x_t - \hat{x}_t) + L_u(u_t - \hat{u}_t) + \Psi(u_t, x_t) \tag{20}$$

where $L_x = \left. \frac{\partial L(\cdot)}{\partial x} \right|_{\{\hat{u}_t, \hat{x}_t\}_{t=0}^{+\infty}}$, $L_u = \left. \frac{\partial L(\cdot)}{\partial u} \right|_{\{\hat{u}_t, \hat{x}_t\}_{t=0}^{+\infty}}$ while $\Psi(\cdot)$ is an error term.

Consider the 3rd addend on the RHS of (20), i.e. $L_u(u_t - \hat{u}_t)$. This term is given by

$$\sum_{t=0}^{+\infty} \left(\frac{1}{1+\rho} \right)^t \frac{\partial H_t(\cdot)}{\partial u_t} (u_t + \hat{u}_t) \qquad t = 0, \ldots, +\infty \tag{21}$$

If (8) is assumed to hold from 0 to infinity, then the 2nd term in (21) is equal to zero for all t. Therefore,

$$L_u(u_t - \hat{u}_t) = 0 \qquad t = 0, \ldots, +\infty \tag{22}$$

Some difficulties might arise with the 2nd addend on the RHS of (20), i.e. $L_x(x_t - \hat{x}_t)$. In the finite case, this term would be equal to

$$\sum_{t=0}^{T-1} \left(\frac{1}{1+\rho} \right)^t \left(\lambda_t - \lambda_{t-1} + \frac{\partial H_t(\cdot)}{\partial x_t} - \rho \lambda_{t-1} \right) (x_t - \hat{x}_t) - \left(\frac{1}{1+\rho} \right)^T \lambda_T (x_t - \hat{x}_t) \tag{23}$$

In the finite-horizon problem, (9) and (11) guarantee that the expression in (23) is equal to zero. Therefore, if $L(\cdot)$ in (19) is concave, then the FOCs in (8)-(11) are sufficient for a maximum. However, in the infinite-horizon case, it is not enough to implement the transversality condition in (11), since x_t may be growing (or declining) too fast.

For problems involving a positive discount rate, the transversality condition

$$\lim_{t \to +\infty} \left(\frac{1}{1+\rho} \right)^t \lambda_t x_t = 0 \tag{24}$$

is necessary and sufficient (e.g. Weitzman 1973).

The continuous-time counterpart of (24) is given by

$$\lim_{t \to +\infty} \exp(-\rho t) \lambda(t) x(t) = 0 \tag{25}$$

It is worth noting that (24) and (25) are not a simple extension, respectively, of (11) and (18) over an infinite-horizon. As suggested by Shell (1969), those transversality conditions express the desire of the maximizing agent to avoid ‘left over’ valuable assets (not only value).

Sometimes (24) and (25) have been interpreted as non-arbitrage conditions (no-Ponzi game conditions, e.g. Barro and Sala-i-Martin 2004); indeed, those transversality conditions point out that the asymptotic actual value of the state variables has to be zero in the feasible manifold (e.g. Shell 1969). Finally, the expressions in (24) and (25) suggest that explosive (or implosive) paths cannot be optimal.

3. Benhabib-Farmer Model in Continuous Time

The Benhabib-Farmer (2004) model investigates the properties of the one-sector IH growth model under the assumption of increasing returns to scale. The possibility of aggregate increasing returns is the guile that allows to break-down the path uniqueness of the RE equilibrium and is reconciled with the competitive behaviour of agents by using two distinct organizational structures, i.e. input externalities and monopolistic competition. On the one hand, the version with input externalities allows for the possibility that in a 'symmetric equilibrium' the social technology might display increasing returns to scale. On the other hand, the version with monopolistic competition proposes a framework similar to the models developed by Dixit and Stiglitz (1977) in order to explore the relationship between market and optimal resource allocation in the presence of non-convexities.

Since the dynamic implications of each organizational structure are qualitatively similar, here I develop a simple continuous version with input externalities by showing the possible sub-optimality of a trajectory chosen by an omniscient social planner and the optimality of the 'symmetric' decentralized market-clearing solution, i.e. a situation in which all the agents take the same actions and in all the markets demand equals supply. Moreover, after the derivation of the bifurcation condition for indeterminacy, I give some insights on how to model intrinsic (or fundamental) and extrinsic uncertainty.

