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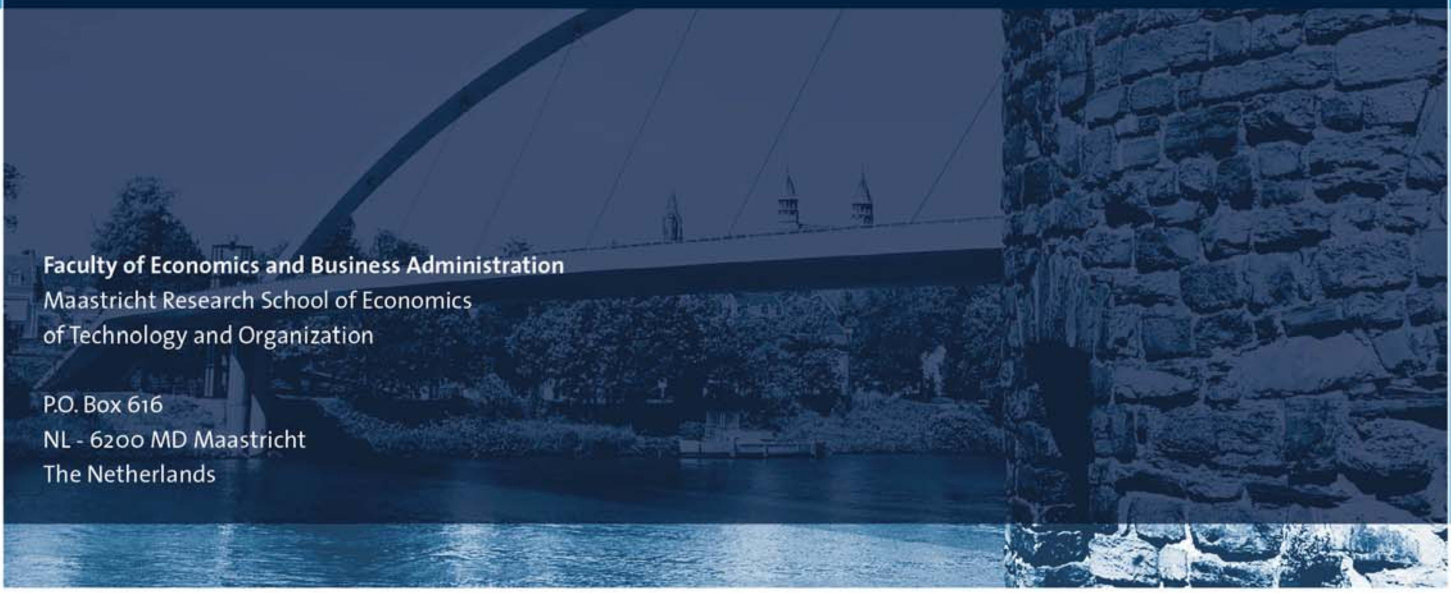

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Dynamic Duopoly with Intertemporal Capacity Constraints*

Anita H.J. van den Berg†, P. Jean-Jacques Herings‡and Hans J.M. Peters†

Abstract

We analyze strategic firm behavior in settings where the production stage is followed by several periods during which only sales take place. We analyze the dynamics of the market structure, the development of prices and sales over time, and the implications for profits and consumer surplus. Two specific settings are analyzed. In the first, a firm can commit up-front to a sales strategy that does not depend on the actual sales of its competitor. In this case there is a unique Nash equilibrium and price increases over time. In the second setting, there is no commitment and firms can adjust their sales in response to observed supply of their competitor in the previous period. It is shown that in this case a subgame perfect Nash equilibrium does not always exist. Equilibria can have surprising features. For some parameter constellations, price may decrease over time. It is also possible that the firm increases its profit by destroying some of its production. When firms have equal size, the equilibrium outcome is the same in both the commitment and the non-commitment setting. In general, the setting without commitment is beneficial to the larger firm, whereas the setting with commitment leads to higher profits for the smaller firm.
Keywords: Dynamic Duopoly, Cournot Competition, Multi-period Capacity Constraints, Commitment
JEL codes: D43, L13

1 Introduction

In most models of dynamic duopoly, it is assumed that production is instantly adjusted to per-period demand. However, in many real-world applications, this is not the case. Take, for instance, an airplane company that is selling seats on a flight, scheduled to take off in a month from now. In the plane, the number of seats is fixed, and is not adjusted to the realized demand. Seats are sold at several moments in time, until the month has elapsed. To maximize profits, the company has to take into account how selling a seat today influences the profits it can make on the remainder of the seats. Moreover, it will have to take into account how its actions today will affect the behavior of its competitors for the rest of the month.

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This paper analyzes competition in situations where production precedes sales and sales take place during a number of periods. As a result, the firm operates under a multi-period capacity constraint. Any production process that involves batch production would fit this description. Other examples concern settings with costly transportation, causing stores to be supplied only every few periods. Another relevant case can be found in the field of exhaustible resources. Firms at the source cannot renew their supply, but have many periods to sell the resource.

We address a number of questions related to the dynamics of the market structure, the development of prices and sales over time, and the implications for profits and consumer surplus. We examine the simplest situation possible: production or resource extraction has already taken place, the commodity is sold during two periods and demand is linear. Firms thereby effectively face a two-period capacity constraint.

In such a multi-period setting, it becomes relevant whether or not firms use current period outcomes before deciding upon their next period actions. We refer to these two possibilities as non-commitment versus commitment. Both the non-commitment and the commitment case are analyzed and related to one another. In the commitment setting, the strategy of a firm specifies the amount it is going to supply at each period. This amount does not depend on the observed sales of the competing firm in the previous periods. In the non-commitment setting, the strategy of a firm describes how much stock to sell in each period, conditional on observed sales in previous periods by the competing firm. We will show that the level of commitment can have a serious influence on the results.

In the commitment case, firms base their plan of action only on the level of initial stock of both the firms. This case is the easiest one to analyze and in the exhaustible resource literature, this has been done so for numerous settings similar to ours. This literature starts with Hotelling (1931). More recently, Loury (1986), Gaudet and Van Long (1994) and Schmalensee (1980), all find results that, basically, coincide with the results we find for the commitment setting. We establish the existence of a unique Nash equilibrium. It is shown that, in this equilibrium, price increases over time and as a consequence, aggregate sales decrease over time. Aggregate sales per period depend on the distribution of initial production over the firms. Also, the firm with more stock will never leave the market before the smaller one does.

In the setting without commitment, a firm's supply is conditional on the amounts sold in the previous period. This makes it possible to adjust the sales path over time in response to observed sales by the competitor. Surprisingly, in the non-commitment setting, a subgame perfect Nash equilibrium does not always exist. However, if an equilibrium exists, it is again unique. In equilibrium, the firm with the larger initial production amount will never leave the market before the smaller firm. Equilibria in the non-commitment setting may exhibit counterintuitive features. For instance, price may decrease over time and therefore aggregate sales may increase over time. In the exhaustible resource literature, Salo and Tahvonen (2001) also analyze a non-commitment setting. To the best of our knowledge, this is the only article in this area. Their model, however, differs a lot from ours, which makes comparing results pointless.

Apart from the literature on exhaustible resources, this paper is related to those papers that analyze models with capacity constraints. Since Edgeworth (1925), it has been widely known that per-period capacity constraints can greatly influence competition. Most of these papers, for instance Levitan and Shubik (1972) and

Osborne and Pitchik (1986), use a static setting in which firms compete in price. More recently, several papers were written in which firms compete in quantity and are constrained in capacity. In Gabszewicz and Poddar (1997), firms choose their level of capacity before demand is known. After true consumers' demand is known, they compete in quantity for one period. It is shown that a symmetric subgame perfect Nash equilibrium exists. Laye and Laye (2008) analyze multi-market Cournot competition with capacity constraints. All firms can produce a limited amount of a homogeneous product. For this product they have to choose which part they will sell at every market. In this situation, a unique Cournot-Nash equilibrium exists.

To the best of our knowledge, the only other paper in the literature that uses an intertemporal capacity constraint, is Biglaiser and Vettas (2004). In their model, the two competing firms have an equal finite amount of product that they can sell in two periods. Demand is in units and growing, and firms compete in prices. The total demand over the two periods is more than one firm can produce, but less than both firms can produce together. An important feature of their model is that not only the sellers, but also the buyers act strategically. One of the results is that, when there is only one consumer, linear pricing implies there is no pure strategy equilibrium. Another paper that shows some resemblance with ours is the two-period model of Saloner (1987). In that paper, there are two periods of production, after which the goods are sold for the market clearing price.

The paper is organized as follows. The next section introduces the model. Section 3 analyzes the equilibria that result in the commitment case. The non-commitment situation is addressed in Section 4. In Section 5, we analyze how the equilibrium outcomes in the commitment setting relate to the equilibrium outcomes in the non-commitment setting. Section 6 concludes. Lengthy and technical proofs are relegated to the appendix.

2 The Model

We consider two profit maximizing firms that have produced (or bought) a homogeneous good. Firm $i = 1, 2$ therefore owns a finite amount $S_i \geq 0$ of the good. Since the goods are produced beforehand, the production costs are sunk and they do not play a role in the model. With their fixed amount of stock as an upperbound, the firms compete in quantity for two periods. A firm may choose to have residual supply at the end of the second period. The quantities sold by firm i in period 1 and period 2 are denoted by q_i and r_i, respectively, so $q_i + r_i \leq S_i$. The inverse demand each period is

$$P(Q) = 1 - Q,$$

where $Q = q_1 + q_2$ in the first period and $Q = r_1 + r_2$ in the second.[1,2] Profits earned in period 2 are discounted with a factor $\delta \in (0, 1]$.

Two cases are analyzed. In the first, firms can commit to a sales strategy that is independent of sales by their competitor. That is, after production has taken place,

[1] For convenience, we allow prices to be negative when the firms together supply more than one unit to the market.

[2] The results in this paper can be extended to inverse demand functions of the form $P(Q) = a - bQ$, where $a, b > 0$, and with firms facing unit costs of c, to be interpreted for instance as handling costs.

both firms unconditionally decide how much they are going to sell in each period. This implies that firm i's strategy space is of the form $\Gamma_i = \{(q_i, r_i) \in \mathbb{R}^2_+ \mid q_i + r_i \leq S_i\}$.

The second case is the one of non-commitment. In this case, the amount a firm is going to offer for sale in a period depends on the realized sales of its competitor in the previous period. As a result, the second-period strategy of a firm is now the specification of a sales quantity conditional on the observation of first-period sales. We define $F_i = \{f_i : [0, S_1] \times [0, S_2] \to [0, S_i] \mid q_i + f_i(q_1, q_2) \leq S_i\}$ as the set of functions that assign a feasible second-period sales quantity to every possible combination of first-period sales. Firm i's strategy space is $\Sigma_i = [0, S_i] \times F_i$.

3 Commitment

In the commitment case firms choose a sales path that does not depend on their competitor's realized sales. Given strategies $(q_1, r_1) \in \Gamma_1$ and $(q_2, r_2) \in \Gamma_2$, the profit $\Pi_i(q_1, r_1, q_2, r_2)$ of firm i is given by

$$\Pi_i(q_1, r_1, q_2, r_2) = q_i P(q_i + q_j) + \delta r_i P(r_i + r_j)$$

When choosing its sales path (q_i, r_i), firm i takes the sales path (q_j, r_j) of firm j as given, where we use the notation i and j for the two competing firms. Firm i therefore solves the problem

$$\max_{q_i, r_i} \Pi_i(q_1, r_1, q_2, r_2)$$

subject to

$$q_i, r_i \geq 0 \text{ and } q_i + r_i \leq S_i.$$

The result is a best response $\gamma_i(q_j, r_j) \in \Gamma_i$ given by

$$\gamma_i(q_j, r_j) = \begin{cases} (S_i, 0) & \begin{cases} \text{if } [q_j - \delta r_j < 1 - \delta - 2S_i, \\ q_j + r_j \leq 2 - 2S_i \text{ and } q_j, r_j \leq 1] \\ \text{or } [q_j \leq 1 - 2S_i \text{ and } r_j > 1] \end{cases} \\ (0, S_i) & \begin{cases} \text{if } [q_j - \delta r_j > 1 - \delta + 2\delta S_i, \\ q_j + r_j \leq 2 - 2S_i \text{ and } q_j, r_j \leq 1] \\ \text{or } [q_j > 1 \text{ and } r_j \leq 1 - 2S_i] \end{cases} \\ \left(\frac{1-\delta+2\delta S_i - q_j + \delta r_j}{2+2\delta}, \frac{2S_i - 1 + \delta + q_j - \delta r_j}{2+2\delta}\right) & \begin{cases} \text{if } 1 - \delta - 2S_i \leq q_j - \delta r_j \leq \\ 1 - \delta + 2\delta S_i, \\ q_j + r_j \leq 2 - 2S_i \text{ and } q_j, r_j \leq 1 \end{cases} \\ (\frac{1}{2} - \frac{1}{2}q_j, \frac{1}{2} - \frac{1}{2}r_j) & \begin{cases} \text{if } q_j + r_j > 2 - 2S_i \\ \text{and } q_j, r_j \leq 1 \end{cases} \\ (\frac{1}{2} - \frac{1}{2}q_j, 0) & \begin{cases} \text{if } 1 - 2S_i < q_j \leq 1 \\ \text{and } r_j > 1 \end{cases} \\ (0, \frac{1}{2} - \frac{1}{2}r_j) & \begin{cases} \text{if } 1 - 2S_i < r_j \leq 1 \\ \text{and } q_j > 1 \end{cases} \\ (0, 0) & \text{if } q_j, r_j > 1. \end{cases}$$

The seven cases for S_i are mutually exclusive and the best responses against (q_j, r_j) are unique. The function γ_i is continuous.

A pair of strategies $(q_1^*, r_1^*, q_2^*, r_2^*)$ is a Nash equilibrium if

$$\begin{aligned}\Pi_1(q_1^*, r_1^*, q_2^*, r_2^*) &\geq \Pi_1(q_1, r_1, q_2^*, r_2^*) \text{ for all } (q_1, r_1) \in \Gamma_1,\\ \Pi_2(q_1^*, r_1^*, q_2^*, r_2^*) &\geq \Pi_2(q_1^*, r_1^*, q_2, r_2) \text{ for all } (q_2, r_2) \in \Gamma_2,\end{aligned}$$

or, equivalently, $\gamma_1(q_2^*, r_2^*) = (q_1^*, r_1^*)$ and $\gamma_2(q_1^*, r_1^*) = (q_2^*, r_2^*)$.

Given any initial combination (S_1, S_2, δ), there is a unique equilibrium, as specified in Table 1 and depicted in Figure 1. In the figure, δ is fixed and S_1, S_2 are variable. A change of δ will not change the shape of the equilibrium areas, only the ratio between them We use the superscript 'c' to refer to equilibria in the commitment case. The two letters in the subscript represent the relative level of stock of respectively firm i and j , where l stands for low, m for medium and h for high. In Figure 1, also the number of active firms in each period is indicated, where $N_1/N_2/N_r$ represents respectively the number of firms that have strictly positive sales in the first period, the number of firms that have strictly positive sales in the second period, and the number of firms that have residual supply at the end of the second period. The figure shows that the number of active firms increases when production increases.

	Parameter conditions	Period 1	Period 2
(X^c_{ll})	$0 \leq S_1 < \frac{1}{2} - \frac{1}{2}S_2 - \frac{1}{2}\delta$ $0 \leq S_2 < \frac{1}{2} - \frac{1}{2}S_1 - \frac{1}{2}\delta$	$q_1^c = S_1$ $q_2^c = S_2$	$r_1^c = 0$ $r_2^c = 0$
(X^c_{lm})	$0 \leq S_1 < \frac{1}{3} - \frac{1}{3}\delta$ $\frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_1 \leq S_2 \leq 1 - \frac{1}{2}S_1$	$q_1^c = S_1$ $q_2^c = \frac{1-S_1-\delta+2\delta S_2}{2+2\delta}$	$r_1^c = 0$ $r_2^c = \frac{2S_2+S_1-1+\delta}{2+2\delta}$
(X^c_{ml})	$\frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_2 \leq S_1 \leq 1 - \frac{1}{2}S_2$ $0 \leq S_2 < \frac{1}{3} - \frac{1}{3}\delta$	$q_1^c = \frac{1-\delta+2\delta S_1-S_2}{2+2\delta}$ $q_2^c = S_2$	$r_1^c = \frac{2S_1+S_2-1+\delta}{2+2\delta}$ $r_2^c = 0$
(X^c_{lh})	$0 \leq S_1 < \frac{1}{3} - \frac{1}{3}\delta$ $1 - \frac{1}{2}S_1 < S_2$	$q_1^c = S_1$ $q_2^c = \frac{1}{2} - \frac{1}{2}S_1$	$r_1^c = 0$ $r_2^c = \frac{1}{2}$
(X^c_{hl})	$1 - \frac{1}{2}S_2 < S_1$ $0 \leq S_2 < \frac{1}{3} - \frac{1}{3}\delta$	$q_1^c = \frac{1}{2} - \frac{1}{2}S_2$ $q_2^c = S_2$	$r_1^c = \frac{1}{2}$ $r_2^c = 0$
(X^c_{mm})	$\frac{1}{3} - \frac{1}{3}\delta \leq S_1 \leq 1 - \frac{1}{2}S_2$ $\frac{1}{3} - \frac{1}{3}\delta \leq S_2 \leq 1 - \frac{1}{2}S_1$	$q_1^c = \frac{1-\delta+3\delta S_1}{3+3\delta}$ $q_2^c = \frac{1-\delta+3\delta S_2}{3+3\delta}$	$r_1^c = \frac{3S_1-1+\delta}{3+3\delta}$ $r_2^c = \frac{3S_2-1+\delta}{3+3\delta}$
(X^c_{mh})	$\frac{1}{3} - \frac{1}{3}\delta \leq S_1 \leq \frac{2}{3}$ $1 - \frac{1}{2}S_1 < S_2$	$q_1^c = \frac{1-\delta+3\delta S_1}{3+3\delta}$ $q_2^c = \frac{2+4\delta-3\delta S_1}{6+6\delta}$	$r_1^c = \frac{3S_1-1+\delta}{3+3\delta}$ $r_2^c = \frac{4+2\delta-3S_1}{6+6\delta}$
(X^c_{hm})	$1 - \frac{1}{2}S_2 < S_1$ $\frac{1}{3} - \frac{1}{3}\delta < S_2 \leq \frac{2}{3}$	$q_1^c = \frac{2+4\delta-3\delta S_2}{6+6\delta}$ $q_2^c = \frac{1-\delta+3\delta S_2}{3+3\delta}$	$r_1^c = \frac{4+2\delta-3S_2}{6+6\delta}$ $r_2^c = \frac{3S_2-1+\delta}{3+3\delta}$
(X^c_{hh})	$\frac{2}{3} < S_1$ $\frac{2}{3} < S_2$	$q_1^c = \frac{1}{3}$ $q_2^c = \frac{1}{3}$	$r_1^c = \frac{1}{3}$ $r_2^c = \frac{1}{3}$

Table 1: Equilibria in the commitment case.

