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Fathali Firoozi
Department of Economics
University of Texas at San Antonio

Department of Economics,
University of Texas at San Antonio,
San Antonio, TX 78249, U.S.A

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Fathali Firoozi*
Department of Economics
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ABSTRACT

Testing inequality hypotheses in econometric models has posed a challenge in terms of identifying an applicable null distribution. This study demonstrates an asymptotic boundary null distribution for testing inequalities and discusses some of the trade offs in terms of test errors.

JEL Codes: C1; C13

Keywords: Boundry distributors, Inequality hypotheses

1 Introduction

Consider the problem of testing joint linear inequalities of the form $R\beta \geq r$ defined on the parameter vector β of the linear model

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \Omega), \tag{1}$$

where X is an $(n \times k)$ matrix of exogenous variables, β is the $(k \times 1)$ vector of parameters, Ω is the $(n \times n)$ matrix of error covariances, R is an $(m \times k)$ matrix with $m \leq k$ and r is an $(m \times 1)$ vector. The classical approach formulates the problem as $H_0 : R\beta = r$ vs $H_1 : R\beta \geq r$ and applies the one-sided version of a standard procedure. As elaborated in the standard references such as Theil (1971), Anderson (1984), Lehmann (1986), and Greene (2008), there are well known and readily applicable F and χ^2 test procedures for this classical formulation where the null distribution is developed under the hypothesis $R\beta = r$. However, it is now well known that for testing $R\beta \geq r$, there is a gain in power when the problem is formulated in a one-sided where the test statistic and its null distribution involve the inequality hypothesis $R\beta \geq r$; see Gourieroux *et al.* (1982), Farebrother (1986), Kodde and Palm (1986), and Shapiro (1988).

The suggested procedures for the one-sided formulation are special cases of the one-sided procedures for the general one-sided problem $H_0 : \mu \in C$ vs $H_1 : \mu \in \mathbb{R}^m$, where C is a closed convex cone in $\mathbb{R}^m$ and μ is an m-vector mean of a multivariate normal density with the covariance matrix Σ. The seminal studies of Kudo (1963) and Perlman (1969) on the general one-sided testing problems in multivariate analysis provided the basis for the one-sided null testing of inequality hypotheses and suggested a number of alternative boundary

* Department of Economics, University of Texas at San Antonio, San Antonio, TX 78249, (210) 458-5395, Fax: (210)458-5837, e-mail: fathali.firoozi@utsa.edu. Extensions of this work are in progress. This work was supported in part by the College of Business summer research grant.

null distributions. Such boundary distributions have been adopted in a number of other contexts in the literature, including the studies stated above for testing $R\beta \geq r$. The central problem is that under the null hypothesis, the parameter μ could take any value from the unbounded set C, thus characterization of a null distribution is obscured. As will be briefly discussed here, the most accurate of the null distributions suggested in the literature is defined on a boundary of C as a weighted sum of χ^2 variables where the weights sample dependent.

A major difficulty in applying the one sided formulations is associated with computing the weights in the stated boundary distributions. Such a difficulty continues to hinder routine uses of the one sided formulations. The literature has provided some approximations. In this study we review some of the suggested approximations, provide a theoretical derivation of an applicable approximation, and discuss some of the trade offs in relation to test errors.

2 Boundary distributions

Consider the problem of testing $H_0 : R\beta \geq r$ vs $H_1 : \beta \in \mathbb{R}^k$ on model (1) where the unconstrained maximum likelihood (ML) estimator of the parameter vector β is $\hat{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$. In reducing the problem to the one-sided testing in multivariate analysis, premultiply model (1) by $R(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}$ and subtract r to generate

$$(R\hat{\beta} - r) = (R\beta - r) + \tau, \tag{2}$$

where $\tau = R(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\varepsilon$. The expectation and covariance matrix of the the $(m \times 1)$ vector τ are then $E(\tau) = 0$ and $Var(\tau) = R(X'\Omega^{-1}X)^{-1}R'$. Define $\mu = R\beta - r$, $\hat{\mu} = R\hat{\beta} - r$, and $\Sigma = R(X'\Omega^{-1}X)^{-1}R'$. Model (1) can then be written equivalently through (2) as

$$\hat{\mu} = \mu + \tau, \;\; \tau \sim N(0, \Sigma). \tag{3}$$

It is now clear that the problem of testing $H_0 : R\beta \geq r$ vs $H_1 : \beta \in \mathbb{R}^k$ on model (1) is equivalent to the problem of testing $H_0 : \mu \geq 0$ vs $H_1 : \mu \in \mathbb{R}^m$ on model (3), which is a special form of the one-sided testing problem in multivariate analysis.