3.1 *Building Blocks*

As in the standard one-sector growth model, the Benhabib-Farmer (1994) model assumes that an infinitely-lived representative agent is called in to choose consumption (C) and employment (L) under the constraint implied by a capital (K) accumulation law. Whenever it does not harm the clarity of the exposition, in the remainder of the paper I will omit the functional dependence of the variables on time. Thereafter, following the notation introduced in section 2, the maximum problem to be solved can be conveyed as

$$\max_{\{C,L,K\}_{t=0}^{+\infty}} \int_0^{+\infty} \exp(-\rho t) J(C,L) dt \qquad \rho > 0 \tag{26}$$

s.to

$$\begin{aligned} \dot{K} &= f(C, L, K) \\ K(0) &= K_0 > 0 \end{aligned} \tag{27}$$

The instantaneous objective function is assumed to take the following form:

$$J(C, L) \equiv \log C - \frac{1}{1+\gamma} L^{1+\gamma} \qquad 0 \leq L \leq L_{\max} \tag{28}$$

where $\gamma \geq -1$ is a measure of the labour supply elasticity while $L_{\max}$ is the maximum amount of labour services that agents would be willing to supply.

The expression in (28) deserves a short comment. First, as it will become apparent later on, Whenever separability between consumption and employment is combined with a Cobb-Douglas production function, the use of a logarithmic utility function over consumption is the only formulation of preferences consistent with a stationary labour supply in a growing economy. Second, the state variable K does not enter directly the instantaneous objective function.

The capital accumulation law for the overall economy takes the usual form. Hence,

$$f(C, L, K) \equiv Y - C - \delta K \qquad 0 < \delta < 1 \tag{29}$$

where δ is the capital depreciation rate.

The dependence of $f(\cdot)$ on both control and state variables occurs through the production function; indeed, aggregate output Y is given by

$$Y = AK^{\alpha} L^{\beta} \qquad \alpha + \beta \geq 1 \tag{30}$$

where A is a common-knowledge productivity shock, i.e. Solow's (1957) residual, while the inequality on the RHS allows for the possibility of increasing returns to scale at the social level. In other words, (30) explains how aggregate output responds whenever all the agents simultaneously expand their use of inputs .

I distinguish the individual problem from the aggregate problem by using an externality argument. Specifically, I assume that the individual technology Y_i is given by a constant-returns-to-scale Cobb-Douglas function. Hence,

$$Y_i = \phi(\cdot) K_i^a L_i^b \qquad a + b = 1 \tag{31}$$

where $\phi(\cdot) \equiv A(\overline{K})^{\alpha - a} (\overline{L})^{\beta - b}$ is a productivity parameter taken as given by the i-th agent and $i \in [0,1]$ is an index for a continuum of identical agents. The terms with the upper bar represent, respectively, the aggregate stock of capital and the aggregate labour input.[3] In other words, I assume that the

[3] It is worth noting that whenever there prevails a symmetric equilibrium, i.e. whenever $K_i = \overline{K} = K$ and $L_i = \overline{L} = L$, (31) reduces to (30).

productivity of an individual firm's inputs is affected by an externality factor defined by the aggregate level of utilization of the same inputs.

3.2 *Centralized Solution*

Suppose that an omniscient social planner is called in to choose the optimal path for the model economy. This planner will try to maximize the discounted sum of the individual welfare in (28) by taking into account that the aggregate technology might be subject to increasing returns to scale.

In order to solve the social problem, I define an Hamiltonian as I did in (14). Taking into account (28) and (29) it is possible to derive

$$H(C, L, K) \equiv \log C - \frac{1}{1+\gamma} L^{1+\gamma} + \lambda f(C, L, K) \tag{33}$$

where λ can be interpreted as the social marginal value of a capital variation.

Implementing (15)-(17), the FOCs for the social maximum problem become

$$\begin{pmatrix} \frac{1}{C} - \lambda \\ L^{\gamma} - \lambda\beta \frac{Y}{L} \end{pmatrix} = 0 \tag{34}$$

$$-\dot{\lambda} = \lambda\alpha \frac{Y}{K} - (\rho + \delta)\lambda \tag{35}$$

$$\dot{K} = Y - C - \delta K \tag{36}$$

In section 2 I stressed that the conditions in (34)-(36) are sufficient for a maximum if and only if $H(\cdot)$ is concave.[4] Simple algebra suggests that this is the case whenever

$$\beta - 1 - \gamma \le 0 \tag{37}$$

The proof is given in Appendix.

As long as aggregate technology displays increasing returns to scale, i.e. $\alpha + \beta > 1$, there is no certainty that (37) is verified. Specifically, this inequality may fail to hold in the case of increasing returns with respect to labour. Moreover, the more rigid the labour supply, the smaller the region in which $H(\cdot)$ is concave. In the limit, if the labour supply is inelastic, i.e. $\gamma \to -1$, and $1 - \alpha > 0$, then there is no value of β such that $H(\cdot)$ is concave. Therefore, even if stable, the social planner's solution couldn't be optimal; indeed, a divergent solution would always violate the 'centralized' transversality condition, i.e. $\lim_{t \to +\infty} \exp(-\rho t)\lambda(t)K(t) = 0$. Specifically, whenever (37) actually fails to hold, the social planner would be better off by pushing labour supply towards $L_{\max}$ settling in a corner solution.

[4] It is worth noting that in the optimal-growth model the marginal value of a capital variation flows into the marginal contribution of the state to the Hamiltonian net of its psychological cost plus depreciation.