When the stock of firm 1 is low, as it is in Regions X^c_{ll}, X^c_{lm}, and X^c_{lh}, it will sell all of its stock in the first period. These regions are non-empty only if the discount rate is strictly below one. The discounting of second-period profits gives firms an incentive to sell in period 1 rather than in period 2. When firm 1 has a low stock S_1, then selling this entirely in the first period will hardly decrease the marginal revenue in the first period. Consequently, as long as δ is not too high, marginal revenue in the second period will be less than the marginal revenue in the first period and firm 1 will sell its entire production in the first period.

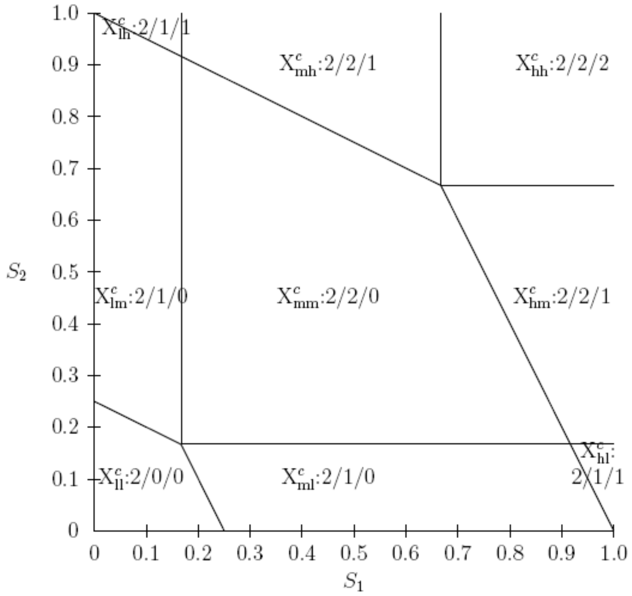

Figure 1: The commitment case: Equilibrium outcome regions and the number of firms that have stock in respectively period 1, period 2 and after period 2, for $\delta = 0.5$.

In Regions X^c_{ml}, X^c_{mm}, and X^c_{mh}, firm 1 has an intermediate amount of the commodity in stock. It then maximizes profit by dividing its sales over the two periods in such a way that marginal revenue in both periods is equal.

In the remaining Regions, X^c_{hl}, X^c_{hm} and X^c_{hh}, firm 1 has a high stock and acts as if it has no capacity constraints. Firm 1 maximizes its profit in each period separately as to maximize total profit. It will have residual stock at the end of period 2.

A similar line of argumentation applies to the equilibrium strategy of firm 2. Note that in both periods in situation X^c_{hh} firms maximize their profit as if there is no capacity limit. This results in both firms choosing their Cournot equilibrium quantities of $\frac{1}{3}$ in both periods.

The next five propositions describe some comparative statics results for the case with commitment.

Proposition 3.1 *In equilibrium, price weakly increases over time.*

Proof. For any given combination of S_1, S_2, and δ, one can verify directly that $q^c_1 + q^c_2 \geq r^c_1 + r^c_2$. Since the aggregate sales in the first period weakly exceed the aggregate sales in the second period, price in the first period is less than or equal to the price in the second period. ■

Notice, in particular, that as long as its capacity doesn't prevent it from doing so, a firm will adjust its sales to achieve equal marginal revenues in both periods. This together with a discount rate which is less than or equal to one implies that price cannot decrease over time.

Also the following proposition describes an intuitive result.

Proposition 3.2 *An increase in S_i leads to a weak increase in firm i's equilibrium profit.*

Proof. The derivative of the equilibrium profit function of firm i with respect to S_i is non-negative in every equilibrium outcome region and the profit function is continuous for all $\delta, S_i, S_j \geq 0$. ■

Notice, of course, that the profits in Proposition 3.2 correspond to sales revenues and do not take into account the costs of production.

The next proposition studies how the relative stock sizes of the two firms affect the commodity price. For fixed aggregate stock size $S_1 + S_2$, we analyze how an increase in asymmetry $|S_1 - S_2|$ influences equilibrium outcomes.

Proposition 3.3 *Given fixed aggregate production $S_1 + S_2$, an increase in $|S_1 - S_2|$ leads to a weak decrease of first-period aggregate equilibrium sales and therefore a weak increase of first-period equilibrium price. It leads (i) to a decrease of second-period aggregate equilibrium sales and an increase of second-period equilibrium price in Regions $\mathrm{X}^{\mathrm{c}}_{mh}$ and $\mathrm{X}^{\mathrm{c}}_{hm}$ and (ii) to an increase of second-period aggregate equilibrium sales and a decrease of second-period equilibrium price in Regions $\mathrm{X}^{\mathrm{c}}_{lm}$ and $\mathrm{X}^{\mathrm{c}}_{ml}$. It has no effect on second-period aggregate equilibrium sales and equilibrium price in the other regions.*

Proof. Let $S = S_1 + S_2$ be fixed and assume without loss of generality that $S_2 \geq S_1$. Then $|S_1 - S_2|$ increases if S_1 decreases. Since $S_2 \geq S_1$, it holds that $(S_1, S_2, \delta) \in \mathrm{X}^{\mathrm{c}}_{\mathrm{ll}} \cup \mathrm{X}^{\mathrm{c}}_{\mathrm{lm}} \cup \mathrm{X}^{\mathrm{c}}_{\mathrm{lh}} \cup \mathrm{X}^{\mathrm{c}}_{\mathrm{mm}} \cup \mathrm{X}^{\mathrm{c}}_{\mathrm{mh}} \cup \mathrm{X}^{\mathrm{c}}_{\mathrm{hh}}$. Let Q_{ab} be the aggregate sales in equilibrium region X_{ab}.

For first-period aggregate sales, we find that

$$\begin{array}{ll}
Q^{\mathrm{c}}_{\mathrm{ll}} = S, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{ll}}}{\partial S_1} = 0, \\
Q^{\mathrm{c}}_{\mathrm{lm}} = \frac{1-\delta+S_1+2\delta S}{2+2\delta}, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{lm}}}{\partial S_1} = \frac{1}{2+2\delta} > 0, \\
Q^{\mathrm{c}}_{\mathrm{lh}} = \frac{1}{2} + \frac{1}{2}S_1, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{lh}}}{\partial S_1} = \frac{1}{2} > 0, \\
Q^{\mathrm{c}}_{\mathrm{mm}} = \frac{2-2\delta+3\delta S}{3+3\delta}, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{mm}}}{\partial S_1} = 0, \\
Q^{\mathrm{c}}_{\mathrm{mh}} = \frac{4+2\delta+3\delta S_1}{6+6\delta}, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{mh}}}{\partial S_1} = \frac{3\delta}{6+6\delta} > 0, \\
Q^{\mathrm{c}}_{\mathrm{hh}} = \frac{2}{3}, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{hh}}}{\partial S_1} = 0.
\end{array}$$

For second-period aggregate sales we have that

$$\begin{array}{ll}
Q^{\mathrm{c}}_{\mathrm{ll}} = 0, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{ll}}}{\partial S_1} = 0, \\
Q^{\mathrm{c}}_{\mathrm{lm}} = \frac{2S-S_1-1+\delta}{2+2\delta}, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{lm}}}{\partial S_1} = -\frac{1}{2+2\delta} < 0, \\
Q^{\mathrm{c}}_{\mathrm{lh}} = \frac{1}{2}, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{lh}}}{\partial S_1} = 0, \\
Q^{\mathrm{c}}_{\mathrm{mm}} = \frac{3S-2+2\delta}{3+3\delta}, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{mm}}}{\partial S_1} = 0, \\
Q^{\mathrm{c}}_{\mathrm{mh}} = \frac{3S_1+2+4\delta}{6+6\delta}, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{mh}}}{\partial S_1} = \frac{3}{6+6\delta} > 0, \\
Q^{\mathrm{c}}_{\mathrm{hh}} = \frac{2}{3}, & \frac{\partial Q^{\mathrm{c}}_{\mathrm{hh}}}{\partial S_1} = 0.
\end{array}$$

■

A larger difference in stocks results in a higher first-period price. This is intuitive: consider the extreme case where one of the firms is a monopolist, resulting in the highest possible first-period price. Surprisingly, the effect of increasing difference between the firms' stocks on second-period prices is ambiguous. In particular, it leads to a weak decrease in second-period price in Regions $\mathrm{X}^{\mathrm{c}}_{\mathrm{lm}}$ and $\mathrm{X}^{\mathrm{c}}_{\mathrm{ml}}$. In these regions, the smaller firm has no stock left at the beginning of period 2. An increase in the size of the bigger firm then simply leads to more sales by this firm in period 2.

The following proposition studies the consequences of increased stocks for consumer surplus. Consumer surplus in the first period and in the second period is respectively $\frac{1}{2}(q_1 + q_2)^2$ and $\frac{1}{2}(r_1 + r_2)^2$. To compute the total consumer surplus we have to discount the second-period consumer surplus by δ. Consumer surplus is therefore given by $\frac{1}{2}(q_1 + q_2)^2 + \frac{1}{2}\delta(r_1 + r_2)^2$.

Proposition 3.4 *Equilibrium consumer surplus weakly increases if the stock of at least one of the firms increases.*

Proof. It follows directly from the equilibrium outcomes that per-period sales weakly increase in S_1 and S_2. ■

Since the effect of an increase in $|S_1 - S_2|$ on second-period sales is ambiguous by Proposition 3.3, it is not a priori clear how such an increase affects consumer surplus. The next proposition states, nevertheless, that this effect is unambiguously negative.

Proposition 3.5 *Given fixed aggregate stock $S_1 + S_2$, an increase in $|S_1 - S_2|$ leads to a weak decrease in equilibrium consumer surplus.*

Proof. Proposition 3.3 implies a weak decrease in sales in both periods when $|S_1 - S_2|$ increases, and therefore a weak decrease in consumer surplus, except possibly in Regions $\mathrm{X}^{\mathrm{c}}_{\mathrm{lm}}$ and $\mathrm{X}^{\mathrm{c}}_{\mathrm{ml}}$.

Consider some (S_1, S_2, δ) in Region $\mathrm{X}^{\mathrm{c}}_{\mathrm{lm}}$ or $\mathrm{X}^{\mathrm{c}}_{\mathrm{ml}}$. Assume without loss of generality that $S_2 \geq S_1$, so $|S_1 - S_2|$ increases if S_1 decreases. Then (S_1, S_2, δ) belongs to Region $\mathrm{X}^{\mathrm{c}}_{\mathrm{lm}}$. Consumer surplus is given by

$$\frac{1}{2}\left(\frac{1 - \delta + S_1 + 2\delta S}{2 + 2\delta}\right)^2 - \frac{1}{2}\delta\left(\frac{2S - S_1 - 1 + \delta}{2 + 2\delta}\right)^2,$$

where, as before, $S = S_1 + S_2$. The derivative of the expression above with respect to S_1 is given by

$$\frac{1 - \delta + S_1 + 2\delta S}{(2 + 2\delta)^2} + \delta\frac{2S - S_1 - 1 + \delta}{(2 + 2\delta)^2},$$

which is easily shown to be non-negative. ■

By the same type of analysis, it can be shown that the results we have found for equilibrium consumer surplus coincide with the results that can be found for equilibrium total surplus. Total surplus is defined as the addition of consumer surplus and both the firms' surplus. In this case, total surplus is

$$(q_1 + q_2)(1 - \frac{1}{2}(q_1 + q_2)) + \delta(r_1 + r_2)(1 - \frac{1}{2}(r_1 + r_2)).$$

Equilibrium total surplus weakly increases if production by at least one of the firms increases and, given fixed aggregate production $S_1 + S_2$, an increase in $|S_1 - S_2|$ leads to a weak decrease in equilibrium total surplus.

Summary of comparative statics results for the commitment case
We find that, when firms have the power to commit to an unconditional sales strategy, price never decreases over time. A firm's profit increases when its stock increases and so does consumer surplus and total surplus. Finally, an increase in the difference between the stocks of the firms leads to lower sales in period 1 and lower consumer surplus and total surplus. The effect on period 2 sales is ambiguous.

	T_i	T_j	r_i	r_j	profits i	profits j
($\mathrm{Y_{hh}}$)	$> \frac{1}{3}$	$> \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$
($\mathrm{Y_{lh}}$)	$\leq \frac{1}{3}$	$> \frac{1-T_i}{2}$	T_i	$\frac{1-T_i}{2}$	$T_i(\frac{1-T_i}{2})$	$(\frac{1-T_i}{2})^2$
($\mathrm{Y_{hl}}$)	$> \frac{1-T_j}{2}$	$\leq \frac{1}{3}$	$\frac{1-T_j}{2}$	T_j	$\frac{(1-T_j)^2}{4}$	$T_j(\frac{1-T_j}{2})$
($\mathrm{Y_{ll}}$)	$\leq \frac{1-T_j}{2}$	$\leq \frac{1-T_i}{2}$	T_i	T_j	$T_i(1-T_i-T_j)$	$T_j(1-T_i-T_j)$

Table 2: Second-period equilibrium outcomes.

4 Non-commitment

We now study the case where the sales strategy of a firm in period 2 depends on the observed first-period sales. Once firms arrive in the second period of the game, they play a one-period game with capacity constraints. We analyze the subgame perfect Nash equilibria of the game. We do this by first analyzing the Nash equilibria of all possible period 2 subgames.

Consider the subgame $q = (q_1, q_2)$ in period 2 that results from first-period sales (q_1, q_2) by the firms. Denote firm i's second-period stock by $T_i = S_i - q_i$. Now we can define $\sigma_{iq} : [0, T_j] \to [0, T_i]$ as firm i's best response function in subgame q. Given sales r_j by firm j, firm i solves the problem

$$\max_{r_i} r_i P(r_i + r_j)$$

subject to

$$0 \leq r_i \leq T_i.$$

The best response for firm i in period 2 is then given by

$$\sigma_{iq}(r_j) = \begin{cases} T_i, & \text{if } 0 \leq T_i \leq \frac{1}{2} - \frac{1}{2} r_j, \\ \max\{0, \frac{1}{2} - \frac{1}{2} r_j\}, & \text{otherwise.} \end{cases}$$

Quantities (r_1^*, r_2^*) are a Nash equilibrium of the second-period subgame q if and only if $\sigma_{1q}(r_2^*) = r_1^*$ and $\sigma_{2q}(r_1^*) = r_2^*$. Each subgame q has a unique Nash equilibrium as specified in Table 2 and depicted in Figure 2.

In Region $\mathrm{Y_{hh}}$, both firms have sufficient residual stock in the second period to choose their unconstrained profit maximizing sales quantity. In Regions $\mathrm{Y_{hl}}$ and $\mathrm{Y_{lh}}$, only one firm is restricted by its residual stock, respectively firm 1 and firm 2. In Region $\mathrm{Y_{ll}}$ both firms are restricted by their residual stock and sell in the second period all they have left.

The equilibrium action chosen by firm i in period 2 is given by the function f_i^* defined by

$$f_i^*(q_i, q_j) = \begin{cases} \frac{1}{3}, & \text{if } T_i, T_j > \frac{1}{3}, \\ \frac{1}{2} - \frac{1}{2} T_j, & \text{if } T_i > \frac{1}{2} - \frac{1}{2} T_j \text{ and } T_j \leq \frac{1}{3}, \\ T_i, & \text{if } T_i \leq \frac{1}{3} \text{ or } T_j \leq 1 - 2T_i. \end{cases} \tag{1}$$

We now replace the second-period subgames by the second-period outcomes as induced by f^*. The result is a one-period reduced game with payoffs given by

$$\Pi_i^{\mathrm{R}}(q_i, q_j) = \Pi_i(q_i, q_j, f_i^*(q_i, q_j), f_j^*(q_i, q_j)), \quad 0 \leq q_i \leq S_i,\ 0 \leq q_j \leq S_j.$$

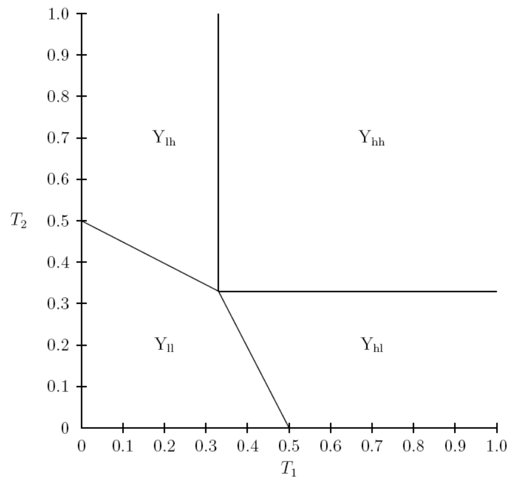

Figure 2: Second-period equilibrium regions for $\delta = 0.5$.