The theoretical foundations for the stated one-sided problem are based on the seminal study of Perlman (1969). The general problem considered by Perlman is $H_0 : \mu \in C$ vs $H_1 : \mu \in \mathbb{R}^m$ where C is a closed convex cone in $\mathbb{R}^m$. In the present context, C is the positive orthant in $\mathbb{R}^m$, i.e., $C = \{\mu \in \mathbb{R}^m : \mu_i \geq 0$ for all $i = 1, \ldots, m\}$. We follow with a brief review of the testing procedure for model (3) and its connection to model (1). Let $\hat{\mu}$ be the ML estimator of μ. As shown by Perlman (1969), the likelihood ratio test statistic $U(C)$ is a function of C and is defined by the following norm in $\mathbb{R}^m$

$$U(C) = \| \hat{\mu} - \tilde{\mu} \|_{\Sigma}^2 = (\hat{\mu} - \tilde{\mu})'\Sigma^{-1}(\hat{\mu} - \tilde{\mu}), \tag{4}$$

where $\tilde{\mu}$ is the projection of $\hat{\mu}$ on C with respect to Σ, i.e., $\tilde{\mu}$ is the solution to the following optimization problem

$$\min_{\mu \in C} \ (\hat{\mu} - \mu)' \Sigma^{-1} (\hat{\mu} - \mu). \tag{5}$$

Since C is a closed and convex set, the solution $\tilde{\mu}$ exists and is unique. It is clear that $\tilde{\mu}$ is the projection of $\hat{\mu}$ on C, hence the statistic $U(C)$ is the distance between $\hat{\mu}$ and the set C. In relation to model (1), it is clear that $\hat{\mu} = R\hat{\beta} - r$ as shown above. Also, it can be shown that $\tilde{\mu} = R\tilde{\beta} - r$ where $\tilde{\beta}$ is the ML estimator of β in model (1) subject to the inequality constraint $R\beta \geq r$.

To implement the stated test $H_0 : \mu \in C$ vs $H_1 : \mu \in \mathbb{R}^m$, one requires a characterization of the distribution of the statistic U defined in (4) under the null hypothesis $H_0 : \mu \in C$. Given such a distribution and a type I error size α, the cut-off value c is the solution to

$$P[U \geq c \mid H_0 : \mu \in C] = \alpha, \tag{6}$$

where the probability measure P stated in (6) corresponds to the null distribution for the stated test procedure. The hypothesis $H_0 : \mu \in C$ is rejected if $U > c$. The main difficulty is associated with characterizing the null distribution represented by P in (6).

Specification of a null distribution for any test typically requires fixing a value for the parameter under the null hypothesis. Therefore, it is clear that the formulation $H_0 : \mu \in C$ raises a problem for specifying the null distribution in (6) above. The usual approach is to adopt a "conservative" approach by choosing the "least favorable" value of μ in C, that is, choosing the value $\mu \in C$ that yields the largest cut-off value c for a given type I error size α. This approach ensures that the true type I error of the test does not exceed the specified α. Thus, based on this approach, the cut-off value c is the solution to

$$\sup_{\mu \in C} P[U \geq c] = \alpha. \tag{7}$$

The stated studies have shown in various contexts that

$$\sup_{\mu \in C} P[U \geq c] = \sum_{i=0}^{m} w(m, m-i, \Sigma) \cdot P[\chi^2(i) \geq c], \tag{8}$$

where each weight $w(m, m-i, \Sigma)$ is the probability that the projection m-vector $\tilde{\mu}$ has exactly $(m-i)$ positive components where the sum of the weights over i is equal to one. Based on the result in (8) above and a type I error size α, the cut-off value c is the solution to

$$\sum_{i=0}^{m} w(m, m-i, \Sigma) \cdot P[\chi^2(i) \geq c] = \alpha. \tag{9}$$

The main difficulty in applying the distribution in (8) is computing the weights $w(m, m-i, \Sigma)$. These weights are sample-dependant thus vary from case to case. Bohrer and Chow (1978) elaborated on the complications involved in computing such weights. These complications in computing the weights have made distributions such as the one in (8) rather unusable in applications. However, it can be shown that an asymptotic upper bound approximation to the null distribution in (8) when the covariance matrix Σ is completely unknown is given by

$$\lim_{n\to\infty} \sup_{\mu\in C,\Sigma>0} P[U \geq c] = \frac{1}{2}P[\chi^2(m-1) \geq c] + \frac{1}{2}P[\chi^2(m) \geq c]. \tag{10}$$

According to this result, for a given size α, the cut-off value c for the test is the solution to

$$\frac{1}{2}P[\chi^2(m-1) \geq c] + \frac{1}{2}P[\chi^2(m) \geq c] = \alpha. \tag{11}$$

The main contribution of the approximation (10) relative to the distribution in (8) is the fact that computing the cut off value via (10) avoids the difficulties associated with computing the weights in (8).

3 Test errors

In the one-sided procedure elaborated above for testing $\mu \in C$, the derivation of a power function for the test involves specification of the distribution of U when μ belongs to the complement of the closed convex cone C in $\mathbb{R}^m$, i.e. when $\mu \in (\mathbb{R}^m \backslash C)$. Note that the set $(\mathbb{R}^m \backslash C)$ has properties that are substantially different from C. For instance, $(\mathbb{R}^m \backslash C)$ is neither closed nor convex, thus the boundary methods applied in deriving the boundary distributions will fail when applied in deriving a power function. However, certain general results can be shown regarding the test power when a true null distribution is replaced by a boundary approximation. As will be shown, there are trade offs in terms of test power when such approximations are used.

The main argument in this section is that when a true null distribution is replaced by an upper bound approximation, the test power declines. This argument intuitively follows from the primary motivation for adopting the upper bound distributions, which is, as stated in the previous section, to ensure that the true type I error of the test does not exceed the specified α. A more elaborate justification for the present argument can be constructed from the fact that $\sup P[U \geq c \mid \mu \in C] \geq P[U \geq c \mid \mu \in C]$.

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