3.3 *Decentralized Solution*

A decentralized solution can be derived by re-formulating the maximum problem as follows. Let L_i^S and L_i^D be, respectively, the labour supply and demand of the i-th representative agent. Moreover, let K_i^S and K_i^D be, respectively, its supply and demand for capital. The i-th agent sells L_i^S units of labour to the market and accumulates K_i^S units of capital that it rents out to other agents. Simultaneously, the i-th agent demands L_i^D units of labour and rents K_i^D units of capital from others for the use in its own firm. As a consequence, in the decentralized solution, capital demand (supply) becomes a control (state) variable.

Under these circumstances, the capital accumulation law for the i-th agent is given by

$$f_D\left(L_i^D, L_i^S, K_i^S, K_i^D\right) \equiv \phi(\cdot)\left(K_i^D\right)^a \left(L_i^D\right)^b - C_i - \delta K_i^S + w\left(L_i^S - L_i^D\right) + r\left(K_i^S - K_i^D\right) \qquad a + b = 1 \qquad (38)$$

where C_i is its consumption, w is the real wage and r is the real rental rate.[5]

Taking into account (28) and (38), the Hamiltonian for the decentralized problem is the following:

$$H_D\left(L_i^S, L_i^D, K_i^S, K_i^D\right) \equiv \log C_i - \frac{1}{1+\gamma}\left(L_i^S\right)^{1+\gamma} + \lambda_i f_D\left(L_i^S, L_i^D, K_i^S, K_i^D\right) \qquad (39)$$

where λ_i is the private marginal value of a capital variation.

Implementing (15)-(17), the FOCs for the decentralized problem become

$$\begin{pmatrix} \frac{1}{C_i} - \lambda_i \\ \left(L_i^S\right)^\gamma - \lambda_i w \\ \phi(\cdot) b\left(K_i^D\right)^a \left(L_i^D\right)^{b-1} - \lambda_i w \\ \phi(\cdot) a\left(K_i^D\right)^{a-1} \left(L_i^D\right)^b - \lambda_i r \end{pmatrix} = 0 \qquad (40)$$

$$-\dot{\lambda}_i = \lambda_i r - (\rho + \delta)\lambda_i \qquad (41)$$

$$\dot{K}_i^S = Y_i^S - C_i - \delta K_i^S + w\left(L_i^S - L_i^D\right) + r\left(K_i^S - K_i^D\right) \qquad (42)$$

where $Y_i^S \equiv \phi(\cdot)\left(K_i^D\right)^a \left(L_i^D\right)^b$.

Moreover, the FOCs in (40)-(42) have to be supported by the following transversality condition:

$$\lim_{t \to +\infty} \exp(-\rho t)\lambda_i(t) K_i^S(t) = 0 \qquad (43)$$

[5] In a market-clearing equilibrium $L_i^S = L_i^D$ and $K_i^S = K_i^D$, for all i.

Simple algebra suggests that in a symmetric market-clearing equilibrium, i.e. whenever $L_i^S = L_i^D = \overline{L} = L$ and $K_i^S = K_i^D = \overline{K} = K$, for all i, $H_D(\cdot)$ is strictly concave.[6] The proof is given in Appendix. Therefore, the symmetric decentralized solution described by (40)-(43) is optimal. However, taking into account the existence of the externality factor in (31), such a solution is not necessarily Pareto-efficient (e.g. Kehoe et al. 1992).

3.4 *Labour Market*

Now I show how the FOC in (40) entails the continuous spot equilibrium in the labour market. In this model economy, as it will be shown later on, the labour market equilibrium is the key to understand why prophecies might be self-fulfilling.

Manipulating the elements in the first-three rows of (40) leads to

$$C_i \left(L_i^S\right)^{\gamma} = \phi(\cdot) b \left(K_i^D\right)^a \left(L_i^D\right)^{b-1} \tag{44}$$

The LHS of (44) is the labour supply schedule of a single agent while the RHS is the corresponding labour demand. Taking the logs of each member it is possible to derive

$$c_i + \gamma l_i^S = \log\phi(\cdot) + \log b + a k_i^D + (b-1) l_i^D \tag{45}$$

where $c_i \equiv \log C_i$, $k_i^D \equiv \log K_i^D$, $l_i^S \equiv \log L_i^S$ and $l_i^D \equiv \log L_i^D$.

Obviously, the intersection of labour demand and supply provides the (log) equilibrium employment ($\log \hat{L}$) and the (log) equilibrium real wage ($\log \hat{w}$). See figure 1.