It follows that the reduced profit function of firm i is given by

$$\Pi_i^{\mathrm{R}}(q_i, q_j) = q_i(1 - q_i - q_j) + \begin{cases} \frac{1}{9}\delta, & \text{if } T_i > \frac{1}{3} \text{ and } T_j > \frac{1}{3}, & (\mathrm{Y_{hh}}) \\ \frac{1}{2}\delta T_i(1 - T_i), & \text{if } 1 - 2T_j < T_i \leq \frac{1}{3}, & (\mathrm{Y_{lh}}) \\ \frac{1}{4}\delta(1 - T_j)^2, & \text{if } T_i > \frac{1}{2} - \frac{1}{2}T_j \text{ and } T_j \leq \frac{1}{3}, & (\mathrm{Y_{hl}}) \\ \delta T_i(1 - T_i - T_j), & \text{if } T_i \leq \min\{\frac{1}{2} - \frac{1}{2}T_j, 1 - 2T_j\}. & (\mathrm{Y_{ll}}) \end{cases} \tag{2}$$

A pair of strategies (q_1^*, q_2^*) is a Nash equilibrium of the reduced game if it holds that

$$\begin{aligned} \Pi_1^{\mathrm{R}}(q_1^*, q_2^*) &\geq \Pi_1^{\mathrm{R}}(q_1, q_2^*), \text{ for all } q_1 \in [0, S_1], \\ \Pi_1^{\mathrm{R}}(q_1^*, q_2^*) &\geq \Pi_1^{\mathrm{R}}(q_1^*, q_2), \text{ for all } q_2 \in [0, S_2]. \end{aligned}$$

A Nash equilibrium (q_1^*, q_2^*) of the reduced game corresponds to a subgame perfect Nash equilibrium $(q_1^*, f_1^*, q_2^*, f_2^*)$ of the complete game and vice versa.

Lemma 4.1 *$q_i^*, q_j^* \leq \frac{1}{2}$ for any Nash equilibrium (q_1^*, q_2^*) of the reduced game.*

Proof. The first-period profit is $q_i(1 - q_i - q_j)$, which is strictly decreasing in q_i if $q_i > \frac{1}{2} - \frac{1}{2}q_j$, so in particular if $q_i > \frac{1}{2}$. If firm i decreases its first-period sales, it increases its second-period stock. As can be seen in Table 2, firm i's second-period profit never decreases when its second-period stock increases. Consequently, firm i strictly increases its profits if it sets $q_i = \frac{1}{2}$ instead of $q_i > \frac{1}{2}$. ■

Using the reduced profit function (2), we determine the reduced best responses, denoting by $\sigma_i^{\mathrm{R}}(q_j)$ the reduced best response of firm i against q_j. Appendix A provides the computational details. Given q_j, the reduced profit function is not always concave, though it is continuous. As a consequence, the reduced best response against q_j does always exist, but may not be unique. We therefore have a reduced best response correspondence rather than a reduced best response function. This correspondence may fail to be convex-valued though it is upper hemi-continuous. The reduced best response correspondence of firm i is presented in Appendix A.

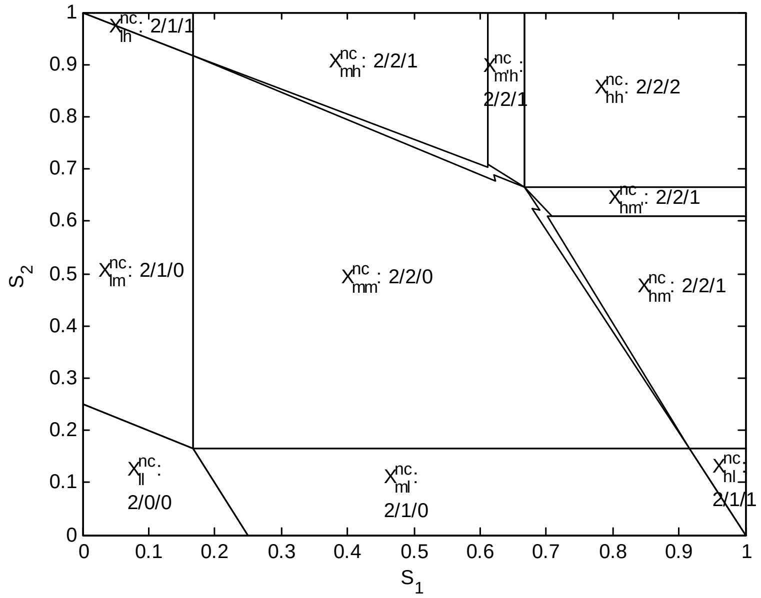

Figure 3: The non-commitment case: Equilibrium outcome regions and the number of firms that have stock in respectively period 1, period 2 and after period 2, for $\delta = 0.5$.

Quantities (q_1^*, q_2^*) are a Nash equilibrium of the reduced game if and only if $q_i^* \in \sigma_i^{\mathrm{R}}(q_j^*)$ and $q_j^* \in \sigma_j^{\mathrm{R}}(q_i^*)$. The Nash equilibria of the reduced game, and thereby the subgame perfect Nash equilibria of the game of interest, are calculated in Appendix B. Since the reduced best response correspondences are not convex-valued, it is not guaranteed that a subgame perfect Nash equilibrium exists. Indeed, it turns out that for some combinations of S_i, S_j and δ a subgame perfect Nash equilibrium fails to exist.

The set of exogenous variables (S_1, S_2, δ) can be partitioned in 11 regions. In each region, the equilibria share the same qualitative features and are differentiable functions of S_1, S_2, and δ. The equilibrium regions are given in Table 3 and depicted in Figure 3 for $\delta = 0.5$. Table 3 also shows the equilibrium outcomes. We use the superscript 'nc' to refer to equilibria in the non-commitment case. The two letters in the subscript represent the relative level of stock of respectively firm i and j , where l stands for low, m for medium, m' for medium-high and h for high.

As is illustrated by Figure 3 for $\delta = 1/2$, the 11 regions are mutually exclusive. This property is generally true, leading to the following theorem.

Theorem 4.2 *There is at most one subgame perfect Nash equilibrium for every combination of S_1, S_2, and δ.*

Proof. It follows from comparing the constraints in Table 3, that all regions are disjoint. Therefore, every combination of S_i, S_j and δ belongs to at most one equilibrium region. The Nash equilibrium of the reduced game is therefore unique for (S_i, S_j, δ) in Regions $\mathrm{X}_{\mathrm{ll}}^{\mathrm{nc}}$ up to and including $\mathrm{X}_{\mathrm{hh}}^{\mathrm{nc}}$. The reduced game has no Nash equilibrium for (S_i, S_j, δ) belonging to Region $\mathrm{X}_{\emptyset}^{\mathrm{nc}}$. Nash equilibria for the reduced

	Parameter conditions	Period 1	Period 2
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{ll}})$	$0 \leq S_1 < \frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_2$ $0 \leq S_2 < \frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_1$	$q_1^{\mathrm{nc}} = S_1$ $q_2^{\mathrm{nc}} = S_2$	$r_1^{\mathrm{nc}} = 0$ $r_2^{\mathrm{nc}} = 0$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{lm}})$	$0 \leq S_1 < \frac{1}{3}(1-\delta)$ $\frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_1 \leq S_2 \leq 1 - \frac{1}{2}S_1$	$q_1^{\mathrm{nc}} = S_1$ $q_2^{\mathrm{nc}} = \frac{1-S_1+2\delta S_2-\delta}{2+2\delta}$	$r_1^{\mathrm{nc}} = 0$ $r_2^{\mathrm{nc}} = \frac{2S_2-1+S_1+\delta}{2+2\delta}$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{ml}})$	$\frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_2 \leq S_1 \leq 1 - \frac{1}{2}S_2$ $0 \leq S_2 < \frac{1}{3}(1-\delta)$	$q_1^{\mathrm{nc}} = \frac{1-S_2+2\delta S_1-\delta}{2+2\delta}$ $q_2^{\mathrm{nc}} = S_2$	$r_2^{\mathrm{nc}} = \frac{2S_1-1+S_2+\delta}{2+2\delta}$ $r_2^{\mathrm{nc}} = 0$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{lh}})$	$0 \leq S_1 < \frac{1}{3}(1-\delta)$ $S_2 > 1 - \frac{1}{2}S_1$	$q_1^{\mathrm{nc}} = S_1$ $q_2^{\mathrm{nc}} = \frac{1}{2} - \frac{1}{2}S_1$	$r_1^{\mathrm{nc}} = 0$ $r_2^{\mathrm{nc}} = \frac{1}{2}$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{hl}})$	$S_1 > 1 - \frac{1}{2}S_2$ $0 \leq S_2 < \frac{1}{3}(1-\delta)$	$q_1^{\mathrm{nc}} = \frac{1}{2} - \frac{1}{2}S_2$ $q_2^{\mathrm{nc}} = S_2$	$r_1^{\mathrm{nc}} = \frac{1}{2}$ $r_2^{\mathrm{nc}} = 0$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}a)$	$\frac{1}{3}(1-\delta) \leq S_1 \leq \beta_1$ $\frac{1}{3}(1-\delta) \leq S_2 \leq \beta_2$	$q_1^{\mathrm{nc}} = \frac{1-\delta+3\delta S_1}{3+3\delta}$ $q_2^{\mathrm{nc}} = \frac{1-\delta+3\delta S_2}{3+3\delta}$	$r_1^{\mathrm{nc}} = \frac{3S_1-1+\delta}{3+3\delta}$ $r_2^{\mathrm{nc}} = \frac{3S_2-1+\delta}{3+3\delta}$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}\mathrm{b})$	$\beta_3 < S_1 \leq 1 - \frac{1}{2}S_2$ $\beta_4 < S_2 \leq 1 - \frac{1}{2}S_1$		
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}})$	$\frac{1}{3}(1-\delta) \leq S_1 \leq \frac{2}{3} - \frac{1}{9}\delta$ $S_2 > \beta_5$	$q_1^{\mathrm{nc}} = \frac{1-\delta+2\delta S_1}{3+2\delta}$ $q_2^{\mathrm{nc}} = \frac{2+3\delta-2\delta S_1}{6+4\delta}$	$r_1^{\mathrm{nc}} = \frac{3S_1-1+\delta}{3+2\delta}$ $r_2^{\mathrm{nc}} = \frac{4+\delta-3S_1}{6+4\delta}$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm}})$	$S_1 > \beta_6$ $\frac{1}{3}(1-\delta) \leq S_2 \leq \frac{2}{3} - \frac{1}{9}\delta$	$q_1^{\mathrm{nc}} = \frac{2+3\delta-2\delta S_2}{6+4\delta}$ $q_2^{\mathrm{nc}} = \frac{1-\delta+2\delta S_2}{3+2\delta}$	$r_1^{\mathrm{nc}} = \frac{4+\delta-3S_2}{6+4\delta}$ $r_2^{\mathrm{nc}} = \frac{3S_2-1+\delta}{3+2\delta}$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}})$	$\frac{2}{3} - \frac{1}{9}\delta < S_1 \leq \frac{2}{3}$ $S_2 > \beta_7$	$q_1^{\mathrm{nc}} = S_1 - \frac{1}{3}$ $q_2^{\mathrm{nc}} = \frac{2}{3} - \frac{1}{2}S_1$	$r_1^{\mathrm{nc}} = \frac{1}{3}$ $r_2^{\mathrm{nc}} = \frac{1}{3}$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm'}})$	$S_1 > \beta_8$ $\frac{2}{3} - \frac{1}{9}\delta < S_2 \leq \frac{2}{3}$	$q_1^{\mathrm{nc}} = \frac{2}{3} - \frac{1}{2}S_2$ $q_2^{\mathrm{nc}} = S_2 - \frac{1}{3}$	$r_1^{\mathrm{nc}} = \frac{1}{3}$ $r_2^{\mathrm{nc}} = \frac{1}{3}$
$(\mathrm{X}^{\mathrm{nc}}_{\mathrm{hh}})$	$S_1 > \frac{2}{3}$ $S_2 > \frac{2}{3}$	$q_1^{\mathrm{nc}} = \frac{1}{3}$ $q_2^{\mathrm{nc}} = \frac{1}{3}$	$r_1^{\mathrm{nc}} = \frac{1}{3}$ $r_2^{\mathrm{nc}} = \frac{1}{3}$
$(\mathrm{X}^{\mathrm{nc}}_{\emptyset})$	All other values of (S_1, S_2, δ)	No equilibrium	

	Explanation of the symbols
β_1	$\frac{7}{6} - S_2 - \frac{1}{6}\delta + \frac{\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta}(\frac{5}{6} + \frac{1}{6}\delta - S_2)$
β_2	$\frac{7}{6} - S_1 - \frac{1}{6}\delta + \frac{\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta}(\frac{5}{6} + \frac{1}{6}\delta - S_1)$
β_3	$\frac{4+5\frac{1}{2}\delta-\frac{1}{2}\delta^2-3\delta S_2}{6+6\delta}$
β_4	$\frac{4+5\frac{1}{2}\delta-\frac{1}{2}\delta^2-3\delta S_1}{6+6\delta}$
β_5	$\frac{7+6\frac{1}{2}\delta+\frac{3}{2}\delta^2-6S_1-5\delta S_1-\delta^2 S_1+(5+5\delta-2\delta S_1)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6+7\delta+2\delta^2+(6+4\delta)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}$
β_6	$\frac{7+6\frac{1}{2}\delta+\frac{3}{2}\delta^2-6S_2-5\delta S_2-\delta^2 S_2+(5+5\delta-2\delta S_2)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6+7\delta+2\delta^2+(6+4\delta)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}$
β_7	$\frac{10+6\delta-7S_1-3\delta S_1}{8+6\delta}$
β_8	$\frac{10+6\delta-7S_2-3\delta S_2}{8+6\delta}$

Table 3: Equilibria in the non-commitment case.

game are in a one to one relationship with subgame perfect Nash equilibria of the complete game. ■

In some cases an equilibrium does not exist.

Corollary 4.3 *For every δ, there is a set of stock levels (S_1, S_2) with non-empty interior for which an equilibrium does not exist.*

Proof. It can be verified that for each δ, the set of stock profiles (S_1, S_2) such that (S_1, S_2, δ) belongs to Region $\mathrm{X}^{\mathrm{nc}}_{\emptyset}$ has a non-empty interior. ■

Figure 3 gives an overview of the dynamic development of the market structure, where again $N_1/N_2/N_r$ represents the number of firms that sell the commodity in the first period, the number of firms that sell the commodity in the second period, and the number of firms that have residual stock by the end of the second period. Just as in the non-commitment case, the number of active firms increases when initial production levels increase.

Some of the regions in the non-commitment case coincide with those in the case with commitment. This holds specifically for the Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{ll}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{lm}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{ml}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{lh}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hl}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$, and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hh}}$.[3] For these regions, the equilibrium outcomes in the commitment and in the non-commitment case are equivalent.

In the commitment case, the price never decreases from period 1 to period 2. The reason is that a decreasing price would make it profitable for a firm to transfer some of its sales from period 2 to period 1. This line of reasoning does not hold when there is no commitment. Indeed, in the non-commitment case a transfer of sales from period 2 to period 1 may trigger a reaction by the competing firm, which renders such a transfer unprofitable, even when the price in period 1 is higher than in period 2.

Proposition 4.4 *In Regions $\mathrm{X}^{\mathrm{nc}}_{ll}, \mathrm{X}^{\mathrm{nc}}_{lm}, \mathrm{X}^{\mathrm{nc}}_{ml}, \mathrm{X}^{\mathrm{nc}}_{lh}, \mathrm{X}^{\mathrm{nc}}_{hl}, \mathrm{X}^{\mathrm{nc}}_{mm}$ and $\mathrm{X}^{\mathrm{nc}}_{hh}$, the equilibrium price weakly increases over time. For any δ, there is a set of stock levels $(S_1, S_2) \in \mathrm{X}^{\mathrm{nc}}_{mh} \cup \mathrm{X}^{\mathrm{nc}}_{m'h} \cup \mathrm{X}^{\mathrm{nc}}_{hm} \cup \mathrm{X}^{\mathrm{nc}}_{hm'}$ with non-empty interior such that the equilibrium price strictly decreases over time. In particular, the equilibrium price strictly decreases over time if and only if $S_i < S_j$ and*

$$\frac{2-2\delta}{3-2\delta} \leq S_i < \frac{2}{3} - \frac{1}{9}\delta,$$

$$S_j > \frac{7 + 6\frac{1}{2}\delta + \frac{3}{2}\delta^2 - 6S_i - 5\delta S_i - \delta^2 S_i + (5 + 5\delta - 2\delta S_i)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6 + 7\delta + 2\delta^2 + (6 + 4\delta)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}$$

or

$$\frac{2}{3} - \frac{1}{9}\delta < S_i < \frac{2}{3},$$

$$S_j > \frac{10 + 6\delta - 7S_i - 3\delta S_i}{8 + 6\delta}.$$

[3] We have named the regions in the non-commitment case in such a way that the names in the commitment and non-commitment case coincide as much as possible.

Proof. By Proposition 3.1, price never decreases over time in the commitment situation. Price decreases in the non-commitment case are therefore only possible in regions where the non-commitment case is different from the case with commitment, i.e. Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$, and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm'}}$. In these regions, one firm has an intermediate and one firm has a high stock level. Let i be the intermediate firm and let j be the large firm. In Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm}}$ it holds that

$$\begin{aligned}\frac{1}{3}(1-\delta) &< S_i \leq \frac{2}{3} - \frac{1}{9}\delta, \\ S_j &> \frac{7 + 6\frac{1}{2}\delta + \frac{3}{2}\delta^2 - 6S_i - 5\delta S_i - \delta^2 S_i + (5 + 5\delta - 2\delta S_i)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6 + 7\delta + 2\delta^2 + (6+4\delta)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}.\end{aligned}$$

The total quantity sold in the first period is

$$q_i^{\mathrm{nc}} + q_j^{\mathrm{nc}} = \frac{1 - \delta + 2\delta S_i}{3 + 2\delta} + \frac{2 + 3\delta - 2\delta S_i}{6 + 4\delta} = \frac{4 + \delta + 2\delta S_i}{6 + 4\delta}.$$

The total quantity sold in the second period is

$$r_i^{\mathrm{nc}} + r_j^{\mathrm{nc}} = \frac{3S_i - 1 + \delta}{3 + 2\delta} + \frac{4 + \delta - 3S_i}{6 + 4\delta} = \frac{3S_i + 2 + 3\delta}{6 + 4\delta}.$$

The price strictly decreases from period 1 to period 2 when $4+\delta+2\delta S_i < 3S_i+2+3\delta$, so when $S_i > \frac{2-2\delta}{3-2\delta}$.

In Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm'}}$ we have

$$\begin{aligned}\frac{2}{3} - \frac{1}{9}\delta &< S_i \leq \frac{2}{3} \\ S_j &> \frac{10 + 6\delta - 7S_i - 3\delta S_i}{8 + 6\delta}.\end{aligned}$$

The total quantity sold in the first period is

$$q_i^{\mathrm{nc}} + q_j^{\mathrm{nc}} = S_i - \frac{1}{3} + \frac{2}{3} - \frac{1}{2}S_i = \frac{1}{2}S_i + \frac{1}{3}.$$

The total quantity sold in the second period is

$$r_i^{\mathrm{nc}} + r_j^{\mathrm{nc}} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Since $\frac{1}{2}S_i + \frac{1}{3} < \frac{2}{3}$ as long as $S_i < \frac{2}{3}$, in this region, price strictly decreases from period one to period two whenever $S_i \neq \frac{2}{3}$. ■

Proposition 4.4 makes clear that price may decrease over time in the non-commitment case. This can happen for the following reason. In the settings where price decreases over time, the larger firm reacts in both periods – unrestricted by its stock – per-period optimal to the sales of the smaller firm. Since the smaller firm has a stock less than 1/3 in the second period, the larger firm cannot deviate in the first period in such a way that the smaller firm will lower its second-period sales. This implies that the larger firm cannot increase profits by deviating. The smaller firm, just as in the commitment case, might want to transfer some of its sales from the second to the first period. However, in the non-commitment situation, if the smaller firm transfers sales from period 2 to period 1, there will be a response by the larger firm. The larger firm reacts to this transfer by increasing its second-period sales, causing the second-period price to fall. Therefore, the second-period profits of the smaller firm drop. The decrease in profits in the second period outweigh the increase in profits in the first period. This makes transferring sales from the second-period to the first not worth the while for the smaller firm.

In the commitment case, an increase in a firm's stock leads to an increase in profits. Is this property still true in the non-commitment case? It is easily shown, with the help of the derivatives of the equilibrium profits, that within each region profit rises when a firm's stock level increases. Moreover, the profit function is continuous on the domain of (S_i, S_j, δ) for which an equilibrium exists. However, it is still possible for the profit to decrease when a firm's stock level increases, namely when a small increase in stock level leads to non-existence of equilibrium. The next proposition confirms that such decreases in profit may occur for specific parameter values. That is, equilibria may not be "destroy-proof".

Proposition 4.5 *An increase in S_i, ceterus paribus, leads to a weak increase of the equilibrium profit of firm i, as long as the increase doesn't change the equilibrium outcome region. If an increase in S_i does change the equilibrium outcome region, there are combinations of S_i, S_j and δ such that an increase in S_i leads to a strict decrease in equilibrium profit of firm i.*

Proof. The derivative of the equilibrium profit function with respect to S_i is non-negative in every equilibrium region. The non-existence of an equilibrium for some combinations of (S_i, S_j, δ) makes it possible that a strict increase in S_i leads to a strict decrease in the equilibrium profit of firm i. Take $\delta = 0.2$, $S_j = 0.69824$, and

$$S_i = \frac{7}{6} - S_j - \frac{1}{6}\delta + \frac{\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta}\left(\frac{5}{6} + \frac{1}{6}\delta - S_j\right) \approx 0.59634.$$

These parameters correspond to a point on the upper boundary of Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$. The equilibrium profit for firm i equals 0.12937. We now let S_i increase to

$$\begin{aligned} S_i' &= \frac{7 + 6\frac{1}{2}\delta + \frac{3}{2}\delta^2 - 6S_j - 78\delta S_j - 2\delta^2 S_j + (5 + 5\delta - 6S_j - 4\delta S_j)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6 + 5\delta + \delta^2 + 2\delta\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}} \\ &\approx 0.61011. \end{aligned}$$

Our parameters now belong to Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$. The equilibrium profit for firm i equals 0.12751. ■

We now study the consequences of increasing difference in stock size on sales.

Proposition 4.6 *Given fixed aggregate stock S_1+S_2, an increase in $|S_1-S_2|$ leads to a weak decrease in first-period aggregate equilibrium sales and a weak increase of first-period equilibrium price. It leads to a decrease in second-period aggregate equilibrium sales and an increase in second-period equilibrium price in Regions $\mathrm{X}^{\mathrm{c}}_{mh}$ and $\mathrm{X}^{\mathrm{c}}_{hm}$ and to an increase in second-period aggregate equilibrium sales and a decrease in second-period equilibrium price in Regions $\mathrm{X}^{\mathrm{c}}_{lm}$ and $\mathrm{X}^{\mathrm{c}}_{ml}$. It has no effect on second-period aggregate equilibrium sales and equilibrium price in the other regions.*

Proof. Let $S = S_1 + S_2$ be fixed and assume without loss of generality that $S_2 \geq S_1$. Then $|S_1 - S_2|$ increases if S_1 decreases. Since $S_2 \geq S_1$, it holds that $(S_1, S_2, \delta) \in \mathrm{X}^{\mathrm{nc}}_{\mathrm{ll}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{lm}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{lh}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{hh}}$. It holds that

$$\begin{array}{ll}
Q^{\mathrm{nc}}_{\mathrm{ll}} = S, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{ll}}}{\partial S_1} = 0, \\
Q^{\mathrm{nc}}_{\mathrm{lm}} = \frac{1-\delta+S_1+2\delta S}{2+2\delta}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{lm}}}{\partial S_1} = \frac{1}{2+2\delta} > 0, \\
Q^{\mathrm{nc}}_{\mathrm{lh}} = \frac{1}{2} + \frac{1}{2}S_1, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{lh}}}{\partial S_1} = \frac{1}{2} > 0, \\
Q^{\mathrm{nc}}_{\mathrm{mm}} = \frac{2-2\delta+3\delta S}{3+3\delta}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{mm}}}{\partial S_1} = 0, \\
Q^{\mathrm{nc}}_{\mathrm{mh}} = \frac{4+\delta+2\delta S_1}{6+4\delta}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{mh}}}{\partial S_1} = \frac{2\delta}{6+4\delta} > 0, \\
Q^{\mathrm{nc}}_{\mathrm{m'h}} = \frac{1}{2}S_1 + \frac{1}{3}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{m'h}}}{\partial S_1} = \frac{1}{2} > 0. \\
Q^{\mathrm{nc}}_{\mathrm{hh}} = \frac{2}{3}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{hh}}}{\partial S_1} = 0,
\end{array}$$

Between Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ there is no equilibrium. Consider an increase in S_1 together with a decrease of the same magnitude in S_2 that leads to a move from Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ to Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$. It holds that $Q^{\mathrm{nc}}_{\mathrm{mm}}$ coincides with $Q^{\mathrm{c}}_{\mathrm{mm}}$ and

$$Q^{\mathrm{nc}}_{\mathrm{mh}} = \frac{4+\delta+2\delta S_1}{6+4\delta} < \frac{4+2\delta+3\delta S_1}{6+6\delta} = Q^{\mathrm{c}}_{\mathrm{mh}}.$$

The desired result for this case now follows from Proposition 3.3.

Between Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ there is no equilibrium. Consider an increase in S_1 together with a decrease of the same magnitude in S_2 that leads to a move from Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ to Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$. Again, it holds that $Q^{\mathrm{nc}}_{\mathrm{mm}}$ coincides with $Q^{\mathrm{c}}_{\mathrm{mm}}$, Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ is a subset of Region $\mathrm{X}^{\mathrm{c}}_{\mathrm{mh}}$, and

$$Q^{\mathrm{nc}}_{\mathrm{m'h}} = \frac{1}{2}S_1 + \frac{1}{3} \leq \frac{4+2\delta+3\delta S_1}{6+6\delta} = Q^{\mathrm{c}}_{\mathrm{mh}},$$

where $S_1 \leq 2/3$ is used to derive the inequality sign. The desired result for this case now follows from Proposition 3.3.

For second-period aggregate sales we have that

$$\begin{array}{ll}
Q^{\mathrm{nc}}_{\mathrm{ll}} = 0, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{ll}}}{\partial S_1} = 0, \\
Q^{\mathrm{nc}}_{\mathrm{lm}} = \frac{2S-S_1-1+\delta}{2+2\delta}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{lm}}}{\partial S_1} = -\frac{1}{2+2\delta} < 0, \\
Q^{\mathrm{nc}}_{\mathrm{lh}} = \frac{1}{2}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{lh}}}{\partial S_1} = 0, \\
Q^{\mathrm{nc}}_{\mathrm{mm}} = \frac{3S-2+2\delta}{3+3\delta}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{mm}}}{\partial S_1} = 0, \\
Q^{\mathrm{nc}}_{\mathrm{mh}} = \frac{3S_1+2+3\delta}{6+4\delta}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{mh}}}{\partial S_1} = \frac{3}{6+4\delta} > 0, \\
Q^{\mathrm{nc}}_{\mathrm{m'h}} = \frac{2}{3}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{m'h}}}{\partial S_1} = 0, \\
Q^{\mathrm{nc}}_{\mathrm{hh}} = \frac{2}{3}, & \frac{\partial Q^{\mathrm{nc}}_{\mathrm{hh}}}{\partial S_1} = 0.
\end{array}$$

■

Between Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ there is no equilibrium. For this region, the consequences of increasing disparity of initial stock on second-period aggregate sales haven't been discussed yet. Consider an increase in S_1 together with a decrease of the same magnitude in S_2 that leads to a move from Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ to Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$. The second-period equilibrium sales may both decrease and increase, depending on the values of S_1, S_2, and δ. For instance, when $S_1 = 5/9$, $S_2 = 3/4$, and $\delta = 1$, we are in Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ and the aggregate second-period sales are equal to $2/3$. After an increase in S_1 accompanied by a decrease in S_2 of the same magnitude resulting in $S_1 = S_2 = 47/72$, we are in Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ and the aggregate second-period sales are equal to $47/72 < 2/3$. We now make the same calculations for a discount rate equal to 1/2. When $(S_1, S_2, \delta) = (5/9, 3/4, 1/2)$ we are in Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ and the aggregate second-period sales are equal to 13/24, whereas at $(S_1, S_2, \delta) = (47/72, 47/72, 1/2)$ we are in Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ and the aggregate second-period sales are equal to $35/54 > 13/24$.

There is also no equilibrium between Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$. An increase in S_1 together with a decrease of the same size in S_2 that leads from Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ to Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ will univocally lead to a decrease in second-period sales. Indeed, since in Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ we have $S_1 + S_2 \leq 4/3$, we have that

$$Q^{\mathrm{nc}}_{\mathrm{mm}} = \frac{3S - 2 + 2\delta}{3 + 3\delta} \leq \frac{2}{3} = Q^{\mathrm{nc}}_{\mathrm{m'h}}.$$

Analogous results hold for comparative statics involving Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm}}$, and Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm'}}$.

We next evaluate the effect of an increase in stock on consumer surplus. We use the same measure for consumer surplus as before.

Proposition 4.7 *An increase in S_i, ceterus paribus, leads to a weak increase in equilibrium consumer surplus, as long as the increase doesn't change the equilibrium outcome region. For some combinations of S_i, S_j and δ, a strict increase in S_i does change the equilibrium outcome region. This can lead to a strict decrease in equilibrium consumer surplus.*

Proof. It follows directly from the equilibrium outcomes that per-period sales in every equilibrium outcome region weakly increase in S_1 and S_2. However, take δ, S_i, S_i' and S_j as defined in the proof of Proposition 4.5. Equilibrium consumer surplus for δ, S_i, S_j is

$$CS^{\mathrm{nc}}_{\mathrm{mm}} \approx 0.25818.$$

An increase from S_i to S_i' results in equilibrium consumer surplus of

$$CS^{\mathrm{nc}}_{\mathrm{mh}} \approx 0.25600.$$

■

That is, just like the firms, consumers usually gain from an increase in stock. There are settings in which consumers are better off if a firm does not increase its stock. However, this can only happen if, for some stock levels in between the old and new stock level of the firm, ceterus paribus, an equilibrium doesn't exist.

The influence of increasing difference in stock level on consumer surplus is given in the following proposition.

Proposition 4.8 *Given fixed aggregate stock $S_1 + S_2$, an increase in $|S_1 - S_2|$ leads to a weak decrease in equilibrium consumer surplus.*

Proof. By Proposition 3.5, this proposition holds for $|S_1 - S_2|$, as long as $(S_1, S_2, \delta) \notin \mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{hm}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{hm'}}$. Assume, without loss of generality, that $S_2 \geq S_1$, so $|S_1 - S_2|$ increases if S_1 decreases. Proposition 4.6 implies a weak decrease in sales in both periods when $|S_1 - S_2|$ increases, and therefore a weak decrease in consumer surplus, for Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$. The remaining cases to check are those where a decrease in S_1 changes the equilibrium outcome from a point in $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ to a point in $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ or from $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$ to $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$.

Let $c = S_1 + S_2$, where c is a constant. Consumer surplus in Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ and Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ is

$$\begin{aligned} CS^{\mathrm{nc}}_{\mathrm{mh}} &= \frac{1}{2}(\frac{4+\delta+2\delta S_1}{6+4\delta})^2 + \frac{1}{2}\delta(\frac{3S_1+2+3\delta}{6+4\delta})^2, \\ CS^{\mathrm{nc}}_{\mathrm{m'h}} &= \frac{1}{2}(\frac{1}{2}S_1 + \frac{1}{3})^2 + \frac{2}{9}\delta, \end{aligned}$$

and in Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$

$$CS^{\mathrm{nc}}_{\mathrm{mm}} = \frac{1}{2}(\frac{2-2\delta+3\delta c}{3+3\delta})^2 + \frac{1}{2}\delta(\frac{3c-2+2\delta}{3+3\delta})^2.$$

As mentioned, consumer surplus in Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ increases with S_1, for fixed c. In Region $CS^{\mathrm{nc}}_{\mathrm{mm}}$, consumer surplus doesn't change if S_1 changes, for fixed c. This implies that, if $CS^{\mathrm{nc}}_{\mathrm{mh}} \leq CS^{\mathrm{nc}}_{\mathrm{mm}}$ for any $(S_1, S_2, \delta) \in \{(S_1, S_2, \delta) \mid S_2 = \beta_5, \frac{1}{3}(1-\delta) \leq S_1 \leq \frac{2}{3} - \frac{1}{9}\delta\}$, consumer surplus decreases when an increase in S_1 changes the equilibrium outcome from period $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ to $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$. Calculations indeed show that $CS^{\mathrm{nc}}_{\mathrm{mh}} \leq CS^{\mathrm{nc}}_{\mathrm{mm}}$ for these values of (S_1, S_2, δ). It also implies that, if $CS^{\mathrm{nc}}_{\mathrm{m'h}} \leq CS^{\mathrm{nc}}_{\mathrm{mm}}$ for any $(S_1, S_2, \delta) \in \{(S_1, S_2, \delta) \mid S_2 = \beta_7, \frac{2}{3} - \frac{1}{9}\delta < S_1 \leq \frac{2}{3}\}$, consumer surplus decreases when an increase in S_1changes the equilibrium outcome from period $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ to $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$. ■

The last part of this section is, again, devoted to total surplus. We have already seen that an increase in a firm's stock can lead to a decrease in its equilibrium profit and in consumer surplus. It will not come as a surprise that, with some extra calculations, the same type of results can be found for total surplus. If an increase in stock of one of the firms doesn't change the equilibrium outcome region, equilibrium total surplus increases with this increase in stock. If an increase in stock of one of the firms does change the equilibrium outcome region, for some combinations of variables, this leads to a decrease in total surplus. And, given fixed aggregate stock $S_1 + S_2$, an increase in $|S_1 - S_2|$ leads to a weak decrease in equilibrium total surplus.

Summary of comparative statics results for the non-commitment case

In this section we have found that there is at most one subgame perfect Nash equilibrium for each combination of S_i, S_j and δ. In contrast to the commitment situation, in the case without commitment it is possible that the equilibrium price decreases over time and that a firm's profit increases when it produces less. Increasing disparity in firm size leads to higher first-period equilibrium prices and lower sales, but has ambiguous effects on second-period equilibrium prices. Within every equilibrium outcome region, an increase in some firm's production level leads to an increase in it's profit, an increase in consumer surplus and an increase in total surplus. However, there are situations in which an increase in some firm's production level can lead to a decrease in its profits, a decrease in consumer surplus and/or a decrease in total surplus.

5 Commitment versus Non-commitment

In this section, we analyze how the equilibrium outcomes of the commitment setting are related to the equilibrium outcomes of the non-commitment case.

For certain regions, as was mentioned before, the equilibrium outcomes coincide. Notice that the equilibrium outcome corresponds to equilibrium sales by the two firms in both periods.