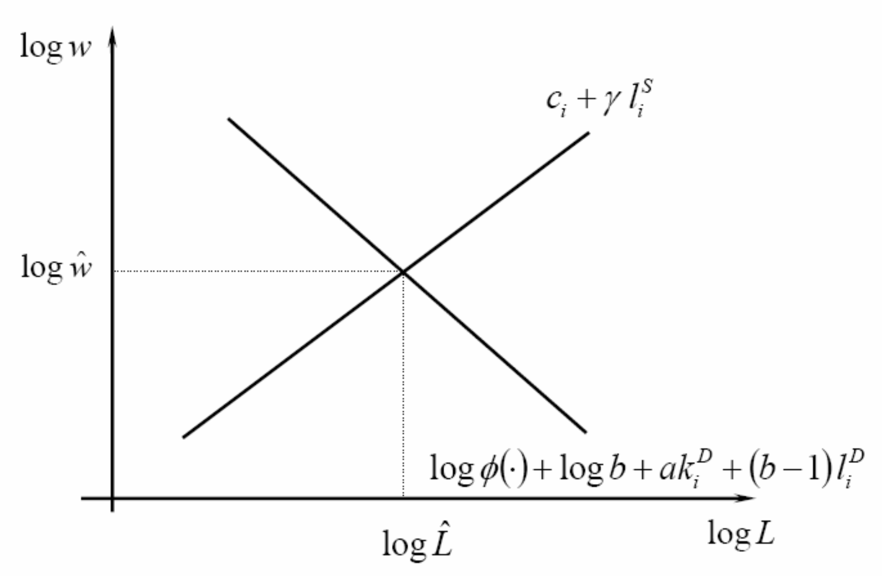

Figure 1: Labour market at the individual level.

[6] It is worth noting that in a decentralized symmetric equilibrium $\lambda_i = \overline{\lambda} = \lambda$, for all i.

Implementing the conditions for a symmetric market-clearing equilibrium, i.e. $L_i^S = L_i^D = \overline{L} = L$ and $K_i^S = K_i^D = \overline{K} = K$, so that $c_i = c \equiv \log C$, for all i, it is possible to derive

$$c + \gamma l = \log A + \log \beta + \alpha k + (\beta - 1) l \tag{46}$$

where $l \equiv \log L$ while $k \equiv \log K$.

The LHS of (46) is the economy-wide labour supply while the RHS is the corresponding labour demand accounting for externalities. Recall that I assumed the possibility of increasing returns at the aggregate level, i.e. $\alpha + \beta \geq 1$. Therefore, whenever $\beta > 1$, the aggregate demand for labour is upward sloped, a possibility theoretically stressed inter alia by Hall (1991). See figure 2.

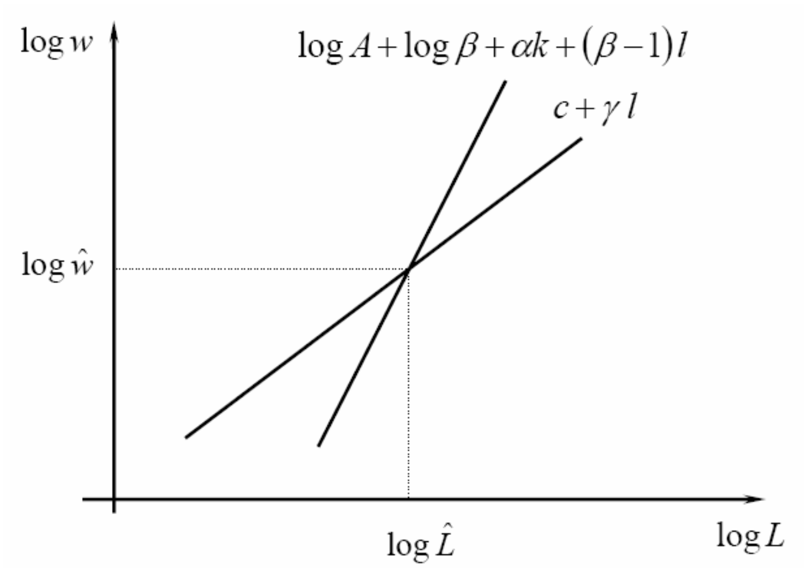

Figure 2: Economy-wide labour market.

Finally, it is worth remarking that for given values of c and k, the equilibrium condition of the labour market provides the corresponding equilibrium level of L. This suggests that I can only focus on one control. As a consequence, following the prevalent approach in the optimal growth literature, the remainder of the analysis is developed in terms of the (log) level of consumption.

3.5 *Local Dynamics under the Decentralized Solution*

In a symmetric market-clearing equilibrium, i.e. whenever $L_i^S = L_i^D = \overline{L} = L$ and $K_i^S = K_i^D = \overline{K} = K$, for all i, the FOCs in (40)-(42) imply that

$$\frac{\dot{C}}{C} = a \frac{Y}{K} - (\rho + \delta) \tag{47}$$

$$\frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{K} - \delta \tag{48}$$