Proposition 5.1 *For every* $(S_1, S_2, \delta) \in \mathrm{X}^{\mathrm{nc}}_{ll} \cup \mathrm{X}^{\mathrm{nc}}_{lm} \cup \mathrm{X}^{\mathrm{nc}}_{ml} \cup \mathrm{X}^{\mathrm{nc}}_{lh} \cup \mathrm{X}^{\mathrm{nc}}_{hl} \cup \mathrm{X}^{\mathrm{nc}}_{mm} \cup \mathrm{X}^{\mathrm{nc}}_{hh}$, *the equilibrium sales in the non-commitment case coincide with those of the commitment setting.*

Proof. From the constraints defining the various regions it follows that $\mathrm{X}^{\mathrm{nc}}_{\mathrm{ll}} \subseteq \mathrm{X}^{\mathrm{c}}_{\mathrm{ll}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{lm}} \subseteq \mathrm{X}^{\mathrm{c}}_{\mathrm{lm}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{ml}} \subseteq \mathrm{X}^{\mathrm{c}}_{\mathrm{ml}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{lh}} \subseteq \mathrm{X}^{\mathrm{c}}_{\mathrm{lh}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hl}} \subseteq \mathrm{X}^{\mathrm{c}}_{\mathrm{hl}}, \mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}} \subseteq \mathrm{X}^{\mathrm{c}}_{\mathrm{mm}}$, and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hh}} \subseteq \mathrm{X}^{\mathrm{c}}_{\mathrm{hh}}$. The equilibrium sales in Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{ll}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{lm}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{ml}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{lh}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hl}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}}$, and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hh}}$ coincide with the equilibrium sales in Region $\mathrm{X}^{\mathrm{c}}_{\mathrm{ll}}$, $\mathrm{X}^{\mathrm{c}}_{\mathrm{lm}}$, $\mathrm{X}^{\mathrm{c}}_{\mathrm{ml}}$, $\mathrm{X}^{\mathrm{c}}_{\mathrm{lh}}$, $\mathrm{X}^{\mathrm{c}}_{\mathrm{hl}}$, $\mathrm{X}^{\mathrm{c}}_{\mathrm{mm}}$, and $\mathrm{X}^{\mathrm{c}}_{\mathrm{hh}}$, respectively. ■

An equilibrium always exists when $S_1 = S_2$. Since, for these production levels, $(S_1, S_1, \delta) \in \mathrm{X}^{\mathrm{nc}}_{\mathrm{ll}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{mm}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{hh}}$ when there is no commitment and $(S_1, S_1, \delta) \in \mathrm{X}^{\mathrm{c}}_{\mathrm{ll}} \cup \mathrm{X}^{\mathrm{c}}_{\mathrm{mm}} \cup \mathrm{X}^{\mathrm{c}}_{\mathrm{hh}}$ when there is commitment, the following corollary follows.

Corollary 5.2 *When firms* 1 *and* 2 *are symmetric, the equilibrium sales in the commitment case coincide with those of the non-commitment setting.*

The equivalence in equilibrium outcomes no longer holds when $(S_i, S_j, \delta) \in \mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{hm}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}} \cup \mathrm{X}^{\mathrm{nc}}_{\mathrm{hm'}} \cup \mathrm{X}^{nc}_{\emptyset}$. In these cases, there is one firm of intermediate size, and one firm that can react almost unrestrictedly to the quantities of its competitor. In the following we refer to these firms as the intermediate firm and the large firm, respectively. We show that the large firm gains and the intermediate firm loses from being in the non-commitment case, whenever we are not in Region $\mathrm{X}^{nc}_{\emptyset}$, i.e. whenever a subgame perfect Nash equilibrium exists in the non-commitment case.

Proposition 5.3 *For every* (S_1, S_2, δ) *outside Region* $\mathrm{X}^{nc}_{\emptyset}$, *the change in equilibrium outcome from the commitment case to the non-commitment case is to the advantage of the larger firm and to the disadvantage of the smaller firm.*

Proof. Assume without loss of generality that $S_2 \geq S_1$. Whenever there is a change in the equilibrium outcome, it holds that (S_1, S_2, δ) belongs to Region $\mathrm{X}^{\mathrm{c}}_{\mathrm{mh}}$. Firm 2 has profits equal to

$$\Pi^{\mathrm{c}}_2 = (\frac{2 + 4\delta - 3\delta S_1}{6 + 6\delta})^2 + \delta(\frac{4 + 2\delta - 3S_1}{6 + 6\delta})^2.$$

It also holds that (S_1, S_2, δ) belongs to Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ or Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$. In Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$, firm 2 has profits equal to

$$\Pi^{\mathrm{nc}}_2 = (\frac{2 + 3\delta - 2\delta S_1}{6 + 4\delta})^2 + \delta(\frac{4 + \delta - 3S_1}{6 + 4\delta})^2.$$

In Region $X^{nc}_{m'h}$ the profits of firm 2 are equal to

$$\Pi^{nc}_2 = \frac{4}{9} + \frac{1}{9}\delta - \frac{2}{3}S_1 + \frac{1}{4}(S_1)^2.$$

We have that $S_1 \leq \frac{2}{3}$ in all these regions, from which it follows that $\Pi^{nc}_2 \geq \Pi^c_2$.

Analogous calculations show the opposite relation for the profits of firm 1. ■

The intuition for this proposition follows from the same line of reasoning as that of Proposition 4.4. The total quantity sold is, in both the settings, the same for each firm. The small firm sells all of its stock in two periods, whereas the large firm reacts per-period optimal. The small firm sells more of its stock in the first period commitment setting than in the first period non-commitment setting and for the large firm it is the other way around. Price is higher in the first period non-commitment setting than in the first period commitment setting (see Proposition 5.4). For the second period, it is the other way around again. Therefore, the large firm makes more profit and the small firm makes less profit in the non-commitment setting, compared to the commitment setting. The small firm cannot change this by selling more of its stock in the first period, since this will induce the large firm to sell extra in the second period, thereby making this deviation unprofitable.

The following proposition describes the consequences of commitment for equilibrium prices and sales.

Proposition 5.4 *For every (S_1, S_2, δ) outside Region $X^{nc}_{\emptyset}$, the first-period equilibrium price in the non-commitment case is greater than or equal to the first-period equilibrium price in the commitment case and the second-period equilibrium price in the non-commitment case is less than or equal to the second-period equilibrium price in the commitment setting. The opposite relationships hold for aggregate sales in the two periods.*

Proof. We assume without loss of generality that $S_2 \geq S_1$. Whenever there is a change in the equilibrium price, (S_1, S_2, δ) belongs to Region X^c_{mh}. In Region X^c_{mh}, prices in the first and second period are respectively

$$p^c_1 = \frac{2 + 4\delta - 3\delta S_1}{6 + 6\delta} \text{ and } p^c_2 = \frac{4 + 2\delta - 3S_1}{6 + 6\delta}.$$

It also holds that (S_1, S_2, δ) belongs to Region X^{nc}_{mh} or $X^{nc}_{m'h}$. In Region X^{nc}_{mh}, prices are

$$p^{nc}_1 = \frac{2 + 3\delta - 2\delta S_1}{6 + 4\delta} \text{ and } p^{nc}_2 = \frac{4 + \delta - 3S_1}{6 + 4\delta}$$

and in Region $X^{nc}_{m'h}$, prices are

$$p^{nc}_1 = \frac{2}{3} - \frac{1}{2}S_1 \text{ and } p^{nc}_2 = \frac{1}{3}.$$

In Region X^c_{mh} it holds that $(1 - \delta)/3 < S_1 \leq \frac{2}{3}$, from which it follows that $p^c_1 \leq p^{nc}_1$ and $p^c_2 \geq p^{nc}_2$. ■

The equilibrium outcome in the commitment case does not always coincide with the equilibrium outcome in the non-commitment setting, in particular when there is one intermediate and one large firm. In these cases, it is the intermediate firm that would deviate if the commitment equilibrium quantities were chosen in the non-commitment setting. By transferring some of its quantity from the first to the second period, the intermediate firm could improve its profit, knowing that it forces the bigger firm to adjust its second-period quantity downwards. This opportunity to deviate profitably leads to the non-existence of an equilibrium in Region $\mathrm{X}^{\mathrm{nc}}_{\emptyset}$.

In Regions $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$, and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm'}}$, the profitable deviation of the intermediate firm results in a change in the equilibrium outcome. Perhaps surprisingly, the equilibrium outcomes change to the disadvantage of the intermediate firm. To avoid a deviation by the intermediate firm, in the non-commitment case the large firm sets a higher first-period quantity than in the commitment case. This increase in sales by the large firm is more than offset by lower first-period sales by the intermediate firm. The first-period equilibrium price is higher in the non-commitment case than in the commitment setting. The intermediate firm still sells all its production, leading to a strong increase in its second-period sales. The second-period equilibrium price is lower in the non-commitment case than in the commitment setting. The large firm reacts per-period optimal to the intermediate firms and has the same total sales as before. It follows that the profit for the intermediate firm is lower in the non-commitment setting than in the commitment setting, whilst it is the other way around for the large firm.

Regarding consumer surplus, we mention the following. One may expect the ability to commit to lead to less competition in the commitment setting than in the case without commitment. However, this only holds for some settings in which future profits are hardly discounted. The non-commitment setting gives the large firm more opportunity to use its power, which, as a result, increases the first period price and decreases the second period price. Due to discounting, in most cases this results in consumer surplus being lower in the non-commitment setting than in the case with commitment.

Proposition 5.5 *For every (S_1, S_2, δ), such that $(S_1, S_2, \delta) \notin \mathrm{X}^{\mathrm{nc}}_{\emptyset}$ and $\delta \leq \frac{24}{25}$, consumers prefer the commitment setting over the non-commitment setting. If $\delta > \frac{24}{25}$, there are combinations of (S_1, S_2, δ) for which consumers prefer the non-commitment setting.*

Proof. We assume, without loss of generality, that $S_2 \geq S_1$. Whenever there is a change in consumer surplus between the settings, (S_1, S_2, δ) belongs to Region $\mathrm{X}^{\mathrm{c}}_{\mathrm{mh}}$, and to Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ or $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$. Consumer surplus in Regions $\mathrm{X}^{\mathrm{c}}_{\mathrm{mh}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ is respectively

$$\begin{aligned}
\mathrm{CS}^{\mathrm{c}}_{\mathrm{mh}} &= (\frac{4+2\delta+3\delta S_1}{6+6\delta})^2 + \delta(\frac{3S_1+2+4\delta}{6+6\delta})^2, \\
\mathrm{CS}^{\mathrm{nc}}_{\mathrm{mh}} &= (\frac{4+\delta+2\delta S_1}{6+4\delta})^2 + \delta(\frac{3S_1+2+3\delta}{6+4\delta})^2, \\
\mathrm{CS}^{\mathrm{nc}}_{\mathrm{m'h}} &= (\frac{1}{2}S_1+\frac{1}{3})^2 + \frac{4}{9}\delta.
\end{aligned}$$

For Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$, it holds respectively that $\frac{1}{3}(1-\delta) \leq S_i \leq \frac{2}{3} - \frac{1}{9}\delta$ and $\frac{2}{3} - \frac{1}{9}\delta < S_i \leq \frac{2}{3}$. Now,

$$\mathrm{CS}^{\mathrm{c}}_{\mathrm{mh}} \geq \mathrm{CS}^{\mathrm{nc}}_{\mathrm{mh}} \text{ for } S_i \in [\frac{1}{3}(1-\delta), \frac{24-7\delta-17\delta^2}{3\delta}]$$

and

$$\mathrm{CS}^{\mathrm{c}}_{\mathrm{mh}} \geq \mathrm{CS}^{\mathrm{nc}}_{\mathrm{m'h}} \text{ for } S_i \in [-2+\frac{8}{3}\delta, \frac{2}{3}].$$

It holds that

$$\frac{24-7\delta-17\delta^2}{3\delta} \geq \frac{2}{3}-\frac{1}{9}\delta$$

iff $\delta \leq \frac{24}{25}$ and

$$-2+\frac{8}{3}\delta \leq \frac{2}{3}-\frac{1}{9}\delta$$

iff $\delta \leq \frac{24}{25}$. So for $\delta \leq \frac{24}{25}$, consumers prefer the commitment setting over the non-commitment setting. ■

The smaller firm prefers the commitment setting, the larger firm prefers the non-commitment setting and consumers prefer in most situations the commitment setting. Which setting then maximizes total surplus is the question yet to answer.

Proposition 5.6 *Total surplus is higher in the commitment setting than in the non-commitment setting.*

Proof. We again assume w.l.o.g. that $S_2 \geq S_1$. Whenever there is a change in total surplus between the commitment and the non-commitment setting, (S_1, S_2, δ) belongs to Region $\mathrm{X}^{\mathrm{c}}_{\mathrm{mh}}$, and to Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ or $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$. Total surplus in Regions $\mathrm{X}^{\mathrm{c}}_{\mathrm{mh}}$, $\mathrm{X}^{\mathrm{nc}}_{\mathrm{mh}}$ and $\mathrm{X}^{\mathrm{nc}}_{\mathrm{m'h}}$ is respectively

$$\begin{aligned}
\mathrm{TS}^{\mathrm{c}}_{\mathrm{mh}} &= (\frac{4+2\delta+3\delta S_1}{6+6\delta})(1\text{-}\frac{1}{2}(\frac{4+2\delta+3\delta S_1}{6+6\delta}))+\delta(\frac{3S_1+2+4\delta}{6+6\delta})(1\text{-}\frac{1}{2}(\frac{3S_1+2+4\delta}{6+6\delta})),\\
\mathrm{TS}^{\mathrm{nc}}_{\mathrm{mh}} &= (\frac{4+\delta+2\delta S_1}{6+4\delta})(1\text{-}\frac{1}{2}(\frac{4+\delta+2\delta S_1}{6+4\delta}))+\delta(\frac{3S_1+2+3\delta}{6+4\delta})(1\text{-}\frac{1}{2}(\frac{3S_1+2+3\delta}{6+4\delta})),\\
\mathrm{TS}^{\mathrm{nc}}_{\mathrm{m'h}} &= (\frac{1}{2}S_1+\frac{1}{3})(1-\frac{1}{2}(\frac{1}{2}S_1+\frac{1}{3}))+\delta(\frac{2}{3})(1-\frac{1}{3}).
\end{aligned}$$

For Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm}}$ and Region $\mathrm{X}^{\mathrm{nc}}_{\mathrm{hm'}}$, it holds respectively that $\frac{1}{3}(1-\delta) \leq S_i \leq \frac{2}{3}-\frac{1}{9}\delta$ and $\frac{2}{3}-\frac{1}{9}\delta < S_i \leq \frac{2}{3}$. Now,

$$\mathrm{TS}^{\mathrm{nc}}_{\mathrm{mh}} \geq \mathrm{TS}^{\mathrm{c}}_{\mathrm{mh}} \text{ iff } S_1 \in [\frac{-12+5\delta+7\delta^2}{3\delta}, \frac{1}{3}(1-\delta)]$$

and

$$\mathrm{TS}^{\mathrm{nc}}_{\mathrm{m'h}} \geq \mathrm{TS}^{\mathrm{c}}_{\mathrm{mh}} \text{ iff } S_1 \in [\frac{2}{3}, 2-\frac{4}{3}\delta].$$

So, for $S_1 \in [\frac{1}{3}(1-\delta), \frac{2}{3}]$, total surplus is the highest in the commitment setting. ■

Summary of comparative statics results for the commitment versus the non-commitment case
The following can be said about the equilibrium outcome regions. There is no difference between the equilibrium outcomes in the commitment and the non-commitment case if the firms are of equal size. Non-commitment is preferred over commitment only by the larger of the two firms. When there is no commitment, the first-period equilibrium price is higher and the second-period equilibrium price is lower than in the case with commitment. Consumer surplus is in most cases highest in the commitment setting and total surplus is in all cases highest in the commitment setting.

6 Concluding Remarks

We have shown that whether firms can or cannot commit to their sales strategy influences prices, sales quantities, profits and surplus. Comparative statics in the case with commitment conform to standard intuition. In the non-commitment situation, however, a number of counterintuitive results were found. First, equilibria may fail to exist. Moreover, in equilibrium prices may decrease over time and higher stocks can lead to lower revenues from sales. Large firms benefit from the absence of commitment, contrary to small firms and, in most cases, consumers.

We have limited the analysis to the case where competition takes place during two periods. We expect our main results to be true in the multi-period setting as well, but we fail to have an analytically tractable model specification for that situation.

We have only analyzed the case where production has already taken place, and firms compete in sales strategies. Such an assumption is valid if the capacity choice is a long-run decision that is not altered easily, as is the case for instance in the example of airplane companies mentioned in the introduction. An extension of the model could be to make the production capacity choice of the firms endogenous, if, again, the tractability issues can be overcome.

Another issue that should be addressed in future research is to what extent the choice for quantity competition affects our outcomes. It is natural to address the questions of this paper for models of price competition. Also here, however, it is not easy to find a model specification that is sufficiently general but still analytically tractable.

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Appendix A The Reduced Best Response Correspondence

We derive the reduced best response correspondence of firm i for the non-commitment case. To keep the appendix within reasonable length, we have omitted the derivation of second-order conditions. In accordance with Proposition 4.1, we can restrict our analysis to best responses against $q_j \leq \frac{1}{2}$. We distinguish three cases:

(A) $q_j < S_j - \frac{1}{2}$,

(B) $S_j - \frac{1}{2} \leq q_j < S_j - \frac{1}{3}$,

(C) $S_j - \frac{1}{3} \leq q_j \leq S_j$.

These three cases correspond to the three cases of residual stock $T_j = S_j - q_j$ of firm j with qualitatively different second-period behavior of firm j.