Let $y \equiv \log Y$. As a consequence, (47) and (48) become

$$\dot{c} = a\exp(y-k) - (\rho+\delta) \tag{49}$$

$$\dot{k} = \exp(y-k) - \exp(c-k) - \delta \tag{50}$$

In order to derive an autonomous system of differential equations I have to express y as a function of c and k. This becomes possible by combining the log-linearizations of (30) and an individual labour market-clearing equation. Hence,

$$y = \log A + \alpha k + \beta l \tag{51}$$

$$c + \gamma l = \log b + y - l \tag{52}$$

Putting together (51) and (52), the subsequent expression follows:

$$y - k = \Phi_0 + \Phi_1 k + \Phi_2 c \tag{53}$$

where $\Phi_0 \equiv -\frac{\beta(\log b + (1+\gamma)\log A)}{\beta - 1 - \gamma}$, $\Phi_1 \equiv \frac{(1+\gamma)(1-\alpha) - \beta}{\beta - 1 - \gamma}$ and $\Phi_2 \equiv \frac{\beta}{\beta - 1 - \gamma}$.

It is worth noting that only Φ_0 depends on productivity shocks.

Using the result in (53), it also becomes possible to write down the pair of (autonomous) differential equations that define Pontryagin paths. Hence,

$$\dot{c} = a\exp(\Phi_0 + \Phi_1 k + \Phi_2 c) - (\rho+\delta) \tag{54}$$

$$\dot{k} = \exp(\Phi_0 + \Phi_1 k + \Phi_2 c) - \exp(c-k) - \delta \tag{55}$$

Any trajectory $\{k(t), c(t)\}_{t=0}^{+\infty}$ that solves (54) and (55) subject to the initial condition $k(0) = \log(K_0)$ and the transversality condition in (43) is an equilibrium path for the model economy; indeed, k is pre-determined since $k(0)$ is given by the initial conditions of the economy (k_0) while $c(0)$ is free to be determined by agents' behaviour.

On the one hand, the unique steady-state solution of the dynamic system is the following:

$$\begin{pmatrix} c^* \\ k^* \end{pmatrix} = \begin{pmatrix} \dfrac{\Phi_1 \log\left(\dfrac{\rho + (1-a)\delta}{a}\right) + \log\left(\dfrac{\rho+\delta}{a}\right) - \Phi_0}{\Phi_1 + \Phi_2} \\ \dfrac{\log\left(\dfrac{\rho+\delta}{a}\right) - \Phi_2 \log\left(\dfrac{\rho + (1-a)\delta}{a}\right) - \Phi_0}{\Phi_1 + \Phi_2} \end{pmatrix} \tag{56}$$

On the other hand, the Taylor first-order approximation of (54) and (55) around (56) is given by

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{bmatrix} \Phi_2(\rho+\delta) & \Phi_1(\rho+\delta) \\ \dfrac{\Phi_2(\rho+\delta) - \rho - (1-a)\delta}{a} & \dfrac{\Phi_1(\rho+\delta) + \rho + (1-a)\delta}{a} \end{bmatrix} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix} \tag{57}$$

It is well-known that the trace ($\mathrm{TR}(M)$) of the $2x2$ Jacobian matrix (M) in (57) measures the sum of its two eigenvalues (r_1 and r_2) while the determinant ($\mathrm{DET}(M)$) measures their product. Moreover, it is also well-known that when a linear transformation of the original variables is concerned, the eigenvalues of M represent the slope of the phase diagram in the stationary solution $(c^* \quad k^*)$ (e.g. Gandolfo 1997). Taking those algebraic results into account, the expressions in (56) and (57) suggest that productivity shocks affect only $(c^* \quad k^*)$ but not its local dynamics; indeed, Φ_0 does not enter the Jacobian matrix M.

More detailed information about local dynamics can be found by analysing the expressions for $\mathrm{TR}(M)$ and $\mathrm{DET}(M)$. Taking the definitions of Φ_1 and Φ_2 into account, it is easy to show that

$$\mathrm{TR}(M) = \rho + \frac{(\rho+\delta)(1+\gamma)(a-\alpha)}{a(\beta-1-\gamma)} \tag{58}$$

$$\mathrm{DET}(M) = \frac{(\rho+\delta)(\rho+\delta(1-a))(1+\gamma)(1-\alpha)}{a(\beta-1-\gamma)} \tag{59}$$

As a consequence of (59), it follows that

$$\mathrm{SGN}[\mathrm{DET}(M)] = \mathrm{SGN}\left[\frac{1-\alpha}{\beta-1-\gamma}\right] \tag{60}$$

On the one hand, in an economy without externalities, i.e. whenever

$$\alpha = a < 1 \text{ and } \beta = b < 1 \tag{61}$$

it holds that $\mathrm{TR}(M) = \rho > 0$ and $\mathrm{SGN}[\mathrm{DET}(M)] < 0$.

In this case r_1 and r_2 are of opposite sign so that the steady-state is a saddle point. This means that there is a one-dimensional manifold in the (c, k) space with the property that trajectories beginning on this manifold converge to the steady-state while all the others diverge. In other words, given $k(0)$, there will be a unique $c(0)$ in the neighbourhood of the stationary solution that generates a trajectory converging to $(c^* \quad k^*)$. This value of $c(0)$ should be selected in order to satisfy the transversality condition in (43) and will place the system exactly on the stable branch of the saddle point $(c^* \quad k^*)$. Therefore, the RE equilibrium path in the neighbourhood of stationary solution is unique or ‘determinate’.

An economy in which the inequalities in (61) are verified is usually termed as a real business cycle (RBC) economy because fluctuations can be driven only by intrinsic uncertainty, i.e. shocks to fundamentals (e.g. Kydland and Prescott 1982). This result would be also obtained from a central planner solution in which the production function is subject to constant returns to scale, i.e. $\alpha + \beta = 1$.

On the other hand, whenever externalities matter, i.e. whenever

$$\alpha > a \text{ and } \beta - 1 - \gamma > 0 \quad (62)$$

it holds that $\text{TR}(M) < 0$ and $\text{SGN}[\text{DET}(M)] > 0$.

This means that there are two negative eigenvalues so that the steady-state is a sink.[7] In other words, all the trajectories satisfying (54) and (55) which begin in the neighbourhood of $(c^* \quad k^*)$ converge back to the steady-state. As a consequence, given $k(0)$, there will be a continuum of equilibrium paths $\{k(t), c(t)\}_{t=0}^{+\infty}$ indexed by $c(0)$, since any path converging to $(c^* \quad k^*)$ necessarily satisfies the transversality condition in (43). Completely stable steady-states giving rise to a continuum of equilibrium paths are termed ‘indeterminate’ because all the trajectories are optimal in spite of the unique stationary solution.

The inequalities in (62) are known as the Benhabib-Farmer (1994) bifurcation condition for indeterminacy and they identify a so-called ‘animal spirits’ economy, i.e. an economy in which fundamentals are not able to pin down a unique RE equilibrium path. It is worth noting that the RHS of (62) is predisposed for a straightforward economic rationalization. Specifically, it simply states that – as in figure 2 – labour demand slopes up more than labour supply. Using a particular identification strategy, Farmer and Guo (1995) show that this unusual characterization of the labour marker can be consistent with US macro evidence over the period 1929-1988. Empirical findings on increasing returns to scale that go in the same direction are also found in Caballero and Lyons (1989) and Baxter and King (1991).

Incidentally, the local dynamics analysis also allows to state two additional interesting results. First, taking the expressions in (53) into account, (61) and (62) suggest that productivity shocks are pro-cyclical in a RBC economy without externalities and counter-cyclical in an ‘animal spirits’ economy. Second, suppose that the condition for the Hamiltonian concavity in the centralized problem is verified but $\alpha > 1$. In this case, i.e. whenever there are increasing returns with respect to capital, taking the result in (60) into account, both r_1 and r_2 would be positive so that the steady-state would be an instable source whose explosive dynamics is inconsistent with the required transversality condition.

[7] Whenever externalities matter it is not possible to rule out the possibility that r_1 and r_2 become complex. In this case, ρ and δ can be used as bifurcation parameters that allow to detect limit cycles (e.g. Benhabib and Farmer 1994).

3.6 *Modelling Intrinsic and Extrinsic Uncertainty: the RBC Economy versus the 'Animal Spirits' Economy*

In a RBC economy fluctuations are driven only by intrinsic (or fundamental) uncertainty. A simple way to model this kind of uncertainty is the definition of an Ornstein-Uhlenbeck process for the log of the productivity shock.[8] Hence,

$$\log \dot{A}(t) = \kappa(\theta - \log(A(t))) + \sigma_A \dot{\chi}(t) \quad \kappa > 0, \theta \geq 0 \text{ and } \sigma_A > 0 \tag{63}$$

where κ is the rate of mean reversion, θ is the long run mean of the process, σ_A is the instantaneous standard deviation and $\chi(t)$ is a standard Brownian motion with zero drift and unit variance.

Obviously, taking the results in (56) and (57) into account, the actual realization of $A(t)$ will alter the stationary solution of the model economy but it won't change its determined local dynamics. Without loss of generality, I assume that r_2 is the negative (convergent) eigenvalues of M. As a consequence, the RE path of a RBC economy is described by

$$\begin{pmatrix} c(t) \\ k(t) \end{pmatrix} = \begin{pmatrix} c^*(A(t)) \\ k^*(A(t)) \end{pmatrix} + \begin{pmatrix} \Omega_0 (k(t) - k^*(A(t))) \\ (k_0 - k^*(A(t)))\exp(r_2 t) \end{pmatrix} \tag{64}$$

where $\Omega_0 \equiv \dfrac{ar_2 - \Phi_2(\rho+\delta) - \rho - (1-a)\delta}{\Phi_1(\rho+\delta) + \rho + (1-a)\delta}$ while $A(t)$ follows the process in (63).