(A) $q_j < S_j - \frac{1}{2}$

Using the reduced profit function (2), for $0 \leq q_i < S_i - \frac{1}{3}$, profit is given by $(\mathrm{Y_{hh}})$, and for $S_i - \frac{1}{3} \leq q_i \leq S_i$, profit is given by $(\mathrm{Y_{lh}})$. Taking the unrestricted first-order condition of the profit function in $(\mathrm{Y_{hh}})$ and $(\mathrm{Y_{lh}})$ and solving for q_i results in q_i^{hh} and q_i^{lh} given by

$$\begin{aligned} q_i^{\mathrm{hh}} &= \frac{1}{2} - \frac{1}{2}q_j, \\ q_i^{\mathrm{lh}} &= \frac{1 - q_j - \frac{1}{2}\delta + \delta S_i}{2+\delta}. \end{aligned}$$

It holds that $q_i^{\mathrm{hh}} \in [0, S_i - \frac{1}{3})$ if and only if $\frac{5}{6} - \frac{1}{2}q_j < S_i$. Similarly, it holds that $q_i^{\mathrm{lh}} \in [S_i - \frac{1}{3}, S_i]$ if and only if $\frac{1}{2} - \frac{1}{2}q_j - \frac{1}{4}\delta \leq S_i \leq \frac{5}{6} - \frac{1}{12}\delta - \frac{1}{2}q_j$. We therefore find that the reduced best response q_i^* of player 1 to q_j is given by

$$q_i^* = \begin{cases} S_i, & \text{if } 0 \leq S_i < \frac{1}{2} - \frac{1}{2}q_j - \frac{1}{4}\delta, \\ \frac{1-q_j-\frac{1}{2}\delta+\delta S_i}{2+\delta}, & \text{if } \frac{1}{2} - \frac{1}{2}q_j - \frac{1}{4}\delta \leq S_i \leq \frac{5}{6} - \frac{1}{12}\delta - \frac{1}{2}q_j, \\ S_i - \frac{1}{3}, & \text{if } \frac{5}{6} - \frac{1}{12}\delta - \frac{1}{2}q_j < S_i \leq \frac{5}{6} - \frac{1}{2}q_j, \\ \frac{1}{2} - \frac{1}{2}q_j, & \text{if } \frac{5}{6} - \frac{1}{2}q_j < S_i. \end{cases}$$

(B) $S_j - \frac{1}{2} \leq q_j < S_j - \frac{1}{3}$

It follows from the reduced profit function (2) that, for $0 \leq q_i < S_i - \frac{1}{3}$, profit is given by $(\mathrm{Y_{hh}})$, for $S_i - \frac{1}{3} \leq q_i < 2T_j - 1 + S_i$, profit is given by $(\mathrm{Y_{lh}})$, and for $2T_j - 1 + S_i \leq q_i \leq S_i$, profit is given by $(\mathrm{Y_{ll}})$. Taking the unrestricted first-order condition of the reduced profit function in $(\mathrm{Y_{hh}})$, $(\mathrm{Y_{lh}})$ and $(\mathrm{Y_{ll}})$ and solving for q_i results in $q_i^{\mathrm{hh}}, q_i^{\mathrm{lh}}$, and q_i^{ll} given by

$$\begin{aligned} q_i^{\mathrm{hh}} &= \frac{1}{2} - \frac{1}{2}q_j, \\ q_i^{\mathrm{lh}} &= \frac{1 - q_j - \frac{1}{2}\delta + \delta S_i}{2+\delta}, \\ q_i^{\mathrm{ll}} &= \frac{1 - q_j + 2\delta S_i - \delta + \delta T_j}{2+2\delta}. \end{aligned}$$

It holds that $q_i^{\mathrm{hh}} \in [0, S_i - \frac{1}{3})$ if and only if $\frac{5}{6} - \frac{1}{2}q_j < S_i$. Similarly, it holds that $q_i^{\mathrm{lh}} \in [\max\{0, S_i - \frac{1}{3}\}, 2T_j - 1 + S_i)$ if and only if $\underline{S}_i^{\mathrm{lh}} < S_i \leq \bar{S}_i^{\mathrm{lh}}$, where

$$\begin{aligned}\underline{S}_i^{\mathrm{lh}} &= \frac{3}{2} + \frac{1}{4}\delta - \frac{1}{2}q_j - (2+\delta)T_j, \\ \bar{S}_i^{\mathrm{lh}} &= \frac{5}{6} - \frac{1}{12}\delta - \frac{1}{2}q_j.\end{aligned}$$

The requirement $q_i^{\mathrm{lh}} \geq 0$ is not binding, since $q_j \leq \frac{1}{2}$ implies q_i^{lh} is positive. It holds that $q_i^{\mathrm{ll}} \in [\max\{0, 2T_j - 1 + S_i\}, S_i]$ if and only if $\max\{\underline{S}_i^{\mathrm{lla}}, \underline{S}_i^{\mathrm{llb}}\} \leq S_i \leq \bar{S}_i^{\mathrm{ll}}$, where

$$\begin{aligned}\underline{S}_i^{\mathrm{lla}} &= \frac{1}{2\delta}(q_j - 1 + \delta - \delta T_j), \\ \underline{S}_i^{\mathrm{llb}} &= \frac{1}{2}(1 - \delta - q_j + \delta T_j), \\ \bar{S}_i^{\mathrm{ll}} &= \frac{3}{2} + \frac{1}{2}\delta - \frac{1}{2}q_j - (2 + \frac{3}{2}\delta)T_j.\end{aligned}$$

Since $S_j - q_j \geq 1/3$, it holds that $\max\{\bar{S}_i^{\mathrm{lh}}, \bar{S}_i^{\mathrm{ll}}\} \leq 5/6 - q_j/2$. The intervals $[\underline{S}_i^{\mathrm{lh}}, \bar{S}_i^{\mathrm{lh}}]$ and $[\max\{\underline{S}_i^{\mathrm{lla}}, \underline{S}_i^{\mathrm{llb}}\}, \bar{S}_i^{\mathrm{ll}}]$ are overlapping. In particular, since $q_j \leq 1/2$, $T_j = S_j - q_j \leq 1/2$, and $\delta \leq 1$, it holds that $\max\{\underline{S}_i^{\mathrm{lla}}, \underline{S}_i^{\mathrm{llb}}\} \leq \underline{S}_i^{\mathrm{lh}} \leq \bar{S}_i^{\mathrm{ll}}$.

The reduced profit function of firm i has two local maxima if $\underline{S}_i^{\mathrm{lh}} \leq S_i \leq \min\{\bar{S}_i^{\mathrm{ll}}, \bar{S}_i^{\mathrm{lh}}\}$. Since $\bar{S}_i^{\mathrm{lh}} \leq \bar{S}_i^{\mathrm{ll}}$ if and only if $q_j \geq S_j - \frac{8+7\delta}{24+18\delta}$, the profit function has two local maxima if

$$S_j - \frac{1}{2} \leq q_j \leq S_j - \frac{8+7\delta}{24+18\delta} \text{ and } \underline{S}_i^{\mathrm{lh}} \leq S_i \leq \bar{S}_i^{\mathrm{ll}}$$

or

$$S_j - \frac{8+7\delta}{24+18\delta} \leq q_j \leq S_j - \frac{1}{3} \text{ and } \underline{S}_i^{\mathrm{lh}} \leq S_i \leq \bar{S}_i^{\mathrm{lh}}.$$

To find the global maximum, we compare the profits in both local maxima. The profits corresponding to q_i^{lh} and q_i^{ll} are respectively

$$\begin{aligned}\Pi_i^{\mathrm{lh}} &= \frac{4 + 4q_j^2 - 4\delta + 16\delta S_i + 8\delta S_i^2 + \delta^2 + -8q_j + 4\delta q_j - 8S_i\delta q_j}{16 + 8\delta}, \\ \Pi_i^{\mathrm{ll}} &= \frac{1\text{-}2\delta+\delta^2\text{-}2q_j+2\delta^2 q_j+q_j^2+2\delta q_j^2+\delta^2 q_j^2+8\delta S_i\text{-}4\delta S_i^2+2\delta S_j\text{-}2\delta^2 S_j\text{-}2\delta q_j S_j\text{-}2\delta^2 q_j S_j\text{-}4\delta S_i S_j+\delta^2 S_j^2}{4 + 4\delta}.\end{aligned}$$

It holds that $\Pi_i^{\mathrm{lh}} \geq \Pi_i^{\mathrm{ll}}$ if and only if $S_i \geq \tilde{S}_i$, where

$$\tilde{S}_i = 1 - \frac{1}{2}q_j - (1 + \frac{1}{2}\delta)T_j + (\frac{1}{2} - T_j)\sqrt{(1+\delta)(1 + \frac{1}{2}\delta)}.$$

Since $\tilde{S}_i > \underline{S}_i^{\mathrm{lh}}$ whenever $q_j \geq S_j - \frac{1}{2}$, q_i^{lh} maximizes profits for $\tilde{S}_i \leq S_i \leq \bar{S}_i^{\mathrm{lh}}$.

Since $\max\{\underline{S}_i^{\mathrm{lla}}, \underline{S}_i^{\mathrm{llb}}\} \leq \tilde{S}_i \leq \bar{S}_i^{\mathrm{ll}}$, q_i^{ll} maximizes profits for $\max\{\underline{S}_i^{\mathrm{lla}}, \underline{S}_i^{\mathrm{llb}}\} \leq S_i \leq \tilde{S}_i$.

When $\max\{\bar{S}_i^{\mathrm{ll}}(q_j), \bar{S}_i^{\mathrm{lh}}(q_j)\} < S_i \leq \frac{5}{6} - \frac{1}{2}q_j$ we have a boundary solution, and profit maximizing sales are given by $q_i^* = S_i - \frac{1}{3}$.

One possibility remains: $\max\{\bar{S}_i^{\mathrm{lh}}, \tilde{S}_i\} < S_i \leq \bar{S}_i^{\mathrm{ll}}$. In this case, the profit maximizing choice is either q_i^{ll} or q_i^*. We argue that q_i^{ll} maximizes profits, so for $\bar{S}_i^{\mathrm{lh}} \leq S_i \leq \bar{S}_i^{\mathrm{ll}}$, the best response of firm i is q_i^{ll}.

It holds that

$$\Pi_i^{\rm ll} \geq \Pi_i^* = (S_i - \frac{1}{3})(\frac{4}{3} - S_i - q_j) + \frac{1}{9}\delta$$

if and only if

$$S_i \in [\frac{5}{6} - \frac{1}{6}\delta - \frac{1}{2}q_j + \frac{1}{2}\delta T_j \pm \frac{1}{3}\sqrt{\delta(1+\delta)(3T_j - 1)}].$$

Since

$$[\bar{S}_i^{\rm lh}, \bar{S}_i^{\rm ll}] \subset [\frac{5}{6} - \frac{1}{6}\delta - \frac{1}{2}q_j + \frac{1}{2}\delta T_j \pm \frac{1}{3}\sqrt{\delta(1+\delta)(3T_j - 1)}]$$

for

$$q_j < S_j - \frac{4 + 5\delta - 4\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6\delta},$$

we have our desired conclusion.

Summarizing, the reduced best response q_i^* of player i against q_j for $S_j - \frac{1}{2} \leq q_j < S_j - \frac{1}{3}$ is given by

$$q_i^* = \begin{cases} 0, & \text{if } 0 \leq S_i < \underline{S}_i^{\rm lla}, \\ S_i, & \text{if } 0 \leq S_i < \underline{S}_i^{\rm llb}, \\ \frac{(1 - q_j + 2\delta S_i - \delta + \delta T_j)}{2 + 2\delta}, & \text{if } \max\{\underline{S}_i^{\rm lla}, \underline{S}_i^{\rm llb}\} \leq S_i \leq \tilde{S}_i, \\ \frac{1 - q_j - \frac{1}{2}\delta + \delta S_i}{2+\delta}, & \text{if } \tilde{S}_i \leq S_i \leq \overline{S}_i^{\rm lh}, \\ \frac{(1 - q_j + 2\delta S_i - \delta + \delta T_j)}{2 + 2\delta}, & \text{if } \max\{\tilde{S}_i, \overline{S}_i^{\rm lh}\} < S_i \leq \overline{S}_i^{\rm ll}, \\ S_i - \frac{1}{3} & \text{if } \max\{\overline{S}_i^{\rm lh}, \overline{S}_i^{\rm ll}\} < S_i \leq \frac{5}{6} - \frac{1}{2}q_j, \\ \frac{1}{2} - \frac{1}{2}q_j, & \text{if } \frac{5}{6} - \frac{1}{2}q_j < S_i, \end{cases}$$

where

$$\begin{aligned} \tilde{S}_i &= 1 - \frac{1}{2}q_j - (1 + \frac{1}{2}\delta)T_j + (\frac{1}{2} - T_j)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}, \\ \underline{S}_i^{\rm lh} &= \frac{3}{2} + \frac{1}{4}\delta - \frac{1}{2}q_j - (2+\delta)T_j, \\ \bar{S}_i^{\rm lh} &= \frac{5}{6} - \frac{1}{12}\delta - \frac{1}{2}q_j, \\ \underline{S}_i^{\rm lla} &= \frac{1}{2\delta}(q_j - 1 + \delta - \delta T_j), \\ \underline{S}_i^{\rm llb} &= \frac{1}{2}(1 - \delta - q_j + \delta T_j), \\ \bar{S}_i^{\rm ll} &= \frac{3}{2} + \frac{1}{2}\delta - \frac{1}{2}q_j - (2 + \frac{3}{2}\delta)T_j. \end{aligned}$$

(C) $S_j - \frac{1}{3} \leq q_j \leq S_j$

It follows from the reduced profit function (2) that, for $0 \leq q_i < S_i - \frac{1}{2} + \frac{1}{2}T_j$, profit is given by ($\text{Y}_{\rm hl}$), and for $S_i - \frac{1}{2} + \frac{1}{2}T_j \leq q_i \leq S_i$, profit is given by ($\text{Y}_{\rm ll}$). Taking the unrestricted first-order condition of the profit function in (Y_{hl}) and (Y_{ll}) and solving for q_i results in q_i^{hl} and q_i^{ll} given by

$$\begin{aligned} q_i^{\mathrm{hl}} &= \frac{1}{2} - \frac{1}{2}q_j, \\ q_i^{\mathrm{ll}} &= \frac{1 - q_j + 2\delta S_i - \delta + \delta T_j}{2 + 2\delta}. \end{aligned}$$

It holds that $q_i^{\mathrm{hl}} \in [0, S_i - \frac{1}{2} + \frac{1}{2}T_j)$ if and only if $1 - \frac{1}{2}S_j < S_i$. Similarly, it holds that $q_i^{\mathrm{ll}} \in [S_i - \frac{1}{2} + \frac{1}{2}T_j, S_i]$ if and only if $\max\{\frac{1}{2}(1 - \delta - q_j + \delta T_j), \frac{1}{2\delta}(-1 + \delta + q_j - \delta T_j)\} \leq S_i \leq 1 - \frac{1}{2}S_j$. We therefore find that the reduced best response q_i^* of player 1 to q_j is given by

$$q_i^* = \begin{cases} S_i, & \text{if } 0 \leq S_i < \frac{1}{2}(1 - \delta - q_j + \delta T_j), \\ 0, & \text{if } 0 \leq S_i < \frac{1}{2\delta}(-1 + \delta + q_j - \delta T_j), \\ \frac{1 - q_j + 2\delta S_i - \delta + \delta T_j}{2 + 2\delta}, & \text{if } \max\{\frac{1}{2}(1 - \delta - q_j + \delta T_j), \frac{1}{2\delta}(-1 + \delta + q_j - \delta T_j)\} \\ & \leq S_i \leq 1 - \frac{1}{2}S_j, \\ \frac{1}{2} - \frac{1}{2}q_j, & \text{if } 1 - \frac{1}{2}S_j < S_i. \end{cases}$$

6.1 The reduced best response correspondence

Table 4 now follows immediately.

Appendix B Subgame Perfect Equilibria

We define the sets $A_j(1), \ldots, A_j(4)$, $B_j(1), \ldots, B_j(7)$, $C_j(1), \ldots, C_j(4)$ as the sets of quantities q_j satisfying the constraints as presented in Table 4. Notice that each of these sets is a subset of $[0, 1/2]$. Moreover, we define $A_j(k_1, \ldots, k_\ell) = A_j(k_1) \cup \cdots \cup A_j(k_\ell)$, and similarly for sets $B_j(k_1, \ldots, k_\ell)$ and $C_j(k_1, \ldots, k_\ell)$. In the proofs we will make use of Table 4. That table presents the reduced best response of firm i to a first-period sales quantity of firm j with the use of coefficients $\alpha_1, \ldots, \alpha_8$. In the sequel we will need the reduced best response of firm j to a first-period sales quantity of firm i, which follows from Table 4 by reversing the roles of firm i and j. The corresponding coefficients are denoted by $\beta_1, \ldots, \beta_8$.

Proposition B.1 If (q_i^*, q_j^*) is a Nash equilibrium of the reduced game and $q_j^* \in A_j(1,2,3) \cup B_j(1,2,3,4,5,6) \cup C_j(1)$, then $S_i - q_i^* \leq \frac{1}{3}$, so $q_i^* \in C_i(1,2,3,4)$.