By contrast, in an 'animal spirits' economy fundamentals cannot pin down a unique RE equilibrium path. This path multiplicity can be solved by modelling extrinsic uncertainty, i.e. by explicitly defining a process for the beliefs of agents that is independent of preferences, endowments and technology. As suggested by Farmer and Guo (1994), a simple way to model a belief function is to augment the differential equation for c with any random variable $V(t)$ endowed with a zero conditional mean and finite variance. Moreover, in order to avoid the counter-factual pro-cyclicality of negative productivity shocks, the level of $A(t)$ is usually set to a constant value; indeed, Kamihigashi (1996) defines 'animal spirits' economies as economies with externalities and sunspots but without productivity shocks. As a consequence, the RE path of an 'animal spirits' economy is described by

$$\begin{pmatrix} c(t) \\ k(t) \end{pmatrix} = \begin{pmatrix} c^* \\ k^* \end{pmatrix} + \begin{bmatrix} \Omega_1 V(t) & \Omega_0 (k(t) - k^*) \\ c(t) - c^* & k_0 - k^* \end{bmatrix} \begin{pmatrix} \exp(r_1 t) \\ \exp(r_2 t) \end{pmatrix} \tag{65}$$

where $\Omega_1 \equiv \dfrac{r_1 - \Phi_2(\rho+\delta)}{\Phi_1(\rho+\delta)}$, $E[V(t)] = 0$, for all t, while $r_1 < 0$ and $r_2 < 0$.

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[8] Another conventional way to model intrinsic uncertainty is to define a stochastic process for a taste-shift in labour supply (e.g. Farmer and Guo 1994).

The actual realization of $V(t)$ is the guile that allows to solve the path multiplicity implied by the Benhabib-Farmer (1994) condition for indeterminacy in (62). In other words, among the infinite equilibrium paths, the realization of $V(t)$ selects in an erratic way the trajectory actually followed by an 'animal spirits' economy. This procedure of randomization over a set of infinite equilibrium paths is also found in OLG models with indeterminate equilibria (e.g. Farmer and Woodford 1997).

It is worth noting that this is a fairly rough-and-ready way to handle beliefs.[9] Indeed, Kamihigashi (1996) argues the observational equivalence between a RBC and an 'animal spirits' economy. Specifically, Kamihigashi (1996) shows that given a realization of $A(t)$ ($V(t)$) it is always possible to define a process for $V(t)$ ($A(t)$) such that the optimal path for a RBC ('animal spirits') economy is an optimal path also for an 'animal spirits' (RBC) economy. Taking into account of this result, it is clearly impossible on a pure empirical ground to assess whether a particular trajectory comes from a RBC or an 'animal spirits' economy. Moreover, as suggested by Boldrin and Rustichini (1994), if indeterminacy is taken seriously, then the interpretation of many simple empirical contributions obtained by pooling together data from different countries can be seriously questioned. Specifically, in case of indeterminacy, there is no reason to believe that countries should be moving along the same equilibrium saddle path.

4. Why are Prophecies Self-Fulfilling? A Discussion

It might seem quite striking that the conditions for indeterminacy should lead to precise implications for the labour market outlook. However, thinking at the inherent logic underlying the standard IH optimal-growth model, the reasons of such a link are straightforward. A suggestive explanation has been suggested by Aiyagari (1995).

Consider (45) and (46). The position of the labour demand schedule is fixed by the stock of capital. By contrast, the position of the labour supply schedule is fixed by the level of consumption. In the absence of shocks, a unique level of consumption determines a unique position for the labour supply schedule. This position, in turn, determines the unique equilibrium level of employment. Therefore, there will be a unique level of output and investment (from the resource constraint), which means a unique level of capital stock for the next period. See (29) and (38). From this argument, it is clear that the key to indeterminacy is that there can't be a unique position of the labour supply schedule, which means that there can't be a unique level of consumption.

[9] More interesting possibilities are explored, for example, by Branch and Evans (2006) who model agents that forecast the future by means of a least square algorithm.

Along the lines put forward in the introduction, optimistic (pessimistic) expectations might lead agents to spend more (less) in consumption. Obviously, this will shift their labour supply schedules. In order to have an equilibrium path driven by self-fulfilling beliefs, these shifts have to lead to labour, output and investment outcomes that ratify the original optimistic (pessimistic) expectations. How this might happen? Presumably, current income and expectations on future income are what influence consumption most. In order for the agents to consume more (less) initially, they have to be optimistic (pessimistic) either that current and future labour incomes will be high (low) or that current and future interest rates will be low (high). In the labour market depicted in figure 1, optimistic (pessimistic) expectations, i.e. positive (negative) realizations of $V(t)$, lead agents to consume more (less) pushing the labour supply to shift inward (outward). Obviously, this lowers (raises) the current level of employment. Thus, current output and investment are lowered (raised). Thereby, future capital stock and, hence, future employment, output, and so on, are lowered (raised). Moreover, future interest rates are raised (lowered) since the capital stock is lowered (raised). These outcomes are clearly inconsistent with the original optimistic (pessimistic) expectations.