Proof. For $q_j^* \in A_j(1,3) \cup B_j(1,6) \cup C_j(1)$ it follows immediately from Table 4 that $S_i - q_i^* \leq \frac{1}{3}$. For $q_j^* \in A_j(2) \cup B_j(4)$,

$$S_i - q_i^* = \frac{2S_i - 1 + q_j^* + \frac{1}{2}\delta}{2 + \delta} \leq \frac{\frac{2}{3} + \frac{1}{3}\delta}{2 + \delta} = \frac{1}{3},$$

where the inequality follows from $q_j^* \leq \alpha_2$. For $q_j^* \in B_j(2)$,

$$S_i - q_i^* < \frac{3}{4} - \frac{1}{4\delta} - \frac{1}{2}S_j < \frac{1}{3},$$

A_j $(q_j < S_j - \frac{1}{2})$	q_i^*	r_i^*
1) $0 \le q_j < \alpha_1$	S_i	0
2) $\alpha_1 \le q_j \le \alpha_2$	$\frac{1-q_j-\frac{1}{2}\delta+\delta S_i}{2+\delta}$	$\frac{2S_i-1+q_j+\frac{1}{2}\delta}{2+\delta}$
3) $\alpha_2 < q_j \le \alpha_3$	$S_i - \frac{1}{3}$	$\frac{1}{3}$
4) $q_j > \alpha_3$	$\frac{1}{2} - \frac{1}{2}q_j$	$\frac{1}{3}$

B_j $(S_j - \frac{1}{2} \le q_j < S_j - \frac{1}{3})$	q_i^*	r_i^*
1) $0 \le q_j < \alpha_4$	S_i	0
2) $q_j > \alpha_5$	0	S_i
3) $\max\{\alpha_4, \alpha_6\} \le q_j \le \alpha_5$	$\frac{1-q_j+2\delta S_i-\delta+\delta T_j}{2+2\delta}$	$\frac{2S_i-1+q_j+\delta-\delta T_j}{2+2\delta}$
4) $q_j \le \alpha_6, q_j \le \alpha_2$	$\frac{1-q_j-\frac{1}{2}\delta+\delta S_i}{2+\delta}$	$\frac{2S_i-1+q_j+\frac{1}{2}\delta}{2+\delta}$
5) $\alpha_2 < q_j < \alpha_6, q_j \ge \alpha_7$	$\frac{1-q_j+2\delta S_i-\delta+\delta T_j}{2+2\delta}$	$\frac{2S_i-1+q_j+\delta-\delta T_j}{2+2\delta}$
6) $\alpha_2 < q_j \le \alpha_3, q_j < \alpha_7$	$S_i - \frac{1}{3}$	$\frac{1}{3}$
7) $q_j > \alpha_3$	$\frac{1}{2} - \frac{1}{2}q_j$	$\frac{1}{3}$

C_j $(q_j \ge S_j - \frac{1}{3})$	q_i^*	r_i^*
1) $0 \le q_j < \alpha_4$	S_i	0
2) $q_j > \alpha_5$	0	S_i
3) $\alpha_4 \le q_j \le \alpha_5, S_i \le \alpha_8$	$\frac{1-q_j+2\delta S_i-\delta+\delta T_j}{2+2\delta}$	$\frac{2S_i-1+q_j+\delta-\delta T_j}{2+2\delta}$
4) $S_i > \alpha_8$	$\frac{1}{2} - \frac{1}{2}q_j$	$\frac{1}{2} - \frac{1}{2}S_j + \frac{1}{2}q_j$

	Explanation of the symbols
α_1	$1 - \frac{1}{2}\delta - 2S_i$
α_2	$\frac{5}{3} - \frac{1}{6}\delta - 2S_i$
α_3	$\frac{5}{3} - 2S_i$
α_4	$\frac{1-\delta+\delta S_j-2S_i}{1+\delta}$
α_5	$\frac{1-\delta+\delta S_j+2\delta S_i}{1+\delta}$
α_6	$\frac{2S_i-2+2S_j+\delta S_j-(1-2S_j)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta+2\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}$
α_7	$\frac{2S_i-3-\delta+4S_j+3\delta S_j}{3+3\delta}$
α_8	$1 - \frac{1}{2}S_j$

Table 4: Reduced best response correspondence $\sigma_i^{\mathrm{R}*}(q_j)$ for $0 \le q_j \le \frac{1}{2}$.

where the first inequality follows from $\alpha_5 < q_j^* \le \frac{1}{2}$ and the second one from $\delta \le 1$ and $S_j > q_j^* + \frac{1}{3} \ge \frac{1}{3}$. For $q_j^* \in B_j(3)$,

$$\begin{aligned} S_i - q_i^* &= \frac{2S_i - 1 + q_j^* + \delta - \delta T_j}{2(1+\delta)} \le \frac{(1 + \delta + \sqrt{(1+\delta)(1+\frac{1}{2}\delta)})(1 - 2T_j)}{2(1+\delta)} \\ &\le \frac{1 + \delta + \sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6(1+\delta)} \le \frac{1}{3}, \end{aligned}$$

where the first inequality follows from $q_j^* \ge \alpha_6$ (i.e. $S_i \le S_i^{\mathrm{c}}$), the second from $\frac{1}{3} < T_j \le \frac{1}{2}$ and the third one from $\delta \in (0,1]$. For $q_j^* \in B_j(5)$,

$$S_i - q_i^* = \frac{2S_i - 1 + q_j^* + \delta - \delta T_j}{2(1+\delta)} \le 1 - 2T_j \le \frac{1}{3},$$

where the first inequality follows from $q_j^* \le \alpha_7$, i.e. $S_i \le \overline{S}_i^{\mathrm{ll}}$, and the second one from $\frac{1}{3} < T_j \le \frac{1}{2}$. ■

Proposition B.2 If (q_i^*, q_j^*) is a Nash equilibrium of the reduced game and $q_j^* \in A_j(4) \cup B_j(7) \cup C_j(4)$, then $S_i - q_i^* > \frac{1}{3}$, so $q_i^* \in A_i(1,2,3,4) \cup B_i(1,2,3,4,5,6,7)$.

Proof. If $q_j^* \in A_j(4) \cup B_j(7)$, then since $q_j^* > \alpha_3$, we have $S_i > \frac{5}{6} - \frac{1}{2}q_j^*$, and $q_i^* = \frac{1}{2} - \frac{1}{2}q_j^*$. Therefore, $S_i - q_i^* > \frac{1}{3}$. If $q_j^* \in C_j(4)$, then $S_j - q_j^* \le \frac{1}{3}$, $S_i > 1 - \frac{1}{2}S_j$, and $q_i^* = \frac{1}{2} - \frac{1}{2}q_j^*$. This implies $S_i - q_i^* > \frac{1}{3}$. ■

We continue by solving for all Nash equilibria (q_i^*, q_j^*) of the reduced game where $q_j^* \in A_j(1)$. Next we consider Nash equilibria (q_i^*, q_j^*) with $q_j^* \in A_j(2)$. We restrict attention to the case with $q_i^* \notin A_i(1)$, since using the symmetry of the firms such equilibria follow already from the first case. We continue with $q_j^* \in A_j(3)$, and so on.

$\mathbf{q_j^* \in A_j(1)}$

It holds that

$$q_j^* < S_j - \frac{1}{2}, \tag{3}$$

$$q_j^* < 1 - \frac{1}{2}\delta - 2S_i, \tag{4}$$

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = S_i. \tag{5}$$

By Proposition B.1, $q_i^* \in C_i(1,2,3,4)$. This gives the following possibilities:

$q_i^* \in C_i(1): \quad q_j^* = S_j,$

$q_i^* \in C_i(2): \quad q_j^* = 0,$

$$q_i^* \in C_i(3): \quad q_j^* = \frac{1 - S_i + 2\delta S_j - \delta}{2 + 2\delta}, \tag{6}$$

$$q_i^* \in C_i(4): \quad q_j^* = \frac{1}{2} - \frac{1}{2}S_i. \tag{7}$$

If $q_i^* \in C_i(2)$, then $q_i^* > \beta_5$ implies $S_j < \frac{1}{2} - \frac{1}{2\delta} + \frac{1}{2\delta}S_i < \frac{1}{2}$ by (5) and Lemma 4.1, so (3) leads to a contradiction.

Next, (3) and (6) imply $S_j > 1 - \frac{1}{2}S_i$, whereas $q_i^* \in C_i(3)$ implies $q_i^* \leq \beta_8$, so $S_j \leq 1 - \frac{1}{2}S_i$, a contradiction.

When $q_i^* \in C_i(4)$, then $q_i^* \geq S_i - \frac{1}{3}$ and $S_j > \beta_8$. These inequalities together with the inequalities (3) and (4) lead to the conclusion that (q_j^*, q_i^*) is a Nash equilibrium with $q_j^* \in A_j(1)$ if and only if $q_j^* = \frac{1}{2} - \frac{1}{2}S_i$, $q_i^* = S_i$, $S_j > 1 - \frac{1}{2}S_i$, and $S_i < \frac{1}{3} - \frac{1}{3}\delta$.

$\mathbf{q_j^* \in A_j(2)}$

It holds that

$$q_j^* < S_j - \frac{1}{2}, \tag{8}$$

$$1 - \frac{1}{2}\delta - 2S_i \leq q_j^* \leq \frac{5}{3} - \frac{1}{6}\delta - 2S_i, \tag{9}$$

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = \frac{1 - q_j^* - \frac{1}{2}\delta + \delta S_i}{2 + \delta} \leq \frac{1}{2}.$$

By Proposition B.1, $q_i^* \in C_i(2, 3, 4)$.[4] This gives the following possibilities:

$$q_i^* \in C_i(2) : q_i^* = \frac{1 - \frac{1}{2}\delta + \delta S_i}{2 + \delta},\ q_j^* = 0, \tag{10}$$

$$q_i^* \in C_i(3) : q_i^* = \frac{1 + 2\delta - \delta^2 + \delta S_i + 2\delta^2 S_i - 2\delta S_j}{3 + 5\delta + 2\delta^2}, \tag{11}$$

$$q_j^* = \frac{2 - 3\delta - \delta^2 + 8\delta S_j + 4\delta^2 S_j + 2\delta S_i}{6 + 10\delta + 4\delta^2},$$

$$q_i^* \in C_i(4) : q_i^* = \frac{1 - \delta + 2\delta S_i}{3 + 2\delta},\ q_j^* = \frac{2 + 3\delta - 2\delta S_i}{6 + 4\delta}. \tag{12}$$

Consider $q_i^* \in C_i(2)$. Then $q_i^* > \beta_5$, so $S_j < \frac{-2+3\delta+\delta^2-2\delta S_i}{8\delta+4\delta^2} < \frac{1}{2}$, and (8) leads to a contradiction.

Consider $q_i^* \in C_i(3)$. It holds that

$$\frac{5 + 2\delta + \delta^2 + 2\delta S_i}{6 + 2\delta} < S_j \leq 1 - \frac{1}{2}S_i, \tag{13}$$

where the first inequality follows from (8) and (11), and the second inequality from $S_j \leq \beta_8$. By rewriting the expression in (13), it follows that $S_i < \frac{1}{3} - \frac{1}{3}\delta$.

However, this is contradicted by

$$S_i \geq \frac{1 + 2\delta - \delta^2 - 2\delta S_j}{3 + 4\delta} \geq \frac{1}{3} - \frac{1}{3}\delta,$$

where the first inequality follows from (9) and (11), and the second inequality from $S_j \leq \beta_8$.

Consider $q_i^* \in C_i(4)$. It is implied by (9) and (12) that

$$\frac{1}{3}(1 - \delta) \leq S_i \leq \frac{1}{3}(2 - \frac{1}{3}\delta).$$

[4] Note that, by Proposition B.1, $q_i^* \notin C_i(1)$. By Proposition B.1, if $q_i^* \in C_i(1)$, then $q_j^* \in C_j(1, 2, 3, 4)$.

From (8) and (12) it follows that $S_j > \frac{5+5\delta-2\delta S_i}{6+4\delta}$. In conclusion, (q_j^*, q_i^*) is a Nash equilibrium with $q_j^* \in A_j(2)$ if and only if $q_j^* = \frac{2+3\delta-2\delta S_i}{6+4\delta}$, $q_i^* = \frac{1-\delta+2\delta S_i}{3+2\delta}$, $\frac{1}{3}(1-\delta) \leq S_i \leq \frac{1}{3}(2-\frac{1}{3}\delta)$, and $S_j > \frac{5+5\delta-2\delta S_i}{6+4\delta}$.

$\mathbf{q_j^* \in A_j(3)}$

It holds that

$$q_j^* < S_j - \frac{1}{2}, \tag{14}$$

$$\frac{5}{3} - \frac{1}{6}\delta - 2S_i < q_j^* \leq \frac{5}{3} - 2S_i, \tag{15}$$

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = S_i - \frac{1}{3}.$$

By Proposition B.1, $q_i^* \in C_i(2,3,4)$. This gives the following possibilities:

$$q_i^* \in C_i(2) : q_j^* = 0,$$

$$q_i^* \in C_i(3) : q_j^* = \frac{\frac{4}{3} - \frac{2}{3}\delta + 2\delta S_j - S_i}{2+2\delta},$$

$$q_i^* \in C_i(4) : q_j^* = \frac{2}{3} - \frac{1}{2}S_i.$$

Consider $q_i^* \in C_i(2)$. Since $q_j^* = 0$, the second inequality in (15) implies $S_i \leq \frac{5}{6}$. We have that

$$\frac{1}{2} < S_j < \frac{S_i - \frac{4}{3} + \frac{2}{3}\delta}{2\delta}, \tag{16}$$

where the first inequality follows from (14) and the second from $q_i^* > \beta_5$. By rewriting the expression (16), we find that $S_i > \frac{4}{3} + \frac{1}{3}\delta$, contradicting $S_i \leq 5/6$.

Consider $q_i^* \in C_i(3)$. By (14), it should hold that

$$S_j > 1\frac{1}{6} + \frac{1}{6}\delta - \frac{1}{2}S_i,$$

which contradicts with $S_j \leq \beta_8$.

Consider $q_i^* \in C_i(4)$. It holds that

$$\frac{2}{3} - \frac{1}{9}\delta < S_i \leq \frac{2}{3},$$

where both inequalities follow from (15). From (14), it follows that

$$S_j > \frac{7}{6} - \frac{1}{2}S_i.$$

The other constraints are redundant. In conclusion, $q_j^* \in A_j(3)$ if and only if $q_i^* \in C_i(4), S_j > \frac{7}{6} - \frac{1}{2}S_i$ and $\frac{2}{3} - \frac{1}{9}\delta < S_i \leq \frac{2}{3}$.

$\mathbf{q_j^* \in A_j(4)}$

It holds that

$$\begin{aligned} q_j^* &< S_j - \frac{1}{2}, \\ q_j^* &> \frac{5}{3} - 2S_i. \end{aligned} \tag{17}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = \frac{1}{2} - \frac{1}{2}q_j \leq \frac{1}{2}.$$

By Proposition B.1 and Proposition B.2, $q_i^* \in A_i(4) \cup B_i(7)$.[5] This gives the following possibilities:

$$\begin{aligned} q_i^* &\in A_i(4) : q_i^* = q_j^* = \frac{1}{3}, \\ q_i^* &\in B_i(7) : q_i^* = q_j^* = \frac{1}{3}, \end{aligned}$$

Consider $q_i^* \in A_i(4) \cup B_i(7)$. It follows from (17) that

$$S_j > \frac{5}{6}.$$

For $q_i^* \in A_i(4)$, it follows from $q_i^* < S_i - \frac{1}{2}$ that $S_i > \frac{5}{6}$. Next, if $q_i^* \in B_i(7)$, it follows from

$$S_i - \frac{1}{2} \leq q_i^* < S_j - \frac{1}{3}$$

that

$$\frac{2}{3} < S_i \leq \frac{5}{6}.$$

The other constraints are redundant. In conclusion, $q_j^* \in A_j(4)$ if and only if $q_i^* \in A_i(4) \cup B_i(7)$ and $S_j > \frac{5}{6}$, $S_i > \frac{2}{3}$.

$\mathbf{q_j^* \in B_j(1)}$

It holds that

$$S_j - \frac{1}{2} \leq q_j^* < S_j - \frac{1}{3}, \tag{18}$$

$$q_j^* < \frac{1 - \delta + \delta S_j - 2S_i}{1 + \delta}. \tag{19}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = S_i \leq \frac{1}{2}.$$

[5] Note that Proposition B.1 excludes that $q_i^* \in A_i(1,2,3) \cup B_i(1,2,3,4,5,6)$ and $q_j^* \in A_j(4)$.

By Proposition B.1, $q_i^* \in C_i(2,3,4)$. This gives the following possibilities:

$$\begin{aligned} q_i^* &\in& C_i(2) : q_j^* = 0, \\ q_i^* &\in& C_i(3) : q_j^* = \frac{1 - S_i + 2\delta S_j - \delta}{2 + 2\delta}, \\ q_i^* &\in& C_i(4) : q_j^* = \frac{1}{2} - \frac{1}{2} S_i. \end{aligned}$$

For $q_i^* \in C_i(2)$, it can be found that

$$S_j < \frac{1}{2\delta}(S_i - 1 + \delta) < \frac{1}{3\delta}(-1 + \delta) \leq 0,$$

where the first inequality follows from $q_i^* > \beta_5$, the second one from (19) and the last one from $\delta \leq 1$.

For $q_i^* \in C_i(3)$, (18) implies $\frac{5}{6} - \frac{1}{6}\delta - \frac{1}{2}S_i < S_j \leq 1 - \frac{1}{2}S_i$. By (19), $S_i < \frac{1}{3} - \frac{1}{3}\delta$. The other constraints are redundant.

Next, $q_i^* \in C_i(4)$ implies $S_j > 1 - \frac{1}{2}S_i$, whereas (18) implies $S_j \leq 1 - \frac{1}{2}S_i$, a contradiction.

In conclusion, $q_j^* \in B_j(1)$ if and only if $q_i^* \in C_i(3)$ and $\frac{5}{6} - \frac{1}{6}\delta - \frac{1}{2}S_i < S_j \leq 1 - \frac{1}{2}S_i, S_i < \frac{1}{3} - \frac{1}{3}\delta$.

$\mathbf{q_j^* \in B_j(2)}$

It holds that

$$\begin{aligned} S_j - \frac{1}{2} &\leq& q_j^* < S_j - \frac{1}{3}, &\quad (20) \\ q_j^* &>& \frac{1 - \delta + \delta S_j + 2\delta S_i}{1 + \delta}. &\quad (21) \end{aligned}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = 0.$$

By Proposition B.1, $q_i^* \in C_i(2,3,4)$. This gives the following possibilities:

$$\begin{aligned} q_i^* &\in& C_i(2) : q_j^* = 0, \\ q_i^* &\in& C_i(3) : q_j^* = \frac{1 + 2\delta S_j - \delta + \delta S_i}{2 + 2\delta}, \\ q_i^* &\in& C_i(4) : q_j^* = \frac{1}{2}. \end{aligned}$$

For $q_i^* \in C_i(2)$, from $q_i^* > \beta_5$ it follows that $S_j < \frac{1}{2\delta}(-1 + \delta - \delta S_i) \leq 0$.

Consider $q_i^* \in C_i(3)$. Inequality (20) implies $S_i < \frac{1}{3\delta}(-1 + \delta) \leq 0$.

If $q_i^* \in C_i(4)$, it holds again that

$$S_i < \frac{1}{4\delta}(-1 + 3\delta - 2\delta S_j) < \frac{1}{3\delta}(-1 + \delta) \leq 0,$$

where the first inequality follows from (21) and the second one from $S_j > \beta_8$.

In conclusion, $q_j^* \notin B_j(2)$.

$\mathbf{q_j^* \in B_j(3)}$

It holds that

$$S_j - \frac{1}{2} \leq q_j^* < S_j - \frac{1}{3}, \tag{22}$$

$$q_j \geq \max\{\frac{1-\delta+\delta S_j-2S_i}{1+\delta}, \frac{2S_i-2+2S_j+\delta S_j-(1-2S_j)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta+2\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}\}, \tag{23}$$

$$q_j \leq \frac{1-\delta+\delta S_j+2\delta S_i}{1+\delta}. \tag{24}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = \frac{1-q_j+2\delta S_i-\delta+\delta S_j-\delta q_j}{2+2\delta}.$$

By Proposition B.1, $q_i^* \in C_i(2,3,4)$. This gives the following possibilities:

$$q_i^* \in C_i(2) : q_i^* = \frac{1+2\delta S_i-\delta+\delta S_j}{2+2\delta}, q_j^* = 0,$$

$$q_i^* \in C_i(3) : q_i^* = \frac{1-\delta+3\delta S_i}{3+3\delta}, q_j^* = \frac{1-\delta+3\delta S_j}{3+3\delta},$$

$$q_i^* \in C_i(4) : q_i^* = \frac{1-3\delta+4\delta S_i+2\delta S_j}{3+3\delta},$$

$$q_j^* = \frac{1+3\delta-2\delta S_i-\delta S_j}{3+3\delta}.$$

For $q_i^* \in C_i(2)$, it follows from $q_i^* > \beta_5$ that $S_j < \frac{1}{3\delta}(-1+\delta) \leq 0$.

If $q_i^* \in C_i(3)$, it is implied by (23) that $\frac{1}{3}(1-\delta) \leq S_i \leq \frac{7}{6} - S_j - \frac{1}{6}\delta + \frac{\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta}(\frac{5}{6}+\frac{1}{6}\delta-S_j)$. It follows from (22) that $S_j > \frac{2}{3}$. The remaining constraints are redundant.

Consider $q_i^* \in C_i(4)$. By $S_j > \beta_8$ and (23) it follows that $2-2S_j < S_i \leq \frac{7+10\delta+3\delta^2-6S_j-10\delta S_j-4\delta^2 S_j+(5+9\delta-6S_j-8\delta S_j)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6+8\delta+2\delta^2+4\delta\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}$. Such an S_i only exists if $S_j > \frac{5+\delta}{6}$. From (22) it follows that $S_i \leq \frac{5+9\delta-6S_j-8\delta S_j}{4\delta}$. Now, there only exists an S_i such that $2-2S_j < S_i \leq \frac{5+9\delta-6S_j-8\delta S_j}{4\delta}$, if $S_j < \frac{5+\delta}{6}$, a contradiction.

In conclusion, $q_j^* \in B_j(3)$ if and only if $q_i^* \in C_i(3)$ and $\frac{1}{3}(1-\delta) \leq S_i \leq \frac{7}{6} - S_j - \frac{1}{6}\delta + \frac{\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta}(\frac{5}{6}+\frac{1}{6}\delta-S_j), S_j > \frac{2}{3}$.

$\mathbf{q_j^* \in B_j(4)}$

It holds that

$$S_j - \frac{1}{2} \leq q_j^* < S_j - \frac{1}{3}, \tag{25}$$

$$q_j < \frac{2S_i-2+2S_j+\delta S_j-(1-2S_j)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta+2\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}, \tag{26}$$

$$q_j \leq \frac{5}{3} - \frac{1}{6}\delta - 2S_i. \tag{27}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = \frac{1 - q_j - \frac{1}{2}\delta + \delta S_i}{2+\delta}.$$

By Proposition B.1, $q_i^* \in C_i(2,3,4)$. This gives the following possibilities:

$$\begin{aligned} q_i^* \in C_i(2) : q_i^* &= \frac{1 - \frac{1}{2}\delta + \delta S_i}{2+\delta}, q_j^* = 0, \\ q_i^* \in C_i(3) : q_i^* &= \frac{1 + 2\delta - \delta^2 + \delta S_i + 2\delta^2 S_i - 2\delta S_j}{3 + 5\delta + 2\delta^2}, \\ q_j^* &= \frac{2 - 3\delta - \delta^2 + 8\delta S_j + 4\delta^2 S_j + 2\delta S_i}{6 + 10\delta + 4\delta^2}, \\ q_i^* \in C_i(4) : q_i^* &= \frac{1 - \delta + 2\delta S_i}{3 + 2\delta}, q_j^* = \frac{2 + 3\delta - 2\delta S_i}{6 + 4\delta}. \end{aligned}$$

For $q_i^* \in C_i(2)$, $q_i > \beta_5$ and inequality (25) imply respectively that $\frac{1}{3} < S_j < \frac{-1+\frac{3}{2}\delta+\frac{1}{2}\delta^2-\delta S_i}{2\delta(2+\delta)}$. However, no such S_j exists, since this would imply that $S_i < \frac{1}{\delta}(-1+\frac{1}{6}\delta - \frac{1}{6}\delta^2) \leq 0$.

Consider $q_i^* \in C_i(3)$. It follows from (26) and $q_i < \beta_5$ that

$$\begin{aligned} &\frac{7+9\frac{1}{2}\delta+2\delta^2\text{-}\frac{1}{2}\delta^3\text{-}6S_j\text{-}9\delta S_j\text{-}3\delta^2 S_j+(5+2\delta+\delta^2\text{-}6S_j\text{-}2\delta S_j)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6 + 9\delta + 3\delta^2 - 2\delta\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}} \\ &< \quad S_i \leq 2 - 2S_j \end{aligned} \tag{28}$$

and from (25) and $q_i < \beta_5$ that

$$\frac{6S_j + 2\delta S_j - 5 - 2\delta - \delta^2}{2\delta} \leq S_i \leq 2 - 2S_j. \tag{29}$$

There exists an S_i such that (28) if and only if $S_j > \frac{5+\delta}{6}$ and there exists an S_i such that (29) holds if and only if $S_j \leq \frac{5+\delta}{6}$, a contradiction.

If $q_i^* \in C_i(4)$, inequality (27) implies $S_i \leq \frac{1}{3}(2 - \frac{1}{3}\delta)$. It follows from (26) and (25) respectively that $S_j > \frac{7+6\frac{1}{2}\delta+\frac{3}{2}\delta^2-6S_i-5\delta S_i-\delta^2 S_i+(5+5\delta-2\delta S_i)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6+7\delta+2\delta^2+(6+4\delta)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}$ and $S_j \leq \frac{5+5\delta-2\delta S_i}{6+4\delta}$. The other constraints are redundant.

In conclusion, $q_j^* \in B_j(4)$ if and only if $q_i^* \in C_i(4)$ and $S_i \leq \frac{1}{3}(2 - \frac{1}{3}\delta)$, $\frac{7+6\frac{1}{2}\delta+\frac{3}{2}\delta^2-6S_i-5\delta S_i-\delta^2 S_i+(5+5\delta-2\delta S_i)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{6+7\delta+2\delta^2+(6+4\delta)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}} < S_j \leq \frac{5+5\delta-2\delta S_i}{6+4\delta)}$.

$\mathbf{q_j^* \in B_j(5)}$

It holds that

$$S_j - \frac{1}{2} \quad \leq \quad q_j^* < S_j - \frac{1}{3}, \tag{30}$$

$$\frac{5}{3} - \frac{1}{6}\delta - 2S_i \quad < \quad q_j < \frac{2S_i - 2 + 2S_j + \delta S_j - (1 - 2S_j)\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1 + \delta + 2\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}, \tag{31}$$

$$q_j \quad \geq \quad \frac{2S_i - 3 - \delta + 4S_j + 3\delta S_j}{3 + 3\delta}. \tag{32}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = \frac{1 - q_j + 2\delta S_i - \delta + \delta S_j - \delta q_j}{2 + 2\delta}.$$

By Proposition B.1, $q_i^* \in C_i(2,3,4)$. This gives the following possibilities:

$$\begin{aligned}
q_i^* \in C_i(2) : q_i^* &= \frac{1 + 2\delta S_i - \delta + \delta S_j}{2 + 2\delta}, q_j^* = 0, \\
q_i^* \in C_i(3) : q_i^* &= \frac{1 - \delta + 3\delta S_i}{3 + 3\delta}, q_j^* = \frac{1 - \delta + 3\delta S_j}{3 + 3\delta}, \\
q_i^* \in C_i(4) : q_i^* &= \frac{1 - 3\delta + 4\delta S_i + 2\delta S_j}{3 + 3\delta}, \\
q_j^* &= \frac{1 + 3\delta - 2\delta S_i - \delta S_j}{3 + 3\delta}.
\end{aligned}$$

For $q_i^* \in C_i(2)$, it follows from $q_i > \beta_6$ that $S_j < \frac{1}{3\delta}(-1 + \delta) \leq 0$.

For $q_i^* \in C_i(3)$, it holds that $S_j \leq 1 - \frac{1}{2}S_i$. It is implied by (31) that $S_i > \max\{\frac{4+5\frac{1}{2}\delta-\frac{1}{2}\delta^2-3\delta S_j}{6+6\delta}, \frac{7}{6} - \frac{1}{6}\delta - S_j + \frac{\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta}(\frac{5}{6} + \frac{1}{6}\delta - S_j)\}$. From (30) it follows that $S_j > \frac{2}{3}$. The other constraints are redundant.

If $q_i^* \in C_i(4)$, it holds that $S_j > 1 - \frac{1}{2}S_i$. From (32) it follows that $S_j \leq 1 - \frac{1}{2}S_i$, a contradiction.

In conclusion, $q_j^* \in B_j(5)$ if and only if $q_i^* \in C_i(3)$ and $S_i > \max\{\frac{4+5\frac{1}{2}\delta-\frac{1}{2}\delta^2-3\delta S_j}{6+6\delta}, \frac{7}{6} - \frac{1}{6}\delta - S_j + \frac{\sqrt{(1+\delta)(1+\frac{1}{2}\delta)}}{1+\delta}(\frac{5}{6} + \frac{1}{6}\delta - S_j)\}$, $\frac{2}{3} < S_j \leq 1 - \frac{1}{2}S_i$.

$\mathbf{q_j^* \in B_j(6)}$

It holds that

$$\begin{aligned}
S_j - \frac{1}{2} &\leq q_j^* < S_j - \frac{1}{3}, && (33) \\
\frac{5}{3} - \frac{1}{6}\delta - 2S_i &< q_j \leq \frac{5}{3} - 2S_i, && (34) \\
q_j &< \frac{2S_i - 3 - \delta + 4S_j + 3\delta S_j}{3 + 3\delta}. && (35)
\end{aligned}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = S_i - \frac{1}{3}.$$

By Proposition B.1, $q_i^* \in C_i(2,3,4)$. This gives the following possibilities:

$$\begin{aligned}
q_i^* &\in C_i(2) : q_j^* = 0, && (36) \\
q_i^* &\in C_i(3) : q_j^* = \frac{\frac{4}{3} - \frac{2}{3}\delta + 2\delta S_j - S_i}{2 + 2\delta}, \\
q_i^* &\in C_i(4) : q_j^* = \frac{2}{3} - \frac{1}{2}S_i.
\end{aligned}$$

For $q_i^* \in C_i(2)$. It holds that

$$S_j \leq \frac{1}{2\delta}(S_i - \frac{4}{3} + \frac{2}{3}\delta) \leq \frac{1}{2\delta}(-\frac{1}{2} + \frac{2}{3}\delta) \leq \frac{1}{3},$$

where the first inequality follows from (36) and the second one from (34). This contradicts with (33).

Consider $q_i^* \in C_i(3)$.It holds that $S_j \leq 1 - \frac{1}{2}S_i$. From (33) it follows that $S_j > 1 - \frac{1}{2}S_i$, a contradiction.

Next, if $q_i^* \in C_i(4)$, it follows from (33) that $S_j \leq \frac{7}{6} - \frac{1}{2}S_i$. By (34) and by (35) it is implied respectively that $\frac{1}{3}(2 - \frac{1}{3}\delta) < S_i \leq \frac{2}{3}$ and $S_j > \frac{10+6\delta-7S_i-3\delta S_i}{8+6\delta}$. The other constraints are redundant.

In conclusion, $q_j^* \in B_j(6)$ if and only if $q_i^* \in C_i(4)$ and $\frac{1}{3}(2 - \frac{1}{3}\delta) < S_i \leq \frac{2}{3}$, $\frac{10+6\delta-7S_i-3\delta S_i}{8+6\delta} < S_j \leq \frac{7}{6} - \frac{1}{2}S_i$.

$\mathbf{q_j^* \in B_j(7)}$

It holds that

$$\begin{aligned} S_j - \frac{1}{2} &\leq q_j^* < S_j - \frac{1}{3}, \\ q_j^* &> \frac{5}{3} - 2S_i. \end{aligned} \tag{37}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = \frac{1}{2} - \frac{1}{2}q_j \leq \frac{1}{2}.$$

By Proposition B.1 and Proposition B.2, $q_i^* \in B_i(7)$. This gives the following possibilities:

$$q_i^* \in B_i(7) : q_i^* = q_j^* = \frac{1}{3}.$$

For $q_i^* \in B_j(7)$, it follows from $S_i - \frac{1}{2} \leq q_i^* < S_j - \frac{1}{2}$ that $\frac{2}{3} < S_i \leq \frac{5}{6}$. From (37) it follows that $\frac{2}{3} < S_j \leq \frac{5}{6}$. The rest of the constraints is redundant.

Therefore, $q_j^* \in B_j(7)$ if $q_i^* \in B_i(7)$ and $\frac{2}{3} < S_i \leq \frac{5}{6}, \frac{2}{3} < S_j \leq \frac{5}{6}$.

$\mathbf{q_j^* \in C_j(1)}$

It holds that

$$q_j^* \geq S_j - \frac{1}{3}, \tag{38}$$

$$q_j^* < \frac{1 - \delta + \delta S_j - 2S_i}{1+\delta}. \tag{39}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = S_i \leq \frac{1}{2}.$$

By Proposition B.1 and Proposition B.2, $q_i^* \in C_i(1, 2, 3)$. This gives the following possibilities:

$$\begin{aligned} q_i^* &\in C_i(1) : q_j^* = S_j, \\ q_i^* &\in C_i(2) : q_j^* = 0, \\ q_i^* &\in C_i(3) : q_j^* = \frac{1 - S_i + 2\delta S_j - \delta}{2 + 2\delta}. \end{aligned}$$

If $q_i^* \in C_i(1)$, then, by $q_i < \beta_4$, it holds that $S_j < \frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_i$. From (39), it follows that $S_i < \frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_j$.

Consider $q_i^* \in C_i(2)$. It holds that

$$S_j < \frac{1}{2\delta}(S_i - 1 + \delta) < \frac{1}{3\delta}(-1+\delta) \leq 0,$$

where the first inequality follows from $q_i > \beta_5$ and the second one from (39).

For $q_i^* \in C_i(3)$, it follows from $q_i^* \geq \beta_4$ that $S_j \geq \frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_i$. By (39), it is implied that $S_i < \frac{1}{3} - \frac{1}{3}\delta$. From (38), it follows that $S_j \leq \frac{5}{6} - \frac{1}{6}\delta - \frac{1}{2}S_i$. The rest of the constraints is redundant.

In conclusion, $q_j^* \in C_j(1)$ if and only if $q_i^* \in C_i(1)$ and $S_i < \frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_j, S_j < \frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_i$ or $q_i^* \in C_i(3)$ and $S_i < \frac{1}{3} - \frac{1}{3}\delta, \frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}S_i \leq S_j \leq \frac{5}{6} - \frac{1}{6}\delta - \frac{1}{2}S_i$.

$\mathbf{q_j^* \in C_j(2)}$

It holds that

$$\begin{aligned} q_j^* &\geq S_j - \frac{1}{3}, \\ q_j^* &> \frac{1 - \delta + \delta S_j + 2\delta S_i}{1+\delta}. \end{aligned} \tag{40}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = 0.$$

By Proposition B.2, $q_i^* \in C_i(2,3)$. This gives the following possibilities:

$$\begin{aligned} q_i^* &\in C_i(2) : q_j^* = 0, \\ q_i^* &\in C_i(3) : q_j^* = \frac{1 + 2\delta S_j - \delta + \delta S_i}{2 + 2\delta}. \end{aligned}$$

For $q_i^* \in C_i(2)$, it follows from (40) that $S_i < \frac{1}{2\delta}(-1 + \delta - \delta S_j) \leq 0$.

For $q_i^* \in C_i(3)$, it is implied by (40) that $S_i < \frac{1}{3\delta}(-1+\delta) \leq 0$.

In conclusion, $q_j^* \notin C_j(2)$ if $q_i^* \in C_i(2,3)$.

$\mathbf{q_j^* \in C_j(3)}$

It holds that

$$q_j^* \geq S_j - \frac{1}{3}, \tag{41}$$

$$\frac{1 - \delta + \delta S_j - 2S_i}{1+\delta} \leq q_j^* \leq \frac{1 - \delta + \delta S_j + 2\delta S_i}{1+\delta}, \tag{42}$$

$$S_i \leq 1 - \frac{1}{2}S_j. \tag{43}$$

We have

$$q_i^* = \sigma_i^{\mathrm{R}}(q_j^*) = \frac{1 - q_j + 2\delta S_i - \delta + \delta S_j - \delta q_j}{1+\delta}.$$

By Proposition B.2, $q_i^* \in C_i(3)$. This gives the following possibility:

$$q_i^* \in C_i(3) : q_i^* = \frac{1 - \delta + 3\delta S_i}{3 + 3\delta}, q_j^* = \frac{1 - \delta + 3\delta S_j}{3 + 3\delta}.$$

For $q_i^* \in C_i(3)$, it follows from $q_i^* \geq S_i - \frac{1}{3}$ and (41) respectively that $S_i \leq \frac{2}{3}$ and $S_j \leq \frac{2}{3}$.

Since $q_i^* \in C_i(3)$ and by (41), $S_i \leq \frac{2}{3}$ and $S_j \leq \frac{2}{3}$. Next, it follows from $q_i^* \geq \beta_4$ that $S_j \geq \frac{1}{3} - \frac{1}{3}\delta$. By (42), $S_i \geq \frac{1}{3} - \frac{1}{3}\delta$. The remaining constraints are redundant.

In conclusion, $q_j^* \in C_j(3)$ if $q_i^* \in C_i(3)$ and $\frac{1}{3}(1 - \delta) \leq S_i \leq \frac{2}{3}, \frac{1}{3}(1 - \delta) \leq S_j \leq \frac{2}{3}$.

$\mathbf{q_j^* \in C_j(4)}$

This case does not need to be calculated here, since, by proposition B.2, it can only be combined with the situations A_i and B_i, and all these situations are already calculated.