The arguments above suggest a way in which optimist (pessimistic) expectations can be self-fulfilling. Consider the labour market in figure 2. In this case, optimistic (pessimistic) expectations will shift the labour supply inward (outward). This will raise (lower) employment and output. Raising (lowering) current output, optimistic (pessimistic) expectations can also raise (lower) the future capital stock and possibly lower (raise) interest rates. These effects are consistent with the higher (lower) initial consumption. Therefore, the original optimistic (pessimistic) expectations are self-validating.

5. Concluding Remarks and Directions for Further Research

This paper provided a self-contained review of the introduction of the animal spirits hypothesis into the IH optimal growth model by updating the ‘classical’ contributions by Dorfman (1969) and Shell (1969) that do not consider the issue of extrinsic uncertainty. The analysis begun with an economic discussion of Pontryagin’s maximum principle. Thereafter, I developed a version of the Benhabib-Farmer (1994) model by showing the problematic nature of a trajectory chosen by an omniscient social planner and the optimality of the ‘symmetric’ decentralized market-clearing equilibrium path. Moreover, after the derivation of the bifurcation condition for indeterminacy, I provided some insights on how to model intrinsic (or fundamental) and extrinsic uncertainty. Finally, analysing the spot equilibrium condition of the labour market, I provided an intuitive rationale for the possibility of self-validating equilibrium paths.

The Benhabib-Farmer (1994) condition for indeterminacy has been widely criticized. On a pure empirical ground, Basu and Fernald (1997) argued that the degree of increasing returns required to generate an upward sloping labour demand seems to be implausible when compared to circumstantial evidence. The macroeconomists of self-fulfilling prophecies tried to address this criticism by developing multi-sector models in which indeterminacy arises with constant returns to scale and very small market imperfections (e.g. Benhabib and Nishimura 1998). However, there is another important theoretical shortcoming in models with indeterminate equilibrium paths, i.e. the assumption of continuous equilibrium in the labour market. In other words, there is no role for non-voluntary unemployment in models that allow for self-fulfilling prophecies. Of course, there are some exceptions to this rule, each of them developed by exploiting the search approach to unemployment popularized by Pissarides (2000). Specifically, I refer to Burda and Weder (2002) and Giammarioli (2003). The former focuses on the complementarity between labour market institutions, the resulting equilibrium unemployment and the propagation of business cycles. The latter shows the possibility of an indeterminate equilibrium path whenever the social matching function displays a certain degree of increasing returns to scale with respect to vacancies.

It is well-known that search unemployment falls in the category of 'frictional' unemployment; indeed, in the matching framework, the responsive for unemployment is the absence of a mechanism (say a market) in which the decisions of workers and firms are efficiently coordinated. With the exception of some attempts in this direction (e.g. Nakajima 2005 and Guerrazzi 2008), the task of building models with indeterminate equilibrium paths and involuntary unemployment is still in progress.

A. Appendix

In this appendix I discuss the conditions for the concavity of the Hamiltonian functions in the centralized and decentralized solution of the Benhabib-Farmer (1994) model developed in section 3.

A.1 *Hamiltonian Concavity in the Centralized Solution*

The FOC in (34) suggests that the Hessian matrix of $H(\cdot)$ is given by

$$\begin{bmatrix} -\frac{1}{C^2} & 0 \\ 0 & -\gamma L^{\gamma-1} - \frac{1}{C}\beta(1-\beta)\frac{Y}{L^2} \end{bmatrix} \quad \text{(A.1)}$$

The matrix in (A.1) ($H(\cdot)$) is negative semidefinite (concave) whenever

$$\beta - 1 - \gamma \leq 0 \tag{A.2}$$

A.2 *Hamiltonian Concavity in the Decentralized Solution*

Considering the conditions for a symmetric market-clearing equilibrium, i.e. $L_i^S = L_i^D = \overline{L} = L$ and, $K_i^S = K_i^D = \overline{K} = K$ so that $C_i = C$, for all i, the FOC in (40) suggests that the Hessian matrix of $H_D(\cdot)$ is given by

$$\begin{bmatrix} -\frac{1}{C^2} & 0 & 0 & 0 \\ 0 & -b(1-b)AK^{\alpha}L^{\beta-2} & 0 & abAK^{\alpha-1}L^{\beta-1} \\ 0 & 0 & -\gamma L^{\gamma-1} & 0 \\ 0 & abAK^{\alpha-1}L^{\beta-1} & 0 & -a(1-a)AK^{\alpha-2}L^{\beta} \end{bmatrix}$$

(A.3)

The matrix in (A.3) ($H_D(\cdot)$) is negative definite (strictly concave) whenever

$$\frac{abL^{\gamma}A^2K^{2\alpha}L^{2\beta}}{K^2L^3C^3}(1-a-b) = 0 \Leftrightarrow a+b=1 \tag{A.4}$$

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